Separability of the Dirac equation in a class of perfect fluid space-times with local rotational symmetry

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(Received 4 June 1984; accepted for publication 12 October 1984)

Chandrasekhar's technique for separation of the Dirac equation in the Kerr background is applied to perfect fluid space-times with local rotational symmetry. These space-times fall into three distinct types. It is found that in case (1) the Dirac equation separates if the space-time is at least "locally static" while in case (3) it separates if the space-time is at least "locally diagonal," in contrast to the massless case where Dhurandhar, Vishveshwara, and Cohen showed that the Hertz potential is separable in all cases. In case (2), however, the Dirac equation is separable in all those cases where the Hertz potential for neutrinos is separable.

I. INTRODUCTION

In a series of papers Dhurandhar, Vishveshwara, and Cohen[1] have been systematically studying massless perturbations of varying spins in perfect fluid space-times with local rotational symmetry. These space-times first discussed by Ellis[2] and Ellis and Stewart[3] form a subclass of the generalized Goldberg–Sachs class. This class includes a wide spectrum of interesting space-times like Friedmann, Godel, Kantowski–Sachs universes, Taub–NUT, anisotropic cosmological models, etc. The above investigation used the Hertz–Debye formalism developed in detail for curved space-times by Cohen and Kegeles.[4]

The above treatment for neutrinos is in need of modification if the neutrinos turn out to be massive.[5] As a preliminary to the study of massive spin half-perturbations, it is interesting to ask whether in the above background space-times wherein the Hertz potential for every massless spin field—in particular the neutrino—is separable,the massive Dirac equation is also separable. This motivates us to look into the question of whether the Chandrasekhar separation for the Dirac equation in Kerr background[6] can be extended to the above class of space-times. In the next section, we write down the general form of the background metric for perfect fluid space-times with local rotational symmetry and collect together geometrical details of relevance. In Sec. III the Dirac equation is written down in the Newman–Penrose[7] spinor form. In Sec. IV, we show that Chandrasekhar's method separates the Dirac equation in a certain subclass of space-times. In the last section we discuss this subclass and obtain decoupled equations for the angular and radial parts. Finally we also briefly discuss the behavior of the angular and radial functions.

II. PERFECT FLUID SPACE-TIMES (PFST) WITH LOCAL ROTATIONAL SYMMETRY (LRS)

Perfect fluid space-times with local rotational symmetry are described by the line element[2,3]

$$ds^2 = (dx^0)^2/F^2 - X^2(dx^1)^2 - Y^2[(dx^2)^2 + t^2(dx^3)^2] - (y/F^2)(2dx^0 - y\,dx^3)dx^3 + hX^2(2dx^1 - h\,dx^3)dx^3, \quad (1)$$

where F, X, Y are functions of x^0 and x^1 and t, y, h are functions of x^2 only. The space-times fall into three different classes as follows:

$$\text{(i)}\quad X=1,\quad Y=Y(x^1),\quad F=F(x^1),\quad h=0. \quad (2a)$$

$$\text{(ii)}\quad h=y=0. \quad (2b)$$

$$\text{(iii)}\quad F=1,\quad X=X(x^0),\quad Y=Y(x^0),\quad y=0. \quad (2c)$$

It is to be noted that t can take one of the following functional forms:

$$\text{(a) } t=\sin(x^2),\quad \text{(b) } t=\sinh(x^2),\quad \text{(c) } t=x^2,\quad \text{(d) } t=\text{const}, \quad (3)$$

while h and y are obtained from t by the relation

$$h_{,2}=ct,\quad y_{,2}=c^1t, \quad (4)$$

where c and c^1 are constants.

A convenient null tetrad for these space-times has been given by Wainwright[8] as

$$k^a=\frac{1}{\sqrt{2}}\left(F,\frac{1}{X},0,0\right), \quad (5a)$$

$$n^a=\frac{1}{\sqrt{2}}\left(F,\frac{-1}{X},0,0\right), \quad (5b)$$

$$m^a=\frac{1}{\sqrt{2}}\left(\frac{-iy}{Yt},\frac{-ih}{Yt},\frac{-1}{Y},\frac{-i}{Yt}\right), \quad (5c)$$

$$\bar{m}^a=\frac{1}{\sqrt{2}}\left(\frac{iy}{Yt},\frac{ih}{Yt},\frac{-1}{Y},\frac{i}{Yt}\right). \quad (5d)$$

