Current Science, September 5, 1986, Vol. 55, No. 17

A GRAVITATIONAL ANALOG OF THE DIRAC MONOPOLE

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ABSTRACT

We present an exact analogy between rotation in stationary geometries and magnetic fields based on a natural decomposition of spacetime into space and time in the spirit of Kaluza-Klein theories. This analogy provides a framework for the discussion of gravitational monopoles. We present a general argument to show that gravitational monopoles violate causality. We also clarify related issues raised recently in the literature.

1. INTRODUCTION

THERE is a remarkable similarity between the laws of electromagnetism and gravitation. For instance, Newton's law of gravitation and Coulomb's law in electrostatics are both inverse square laws. The source of the field in one case is mass and in the other, electric charge. In electromagnetism, a moving charge produces both magnetic and electric fields. However, in Newtonian gravitation there is no analogue of the magnetic field; a moving mass produces the same gravitational field as a mass at rest, if the instantaneous mass distributions are identical. This is a reflection of the fact that, unlike electromagnetism Newtonian gravitation is not relativistically invariant. The situation in Einstein's general theory of relativity is quite different from the Newtonian case, and quite similar to electromagnetic theory. In Einstein's theory the gravitational field produced by a body depends not only on the distribution of matter but also on its state of motion. The mathematical expression of this fact is that the source of gravitational field is the energy-momentum tensor ($T_{\mu\nu}$) which has components corresponding to mass (T_{00}) as well as motion (T_{0i}). The gravitational analogue of the magnetic force—gravimagnetism—is like the Lorentz force in electrodynamics and depends on the velocity of the test particle. It has physical consequences like the dragging of inertial frames and the Lense-Thirring effect. Experimental tests of general relativity so far measure only the gravielectric part of the field. These are small but the gravimagnetic component is smaller still by a factor of (v/c) where v is a typical velocity in the problem. Experiments to detect this component are in progress[1].

To proceed with the analogy we notice that the Lorentz force $(e\bar{v} \times \bar{B})$ is similar in form to the Coriolis force $m(2\bar{v} \times \bar{\omega})$ in a rotating frame of reference. From the principle of equivalence between inertial and gravitational forces it is plausible that the gravimagnetic force is similar to rotation. In this paper we develop this analogy both from the physical and mathematical points of view and exhibit precise and detailed correspondence between rotation and magnetic fields. In electromagnetism there has long been a conjecture about the possible existence of magnetic monopoles. Given the detailed similarity between rotation and magnetic fields we ask if, in gravitation there is such a thing as a gravimagnetic monopole. This question has been discussed earlier in the literature[2-5]. We present a mathematically simple and physically motivated discussion of this question based on the analogy between rotation and magnetic fields.

The material to be presented is organized as follows. In §2 we discuss physical aspects of rotation and operational ways of detecting it. In §3 by restricting to stationary gravitational fields we exhibit an exact analogy between rotation in general relativity and magnetic fields. In the next section the geometrical aspects of this analogy are brought out. In §5 we search for a monopole-like solution of Einstein's field equations and are uniquely led to the NUT spacetime. We discuss this solution and bring out a complete parallel with the electromagnetic case. In §6 we discuss gravitational monopoles in general and show that they necessarily lead to causality violation. The last section is a concluding discussion.

2. ROTATING FRAMES OF REFERENCE

Many general relativistic effects can be understood by invoking the equivalence between gravitational and inertial forces—the equivalence principle. Thus, a uniformly accelerated elevator cannot be distinguished from an elevator in a constant gravitational field, by means of local experiments performed in the elevator. From this one can for instance, deduce the bending of light in a gravitational field. In order to understand the effects of rotation let us now consider a uniformly rotating elevator. How would an observer confined to experiments in such an elevator detect the presence of rotation? One way to do this is by using the Sagnac effect[6]. The observer takes a toroidal tube stationary relative to the elevator. Using a half silvered mirror he sends two rays of monochromatic light in opposite directions around the tube. These are then made to interfere after each ray has gone round once. The co-rotating ray will take longer to come round than the counter-rotating ray leading to a time delay which can be observed as a fringe shift. The time delay will be given by

$$\Delta t = -4A\omega/c^2, \tag{1}$$

where A is the area enclosed by the tube, projected onto the plane normal to the axis of rotation. Note that the Coriolis force and Sagnac effect depend linearly on ω unlike the centrifugal force which is quadratic in ω. The Sagnac effect is dominant at lower angular velocities and finds practical application in inertial navigation systems[1].

What would be the metric the rotating observer uses to describe spacetime? The answer follows by starting with flat spacetime in polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2}$$

and performing a coordinate transformation

$$\phi \to \phi + \omega t. \tag{3}$$

This yields

$$ds^2 = -(1-\omega^2 r^2)\,dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + 2\omega\, d\phi\, dt. \tag{4}$$

Notice the appearance of a cross term $d\phi\, dt$ in the metric proportional to ω. This as we shall see later is a signature of rotation in any spacetime.

3. ANALOGY BETWEEN ROTATION IN GENERAL RELATIVITY AND MAGNETIC FIELDS

As seen in §2 the cross term in the metric is related to rotation. However in that case they represent inertial forces. Genuine gravitational fields also lead to similar effects; we now move on to discuss these, restricting ourselves to stationary spacetimes, which correspond to uniform rotation.

