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Path integral quantization of parametrized field theory

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Free scalar field theory on a flat spacetime can be cast into a generally covariant form known as parametrized field theory in which the action is a functional of the scalar field as well as the embedding variables which describe arbitrary, in general curved, foliations of the flat spacetime. We construct the path integral quantization of parametrized field theory in order to analyze issues at the interface of quantum field theory and general covariance in a path integral context. We show that the measure in the Lorentzian path integral is nontrivial and is the analog of the Fradkin-Vilkovisky measure for quantum gravity. We construct Euclidean functional integrals in the generally covariant setting of parametrized field theory using key ideas of Schleich and show that our constructions imply the existence of nonstandard "Wick rotations" of the standard free scalar field two-point function. We develop a framework to study the problem of time through computations of scalar field two-point functions. We illustrate our ideas through explicit computation for a time independent $(1 + 1)$-dimensional foliation. Although the problem of time seems to be absent in this simple example, the general case is still open. We discuss our results in the contexts of the path integral formulation of quantum gravity and the canonical quantization of parametrized field theory.

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I. INTRODUCTION

Most treatments of quantum fields on a flat spacetime are based on the existence of foliations of the spacetime by flat slices of constant inertial time. In generally covariant systems like general relativity, no preferred foliations exist. Indeed, general covariance requires that all spacelike foliations should be allowed in the description of dynamics. This is only one aspect of the many new conceptual and technical issues which arise in attempts to quantize the gravitational field. In order to isolate and understand this aspect better, it is useful to study quantum field theory on *curved* foliations of flat spacetime as a toy model. Since we are interested primarily in the intertwining of general covariance with quantum field theory, the detailed dynamics of the quantum field itself is a further complication which we may ignore in a first treatment.

Thus, we shall focus on the quantization of a free massive scalar field on arbitrary foliations of flat spacetime. An elegant way to view classical free scalar field theory on arbitrary foliations is to cast it in a generally covariant form known as parametrized field theory [1]. In this form the theory can be used as testing ground for various aspects of general covariance encountered in gravity. Indeed, certain midisuperspace reductions of gravity such as cylindrical waves [2], as well as theories of gravity in lower dimensions [3], can be mapped onto parametrized field theory by suitable variable redefinitions, thus providing an even stronger motivation for studying parametrized field theory.

The canonical quantization of parametrized free field theory was studied in [4–6] with interesting consequences such as the necessity of an anomaly potential in the functional Schroedinger equation in two dimensions [4] and the nonexistence of the functional Schroedinger picture as the unitary image of the Heisenberg picture in spacetime dimensions greater than two [6]. Such results underline the importance of the study of parametrized field theory both in itself and in its role as a toy model for *canonical* quantum gravity. The next logical step is to examine which, if any, aspects of the *path integral* approach to gravity may be better understood by an analysis of the path integral quantization of parametrized field theory.

As emphasized earlier, one of the problems of defining the quantization of a generally covariant theory such as gravity is the absence of a preferred choice of time [7]. This "problem of time" has been studied, most often, in the canonical quantization context. In a path integral formulation it is most directly encountered in the construction of vacuum wave functions. In Poincaré invariant theories, vacuum wave functions are constructed as Euclidean path integrals which, in turn, are constructed from their Lorentzian counterparts by a Wick rotation of the preferred inertial time. In a generally covariant context no preferred time, and hence, no preferred Wick rotation, is available to define the Euclidean theory.

Another aspect of the problem of time is that of inequivalent quantizations. In canonical treatments of gravity the choice of time is very often made by breaking the time reparametrization invariance of the theory via a choice of gauge fixing. Different choices of gauge fixing lead to different choices of time which in turn may lead to inequivalent quantizations. In a path integral for a theory with gauge invariances, gauge fixing terms must be included [8] so as to avoid infinities coming from summing

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over gauge equivalent configurations. Thus we expect that the problem of time could manifest in different choices of gauge fixings in the gravitational path integral.[1]

In this work we examine the above facets of the problem of time in the context of a path integral quantization of parametrized field theory. In Sec. II we construct the (Lorentzian) configuration space path integral from the phase space path integral and hence obtain the correct nontrivial measure. It is clearly seen that different time slicings correspond to appropriately different choices of gauge fixing terms. In Sec. III, we examine the issue of Euclideanization. As mentioned earlier, in the generally covariant context of gravity, no preferred Wick rotation to a Euclidean theory is available. Instead *ad hoc* prescriptions have been proposed [9] which have no clear connection to the Lorentzian theory. An exception is the proposal of Schleich [10], wherein the vacuum wave function is defined from a reduced phase space path integral. We use her ideas to define "Euclidean" path integrals in the generally covariant context of parametrized field theory. We find that her unambiguous definition of Euclideanization implies the existence of nonstandard Wick rotations of the standard free scalar field two-point function. We confirm by direct inspection that such Wick rotated two-point functions indeed exist.

We initiate our investigation into the existence (or absence) of inequivalent quantizations for nonstandard choices of time in Sec. IV. We show how computations of the scalar field two-point function may be used to illuminate this issue. We work through, in some detail, the case of a time independent foliation in $(1+1)$ dimensions. Since there is reason to expect that this simple choice of time reproduces the standard quantization,[2] we provide explicit calculations primarily to illustrate our general framework. Indeed, the case of a general foliation is still open. In Sec. V we discuss our results in the context of the path integral approach to quantum gravity as well as in the context of canonical quantization of parametrized field theory and indicate open issues. Details of some of our considerations are collected in the Appendix.

[1]The Fadeev-Popov determinants in the path integral are supposed to ensure gauge independence. Note, however, that the formal proof of gauge independence assumes that the reduced phase space path integral implements a unique quantization. In theories without extra structures such as global Poincaré invariance or in quantizations which do not assign an explicit role to Poincaré invariance, it is by no means clear that there exists a unique quantization at the reduced phase space level. Thus, a particular gauge choice may present the theory in a guise which is amenable to a particular choice of quantization.

[2]Our choice of foliation is such that the orbits of the time vector field defined by the foliation agree with the orbits of the time isometry of the flat spacetime metric.

Note that the flat $(n+1)$-dimensional spacetime manifold is R^{n+1}. $\alpha, \beta, \gamma = 0 \cdots n$ are spacetime indices in an arbitrary coordinate system x^α. We shall set $x^0 = t$. The $t = \text{const}$ submanifolds are assumed to be n dimensional, spatial hypersurfaces diffeomorphic to R^n. $i, j, k = 1 \cdots n$ are spatial indices on this hypersurface. $A, B, C = 0 \cdots n$ are spacetime indices in inertial coordinates X^A with $X^0 =: T$. The spatial inertial coordinates are $X^{\hat{A}}, \hat{A} = 1 \cdots n$. The Minkowski metric of signature $(-, + + \cdots +)$ is $\eta_{\alpha\beta}$. $\partial_\alpha, \partial_A, \partial_i$ are the partial derivative operators with respect to x^α, X^A, x^i, respectively. The dot "$\cdot$" denotes $\frac{\partial}{\partial t}$.

II. THE PATH INTEGRAL

In this section we derive the classical phase space action, define the phase space path integral, and integrate over the momenta to obtain the configuration space path integral.

A. The classical formulation

The Minkowski metric in an arbitrary coordinate system is given by

$$\eta_{\alpha\beta} = \eta_{AB} \partial_\alpha X^A \partial_\beta X^B, \tag{1}$$

where η_{AB} is the standard Minkowski metric in inertial coordinates. From (1) the spacetime element in an arbitrary coordinate system is

$$\begin{aligned} ds^2 = (-\dot{T}^2 + \dot{X}^2)dt^2 + 2(-\dot{T}\partial_i T + \dot{X}^{\hat{A}} \partial_i X^{\hat{A}}) dt dx^i \\ + (\partial_i X^{\hat{A}} \partial_j X_{\hat{A}} - \partial_i T \partial_j T) dx^i dx^j. \end{aligned} \tag{2}$$

The line element may also be written in the standard Arnowitt-Deser-Misner (ADM) form in terms of the lapse N, shift N^i, and spatial metric q_{ij} as

$$ds^2 = (-N^2 + N_i N^i) dt^2 + 2N_i dt dx^i + q_{ij} dx^i dx^j. \tag{3}$$

X_i^A are the projectors into the hypersurface and are defined by

$$X_i^A = \partial_i X^A. \tag{4}$$

It is straightforward to show the following useful identities:

$$X_i^A X_{Aj} = q_{ij}. \tag{5}$$

$$\epsilon_{AA_1 \cdots A_n} X_{i_1}^{A_1} \cdots X_{i_n}^{A_n} = -n_A \epsilon_{i_1 i_2 \cdots i_n} \tag{6}$$

where $\epsilon_{AA_1 \cdots A_n}$ is the spacetime volume form, $\epsilon_{i_1 i_2 \cdots i_n}$ is the spatial volume form on the $t = \text{const}$ spatial hypersurface, and n_A is the unit, future pointing, timelike normal to this hypersurface. From (6) and $\sqrt{\eta} = N\sqrt{q}$ (η, q are, respectively, the determinants of the spacetime and spatial metrics) it follows that

$$\frac{\partial N}{\partial \dot{X}^A} = -n_A. \tag{7}$$

Equations (2) and (3) imply that

$$\frac{\partial N^i}{\partial \dot{X}^A} = q^{ij} X_{Aj}. \tag{8}$$

The action for a free scalar field ϕ of mass m on Minkowski spacetime expressed in inertial coordinates is

$$S[\phi] = -\frac{1}{2}\int d^{n+1}X(\eta^{AB}\partial_A\phi\partial_B\phi + m^2\phi^2). \tag{9}$$