As usual we have for the only nonvanishing innerproducts of the above null vectors

$$k_an^a=1,\quad m_a\bar{m}^a=-1. \quad (6)$$

The associated directional derivatives are given by

$$D\equiv k^a\partial_a=\frac{1}{\sqrt{2}}\left(F\partial_0+\frac{1}{X}\partial_1\right), \quad (6a)$$

$$\Delta\equiv n^a\partial_a=\frac{1}{\sqrt{2}}\left(F\partial_0-\frac{1}{X}\partial_1\right), \quad (6b)$$

$$\delta\equiv m^a\partial_a=\frac{-1}{\sqrt{2}Y}\left(\frac{iy}{t}\partial_0+\frac{ih}{t}\partial_1+\partial_2+\frac{i}{t}\partial_3\right), \quad (6c)$$

$$\delta^*\equiv\bar{m}^a\partial_a=\frac{1}{\sqrt{2}Y}\left(\frac{iy}{t}\partial_0+\frac{ih}{t}\partial_1-\partial_2+\frac{i}{t}\partial_3\right). \quad (6d)$$

After a lengthy computation the spin coefficients are computed straightforwardly and are given by

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$$\alpha = -\beta^* = \frac{1}{2\sqrt{2}Yt}\left[t_{,2} - i\frac{(yY_{,0} + hY_{,1})}{Y}\right],$$

$$\gamma = -\frac{1}{2\sqrt{2}}\left[\frac{F}{X}\left(X_{,0} + \frac{F_{,1}}{F^2}\right) + \frac{i}{2Y^2t}\left(Xh_{,2} + \frac{y_{,2}}{F}\right)\right],$$

$$\epsilon = \frac{1}{2\sqrt{2}}\left[\frac{F}{X}\left(X_{,0} - \frac{F_{,1}}{F^2}\right) + \frac{i}{2Y^2t}\left(Xh_{,2} - \frac{y_{,2}}{F}\right)\right],$$

$$\mu = -\frac{1}{\sqrt{2}Y}\left[-\left(FY_{,0} - \frac{Y_{,1}}{X}\right) + \frac{i}{2Yt}\left(Xh_{,2} + \frac{y_{,2}}{F}\right)\right], \quad (7)$$

$$\rho = -\frac{1}{\sqrt{2}Y}\left[\left(FY_{,0} + \frac{Y_{,1}}{X}\right) - \frac{i}{2Yt}\left(Xh_{,2} - \frac{y_{,2}}{F}\right)\right],$$

$$\nu = \kappa = -\frac{i}{2\sqrt{2}Yt}\left[h\left(\frac{F_{,1}}{F} + \frac{X_{,1}}{X}\right) + y\left(\frac{F_{,0}}{F} + \frac{X_{,0}}{X}\right)\right],$$

$$\pi = \tau = -\frac{i}{2\sqrt{2}Yt}\left[h\left(\frac{F_{,1}}{F} - \frac{X_{,1}}{X}\right) + y\left(\frac{F_{,0}}{F} - \frac{X_{,0}}{X}\right)\right],$$

$$\lambda = \sigma = 0.$$

III. THE DIRAC EQUATION IN PFST WITH LRS

Following Chandrasekhar[6] the Dirac equation in curved space-time is written as a set of four coupled first-order differential equations

$$(D + \epsilon - \rho)F_1 + (\delta^* + \pi - \alpha)F_2 = i\bar{\mu}_e G_1, \quad (8a)$$

$$(\Delta + \mu - \gamma)F_2 + (\delta + \beta - \tau)F_1 = i\bar{\mu}_e G_2, \quad (8b)$$

$$(D + \epsilon^* - \rho^*)G_2 - (\delta + \pi^* - \alpha^*)G_1 = i\bar{\mu}_e F_2, \quad (8c)$$

$$(\Delta + \mu^* - \gamma^*)G_1 - (\delta^* + \beta^* - \tau^*)G_2 = i\bar{\mu}_e F_1, \quad (8d)$$

where

$$F_1 = P^0, \quad F_2 = P^1, \quad G_1 = \bar{Q}^{1'}, \quad G_2 = -\bar{Q}^{0'}. \quad (9)$$

The four-component Dirac wave function ψ is given by $\psi = (P^A, \bar{Q}_{A'})^T$ and mass of the particle is $\mu_e = \sqrt{2}\bar{\mu}_e$. As explained earlier the PFST with LRS fall into three distinct classes and we shall write down the Dirac equation in each of the three cases.

