Hidden symmetries in deformed microwave resonators

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We explain the "hidden symmetries" observed in wave functions of deformed microwave resonators in recent experiments. We also predict that other such symmetries can be seen in microwave resonators.

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Lauber *et al.* [1] experimentally studied the pattern of Berry phases that emerges when a microwave cavity is cyclically deformed around a rectangular shape. Standing electromagnetic waves in the cavity can be mapped and the "wave functions" followed through the cyclic deformation to measure the Berry phase. Apart from the Berry phases, which were primarily of interest in Ref. [1], those authors also noticed a curious symmetry: the standing wave patterns at different deformations are related. Subsequent theoretical work [2,3] has clarified the pattern of Berry phases seen in the experiment. However, the "hidden symmetry" has not been explained so far. The purpose of this Brief Report is to provide an understanding of the "hidden symmetry" and thus a complete and correct interpretation of the experiment described in [1].

Consider a rectangular cavity (see Fig. 1) with sides (a,b) having n degenerate modes: the scalar Laplacian $-\nabla^2$ has n degenerate eigenfunctions. If the cavity is deformed, the degeneracy will in general be broken. Let us suppose that the deformation consists (as in the experiment of Ref. [1]) of moving the corner around its undeformed position so that the rectangle is deformed to a quadrilateral. This deformation can be effected in the formalism by performing a coordinate transformation $x=u(1+\alpha v), y=v(1+\beta u)$ [where (α,β) are the deformation parameters] which maps the deformed rectangle in the (x,y) plane to an undeformed rectangle in the (u,v) plane. Transforming the Laplacian to curvilinear (u,v) coordinates, we find $H=-\nabla^2=(-1/\sqrt{g})(\partial/\partial x^\mu)\sqrt{g}g^{\mu\nu}\partial/\partial x^\nu$. Matrix elements of H have the form $\langle\psi_1|H|\psi_2\rangle=-\int d^2x\sqrt{g}\psi_1^*\nabla^2\psi_2=-\int d^2x\,\psi_1^*(\partial/\partial x^\mu)\sqrt{g}g^{\mu\nu}(\partial/\partial x^\nu)\psi_2$. Expanding to first order in α,β, we then get $H=H_0+H_1$, where $H_0=-(\partial_u\partial_u+\partial_v\partial_v)$ and $H_1=\alpha f+\beta g$, with $f=vX+uY$ and $g=-uX+vY$, expressed in terms of the differential operators $X=\partial_u\partial_u-\partial_v\partial_v$ and $Y=2\partial_u\partial_v$.

The unperturbed Hamiltonian H_0 has the discrete symmetries $P_1: u\rightarrow a-u$, $P_2: v\rightarrow b-v$, the mirror planes of the rectangular box. We now restrict attention to the n-dimensional degenerate subspace $\mathcal{H}_n$ of H_0 and choose eigenstates of H_0 to have definite parity with respect to both these reflections. In fact, we choose these in the form $|i\rangle=|n_i m_i\rangle=(2/\sqrt{ab})\sin(n_i u\pi/a)\sin(m_i v\pi/b)$, where n_i, m_i are positive integers. Since the states are all degenerate eigenstates of H_0, we have $n_i^2/a^2+m_i^2/b^2=n_j^2/a^2+m_j^2/b^2$ for all i,j. In particular $n_i=n_j\Rightarrow m_i=m_j$. These states are also eigenstates of X with eigenvalues $\lambda_i=(n_i^2\pi^2/a^2-m_i^2\pi^2/b^2)$. It follows that $\langle i|vX|j\rangle=\lambda_j\langle i|v|j\rangle=\lambda_j\langle n_i|n_j\rangle\langle m_i|v|m_j\rangle=\lambda_i\delta_{ij}\langle m_i|v|m_i\rangle$. From $P_2vP_2=(b-v)$, it follows that $\langle m_i|v|m_i\rangle=\langle m_i|P_2vP_2|m_i\rangle=b\langle m_i|m_i\rangle-\langle m_i|v|m_i\rangle$. So we conclude that $\langle m_i|v|m_i\rangle=b/2$ and so, in $\mathcal{H}_n$, $vX=bX/2$ and similarly $uX=aX/2$. The form of the perturbations is thus $f=bX/2+uY$, $g=-aX/2+vY$.

The "mirror symmetry" observed by Lauber *et al.* in their experiment is related to the way the unperturbed levels transform under parity. We consider all possible cases and thus find the necessary and sufficient conditions for this symmetry to be observed. Let us introduce σ_{1i} as the P_1 parity of the ith state $(P_1|i\rangle=\sigma_{1i}|i\rangle)$ and similarly σ_{2i} as the P_2 parity of the ith state. The different cases are listed below with an example (for $n=3$) illustrating each nontrivial case.

(1) $\sigma_{1i}=\sigma$ and $\sigma_{2i}=\sigma'$ for all $i=1,2,\ldots,n$ where σ,σ' can take values ±1 [Example: $a=\sqrt{3}$, $b=1$, and levels (2,6), (8,4), (10,2)]. In this case $\langle i|uY|j\rangle=\langle i|P_2(P_2uYP_2)P_2|j\rangle=-\langle i|uY|j\rangle=0$ and similarly $\langle i|vY|j\rangle=\langle i|P_1(P_1vYP_1)P_1|j\rangle=-\langle i|vY|j\rangle=0$. Thus $f=bX/2$ and $g=-aX/2$ and this is an uninteresting case because the perturbations do not span a two-dimensional space.