The action for parametrized field theory is obtained by expressing (9) in arbitrary coordinates x^α and treating the action as a functional of ϕ as well as the embedding variables $X^A(x^i, t)$. Thus

$$S[\phi, X^A] = -\frac{1}{2}\int d^{n+1}x\sqrt{\eta}(\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + m^2\phi^2), \tag{10}$$

with $\eta_{\alpha\beta}$ interpreted as a functional of X^A via (1). In this form, the action is a manifestly diffeomorphism invariant functional of the $(n+1)$ scalar fields X^A and the scalar field ϕ. A straightforward Hamiltonian analysis of (10) using (1)–(8) yields the Hamiltonian form of the action given by

$$S = \int dt d^n x (P_A \dot{X}^A + \pi\dot{\phi} - M^A C_A). \tag{11}$$

Here P_A and π are the momenta canonically conjugate to X^A and ϕ. M^A are the Lagrange multipliers for the first class constraints C_A with

$$C_A = P_A - n_A h + q^{ij} X_{Aj} h_i, \tag{12}$$

$$h := \frac{\pi^2}{2\sqrt{q}} + \frac{\sqrt{q}(q^{ij}\partial_i\phi\partial_j\phi + m^2\phi^2)}{2}, \qquad h_i = \pi\partial_i\phi. \tag{13}$$

Note that n_A and q_{ij} are to be considered as functionals of X^A through (5) and (6). The constraint algebra is Abelian, i.e., $\{C_A, C_B\} = 0$. The algebra of diffeomorphisms can be recovered by smearing the constraints with vector fields ξ_1, ξ_2 which depend on the embedding variables X^A so that

$$\left\{\int d^n x_1 \xi_1^A C_A, \int d^n x_2 \xi_2^B C_B\right\} = \int d^n x(\xi_2^B \partial_B \xi_1^A - \xi_1^B \partial_B \xi_2^A) C_A. \tag{14}$$

We restrict attention to asymptotically inertial embeddings by imposing the following boundary conditions as $\sum_{i=1}^n x^i x^i \rightarrow \infty$:

$$\begin{aligned} X^0(x,t) &= t, \qquad X^1(x,t) = x^1, \\ X^2(x,t) &= x^2, \ldots, X^n(x,t) = x^n, \end{aligned} \tag{15}$$

$$M^0(x,t) = 1, \qquad M^{\hat{A}} = 0, \qquad \hat{A} = 1, \ldots, n. \tag{16}$$

We also impose that P_A, π, ϕ be of compact support on the spatial slice.

B. The path integral

In addition to the classical action (11), a choice of gauge fixing is needed to define the phase space path integral. Since this work constitutes a first attempt to analyze the problem of time in a path integral context in parametrized field theory, we restrict attention to choices of gauge fixing which have the clear geometric meaning of fixing corresponding choices of time functions (i.e., foliations by spacelike surfaces of constant time) on the flat spacetime. The gauge fixing term $\delta[\chi^A]$, where

$$\chi^A = X^A(x,t) - f^A(x,t), \tag{17}$$

corresponds to choosing a foliation of the spacetime defined by the embedding variables $X^A(x,t)$ taking the values $f^A(x,t)$. With this choice of gauge fixing it is easily checked that the Fadeev-Popov determinant (see, for example, [11]) is unity. Hence, the phase space path integral is given by

$$Z = \int \mathcal{D}\phi\mathcal{D}\pi\mathcal{D}M^A\mathcal{D}P_A\mathcal{D}X^A\delta[X^A = f^A]\exp i\int dt d^n x(P_A\dot{X}^A + \pi\dot{\phi} - M^A C_A). \tag{18}$$

In the above equation it is understood that the configuration variables $\phi(x,t)$, $X^A(x,t)$ interpolate between fixed initial and final values at some initial and final instants of time $t = t_I$ and $t = t_F$. The end point values of X^A are assumed to be consistent with the gauge choice (17). We shall not explicitly specify the end point dependence of Z in our notation.

We integrate (18) over M^A and P_A to obtain

$$Z = \int \mathcal{D}\phi\mathcal{D}\pi\mathcal{D}X^A\delta[X^A = f^A]\exp i\int dt d^n x(\pi\dot{\phi} - Nh - N^i h_i) \tag{19}$$

with

$$N = -\dot{X}^A n_A, \qquad N^i = q^{ij} X_{Ai} \dot{X}^A \tag{20}$$

[notice the consistency of these expressions with (7) and (8)], and h, h_i given by (13). In this form it is clear that our choice of gauge fixing presents the parametrized field theory in the form of free scalar field theory on the (in general, curved) foliation $f^A(x, t)$. A further integration over the momenta π yields the configuration space path integral

$$Z = \int \mathcal{D}\phi \mathcal{D}X^A \left[\det \frac{iN}{\sqrt{q}} \right]^{-1/2} \delta[X^A = f^A] \exp iS[\phi, X^A] \tag{21}$$

where $S[\phi, X^A]$ is the classical action given by (10).

Note that the path integral measure has a factor of $[\det(iN/\sqrt{q})]^{-1/2}$. In our specific gauge choice, this determinant factor reduces to an irrelevant c number depending on $f^A(x, t)$. However, with a more general choice of gauge fixing term, we expect this term to persist and, as a consequence, contribute nontrivially to the path integral measure. Although the treatment of the most general gauge choice can be done via Becchi-Rouet-Stora-Tyutin (BRST) methods (see [10,12]), such a treatment is beyond the scope of this paper. Instead, in the Appendix, we have extended our treatment to slightly more general (ϕ-dependent) gauge choices than those of (17). We find that the determinant factor persists and contributes nontrivially to the measure. We believe that this measure is the exact analog of the measure found by Fradkin and Vilkovisky in [13] for quantum gravity. This lends added credence to their measure being the correct one rather than the more commonly used measure proposed by DeWitt in [14]. We shall comment further on this in Sec. V.

III. EUCLIDEANIZATION

Our aim is to construct convergent path integrals in order to evaluate vacuum wave functions. Indeed, we shall *define* this construction to be Euclideanization. For the case of a flat inertial foliation, it will be seen that the construction reproduces the standard Wick rotated path integral.

We are motivated by the remark of Schleich in [10] to the effect that there is no obstruction to constructing a convergent path integral for the vacuum wave function in any theory with a positive definite Hamiltonian. To illustrate this remark, consider such a theory in the absence of constraints with a time independent Hamiltonian H and a single configuration space degree of freedom q. The vacuum is defined as the eigenfunction of $\hat{H}$ with lowest eigenvalue. Under the assumption that the vacuum is unique and that the zero of energy has been chosen so that the vacuum energy vanishes, the vacuum wave function (in obvious notation) may be obtained from the Feynman-Kac–type formula:

$$\psi_0(q_F, t_F)\psi_0^*(q_I, t_I) = \lim_{t_I \to -\infty} \langle q_F, t_F | \exp(-ia\hat{H} \times (t_F - t_I) | q_I, t_I \rangle). \tag{22}$$

Here a is any complex number with negative imaginary part and t_F, t_I are final and initial times. The above identity is obtained by expanding $|q_F, t_F\rangle, |q_I, t_I\rangle$ in a complete set of energy eigenstates. The negative imaginary part of a and the $t_I \to -\infty$ limit conspire to project the initial and final states onto the vacuum state, resulting in Eq. (22). It is straightforward to check that the matrix element on the right-hand side of this identity may be written as a phase space path integral:

$$\langle q_F, t_F | \exp(-ia\hat{H}(t_F - t_I) | q_I, t_I) \rangle = \int \mathcal{D}q \mathcal{D}p \exp\left\{ i \int dt [p_i \dot{q}_i - (a+1)H] \right\}. \tag{23}$$

Since H is positive and a has negative imaginary part, the path integral is (formally) convergent. Equations (22) and (23) illustrate Schleich's remarks in the context of systems without constraints.

In Appendix A 2, we develop similar expressions for vacuum wave functions in terms of phase space path integrals for systems with first class constraints. We restrict attention to cases in which time evolution is generated by a nonvanishing positive definite Hamiltonian. In Sec. III A, we extend the considerations of Appendix A 2 to parametrized field theory so as to write vacuum functions as convergent phase space path integrals. In Sec. III B, we integrate these expressions over momenta to obtain convergent path integrals which we refer to as Euclidean path integrals for want of a better name. The results of Sec. III B imply that the standard Minkowskian two-point function can be continued to a Euclidean two-point function through nonstandard Wick rotations. We show that this is indeed true in Sec. III C.

A. Vacuum wave functions as phase space path integrals

In a theory with coordinates q_i, momenta p_i ($i = 1 \cdots n$), first class constraints $C_\alpha, \alpha = 1 \cdots m$, Lagrange multipliers λ^α, gauge fixing constraints χ_α, and Hamiltonian H, the transition amplitude can be written as

$$Z(q_{iI}, t_I, q_{iF}, t_F) = \int \mathcal{D}q \mathcal{D}p \mathcal{D}\lambda \delta(\chi_\alpha) \det[\{C_\alpha, \chi_\beta\}] \times \exp\left(i \int p_i \dot{q}_i - \lambda^\alpha C_\alpha - H \right). \tag{24}$$

The integral is over all paths which have end points at times t_I, t_F specified by initial values $q_i = q_{iI}$ and final values $q_i = q_{iF}$. The end point configurations must satisfy the gauge fixing constraints.