A. Case (1)

Employing the particular values $X = 1$, $Y = Y(x^1)$, $F = F(x^1)$, and $h = 0$ in Eqs. (6) and (7) and substituting in Eqs. (8), we obtain the Dirac equation for this case:

$$\left(F\partial_0 + \partial_1 - \frac{F_{,1}}{2F} + \frac{Y_{,1}}{Y} + \frac{iy_{,2}}{4Y^2tF}\right)F_1 + \frac{1}{Y}\left(\frac{iy}{t}\partial_0 - \partial_2 + \frac{i}{t}\partial_3 - \frac{t_{,2}}{2t}\right)F_2 = i\mu_e G_1, \quad (10a)$$

$$\left(F\partial_0 - \partial_1 + \frac{F_{,1}}{2F} - \frac{Y_{,1}}{Y} - \frac{iy_{,2}}{4Y^2tF}\right)F_2 - \frac{1}{Y}\left(\frac{iy}{t}\partial_0 + \partial_2 + \frac{i}{t}\partial_3 + \frac{t_{,2}}{2t}\right)F_1 = i\mu_e G_2, \quad (10b)$$

$$\left(F\partial_0 + \partial_1 - \frac{F_{,1}}{2F} + \frac{Y_{,1}}{Y} - \frac{iy_{,2}}{4Y^2tF}\right)G_2 + \frac{1}{Y}\left(\frac{iy}{t}\partial_0 + \partial_2 + \frac{i}{t}\partial_3 + \frac{t_{,2}}{2t}\right)G_1 = i\mu_e F_2, \quad (10c)$$

$$\left(F\partial_0 - \partial_1 + \frac{F_{,1}}{2F} - \frac{Y_{,1}}{Y} + \frac{iy_{,2}}{4Y^2tF}\right)G_1 - \frac{1}{Y}\left(\frac{iy}{t}\partial_0 - \partial_2 + \frac{i}{t}\partial_3 - \frac{t_{,2}}{2t}\right)G_2 = i\mu_e F_1. \quad (10d)$$

The above equations do not depend on x^0 and x^3 and hence the x^0, x^3 dependence is of the form

$$\exp i(wx^0 + mx^3). \quad (11)$$

Writing

$$F_i = \exp i(wx^0 + mx^3)F_i(x^1,x^2), \quad i = 1,2,$$
$$G_i = \exp i(wx^0 + mx^3)G_i(x^1,x^2), \quad i = 1,2, \quad (12)$$

and noting that $y_{,2}/t = c^1$, a constant, Eq. (10) can be rewritten as

$$(\mathscr{D}_1 + ic^1/4YF)F_1 - \mathscr{L}_1F_2 = i\mu_e YG_1, \quad (13a)$$

$$(\mathscr{D}_1^\dagger + ic^1/4YF)F_2 + \mathscr{L}_1^\dagger F_1 = -i\mu_e YG_2, \quad (13b)$$

$$(\mathscr{D}_1 - ic^1/4YF)G_2 + \mathscr{L}_1^\dagger G_1 = i\mu_e YF_2, \quad (13c)$$

$$(\mathscr{D}_1^\dagger - ic^1/4YF)G_1 - \mathscr{L}_1G_2 = -i\mu_e YF_1, \quad (13d)$$

where

$$\mathscr{D}_1 \equiv Y(\partial_1 + iwF + Y_{,1}/Y - F_{,1}/2F), \quad (14a)$$

$$\mathscr{D}_1^\dagger \equiv Y(\partial_1 - iwF + Y_{,1}/Y - F_{,1}/2F), \quad (14b)$$

$$\mathscr{L}_1 \equiv \partial_2 + (wy + m)/t + t_{,2}/2t, \quad (15a)$$

$$\mathscr{L}_1^\dagger \equiv \partial_2 - (wy + m)/t + t_{,2}/2t. \quad (15b)$$

To separate the x^1 and x^2 dependence in Eqs. (13) one introduces

$$F_1 = R_-(x^1)S_-(x^2), \quad F_2 = R_+(x^1)S_+(x^2),$$
$$G_1 = R_+(x^1)S_-(x^2), \quad G_2 = R_-(x^1)S_+(x^2), \quad (16)$$

and rearranges terms to obtain

$$(\mathscr{D}_1 + ic^1/4YF)R_- = (\lambda_1 + i\mu_e Y)R_+, \quad (17a)$$

$$\mathscr{L}_1S_+ = \lambda_1S_-, \quad (17b)$$

$$(\mathscr{D}_1^\dagger + ic^1/4YF)R_+ = -(\lambda_2 + i\mu_e Y)R_-, \quad (18a)$$

$$\mathscr{L}_1^\dagger S_- = \lambda_2S_+, \quad (18b)$$

$$(\mathscr{D}_1 - ic^1/4YF)R_- = (-\lambda_3 + i\mu_e Y)R_+, \quad (19a)$$

$$\mathscr{L}_1^\dagger S_- = \lambda_3S_+, \quad (19b)$$

$$(\mathscr{D}_1^\dagger - ic^1/4YF)R_+ = (\lambda_4 - i\mu_e Y)R_-, \quad (20a)$$