(2) The product $\sigma_{1i}\sigma_{2i}=\sigma$ for all i, but σ_{1i} and σ_{2i} are not the same for all i [Example: $a=\sqrt{3}$, $b=1$, and levels (1,3),(4,2),(5,1)]. In this case $\langle i|uY|j\rangle=\langle i|P_2P_1(P_1P_2uYP_2P_1)P_1P_2|j\rangle=\langle i|(a-u)Y|j\rangle$, which implies $uY=aY/2$. Also $\langle i|vY|j\rangle=\langle i|P_2P_1(P_1P_2vYP_2P_1)P_1P_2|j\rangle=\langle i|(b-v)Yj\rangle$ and this gives $vY=bY/2$. Thus in this case $f=bX/2+aY/2$ and $g=-aX/2+bY/2$. Defining new coordinates $\alpha=b\alpha'+a\beta'$, $\beta=-a\alpha'+b\beta'$, we have $H_1=\alpha f+\beta g=\alpha'(bf-ag)+\beta'(af+bg)=(a^2+b^2)(\alpha' X/2+\beta' Y/2)$. Since $PXP=X$, $PYP=-Y$ for $P=P_1,P_2$; hence we see that

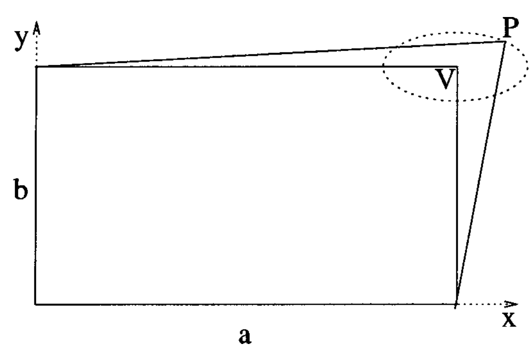

FIG. 1. A deformation of a rectangle into a quadrilateral. The vertex $V=(a,b)$ is moved to the point $P=V+(\delta x,\delta y)=V+ab(\alpha,\beta)$. We consider an experiment where P is moved around the elliptic path shown in the figure.

wave functions at points $p(\alpha',\beta')$ and $p'(\alpha',-\beta')$ can be related by either P_1 or P_2. Proof: Let $H_p|\psi_p\rangle = e|\psi_p\rangle$. Then $P_1H_{1p}P_1P_1|\psi_p\rangle = eP_1|\psi_p\rangle$ or $H_{1p'}P_1|\psi_p\rangle = eP_1|\psi_p\rangle$, which implies, assuming all degeneracies have been lifted, that $|\psi_{p'}\rangle = \pm P_1|\psi_p\rangle$. This is the case studied by Lauber *et al.* [1]. Note that the β' axis is along the long diagonal of the rectangular cavity.

(3) $\sigma_{1i}=\sigma$ for all i, but σ_{2i} is not the same for all i [Example: $a=2$, $b=1$, and levels $(2,18),(12,17),(20,15)$]. In this case $f=bX/2+uY$ and $g=-aX/2$. The coordinate transformation $\alpha=a\beta'$, $\beta=\alpha'+b\beta'$ gives $H_1=\alpha f+\beta g=\alpha' g+\beta'(af+bg)=-\alpha' aX/2+\beta' auY$. Since $P_2XP_2=X$ and $P_2uYP_2=-uY$, it follows that wave functions at points $p(\alpha',\beta')$ and $p'(\alpha',-\beta')$ can be related by P_2.

(4) $\sigma_{2i}=\sigma$ for all i, but σ_{1i} is not the same for all i. This case is similar to (3).

(5) Neither of $\sigma_{1i},\sigma_{2i},\sigma_{1i}\sigma_{2i}$ is the same for all i. It can be proved that this case cannot be realized for any choice of a,b,n_i,m_i. Proof: We enumerate all the possibilities. We can have $n_i^2/a^2+m_i^2/b^2=n_j^2/a^2+m_j^2/b^2$ only if b^2/a^2 is rational. Let $b^2/a^2=p/q$, where p and q are relatively prime. We find that $n_i^2p+m_i^2q=n_j^2p+m_j^2q=N$ for all i,j. Thus we need to consider the following cases classified according to the parity (odd or even) of (p,q): (a) (o,o), (b) (o,e), (c) (e,o), where o and e denote odd and even parities, respectively. For case (a), if N is even then the states can have parities (P_1,P_2) either $(-,-)$ or $(+,+)$. If N is odd then they can have parity $(+,-)$ or $(-,+)$. Thus the only combinations we can get belong to type 1 or 2. For case (b), if N is even then the states can have parity $(+,+)$ or $(+,-)$. If N is odd then they can have parity $(-,+)$ or $(-,-)$. In this case the possible combinations belong to type 3. The case (c) leads to type (3).

Thus there are no examples of type 5.

In summary, we have explained the mirror symmetry of [1] in the framework of first-order perturbation theory (see [3,4] for the limitations of this theory) and noticed other situations where such symmetry may be observed.

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