Standard arguments [11] using canonical transformations to appropriate variables indicate that the transition amplitude (24) is independent of the gauge choice. χ_α. These arguments are not without shortcomings. First, any application of canonical transformations to the path integral is fraught with problems related to the "roughness of paths" [15]. Second, many of the standard arguments ([16] is an exception) ignore end point contributions in the canonically transformed action, as well as the possible differences in the specification of end point values in terms of the old configuration variables as compared to their specification in terms of the new canonically transformed configuration variables.

Here (and in Appendix A 2), we shall also ignore the issues mentioned above. Although we shall further justify our constructions for parametrized field theory, these issues require a careful treatment in more complicated systems such as quantum gravity. With these caveats in mind, we consider the gauge independent quantity Z_a given by

$$Z_a(q_{iI}, t_I; q_{iF}, t_F) = \int \mathcal{D}q\mathcal{D}p\mathcal{D}\lambda\delta(\chi_\alpha)\det[\{C_\alpha, \chi_\beta\}] \times \exp\left[i\int dt(p_i\dot{q}_i - \lambda^\alpha C_\alpha - aH)\right], \tag{25}$$

where a is an arbitrary complex number. As shown in the Appendix A 2, if we choose a to have negative imaginary part and the vacuum to have vanishing energy, it follows that

$$Z_a(q_{iI}, t_I = -\infty; q_{iF}, t_F) = \psi_0(q_F, t_F)\psi_0^*(q_I, t_I), \tag{26}$$

where ψ_0 denotes the vacuum wave function. For a fixed initial configuration, this is an expression for the vacuum wave function as a function of the final configuration.

We aim to construct similar phase space path integrals to express the vacuum wave functionals of parametrized field theory. Our strategy is to first construct the counterpart of Eq. (25) and then to argue that its $t_I \to -\infty$ limit is the correct parametrized field theory counterpart of Eq. (26). There are two differences between parametrized field theory and the system considered in Eq. (25). First, the action for parametrized field theory given by Eq. (11) does not exhibit a nonvanishing Hamiltonian, and, second, the gauge fixing conditions which define a foliation are *time dependent.*[3] Our considerations for the system defined by (25) generalize straightforwardly to parametrized field theory in spite of this time dependence. The first point of difference remains, however, and we need to construct an analog of the Hamiltonian in (25). To do so, we note that in standard free scalar field theory, the vacuum is defined as the ground state of the operator corresponding to the conserved, positive definite scalar field energy E given by

$$E = \frac{1}{2}\int_{T=\mathrm{const}} d^nX\left[\left(\frac{\partial\phi}{\partial T}\right)^2 + \sum_{\hat{A}}\partial_{\hat{A}}\phi\partial_{\hat{A}}\phi + m^2\phi^2\right]. \tag{27}$$

In the context of parametrized field theory it is straightforward to verify that E is simply the evaluation, on a classical solution, of the Dirac observable H, given by

$$H = \int_{R^n} d^nxP_T, \tag{28}$$

where $P_T := P_{A=0}$. Therefore, we define the vacuum state in parametrized field theory to be the ground state of the operator corresponding to H [see (28)]. Since $E = H$ on the constraint surface, our definition is consistent with the usual definition of the vacuum in free scalar field theory.

The above considerations imply that the correct counterpart of (25) is

$$Z_a[\phi_I, X_I^A, t_I; \phi_F, X_F^A, t_F] := \int \mathcal{D}\phi\mathcal{D}\pi\mathcal{D}M^A\mathcal{D}P_A\mathcal{D}X^A\delta[X^A = f^A] \times \exp\left[i\int dtd^nx(P_A\dot{X}^A + \pi\dot{\phi} - M^AC_A) - ia\int dtH\right], \tag{29}$$

with H defined by (28). Using the fact that H is a Dirac observable, it can be checked that the methods of [11] mentioned in Appendix A 2 show gauge independence (with respect to appropriately defined infinitesimal changes of gauge [11]) of the above expression.

To see that the $t_I \to -\infty$ limit of (29) indeed yields the vacuum wave function, we note that the integration of Eq. (29) over X^A and M^A gives

$$Z_a = \int \mathcal{D}\phi\mathcal{D}\pi\exp i\int dtd^nx(\pi\dot{\phi} - \mathcal{H} - aH). \tag{30}$$

[3]Note that the conditions $X^A(x, t) = f^A(x, t)$ constitute a *one parameter family* of gauge fixing conditions; i.e., for every instant of time one has a complete gauge fixing. Hence, strictly speaking, these conditions define a *deparametrization* of the theory rather than a gauge fixing. We shall, however, continue to refer to them as gauge fixing conditions.

Here $\mathcal{H} = \dot{F}^A h_A$ is the generator of evolution in time t along the foliation $F^A(x^i, t)$. Therefore, in operator language (with a suitable operator ordering prescription) the above path integral expression corresponds to, in obvious notation,

$$Z_a = \lim_{t_I \to -\infty} \langle \phi_F, t_F | e^{-ia\hat{H}(t_F - t_I)} | \phi_I, t_I \rangle \tag{31}$$

$$= \psi_0[\phi_F; t_F] \psi_0^*[\phi_I; t_I], \tag{32}$$

where ψ_0 denotes the vacuum state and we have evaluated the action of $e^{-ia\hat{H}(t_F - t_I)}$ via a spectral decomposition of $\hat{H}$ under the assumption that its lowest eigenvalue is normalized to zero and that a has negative imaginary part. Equation (32) further justifies our constructions for Euclideanization in parametrized field theory.

Finally, we note that on the flat foliation, $f^A = x^\alpha \delta^A_\alpha$, Z_a evaluates to

$$Z_a = \int \mathcal{D}\phi \mathcal{D}\pi \exp i \int dt d^n x \bigg[\pi \dot{\phi} - \frac{a+1}{2} \times \bigg(\pi^2 + \sum_i \partial_i \phi \partial_i \phi + m^2 \phi^2 \bigg) \bigg]. \tag{33}$$

The choice $a = -1 - i$ reproduces the usual expression for the vacuum wave functional. Henceforth we shall set $a = -1 - i$ and define the Euclidean phase space path integral to be

$$Z_E[\phi_I, X^A_I, t_I = -\infty; \phi_F, X^A_F, t_F] = \int \mathcal{D}\phi \mathcal{D}\pi \mathcal{D}M^A \mathcal{D}P_A \mathcal{D}X^A \delta[X^A = f^A] \exp i \int dt d^n x \times [P_A \dot{X}^A + \pi \dot{\phi} - M^A C_A - (-1 - i) P_T]. \tag{34}$$

Because of the positivity of (28) on the constraint surface, the above expression is (formally) convergent. Our strategy is to define the Euclidean theory in configuration space by integrating over the momenta in (34).

B. Euclidean path integral

Equation (34) can be written in the form

$$Z_E[\phi_I, X^A_I, t_I = -\infty; \phi_F, X^A_F, t_F] = \int \mathcal{D}\phi \mathcal{D}\pi \mathcal{D}M^A \mathcal{D}P_A \mathcal{D}X^A \delta[X^A = f^A] \exp i \int dt d^n x \times (P_T \dot{T}_E + P_{\hat{A}} \dot{X}^{\hat{A}} + \pi \dot{\phi} - M^A C_A), \tag{35}$$

where we have defined

$$T_E = T - t - it. \tag{36}$$

Integration over M^A, P_A yields

$$Z_E = \int \mathcal{D}\phi \mathcal{D}\pi \mathcal{D}X^A \delta[X^A = f^A] \exp i \int dt d^n x (\pi \dot{\phi} - N_E h - N^i_E h_i), \tag{37}$$

where h, h_i are given by (13) and we define

$$N_E = -\dot{X}^A n_A + (1 + i) n_{A=0}, \tag{38}$$

$$N^i_E = q^{ij} [X_{Aj} \dot{X}^A - (1 + i) \partial_j T]. \tag{39}$$

Notice that the above equations can be obtained from Eqs. (19) and (20) by replacing T with T_E.

Next, we integrate over π. After "completing the square," the π-dependent term to be integrated over is

$$\exp - i \frac{N_E \sqrt{q}}{2} \bigg(\pi - \frac{\dot{\phi} - N^i_E \partial_i \phi}{N_E} \bigg)^2.$$

In what follows, we shall denote the real and imaginary parts of a complex number a by a_R and a_I so that $a = a_R + i a_I$. The absolute value of a will be denoted by $|a|$. From (38) and from the fact that n^A is a future-pointing, timelike vector, we have that

$$N_{E_I} = n_{A=0} = n_A \bigg(\frac{\partial}{\partial T} \bigg)^A < 0. \tag{40}$$

This ensures that the exponential is convergent and can be integrated over π. Performing this integration yields

$$Z_E = \int \mathcal{D}\phi \mathcal{D}X^A \delta[X^A = f^A] \exp - \int d^{n+1} x L_E, \tag{41}$$

where L_E is given by

$$L_E = -\frac{i}{2} N_E \sqrt{q} \bigg[\bigg(\frac{\dot{\phi} - N^i_E \partial_i \phi}{N_E} \bigg)^2 - q^{ij} \partial_i \phi \partial_j \phi - m^2 \phi^2 \bigg]. \tag{42}$$

This is our final expression for what we call the Euclidean path integral.