$$\mathscr{L}_1S_+ = \lambda_4S_-, \quad (20b)$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are four constants of separation. Consistency of Eqs. (17b) and (20b) imply $\lambda_1 = \lambda_4$ while that of (18b) and (19b) give $\lambda_2 = \lambda_3$. Substituting this in the equation for $R_\pm$ we get

$$(\mathscr{D}_1 + ic^1/4YF)R_- = (\lambda_1 + i\mu_e Y)R_+, \quad (21a)$$

$$(\mathscr{D}_1^\dagger + ic^1/4YF)R_+ = -(\lambda_2 + i\mu_e Y)R_-, \quad (21b)$$

$$(\mathscr{D}_1 - ic^1/4YF)R_- = (-\lambda_2 + i\mu_e Y)R_+, \quad (21c)$$

$$(\mathscr{D}_1^\dagger - ic^1/4YF)R_+ = (\lambda_1 - i\mu_e Y)R_-. \quad (21d)$$

Consistency of the above equations is possible only if $\lambda_1 = -\lambda_2$ and $c^1 = 0$. Thus in space-times of type I, the Dirac equation is separable in the subclass given by $c^1 = 0$. The above analysis however, does not preclude separability by some other method when $c^1 \neq 0$. For $c^1 = 0$, the Dirac equation becomes

$$\mathscr{D}_1 R_- = (\lambda + i\mu_e Y) R_+, \tag{22a}$$

$$\mathscr{D}_1^\dagger R_+ = (\lambda - i\mu_e Y) R_-, \tag{22b}$$

$$\mathscr{L}_1 S_+ = \lambda S_-, \tag{23a}$$

$$\mathscr{L}_1^\dagger S_- = -\lambda S_+. \tag{23b}$$

B. Case (3)

Before going to case (2) we treat case (3) since the treatment is very similar to that of case (1). In this case, $F = 1$, $X = X(x^0)$, $Y = Y(x^0)$, and $y = 0$. Employing the directional derivatives and spin coefficients appropriate to this instance the Dirac equation may be written down as before. The equations, however, now are translationally invariant with respect to x^1 and x^3 and further $h_{,2}/t = c$, a constant. Writing

$$F_i = \exp i(k_1 x^1 + k_3 x^3) F_i(x^0, x^2), \quad i = 1,2,$$
$$G_i = \exp i(k_1 x^1 + k_3 x^3) G_i(x^0, x^2), \quad i = 1,2, \tag{24}$$

the equations may be written in the form

$$(\mathscr{D}_3 - icX/4Y) F_1 - \mathscr{L}_3 F_2 = i\mu_e Y G_1, \tag{25a}$$

$$(\mathscr{D}_3^\dagger - icX/4Y) F_2 - \mathscr{L}_3^\dagger F_1 = i\mu_e Y G_2, \tag{25b}$$

$$(\mathscr{D}_3 + icX/4Y) G_2 + \mathscr{L}_3^\dagger G_1 = i\mu_e Y F_2, \tag{25c}$$

$$(\mathscr{D}_3^\dagger + icX/4Y) G_1 + \mathscr{L}_3 G_2 = i\mu_e Y F_1, \tag{25d}$$

where

$$\mathscr{D}_3 \equiv Y\left(\partial_0 + \frac{ik_1}{X} + \frac{X_{,0}}{2X} + \frac{Y_{,0}}{Y}\right), \tag{26a}$$

$$\mathscr{D}_3^\dagger \equiv Y\left(\partial_0 - \frac{ik_1}{X} + \frac{X_{,0}}{2X} + \frac{Y_{,0}}{Y}\right), \tag{26b}$$

$$\mathscr{L}_3 \equiv \partial_2 + \frac{hk_1 + k_3}{t} + \frac{t_{,2}}{2t}, \tag{27a}$$

$$\mathscr{L}_3^\dagger \equiv \partial_2 - \frac{hk_1 + k_3}{t} + \frac{t_{,2}}{2t}. \tag{27b}$$

Writing

$$F_1 = T_-(x^0) S_-(x^2), \quad F_2 = T_+(x^0) S_+(x^2),$$
$$G_1 = T_+(x^0) S_-(x^2), \quad G_2 = T_-(x^0) S_+(x^2) \tag{28}$$

and repeating consistency arguments similar to the previous case, one obtains the result that the Dirac equation is separable if we have $c = 0$. As before, this does not rule out separability by different procedures when $c \neq 0$. For $c = 0$, the separated equations are of the form

$$\mathscr{D}_3 T_- = (\lambda + i\mu_e Y) T_+, \tag{29a}$$

$$\mathscr{D}_3^\dagger T_+ = (-\lambda + i\mu_e Y) T_-, \tag{29b}$$

$$\mathscr{L}_3 S_+ = \lambda S_-, \tag{30a}$$

$$\mathscr{L}_3^\dagger S_- = -\lambda S_+. \tag{30b}$$

C. Case (2)

Finally we look into case (2) which is characterized by $h = y = 0$. In this case consequently the Dirac equation may be written down using Eqs. (6), (7), and (8). Further noting that in general only ∂_3 is a Killing vector the x^3 dependence of F and G is given by $\exp ik_3 x^3$. Consequently, the equation may be simplified to the form

$$\mathscr{D}_2 F_1 - \mathscr{L}_2 F_2 = i\mu_e Y G_1, \tag{31a}$$

$$\mathscr{D}_2^\dagger F_2 - \mathscr{L}_2^\dagger F_1 = i\mu_e Y G_2, \tag{31b}$$

$$\mathscr{D}_2 G_2 + \mathscr{L}_2^\dagger G_1 = i\mu_e Y F_2, \tag{31c}$$

$$\mathscr{D}_2^\dagger G_1 + \mathscr{L}_2 G_2 = i\mu_e Y F_1, \tag{31d}$$

where $\mathscr{L}_2$ and $\mathscr{L}_2^\dagger$ are the "angular" operators depending on x^2 only while $\mathscr{D}_2$ and $\mathscr{D}_2^\dagger$ are in general "radial-temporal" operators depending on both x^0 and x^1

$$\mathscr{D}_2 \equiv Y\left[F\partial_0 + \frac{1}{X}\partial_1 + \frac{F}{2X}\left(X_{,0} - \frac{F_{,1}}{F^2}\right) + \frac{1}{Y}\left(FY_{,0} + \frac{Y_{,1}}{X}\right)\right], \tag{32a}$$

$$\mathscr{D}_2^\dagger \equiv Y\left[F\partial_0 - \frac{1}{X}\partial_1 + \frac{F}{2X}\left(X_{,0} + \frac{F_{,1}}{F^2}\right) + \frac{1}{Y}\left(FY_{,0} - \frac{Y_{,1}}{X}\right)\right], \tag{32b}$$

$$\mathscr{L}_2 \equiv \partial_2 + k_3/t + t_{,2}/2t, \tag{33a}$$

$$\mathscr{L}_2^\dagger \equiv \partial_2 - k_3/t + t_{,2}/2t. \tag{33b}$$

The "angular" dependence may be extracted out by introducing

$$F_1 = Z_-(x^0,x^1) S_-(x^2), \quad F_2 = Z_+(x^0,x^1) S_+(x^2),$$
$$G_1 = Z_+(x^0,x^1) S_-(x^2), \quad G_2 = Z_-(x^0,x^1) S_+(x^2). \tag{34}$$

Consistency arguments along the lines of case (1) now show that the angular part separates in all cases. We then obtain for the following system of equations for the angular part and the radial-temporal part:

$$\mathscr{D}_2 Z_- = (\lambda + i\mu_e Y) Z_+, \tag{35a}$$

$$\mathscr{D}_2^\dagger Z_+ = (-\lambda + i\mu_e Y) Z_-, \tag{35b}$$

$$\mathscr{L}_2 S_+ = \lambda S_-, \tag{36a}$$

$$\mathscr{L}_2^\dagger S_- = -\lambda S_+. \tag{36b}$$

Equations (35) are more complicated than in the earlier cases since the time and spatial dependence are still coupled. Though in general X, Y, and F are functions of x^0 and x^1 a useful restriction obtains if one assumes

$$X = X(x^1), \quad Y = Y(x^1), \quad \text{and } F = F(x^0). \tag{37}$$

In this case, Eqs. (35) become

$$F\partial_0 Z_- = \frac{\lambda + i\mu_e Y}{Y} Z_+ - \frac{1}{X}\left(\partial_1 + \frac{Y_{,1}}{Y}\right) Z_-, \tag{38a}$$

$$F\partial_0 Z_+ = -\left(\frac{\lambda - i\mu_e Y}{Y}\right) Z_- + \frac{1}{X}\left(\partial_1 + \frac{Y_{,1}}{Y}\right) Z_+. \tag{38b}$$

The above equations can be separated by writing

$$Z_- = T(x^0) R_-(x^1), \quad Z_+ = T(x^0) R_+(x^1), \tag{39}$$

whence one obtains

$$F\frac{\partial_0 T}{T}=\frac{1}{R_-}\left[\frac{\lambda+i\mu_e Y}{Y}R_+-\frac{1}{X}\left(\partial_1+\frac{Y_{,1}}{Y}\right)R_-\right]$$
$$=\frac{1}{R_+}\left[-\frac{\lambda-i\mu_e Y}{Y}R_-+\frac{1}{X}\left(\partial_1+\frac{Y_{,1}}{Y}\right)R_+\right] \tag{40}$$
$$= \text{constant } w \text{ say.}$$

Integrating the equation for T one obtains

$$T=\exp\left[w\int\frac{dx^0}{F}\right] \tag{41}$$

while the x^1 dependence is given by the coupled equations

$$((1/X)\partial_1+w)YR_-=(\lambda+i\mu_e Y)R_+, \tag{42a}$$

$$((1/X)\partial_1-w)YR_+=(\lambda-i\mu_e Y)R_-. \tag{42b}$$

Thus in this restricted case the Dirac equation is completely separable.