Next, we show that the Euclidean path integral is indeed convergent by showing that the real part of L_E is positive. From (42), the real part of L_E is determined by the expression

$$-\frac{2L_{E_R}}{\sqrt{q}} = N_{E_I}\left(q^{ij} - \frac{N^i_{E_I}N^j_{E_I}}{|N_E|^2}\right)A_iA_j + \frac{N_{E_I}}{|N_E|^2}B^2 + 2\frac{N^i_{E_I}N_{E_R}}{|N_E|^2}A_iB + N_{E_I}m^2\phi^2, \tag{43}$$

where

$$A_i = \partial_i\phi, \qquad B = \dot{\phi} - N^i_{E_R}\partial_i\phi. \tag{44}$$

It is straightforward to show, using Eqs. (5), (20), (38), and (39), that

$$N^2_{E_I} - q_{ij}N^i_{E_I}N^j_{E_I} = \frac{1}{2}\frac{\partial^2}{\partial\dot{T}^2}(N^2 - q_{ij}N^iN^j). \tag{45}$$

Using this in conjunction with Eq. (3) gives the key inequality

$$N^2_{E_I} - q_{ij}N^i_{E_I}N^j_{E_I} = \frac{1}{2}\frac{\partial^2}{\partial\dot{T}^2}(\dot{T}^2 - \dot{X}^2) = 1 > 0. \tag{46}$$

Note that a trivial application of the Schwarz inequality shows that

$$N^2_{E_I}q^{ij}A_iA_j > (N^i_{E_I}A_i)^2. \tag{47}$$

Straightforward manipulations using the above inequalities in conjunction with (40) imply that, when A_i and B are not both identically zero,

$$-\frac{2L_{E_R}}{\sqrt{q}} < \frac{N_{E_I}}{|N_E|^2}(|B| - |N_{E_R}|\sqrt{q^{ij}A_iA_j})^2 + N_{E_I}m^2\phi^2 < 0. \tag{48}$$

Thus, as expected, the Euclidean path integral is convergent.

For a flat foliation with $T = t$, Eq. (36) defines a Euclidean time via the standard Wick rotation. Note that we did *not* first define a Lorentzian configuration space path integral and then make a Wick rotation. Rather, we defined a Euclidean phase space integral and Eq. (36) emerged as a consequence of this. For arbitrary foliations, we have that $t \neq T$ and consequently that Eq. (36) *differs* from the standard Wick rotation. This suggests that the two-point functions of the theory can be continued through this nonstandard Wick rotation. We shall confirm the existence of these Wick rotated two-point functions in Sec. III C. The Euclidean action (42) is in general complex and depends on the choice of foliation as does the Wick rotation. This is reminiscent of 't Hooft's discussion of Wick rotations in perturbative quantum gravity [17] wherein he states that the details of the Wick rotation depend on the gauge chosen.

C. "Wick rotated" two-point functions

In order to discuss Wick rotations of the form (36), it is useful to express the embedding time $X^0 = T$ in terms of x^α through a function $h(x^\alpha)$ defined by

$$T = t + h(x^\alpha). \tag{49}$$

From (36), the Euclidean time T_E is

$$T_E = -it + h(x^\alpha). \tag{50}$$

Denote the standard time-ordered Minkowski spacetime two-point function by $G(T_1, X^{\hat{A}}_1; T_2, X^{\hat{A}}_2)$. G is a function of x^α_i, $i = 1, 2$ through the dependence on x^α of the embeddings, i.e., $X^A \equiv X^A(x^\alpha)$. The Wick rotated two-point function G_E is given by

$$G_E(x^\alpha_1, x^\alpha_2) = G(T_{1E}(x^\alpha_1), X^{\hat{A}}_1(x^\alpha_1); T_{2E}(x^\alpha_2), X^{\hat{A}}_2(x^\alpha_2)), \tag{51}$$

where the right-hand side denotes a continuation of G to the complex arguments defined by (50).

To show that G_E exists, recall that the standard Minkowskian two-point function is defined by

$$G(X^A_1, X^A_2) = \langle 0|\theta(T_1 - T_2)\hat{\phi}(X^A_1)\hat{\phi}(X^A_2) + \theta(T_2 - T_1)\hat{\phi}(X^A_2)\hat{\phi}(X^A_1)|0\rangle, \tag{52}$$

where $|0\rangle$ denotes the vacuum state. As shown in Sec. IV [see the discussion after Eq. (64)], due to Lorentz invariance and the spacelike nature of $t = \text{const}$ slices, we can equally well write the two-point function as

$$\langle 0|\theta(t_1 - t_2)\hat{\phi}(X^A_1(x^\alpha_1))\hat{\phi}(X^A_2(x^\alpha_2)) + \theta(t_2 - t_1)\hat{\phi}(X^A_2(x^\alpha_2))\hat{\phi}(X^A_1(x^\alpha_1))|0\rangle. \tag{53}$$

Denoting $\langle 0|\hat{\phi}(X^A_1(x^\alpha_1))\hat{\phi}(X^A_2(x^\alpha_2))|0\rangle$ by $D(X^A_1(x^\alpha_1), X^A_2(x^\alpha_2))$ we can write the above equation as

$$G(X^A_1(x^\alpha_1), X^A_2(x^\alpha_2))|_{t_1>t_2} = D(X^A_1(x^\alpha_1), X^A_2(x^\alpha_2))|, \tag{54}$$

$$G(X^A_1(x^\alpha_1), X^A_2(x^\alpha_2))|_{t_2>t_1} = D(X^A_2(x^\alpha_2), X^A_1(x^\alpha_1))|_{t_2>t_1}, \tag{55}$$

$$G(X^A_1(x^\alpha_1), X^A_2(x^\alpha_2))|_{t_1=t_2} = D(X^A_1(x^\alpha_1), X^A_2(x^\alpha_2))|_{t_1=t_2} \tag{56}$$

$$= D(X^A_2(x^\alpha_2), X^A_1(x^\alpha_1))|_{t_1=t_2}. \tag{57}$$

To obtain the last equation, note that $t_2 = t_1$ implies that events 1 and 2 are spacelike and hence the field operators at these points commute. We use the standard expression for $D(X^A_1, X^A_2)$ and Eq. (49) to obtain

$$D(X_1^A, X_2^A) = \left(\frac{1}{2\pi}\right)^n \int \frac{d^n k}{2\omega_k} e^{-i\omega_k(T_1-T_2)+ik_i(X_1^i - X_2^i)} = \left(\frac{1}{2\pi}\right)^n \int \frac{d^n k}{2\omega_k} e^{-i\omega_k(t_1-t_2)+i[k_i(X_1^i(x_1^\alpha)-X_2^i(x_2^\alpha))-\omega_k(h(x_1^\alpha)-h(x_2^\alpha))]}. \quad (58)$$

where $\omega_k = \sqrt{\sum_{i=1}^n (k_i)^2 + m^2}$. From (51) and the above equations we obtain

$$G_E(x_1^\alpha, x_2^\alpha)|_{t_1>t_2} = \left(\frac{1}{2\pi}\right)^n \int \frac{d^n k}{2\omega_k} e^{-\omega_k(t_1-t_2)+i[k_i(X_1^i(x_1^\alpha)-X_2^i(x_2^\alpha))-\omega_k(h(x_1^\alpha)-h(x_2^\alpha))]}, \quad (59)$$

$$G_E(x_1^\alpha, x_2^\alpha)|_{t_2>t_1} = \left(\frac{1}{2\pi}\right)^n \int \frac{d^n k}{2\omega_k} e^{-\omega_k(t_2-t_1)+i[k_i(X_2^i(x_2^\alpha)-X_1^i(x_1^\alpha))-\omega_k(h(x_2^\alpha)-h(x_1^\alpha))]}, \quad (60)$$

$$G_E(x_1^\alpha, x_2^\alpha)|_{t_1=t_2} = \left(\frac{1}{2\pi}\right)^n \int \frac{d^n k}{2\omega_k} e^{i[k_i(X_1^i(x_1^\alpha)-X_2^i(x_2^\alpha))-\omega_k(h(x_1^\alpha)-h(x_2^\alpha))]} \quad (61)$$

$$= \left(\frac{1}{2\pi}\right)^n \int \frac{d^n k}{2\omega_k} e^{i[k_i(X_2^i(x_2^\alpha)-X_1^i(x_1^\alpha))-\omega_k(h(x_2^\alpha)-h(x_1^\alpha))]}. \quad (62)$$

Clearly the first two equations above define convergent integrals whereas the expressions for $t_1 = t_2$ agree with their Lorentzian counterparts and hence exist as well defined distributions.

Thus, direct evaluation of (51) indeed shows that the Wick rotated two-point functions do exist. Note that in the generally covariant formulation of parametrized field theory, the coordinate t, which is crucial to the definition of the Wick rotation (50), has no intrinsically distinguished role. Indeed, when confronted with parametrized field theory, it is difficult to guess the existence of this foliation-dependent Wick rotation and the consequent Euclideanization of the theory. From this perspective it is very satisfying to see that the well motivated definition of Euclideanization which we have used is in harmony with the properties of the standard Minkowskian two-point function.

IV. PATH INTEGRAL QUANTIZATION WITH NONSTANDARD CHOICE OF TIME

The form of the scalar field action appropriate to the f^A foliation may be obtained either by integrating the path integral (21) over the embedding variables or by performing a coordinate transformation from inertial coordinates X^A to x^α in the action (9). The action describes a scalar field on a flat Minkowski spacetime *with inertial coordinates* x^α interacting with an external field determined by $f^A(x, t)$. The issue of interest is whether, despite being classically equivalent to the standard action (9), the action in this form naturally suggests a quantization procedure based on the x^α flat spacetime which is inequivalent to the standard quantization. Since the action is quadratic in the scalar field, most of the physics is in the two-point function. An application of standard perturbative quantum field theory to this form of the action results in a computation of the two-point function in an expansion in powers of the external field. The result can be compared to the standard Minkowskian two-point function.

Let p_i, $i = 1, 2$ be a pair of events on the flat spacetime. We denote their spacetime coordinates in the x^α coordinate system by $\vec{x}_i$ and their coordinates in the inertial X^A system by $\vec{X}_i$. Let the two-point function in the f^A formulation be $G_f(\vec{x}_1, \vec{x}_2)$ and let the standard two-point function be $G(\vec{X}_1, \vec{X}_2)$. In the case of the nonstandard foliation, the two-point operator is

$$\hat{O}_f = \theta(t_1 - t_2)\hat{\phi}(p_1)\hat{\phi}(p_2) + \theta(t_2 - t_1)\hat{\phi}(p_2)\hat{\phi}(p_1), \quad (63)$$

whereas in the usual inertial foliation the two-point operator is

$$\hat{O} = \theta(T_1 - T_2)\hat{\phi}(p_1)\hat{\phi}(p_2) + \theta(T_2 - T_1)\hat{\phi}(p_2)\hat{\phi}(p_1). \quad (64)$$

Here θ is the usual step function which implements time ordering. It is readily verified, using the fact that the $t =$ const slices are spacelike with future-pointing timelike normal $(dt)_\alpha$, that if p_1 and p_2 are causally related then they have the same time ordering with respect to t as with T. Further, in the standard quantization, if p_1 and p_2 are not causally related then $[\hat{\phi}(p_1), \hat{\phi}(p_2)]$ vanishes and the ordering does not matter. Thus, we can as well replace $\hat{O}$ by $\hat{O}_f$ in the standard quantization.

If $G_f(\vec{x}_1, \vec{x}_2) \neq G(\vec{X}_1, \vec{X}_2)$, we may conclude that either the f-dependent quantization and the standard quantization of the operator $\hat{O}_f$ are inequivalent or that the vacuum state selected by the procedure to calculate G_f is different from the standard vacuum state. Thus, in both cases the choice of foliation affects the quantum theory either in its representation of operators or in its identification of the vacuum state. To illustrate our ideas, we work through a simple $(1+1)$-dimensional example in Sec. IV A. It is worth emphasizing that the example represents nothing more or less than the quantization of a scalar field on a flat background in a specific and unusual coordinate system. The issue is whether such a choice of space and *time* coordinates presents the theory in a form which suggests a natural quantization which is inequivalent to the standard Poincaré invariant quantum theory.

In Sec. IV B we describe our framework for a general foliation in $(n+1)$ dimensions. The actual computations which would show existence (or lack thereof) of inequivalent quantizations are left for future work.

A. Two-dimensional example

In two spacetime dimensions denote the inertial time by T and the inertial space coordinate by X and specify the foliation by $T(x,t) = t + f(x)$ and $X(x,t) = x$, where $f(x)$ is a function of compact support. The path integral (21) can be integrated over X^A to give

$$Z = \int \mathcal{D}\phi \exp iS[\phi(x,t)]. \tag{65}$$

The irrelevant c-number determinant has been dropped and $S[\phi(x,t)]$ is defined as

$$S[\phi(x,t)] = \frac{1}{2}\int dxdt(\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + f^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - m^2\phi^2). \tag{66}$$

Here $\eta_{\alpha\beta}$ denotes the flat metric with line element $ds^2 = (dt)^2 - (dx)^2$ and $f^{\alpha\beta}$ is defined as

$$f^{00} = -\left(\frac{df}{dx}\right)^2, \qquad f^{01} = f^{10} = \frac{df}{dx}, \qquad f^{11} = 0. \tag{67}$$

The reader is requested to bear with us, in that we have changed our conventions for the metric signature from $(-+)$ to $(+-)$ *only in this subsection*. The reason is to ensure easy cross-checking of numerical factors for Feynman diagrams with standard field theory references (see, for example, [18]) which use the $(+-)$ conventions.

The action (65) describes a scalar field interacting with a static potential on the (x,t) Minkowski spacetime and G_f may be computed via standard Feynman diagrammatics. In momentum space, we have

$$G_f(\vec{p},\vec{q}) := \int d^2x d^2y e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}\cdot\vec{y}} G_f(\vec{x},\vec{y}), \tag{68}$$

where we have used the notation $\vec{x}$ for (x^0, x), $\vec{p}$ for (p^0, p), and $\vec{p}\cdot\vec{x}$ for $(p^0x^0 - px)$. The Fourier transform of $f^{\alpha\beta}(\vec{x})$ is defined as

$$f^{\alpha\beta}(\vec{k}) = \int d^2x f^{\alpha\beta}(\vec{x}) e^{i\vec{q}\cdot\vec{x}}. \tag{69}$$

No loops are encountered in the Feynman diagrams since (66) has only two-point interactions. Each vertex contributes a factor of

$$iC(\vec{k}_1,\vec{k}_2) = if^{\mu\nu}(\vec{k}_1 - \vec{k}_2)k_{1\mu}k_{2\nu} \tag{70}$$

with incoming momentum $\vec{k}_1$ and outgoing momentum $\vec{k}_2$. Each propagator contributes a factor of $-i/(\vec{k}^2 - m^2 + i\epsilon)$. Here $\vec{k}^2 = \vec{k}\cdot\vec{k}$. Notice that the choice of time t dictates the $i\epsilon$ prescription in the propagator.

With the correct factors of i and 2π we have

$$G_f(\vec{p},\vec{p}) = \frac{i(2\pi)^2}{\vec{p}^2 - m^2 + i\epsilon}\delta(\vec{p},\vec{q}) + \frac{-i}{\vec{p}^2 - m^2 + i\epsilon}C(\vec{p},\vec{q})\frac{1}{\vec{q}^2 - m^2 + i\epsilon} + \sum_{n=2}^{\infty}\frac{(-1)^n i}{(2\pi)^{2n-2}}\frac{1}{\vec{p}^2 - m^2 + i\epsilon}\int\prod_{j=1}^{n-1} d^2k_j \times\left[C(\vec{p},\vec{k}_1)\frac{1}{\vec{k}_1^2 - m^2 + i\epsilon}C(\vec{k}_1,\vec{k}_2)\cdots C(\vec{k}_{n-2},\vec{k}_{n-1})\frac{1}{\vec{k}_{n-1}^2 - m^2 + i\epsilon}C(\vec{k}_{n-1},\vec{q})\right]\frac{1}{\vec{q}^2 - m^2 + i\epsilon}. \tag{71}$$

From (67) and (70) we have that

$$C(\vec{k},\vec{l}) = C^{(1)}(\vec{k},\vec{l}) + C^{(2)}(\vec{k},\vec{l}), \tag{72}$$

$$C^{(1)}(\vec{k},\vec{l}) = 2\pi i\delta(k_0,l_0)f(k-l)k_0(k^2 - l^2), \tag{73}$$

$$C^{(2)}(\vec{k},\vec{l}) = k_0^2\delta(k_0,l_0)\int dsf(s)f(k-l-s)s(k-l-s). \tag{74}$$

The $\delta(k_0,l_0)$ factors ensure that, as expected, the static potential conserves energy. Of course, momentum is not conserved due to the lack of translational invariance in the presence of the potential. Using (73) and (74), we can write $G_f(\vec{p},\vec{q})$ as an expansion in orders of f so that

$$G_f(\vec{p},\vec{q}) = \sum_{N=0}^{\infty} G_f^{(N)}(\vec{p},\vec{q}), \tag{75}$$

where $G_f^{(N)}(\vec{p},\vec{q})$ is of order f^N.