Finally we consider another particular instance of case (2) which also yields a completely separable system. In this case

$$X=X(x^1),\quad Y=Y(x^0),\quad F=F(x^0). \tag{43}$$

Proceeding as before in this case, we are led to the equations

$$Z_-=T_-(x^0)R(x^1),\quad Z_+=T_+(x^0)R(x^1), \tag{44}$$

where

$$R(x^1)=\exp\left[k_1\int X\,dx^1\right]. \tag{45}$$

The x^0 dependence in this instance is given by the coupled system

$$Y(F\partial_0+F(Y_{,0}/Y)+k_1)T_-=(\lambda+i\mu_e Y)T_+, \tag{46a}$$

$$Y(F\partial_0+F(Y_{,0}/Y)-k_1)T_+=(-\lambda+i\mu_e Y)T_-. \tag{46b}$$

Thus in the case (2) the Dirac equation is completely separable in the two subclasses specified by Eqs. (37) and (43).

IV. THE PARTICULAR SPACE-TIMES AND DECOUPLED EQUATIONS

In this section we examine the subclasses of space-times in which the Dirac equation separates. We shall mention some characteristics of such space-times based essentially on the acceleration, rotation, expansion, and shear of the fluid world-lines. As demonstrated in the previous section the Dirac equation separates in case (1) if $c^1=0$. This implies, using Eq. (4), that y is a constant. Thus the space-times where the Dirac equation is separable are of the form

$$ds^2=(1/F^2)(dx^0-y\,dx^3)^2-(dx^1)^2-Y^2((dx^2)^2+t^2(dx^3)^2). \tag{47}$$

Since y is a constant, introducing

$$\bar{x}^0=x^0-yx^3 \tag{48}$$

puts the above metric in the form

$$ds^2=(1/F^2)(d\bar{x}^0)^2-(dx^1)^2-Y^2((dx^2)^2+t^2(dx^3)^2). \tag{49}$$

If x^3 is a cyclic coordinate (the usual spherical coordinate ϕ) the above transformation is an allowed transformation only locally. In this case $c^1=0$ represents space-times which can be called "locally static." We have thus proved that for metrics of type I the Dirac equation is separable if the metric is at least locally static. A straightforward calculation shows that the fluid world-lines have nonvanishing acceleration with the other parameters, viz., rotation, expansion, and shear, being zero.

We next proceed to obtain the decoupled equation for the angular and radial parts. Operating on Eq. (23b) by $\mathscr{L}_1$ and employing Eq. (23a) yields the equation satisfied by S_-:

$$(\mathscr{L}_1\mathscr{L}_1^\dagger+\lambda^2)S_-=0. \tag{50a}$$

Employing Eqs. (15) $\mathscr{L}_1\mathscr{L}_1^\dagger$ can be explicitly obtained. Thus

$$\mathscr{L}_1\mathscr{L}_1^\dagger=\partial_2^2+\frac{t_{,2}}{t}\left(\partial_2+\frac{wy+m}{t}-\frac{t_{,2}}{4t}\right)+\frac{t_{,22}}{2t}-\left(\frac{wy+m}{t}\right)^2 \tag{50b}$$

so that Eqs. (50) yield the decoupled equation satisfied by S_-. Similarly, $S_+(w,m,\lambda;x^2)$ satisfies the same equation as $S_-(-w,-m,\lambda;x^2)$.

To obtain the decoupled eqution for the radial part we operate on Eq. (22a) by $\mathscr{D}_1^\dagger$ and using Eq. (22b) we obtain

$$\left[\mathscr{D}_1^\dagger\mathscr{D}_1-\frac{i\mu_e Y_{,1}Y}{\lambda+i\mu_e Y}\mathscr{D}_1-(\lambda^2+\mu_e^2Y^2)\right]R_-(w,\lambda;x^1)=0. \tag{51}$$