On the other hand, the standard two-point function is

$$G(\vec{X}, \vec{Y}) = \frac{i}{(2\pi)^2} \int d^2k \frac{e^{-i\vec{k}\cdot(\vec{X}-\vec{Y})}}{\vec{k}^2 - m^2 + i\epsilon}, \tag{76}$$

with Fourier transform

$$G(\vec{p}, \vec{q}) = \int d^2x d^2y e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{y}} G(\vec{X}(\vec{x}), \vec{Y}(\vec{y})) \tag{77}$$

$$= i\delta(p_0, q_0) \int dxdydk \frac{e^{i(p+k)x} e^{-i(q+k)y}}{\vec{k}^2 - m^2 + i\epsilon} e^{iq_0[f(x)-f(y)]}, \tag{78}$$

where we have substituted for $\vec{X}, \vec{Y}$ in terms of $\vec{x}, \vec{y}$. We can expand the last exponential in (78) in a power series and hence obtain G as an expansion in powers of f, i.e.,

$$G(\vec{p}, \vec{q}) = \sum_{N=0}^{\infty} G^{(N)}(\vec{p}, \vec{q}), \tag{79}$$

where $G^{(N)}(\vec{p}, \vec{q})$ is of order f^N. To second order we have, from (78),

$$G^{(0)}(\vec{p}, \vec{q}) = \frac{i(2\pi)^2}{\vec{p}^2 - m^2 + i\epsilon} \delta(\vec{p}, \vec{q}), \tag{80}$$

$$G^{(1)}(\vec{p}, \vec{q}) = 2\pi q_0 \delta(p_0, q_0) f(p-q) \left(\frac{1}{\vec{p}^2 - m^2 + i\epsilon} - \frac{1}{\vec{q}^2 - m^2 + i\epsilon} \right), \tag{81}$$

$$G^{(2)}(\vec{p}, \vec{q}) = \frac{-iq_0^2}{2} \delta(p_0, q_0) \int dk f(p+k) f(-q-k) \times \left(\frac{1}{\vec{q}^2 - m^2 + i\epsilon} + \frac{1}{\vec{p}^2 - m^2 + i\epsilon} - \frac{2}{q_0^2 - k^2 - m^2 + i\epsilon} \right). \tag{82}$$

Clearly, $G^{(0)}(\vec{p}, \vec{q}) = G_f^{(0)}(\vec{p}, \vec{q})$. From (71) and (73), we have

$$G_f^{(1)}(\vec{p}, \vec{q}) = \frac{-i}{\vec{p}^2 - m^2 + i\epsilon} C^{(1)}(\vec{p}, \vec{q}) \frac{1}{\vec{q}^2 - m^2 + i\epsilon} = \frac{-i}{\vec{p}^2 - m^2 + i\epsilon} 2\pi i \delta(p_0, q_0) f(p-q) p_0 (p^2 - q^2) \frac{1}{\vec{q}^2 - m^2 + i\epsilon}$$
$$= 2\pi p_0 \delta(p_0, q_0) f(p-q) \left(\frac{1}{\vec{p}^2 - m^2 + i\epsilon} - \frac{1}{\vec{q}^2 - m^2 + i\epsilon} \right) = G^{(1)}(\vec{p}, \vec{q}). \tag{83}$$

Finally, from (71), (73), and (74) we have

$$G_f^{(2)}(\vec{p}, \vec{q}) = \frac{i}{(2\pi)^2} \frac{1}{\vec{p}^2 - m^2 + i\epsilon} \int d^2k C^{(1)}(\vec{p}, \vec{k}) \frac{1}{\vec{k}^2 - m^2 + i\epsilon} C^{(1)}(\vec{k}, \vec{q}) \frac{1}{\vec{q}^2 - m^2 + i\epsilon} + \frac{(-i)}{\vec{p}^2 - m^2 + i\epsilon} C^{(2)}(\vec{p}, \vec{q}) \frac{1}{\vec{q}^2 - m^2 + i\epsilon}$$
$$= -iq_0^2 \delta(q_0, p_0) \int dk \left\{ \frac{f(p-k) f(-q+k)}{(\vec{p}^2 - m^2 + i\epsilon)(\vec{q}^2 - m^2 + i\epsilon)} \left[\frac{(p^2 - k^2)(k^2 - q^2)}{q_0^2 - k^2 - m^2 + i\epsilon} + (p-k)(k-q) \right] \right\}. \tag{84}$$

After some algebra it can be shown that

$$G_f^{(2)}(\vec{p}, \vec{q}) = G^{(2)}(\vec{p}, \vec{q}) + \frac{iq_0^2 (p+q)}{(\vec{p}^2 - m^2 + i\epsilon)(\vec{q}^2 - m^2 + i\epsilon)} \times \delta(q_0, p_0) \int dk f(p+k) f(-q-k) \times \left(\frac{p+q}{2} + k \right). \tag{85}$$

Setting $(p+q)/2 + k = l$ we have

$$\int_{-\infty}^{\infty} dk f(p+k) f(-q-k) \left(\frac{p+q}{2} + k \right) = \int_{-\infty}^{\infty} dl l f\left(l + \frac{p-q}{2} \right) f\left(-l + \frac{p-q}{2} \right), \tag{86}$$

which vanishes by virtue of the integrand being odd in l.

Hence G and G_f are identical to second order. We have exhibited the calculations in some detail to show that this agreement is not entirely trivial as well as to illustrate our ideas in a concrete setting. In the next section we discuss the case of an arbitrary foliation in $(n+1)$ dimensions.

B. The general case

Integration of (21) over X^A and dropping of the irrelevant c-number determinant term gives

$$Z = \int \mathcal{D}\phi \exp iS[\phi(x^i, t)]. \tag{87}$$

Here

$$S[\phi(x,t)] = -\frac{1}{2} \int d^{n+1}x \sqrt{\eta} (\eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + m^2 \phi^2), \tag{88}$$

where $\eta^{\mu\nu}$ is defined by (1) with $X^A = f^A$. In the above equation, the coordinates x^α are *fixed once and for all* by the choice of the embedding $f^A(x, t)$. Just as for the 2D example in Sec. IVA, the action (88) can be written as the sum of a free part and an interaction term describing interaction with external fields.

The free part of the action describes a scalar field propagating on a flat spacetime with x^α *as inertial coordinates* so that the line element of this spacetime is $ds^2 = -(dt)^2 + \sum_{i=1}^n (dx^i)^2$. We denote the flat spacetime metric defined by this line element by $\eta_f^{\mu\nu}$. The analog of $f(x, t)$ in Sec. IVA is

$$h^A(x^i, t) := X^A(x^i, t) - x^\alpha \delta^A_\alpha, \tag{89}$$

i.e., $h^0 = T - t$, $h^1 = X^1 - x^1$, etc. We define the external fields $f^{\mu\nu}$ and α by

$$f^{\mu\nu} = \sqrt{\eta}\eta^{\mu\nu} - \eta_f^{\mu\nu}, \qquad \alpha = \sqrt{\eta} - 1. \tag{90}$$

The above equation is defined in the *fixed* (x^i, t) coordinate system and $\sqrt{\eta}$ is calculated in this coordinate system. Note that in analogy to (67), $f^{\mu\nu}$ and α can be obtained as a series expansion in powers of $\partial_\alpha h^A$.

The action (88) takes the form

$$S[\phi(x, t)] = -\frac{1}{2}\int d^{n+1}x(\eta_f^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + m^2\phi^2 + f^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \alpha m^2\phi^2). \tag{91}$$

G_f can be defined as a standard perturbative quantum field theory expansion in powers of the external field $f^{\mu\nu}$, α. This expansion makes use of the propagator defined from the free part of the action which in turn derives its structure from the flat metric $\eta_f^{\mu\nu}$.

Our general strategy is as follows. The momentum space two-point function $G_f(\vec{p}, \vec{q})$ is defined by an $(n+1)$-dimensional analog of the expression (71) with $C(\vec{k}, \vec{l})$ defined by an appropriate generalization of (70). $C(\vec{k}, \vec{l})$ itself is a sum over $C^{(N)}(\vec{k}, \vec{l})$ where the $C^{(N)}$ are of order $(h^A)^N$. Unlike the specific two-dimensional example discussed in Sec. IVA where $N = 1, 2$, here N can in general range from 1 to ∞. The Nth order (in h^A) contribution $G_f^{(N)}$ to G_f can be calculated. The standard two-point function $G(\vec{X}(\vec{x}), \vec{Y}(\vec{y}))$ can be Fourier transformed in analogy to (77) to give $G(\vec{p}, \vec{q})$. The latter can be expanded in powers of h^A in analogy to the expansion defined by Eq. (79). Finally, the Nth order contributions $G^{(N)}(\vec{p}, \vec{q})$ can be compared with $G_f^{(N)}(\vec{p}, \vec{q})$.

Though the general strategy seems straightforward, the following discussion indicates that there are complications in defining $G_f(\vec{x}_1, \vec{x}_2)$ in the manner sketched above (similar complications, arising from "illegal" expansions of the relevant exponential inside the $(n+1)$-dimensional analog of (78) for a general foliation, may exist for the computation of $G^{(N)}(\vec{p}, \vec{q})$). If $h^A(x^i, t)$ are of compact support in x^α, one can check that contributions to $G_f(\vec{p}, \vec{q})$ to any order in $f^{\mu\nu}$ are UV finite. For an arbitrary choice of f^A, h^A is restricted by the boundary conditions (15) to be of compact support only in x^i and not in both x^i and t. For generic choices of h^A, UV divergences may possibly exist. Further, even if there are no UV divergences, $G_f(\vec{x}_1, \vec{x}_2)$ is defined in position space via the inverse Fourier transform of $G_f(\vec{p}, \vec{q})$, the latter being the sum of contributions at every order of perturbation theory. Whether this sum converges well enough for its inverse Fourier transform to exist (as a distribution) is also not clear. We shall concern ourselves with an investigation of these issues in future work.

Our computations in Sec. IVA indicate that a brute force term by term analysis would probably be quite involved. In the remainder of this section we propose a line of attack springing from considerations of a more general nature. In what follows, we shall simply assume that there is some way to define $G_f(\vec{x}_1, \vec{x}_2)$ as a distribution by using standard perturbative quantum field theory techniques on the flat $\eta_f^{\mu\nu}$ spacetime such that its only singularities are when $\vec{x}_1 = \vec{x}_2$. Once this fairly restrictive assumption has been made, $G_f(\vec{x}_1, \vec{x}_2)$ is constrained by the following argument.

The equations of motion from the action (88) are

$$\partial_\mu(\sqrt{\eta}\eta^{\mu\nu}\partial_\nu\phi) - \sqrt{\eta}m^2\phi = \sqrt{\eta}(\Box - m^2)\phi = 0. \tag{92}$$

Here $\Box - m^2 = \eta^{AB}\partial_A\partial_B - m^2$ is a *scalar* differential operator. From its definition, it follows that $G_f(\vec{x}_1, \vec{x}_2)$ is Green's function for the operator $\partial_\mu(\sqrt{\eta}\eta^{\mu\nu}\partial_\nu) - \sqrt{\eta}m^2$ so that

$$\left[\frac{\partial}{\partial x^\mu}\sqrt{\eta(x)}\eta^{\mu\nu}(x)\frac{\partial}{\partial x^\nu} - \sqrt{\eta}m^2\right]G_f(\vec{x}, \vec{y}) = i\delta(\vec{x}, \vec{y}). \tag{93}$$

Using the fact that $\delta(\vec{x}, \vec{y})$ transforms as a unit density in its first argument and as a scalar in its second, we have

$$(\Box - m^2)G_f(\vec{x}, \vec{y}) = i\delta(\vec{X}, \vec{Y}). \tag{94}$$

Thus G_f and G are constrained by virtue of their being Green's functions for the same differential operator. In the context of our restrictive assumptions this implies that their difference, $\Delta G_f(\vec{x}_1, \vec{x}_2) = G(\vec{X}_1(\vec{x}_1), \vec{X}_2(\vec{x}_2)) - G_f(\vec{x}_1, \vec{x}_2)$, must be a smooth solution to the Klein-Gordon equation.

First consider the case when h^A are of compact support. This means that x^α agree with X^A outside a compact region $K \subset R^{n+1}$ and that $f^{\mu\nu}, \alpha$ vanish outside K. Suppose that we could show that $\Delta G_f(\vec{x}_1, \vec{x}_2) = 0$ for $\vec{x}_1, \vec{x}_2 \in R^{n+1} - K$ (here $R^{n+1} - K$ refers to the complement of K). Then the following argument shows that

$\Delta G_f(\vec{x}_1, \vec{x}_2) = 0$ everywhere. Fix the point p_1 such that $p_1 \in R^{n+1} - K$. Then, we have that $\Delta G_f(\vec{x}_1, \vec{x}_2) = 0$ for $\vec{x}_2 \in R^{n+1} - K$ and that $\Delta G_f(\vec{x}_1, \vec{x}_2)$ satisfies the Klein-Gordon equation. From the uniqueness of evolution from initial data on a Cauchy slice contained in $R^{n+1} - K$, it follows that $\Delta G_f(\vec{x}_1, \vec{x}_2)$ vanishes for all $\vec{x}_2 \in R^{n+1}$. Since G and G_f are symmetric in their arguments, it follows that $\Delta G_f(\vec{x}_1, \vec{x}_2)$ vanishes also for all $\vec{x}_1 \in R^{n+1}$ and all $\vec{x}_2 \in R^{n+1} - K$. Again using uniqueness of evolution from a Cauchy slice in $R^{n+1} - K$, it follows that $\Delta G_f(\vec{x}_1, \vec{x}_2)$ vanishes for all $\vec{x}_1, \vec{x}_2 \in R^{n+1}$.

If h^A is not of compact support, it must still be true from the boundary conditions (15) that x^α agrees with X^A outside a timelike tube τ and that $f^{\mu\nu}$, α vanish outside τ. Again, suppose that we could show that $\Delta G_f(\vec{x}_1, \vec{x}_2) = 0$ for $\vec{x}_1, \vec{x}_2 \in R^{n+1} - \tau$. Then, using the fact that the only smooth solution of the Klein-Gordon equation with support restricted to τ is the trivial solution,[4] arguments similar to those used for the case of h^A having compact spacetime support show that, once again, $\Delta G_f(\vec{x}_1, \vec{x}_2)$ vanishes for all $\vec{x}_1, \vec{x}_2 \in R^{n+1}$.

Thus, the absence (or existence) of inequivalent quantizations has been reduced to the vanishing (or not) of $\Delta G_f(\vec{x}_1, \vec{x}_2)$ for $\vec{x}_1, \vec{x}_2$ both outside the support of h^A. We propose to analyze this behavior of $\Delta G_f(\vec{x}_1, \vec{x}_2)$ through a *position space* perturbative expansion of $G_f(\vec{x}_1, \vec{x}_2)$ in future work.

V. DISCUSSION

The three issues dealt with in this work are the correct measure for the Lorentzian path integral, the construction of a convergent Euclidean path integral to compute the vacuum wave function, and the possibility of inequivalent quantizations based on different choices of time. These issues arise in the quantization of any generally covariant theory. Below, we remark on each of them in view of the results obtained in Secs. II, III, and IV.

The Lorentzian path integral measure obtained in Appendix A 1 is very similar to the one obtained by Fradkin and Vilkovisky in [13] (and anticipated even earlier by Leutwyler in [19]) for quantum gravity. In both cases (i.e., parametrized field theory and gravity), the measure appears noncovariant in that it explicitly refers to the coordinate time t. In the case of gravity, Fradkin and Vilkovisky argue that their measure is, despite appearances to the contrary, diffeomorphism invariant. The key point is that diffeomorphisms corresponding to time reparametrizations must be handled with extreme care because of the nontrivial contribution of nonsmooth paths to the transition amplitude. The reason for the noncovariant factors in the measure is that the path integral is defined in terms of the limit of a discretization which itself depends on the choice of time. The noncovariant factor in the measure exactly compensates for this intrinsic discretization dependent noncovariance, so as to make the measure diffeomorphism invariant. Since parametrized field theory is also a *spacetime* diffeomorphism invariant theory, the arguments in [13] apply to it. Since we have derived the measure for this simple system from the correct Liouville measure in the phase space path integral, we believe[5] in its validity and interpret our results as supportive of the Fradkin-Vilkovisky measure being the correct one for quantum gravity as opposed to the more commonly used de Witt measure [14].[6] We also would like to note that contrary to what Fradkin and Vilkovisky state in [12], the above subtleties regarding the measure do not have anything to do with the appearance of structure functions in the constraint algebra; clearly in parametrized field theory the constraint algebra for the constraints C_A is Abelian and no structure functions appear. Rather, these subtleties seem to be entirely due to the property of general covariance.

Our definition of Euclideanization was motivated by the work of Schleich [10]. She was interested in constructing diffeomorphism invariant, convergent Euclidean path integrals from the correct reduced phase space path integral expression for the vacuum wave function. Her strategy was to start from the explicit reduced phase space path integral expression and rewrite it as a convergent, diffeomorphism invariant, configuration space path integral. For a treatment of gravity beyond perturbation theory, an explicit characterization of the reduced phase space is not available and Schleich's strategy is hard to implement. In this work we have suggested a strategy which does not require an explicit parametrization of the reduced phase space. We start from a gauge fixed expression in phase space and integrate out the momenta. We are *not* concerned with maintaining diffeomorphism invariance and indeed this is an aspect of our constructions which we need to understand better. We have verified that the Euclidean action (42) is spatially diffeomorphism invariant. We do not know if it displays any sort of invariance related to Lorentzian time reparametrizations. Although in general the Euclidean action is complex, it is easily verified that for the case of an inertial foliation, the action turns out to be the standard, real Euclidean action. The difference between Schleich's aims (as we under-

[4]We shall display our proof of this assertion elsewhere.

[5]We again emphasize that we have neglected end point terms and used only canonical gauges in our arguments (see [20,21] in this regard). Nevertheless, our intuition is that our result for the measure is sufficiently robust to survive a more careful treatment of end point contributions.

[6]For perturbative quantum gravity calculations in a *dimensional regularization scheme*, contributions of the nontrivial local factor in the measure are regulated away to zero because they are proportional to $\delta^4(0)$. However for any nonperturbative treatment, this factor should be important.

stand them—of course we could be in error in our understanding) and ours may be stated in this context as follows. Whereas we are content with the form of the action given by (42), Schleich would take the flat foliation related standard Euclidean action and construct parametrized Euclidean field theory to obtain an explicitly diffeomorphism invariant, convergent path integral expression for the vacuum wave function.