Using Eqs. (14) $\mathscr{D}_1^\dagger\mathscr{D}_1$ may be explicitly computed. Substituting this in Eq. (51) and after some simplifications we obtain the decoupled equation satisfied by $R_-(w,\lambda;x^1)$:

$$Y^2\left[\partial_1^2+\left(2\frac{Y_{,1}}{Y}-\frac{F_{,1}}{F}+Q_1^-\right)+iwF\left(Q_1^-+\frac{F_{,1}}{F}\right)+\frac{Y_{,11}}{Y}-\frac{F_{,11}}{F}+\frac{3}{4}\frac{F_{,1}^2}{F^2}-\frac{F_{,1}}{2F}\left(2\frac{Y_{,1}}{Y}+Q_1^-\right)+\frac{Y_{,1}}{Y}Q_1^-+w^2F^2-\frac{\lambda^2+\mu_e^2Y^2}{Y^2}\right]R_-(w,\lambda;x^1)=0, \tag{52a}$$

where

$$Q_1^-\equiv Q_1^-(\lambda,x^1)=\frac{Y_{,1}}{Y}\left(1-\frac{i\mu_e Y}{\lambda+i\mu_e Y}\right). \tag{52b}$$

$R_+(w,\lambda;x^1)$ satisfies the same equation as $R_-(-w,-\lambda;x^1)$.

Case (3): In this case the particular space-time for which the Dirac equation separates are given by $c=0$ so that Eq. (4) implies h is a constant. Restricting to this subclass the space-times are characterized by the line element of the form

$$ds^2=(dx^0)^2-X^2(dx^1-h\,dx^3)^2-Y^2(dx^2)^2-t^2Y^2(dx^3)^2. \tag{53}$$

Introducing, as before,

$$\bar{x}^1=x^1-hx^3 \tag{54}$$

reduces the metric to a diagonal form. If x^3 is a cyclic coordinate this is possible only locally and we may call the metric "locally diagonal." In this case fluid lines are geodetic and nonrotating, but have nonzero expansion and shear.

Applying the procedure of the previous section to Eqs. (30) and (29) and using Eqs. (21) and (26) we obtain the decoupled angular radial equations. Thus,

$$\Big[\partial_2^2 + \frac{t_{,2}}{t}\Big(\partial_2 + \frac{hk_1 + k_3}{t} - \frac{t_{,2}}{4t}\Big) + \frac{t_{,22}}{2t} - \Big(\frac{hk_1 + k_3}{t}\Big)^2 + \lambda^2\Big] S_- = 0; \quad (55)$$

$$Y^2\Big[\partial_0^2 + \Big(\frac{2Y_{,0}}{Y} + \frac{X_{,0}}{X} + Q_3^-\Big)\partial_0 + \frac{ik_1}{X}\Big(Q_3^- - \frac{X_{,0}}{X}\Big) + \frac{X_{,00}}{2X} + \frac{Y_{,00}}{Y} - \frac{X_{,0}^2}{4X^2} + \frac{X_{,0}}{2X}\Big(\frac{2Y_{,0}}{Y} + Q_3^-\Big) + \frac{Y_{,0}}{Y}Q_3^- + \frac{k_1^2}{X^2} + \frac{\lambda^2 + \mu_e^2 Y^2}{Y^2}\Big] T_-(k_1,\lambda;x^0) = 0; \quad (56a)$$

$$Q_3^-(\lambda) = \frac{Y_{,0}}{Y}\Big(1 - \frac{i\mu_e Y}{\lambda + i\mu_e Y}\Big); \quad (56b)$$

$S_+(k_1,k_3,\lambda;x^2)$ satisfies the same equation as $S_-(-k_1,-k_3,\lambda;x^2)$ while $T_+(k_1,\lambda;x^0)$ satisfies the same equation as $T_-(-k_1,-\lambda;x^0)$.

Case (2): In this case the angular part is separable in all the cases. The decoupled equation for S_- by following a procedure indicated previously is given by

$$\Big[\partial_2^2 + \frac{t_{,2}}{t}\Big(\partial_2 - \frac{t_{,2}}{4t} + \frac{k_3}{t}\Big) + \frac{t_{,22}}{2t} - \frac{k_3^2}{t^2} + \lambda^2\Big] S_-(k_3,\lambda;x^2) = 0. \quad (57)$$

$S_+(k_3,\lambda;x^2)$ satisfies the same equation as $S_-(-k_3,\lambda;x^2)$.

The radial (temporal) part as shown in the previous section is separable in two particular cases. In the first case given by Eq. (37) the metric is

$$ds^2 = \frac{(dx^0)^2}{F^2(x_0)} - X^2(x^1)(dx^1)^2 - Y^2(x^1) \times [(dx^2)^2 + t^2(x^2)(dx^3)^2]. \quad (58)$$

This is a static metric with the fluid lines having all the parameters zero.