We would like to emphasize again that in our arguments for the phase space path integral [Eq. (25) in Sec. III A and Eq. (A11) in Appendix A 1], we have neglected end point contributions. These contributions are important (see [16,20–22]). A more careful treatment of these end point contributions is desirable. Indeed, this seems to be the only possible obstacle to an application of our ideas to quantum gravity in the asymptotically flat case where a true, nonvanishing, positive Hamiltonian exists. If we neglect the end point contributions and use a canonical gauge independent of momenta, it does seem possible to try out our proposal for asymptotically flat quantum gravity. If end point contributions could be taken care of and if we could do the relevant calculations, we would expect to get an, in general complex, convergent Euclidean action for gravity. It is conceivable that progress could then be made towards numerical evaluation of the vacuum wave function. In fact, Loll and coworkers [23] have embarked on a program of numerical evaluation of Euclidean path integrals, but their analytical justification [24] seems to have as an input the de Witt measure, which we suspect is incorrect. It is worth emphasizing that our proposal *does not* apply to configurations of the gravitational field which admit no preferred sense of time at all (not even an asymptotic one) and consequently no true, nonvanishing Hamiltonian. This is the case for cosmological solutions with compact (without boundary) spatial topology. Note that in the absence of a true Hamiltonian, it is not possible to define a lowest energy vacuum state, let alone display vacuum wave functions via path integrals.

Finally we turn to a discussion of the issue of inequivalent quantizations. As shown in Sec. IV, we have connected this issue to that of gauge independence of the time-ordered two-point function. Note that by virtue of our boundary conditions (15) we have disallowed all global Poincaré transformations (with the exception of time translation). The two-point function in different gauges actually corresponds to (in the Hamiltonian framework) the evaluation of the vacuum expectation value of the same Dirac observable in different gauges. The Dirac observables can be constructed as "evolving constants of motion" [25] from observables corresponding to initial data on a fixed $T = 0$ slice. The latter observables can be constructed from the data (ϕ, π, X^A, P^A) by a Hamilton-Jacobi type of canonical transformation [1]. Note that the sort of gauge independence which would ensure the absence of inequivalent quantizations is qualitatively different from the more commonly encountered gauge independence of the S-matrix in Poincaré invariant quantum field theory. It may well turn out that there is no inequivalent quantization as far as the two-point function is concerned but, as we have tried to argue, the verification of this is nontrivial. We would also like to make contact with the existence of unitarily inequivalent quantizations in higher dimensions noted in [6] in the context of canonical quantization.

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APPENDIX

1. The path integral measure for ϕ-dependent gauges.

Consider the following momenta-independent but otherwise arbitrary gauge fixing conditions $\chi_A[X^B, \phi; y) = 0$. The notation indicates that χ^A is a functional of X^B and ϕ and a function of the point with coordinates y^α. In what follows we shall suppress the X^B, ϕ dependence in our notation. The contribution of the ghosts (ω^A, ω^{*A}) to the phase space action (11) is

$$S_{gh} = \int d^{n+1}x d^{n+1}y \omega^{*A}(x)\{C_A(x), \chi_B(y)\}\omega^B(y), \quad \text{(A1)}$$

where

$$\{C_A(x), \chi_B(y)\} = -\frac{\delta\chi_B(y)}{\delta X^A(x)} - \frac{\delta\chi_B(y)}{\delta\phi(x)}\bigg[-n_A(x)\frac{\pi(x)}{\sqrt{q(x)}} + q^{ij}(x)X_{Aj}(x)\partial_j\phi(x)\bigg]. \quad \text{(A2)}$$

Integration of the phase space path integral over M^A, P_A can be done as before to obtain

$$Z = \int \mathcal{D}\phi\mathcal{D}\pi\mathcal{D}X\mathcal{D}\omega^*\mathcal{D}\omega\,\delta[\chi] \times \exp\bigg[i\int dtd^nx(\pi\dot{\phi} - Nh - N^ih_i) + iS_{gh}\bigg]. \quad \text{(A3)}$$

Notice that from (A2), S_{gh} is linear in π and hence the total action is still quadratic in π. It is straightforward to integrate (A3) over π to obtain

$$Z = \int \mathcal{D}\phi \mathcal{D}X \mathcal{D}\omega^* \mathcal{D}\omega \left[\det \frac{iN}{\sqrt{q}}\right]^{-1/2} \delta[\chi] \times \exp\left[i \int dt d^n x(\pi\dot{\phi} - Nh - N^i h_i) + iS_{gh}\right], \tag{A4}$$

where S_{gh} is evaluated at the classical value of π given by

$$\pi_{\text{class}} = \sqrt{q}\left(\frac{\dot{\phi} - N^i \partial_i \phi}{N}\right). \tag{A5}$$

The only nontrivial step in the computation is to use the Grassmanian nature of ω^{*A} to conclude that $(\omega^{*A}(x)n_A(x))^2$ vanishes. As usual, q^{ij}, N, N^i are interpreted as functions of X^A. The ghost variables can be integrated over to give the determinant of $\{C_A(x), \chi_B(y)\}$ where the latter is given by the right-hand side of Eq. (A2) evaluated at $\pi = \pi_{\text{class}}$ given by (A5). It is straightforward to verify that

$$\{C_A(x), \chi_B(y)\}|_{\pi=\pi_{\text{class}}} = -\left[\frac{\delta \chi_B(y)}{\delta X^A(x)} + \frac{\delta \chi_B(y)}{\delta \phi(x)} \frac{\partial \phi(x)}{\partial X^A(x)}\right]. \tag{A6}$$

The operator $\frac{\partial}{\partial X^A(x)}$ is defined via the invertible dependence of the embeddings $X^A(x)$ on the coordinates x^α. Thus, Eq. (A6) can be rewritten as

$$\begin{aligned}\{C_A(x), \chi_B(y)\}|_{\pi=\pi_{\text{class}}} &= -\frac{\delta \mathcal{L}_\xi \chi_B(y)}{\delta \xi^A(x)} \\ &= -\frac{\delta \mathcal{L}_\xi \chi_B(y)}{\delta \xi^\alpha(x)} \frac{\partial x^\alpha}{\partial X^A(x)}.\end{aligned} \tag{A7}$$

Using this and the fact that the Jacobian of the coordinate transformation from $X^A \to x^\alpha$ is $\sqrt{\eta}$, we have

$$Z = \int d\mu[\phi, X^A] \delta[\chi_B] \det\left(\frac{\delta \mathcal{L}_\xi \chi_B(y)}{\delta \xi^\alpha(x)}\right) \exp iS[\phi, X^A]. \tag{A8}$$

Here $S[\phi, X^A]$ is the classical action (10), and the path integral measure can be written in the context of an appropriate discretization as

$$d\mu[\phi, X^A] = \prod_x \eta^{tt}(x) \eta^{-1/4}(x) d\phi(x) dX^A(x). \tag{A9}$$

2. Gauge independence of Z_a and its relation to the vacuum wave function.

In the case where the gauge fixing constraints are independent of momenta, gauge independence of the transition amplitude (24) may be shown (modulo the caveats mentioned in Sec. III A) through methods similar to those employed in [11]. We make a canonical transformation from (q_i, p_i), $i = 1 \cdots n$ to new conjugate pairs $(\bar{q}_l, \bar{p}_l)$, $l = 1 \cdots n - m$ and (Q_α, P_α), $\alpha = 1 \cdots m$. Here $Q_\alpha = \chi_\alpha$ and $(\bar{q}_l, Q_\alpha)$ encode the same information as q_i. On the surface defined by $Q_\alpha = C_\alpha = 0$, P_α is a function of $\bar{q}_l, \bar{p}_l$. It can be checked that (24) reduces to

$$Z(\bar{q}_{lI}, t_I; \bar{q}_{lF}, t_F) = \int \mathcal{D}\bar{q} \mathcal{D}\bar{p} \exp\left[i \int \bar{p}_l \dot{\bar{q}}_l - H(\bar{p}, \bar{q})\right]. \tag{A10}$$

Gauge independence of (24) under infinitesimal changes of gauge can be checked [11] by subjecting the gauge condition to a canonical transformation generated by the constraints. In this treatment, end point contributions arising from the canonical transformations encountered are ignored and (24) [as well as (A10)] is identified with the transition amplitude between the gauge fixed end points q_{iI} and q_{iF}.

An identical treatment can also be applied to show the gauge independence of the expression (25). It is straightforward to check that (25) reduces to

$$Z_a(\bar{q}_{lI}, t_I; \bar{q}_{lF}, t_F) = \int \mathcal{D}\bar{q} \mathcal{D}\bar{p} \exp\left[i \int \bar{p}_l \dot{\bar{q}}_l - aH(\bar{p}, \bar{q})\right], \tag{A11}$$

and that its gauge independence is ensured by virtue of the fact that $H(q_i, p_i)$ commutes (weakly) with the constraints. Again, we disregard various end point contributions coming from canonical transformations. It is straightforward to see that in operator language,

$$Z_a = \langle \bar{q}_{lF}, t_F | \exp[-i(a-1)\hat{H}(t_F - t_I)] | \bar{q}_{lI}, t_I \rangle. \tag{A12}$$

Since (q_{lI}, q_{lF}) satisfy the gauge conditions, we may identify them with $(\bar{q}_{lI}, \bar{q}_{lF})$. Then, under the assumption that the ground state energy vanishes and with a chosen such that it has negative imaginary part, the usual Feynman-Kac–type arguments show that

$$\begin{aligned}Z_a(\bar{q}_{lI}, t_I = -\infty; \bar{q}_{lF}, t_F) &= Z_a(q_{iI}, t_I = -\infty; q_{iF}, t_F) \\ &= \psi_0(q_F, t_F)\psi_0^*(q_I, t_I).\end{aligned} \tag{A13}$$

Here ψ_0 is the vacuum wave function, the vacuum being defined as the lowest energy state of the quantum operator corresponding to the classical Hamiltonian H.

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