In this subclass $R_-(w,\lambda;x^1)$ satisfies

$$Y\Big[\frac{1}{X^2}\partial_1^2 + \frac{1}{X^2}\Big(\frac{2Y_{,1}}{Y} - \frac{X_{,1}}{X} + Q_3^-\Big)\partial_1 + \frac{Y_{,11}}{X^2 Y} - \frac{Y_{,1}}{X^2 Y}\Big(\frac{X_{,1}}{X} - Q_3^-\Big) + \frac{wQ_3^-}{X} - w^2 - \frac{\lambda^2 + \mu_e^2 Y^2}{Y^2}\Big] R_-(w,\lambda;x^1) = 0, \quad (59a)$$

$$Q_3^- = \lambda Y_{,1}/Y(\lambda + i\mu_e Y). \quad (59b)$$

$R_+(w,\lambda;x^1)$ satisfies the same equation as $R_-(-w,-\lambda;x^1)$.

The second subclass [Eq. (4)] corresponds to the line element

$$ds^2 = \frac{(dx^0)^2}{F^2(x^0)} - X^2(x^1)(dx^1)^2 - Y^2(x^0)((dx^2)^2 + t^2(x^2)(dx^3)^2). \quad (60)$$

Here the space-time is nonexpanding in the x^1 direction. In contrast to the first subclass, the fluid world-lines, though geodetic and nonrotating, have nonvanishing expansion and shear. The decoupled temporal equation in this instance is given by

$$Y\Big[F^2\partial_0^2 + F^2\Big(\frac{2Y_{,0}}{Y} + \frac{F_{,0}}{F} + Q_4^- F\Big)\partial_0 + \frac{Y_{,00}F^2}{Y} + \frac{F^2 Y_{,0}}{Y}\Big(\frac{F_{,0}}{F} + Q_4^-\Big) + k_1 F Q_4^- - k_1^2 + \frac{\lambda^2 + \mu_e^2 Y^2}{Y^2}\Big] T_-(k_1,\lambda;x^0) = 0, \quad (61a)$$

$$Q_4^- = \lambda Y_{,0}/Y(\lambda + i\mu_e Y). \quad (61b)$$

$T_+(k_1,\lambda;x^0)$ satisfies the same equation as $T_-(-k_1,-\lambda;x^0)$.

V. DISCUSSION

In the previous sections we have obtained the subclass of perfect fluid space-times with local rotational symmetry wherein the Dirac equation is separable. For space-times belonging to case (1), the Dirac equation is separable if the background is at least "locally static" while in case (3), it separates if the space-time is at least "locally diagonal." In case (2) the massive Dirac equation is separable in those cases where the Hertz potential for the massless spin-$\frac{1}{2}$ equation is separable.[1] We may mention in passing that though cases (1) and (3) described, respectively, by Eqs. (49) and (53), resemble case (2) they are distinct from it. This is because $F = F(x^1)$ in case (1) while $F = F(x^0)$ in case (2) and similarly $X = X(x^0)$ in case (3) while $X = X(x^1)$ in case (2). However, whether the Dirac equation separates out in other cases for different, judicious choices of tetrads and variables remains an open question.

We have also obtained the second-order decoupled equations satisfied by the angular and radial (temporal) parts of the wave function. To discuss further the angular and radial equations, we note that the form of the various equations in different cases is of the same nature so that we need to discuss in detail only a prototype for each. For instance, comparing the radial and angular parts in cases (1) and (3), i.e., Eqs. (50) and (55), Eqs. (52) and (56), respectively, we find that the equation for case (3) may be obtained for those of case (1) by the identifications

$$x^0 \leftrightarrow x^1, \quad w \leftrightarrow k_1, \quad m \leftrightarrow k_3, \quad y \leftrightarrow h, \quad F \leftrightarrow 1/X, \quad (\lambda^2 + \mu_e^2 Y^2) \leftrightarrow -(\lambda^2 + \mu_e^2 Y^2). \quad (62)$$

Further, the angular equation in case (2) also is of the same form as for case (1) since y is a constant. They may be obtained by the identification $wy + m \leftrightarrow k_3$. One caution, however. In case x^3 is a cyclic coordinate, boundary conditions—like single valuedness of ψ—would imply a different spectrum for m as compared to k_3. In the two subclasses of case (2) where the Dirac equation is separable the radial (temporal) equations have a similar structure. From Eqs. (59) and (61) it can be seen that the equations go into one another with the identifications

$$x^0 \leftrightarrow x^1, \quad F \leftrightarrow 1/X, \quad (\lambda^2 + \mu_e^2 Y^2) \leftrightarrow (\lambda^2 + \mu_e^2 Y^2). \quad (63)$$

Further studies of the explicit solutions of the angular and radial equations are in progress and will be published elsewhere.

ACKNOWLEDGMENTS

We thank Joseph Samuel for useful discussions.

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