Consider a stationary spacetime $\mathscr{M}$, i.e. one in which the metric tensor $g_{\mu\nu}(x)[(\mu, \nu = 0, 1, 2, 3)$, signature $(- + + +)]$ admits a time-like Killing vector field ξ:

$$\underset{\xi}{\mathscr{L}} g_{\mu\nu} = 0. \tag{5}$$

The family of integral curves of this vector field form a three-dimensional manifold, Σ. Each curve represents the world line of a stationary observer and the family of curves, Σ is space. Let us choose local co-ordinates x^i, $(i = 1, 2, 3)$ on Σ and a time co-ordinate t adapted to the time-like Killing vector: $\xi = \partial/\partial t$. In co-ordinates (t, x^i) on $\mathscr{M}$, (5) reads

$$\frac{\partial}{\partial t}(g_{\mu\nu}) = 0. \tag{6}$$

The co-ordinate transformations on $\mathscr{M}$ that respect (6) are

$$\text{(a)}\ x^i \to x^i(x^i) \qquad \text{(b)}\ t \to ct$$

$$\text{and} \quad \text{(c)}\ t \to t + \alpha(x^i) \tag{7}$$

(a) corresponds to a co-ordinate transformation on Σ, while (b) is a global rescaling of the time coordinate. (c) represents the freedom of each stationary observer, labelled by x^i, to independently reset his origin of time. Under (7), the combination of metric coefficients $A_i = g_{0i}/g_{00}$ transforms as

$$A_i \to A_i + \alpha_{,i}, \tag{8}$$

as seen easily by writing the line element in the form

$$ds^2 = \gamma_{ij}\, dx^i\, dx^j - \psi(dt + A_i dx^i)^2.$$

This also shows that

$$\gamma_{ij} \equiv g_{ij} - g_{00} A_i A_j,$$

are invariant under (7). Transformation (8) is identical to gauge transformations for the electromagnetic potential. We will refer to (7) as a "gauge transformation" in quotes. If we introduce the "field strength" corresponding to A

$$F_{ij} \equiv A_{i,j} - A_{j,i},$$

we find that it is 'gauge invariant'. From the definition, F satisfies the Bianchi identity

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0,$$

which corresponds to Gauss' law. If F vanishes, A_i can locally be set to zero by a 'gauge transformation' (7). The Killing vector field ξ is then orthogonal to the space-like hypersurfaces t = constant. Such a space-time is static or non-rotating. In the general case, F is non-zero and A cannot be transformed away. This situation is stationary and the Killing vector ξ is not hypersurface orthogonal. The stationary case corresponds, in electromagnetism, to a time-independent configuration of electric and magnetic fields and the static one to a purely electric, time-independent field.

As seen in §2 the Sagnac experiment provides an operational definition of rotation. In a space-time which is stationary but not static even a tube at rest will register a Sagnac shift given by $2\nu_0\Phi(c)$ where ν_0 is the conserved frequency of the radiation and $\Phi(c)$ is given by

$$\Phi(c) = \oint_c A_i\, dx^i = \int_s F_{ij}\, dx^i \wedge dx^j, \tag{9}$$

where s is any surface bounded by c. (The timelike Killing vector, ξ, provides a natural definition of rest.) This is the general relativistic Sagnac effect[7,8] and is analogous to the Bohm-Aharonov effect. The non-vanishing of $\Phi(c)$ is also related to the impossibility of synchronizing clocks on a closed curve[9].

To pursue the analogy further, consider a test particle of rest mass m moving in a background space-time. Its Hamiltonian is

$$H = \tfrac{1}{2}(g^{\mu\nu} p_\mu p_\nu + m^2) \simeq 0. \tag{10}$$

Since the metric is assumed stationary, t is a cyclic co-ordinate and its conjugate momentum

$$E \equiv \xi^\mu p_\mu = p_t \tag{11}$$

is conserved in the motion. E is a general co-ordinate scalar and represents the energy of the test particle in any reference frame in which the metric is stationary. Putting (11) back into (10), we can rearrange the Hamiltonian in three-dimensional form as

$$H = \tfrac{1}{2}\left[\gamma^{ij}(p_i - EA_i)(p_j - EA_j) - \frac{E^2}{\psi} + m^2\right] \approx 0, \tag{12}$$

where $\gamma^{ij} = g^{ij} = (\gamma_{ij})^{-1}$ is the three-dimensional metric on Σ. This closely resembles the Hamiltonian for a charged particle in a magnetic field, with minimal coupling. From (12), we identify the conserved energy (11) in four dimensions with the coupling constant in three dimensions. This reduction from four to three dimensions (spacetime to space) is entirely in the spirit of the Kaluza-Klein reduction from 4 + D to 4 dimensions[10]. In fact, the Einstein action can be written in three-dimensional form as

$$S = \int d^4x \sqrt{-g} R$$

$$= \left[\int dt\right] \int d^3x \sqrt{\gamma}\sqrt{\psi}\left({}^3R + \frac{\psi}{4} F_{ij}F^{ij}\right). \tag{13}$$

The gravitational field splits up into a scalar potential ψ, a vector field, A_i, describing rotation and a second rank tensor field γ_{ij} describing the curvature of space. The reduced form of action (13) shows that the rotation field obeys equations similar to those of a magnetic field.

4. GEOMETRICAL FORMULATION OF THE ANALOGY

The theory of fibre bundles provides an elegant and economic description of the analogy desribed in the previous section. This has the virtue of being entirely coordinate free and ideally suited to describe monopoles.

Consider the triplet $(\mathcal{M}, \Sigma, \Pi)$ where Π is the natural map which assigns each point $m \in \mathcal{M}$ to the curve on which it lies. The triplet forms a fibre bundle with base space Σ, the fibres being the integral curves of ξ. Since the spacetime is stationary, there is a one parameter group of motions G on $\mathcal{M}$, generated by ξ, which maps each fibre onto itself. Let us choose a point p_σ from each fibre $\Pi^{-1}(\sigma)$; $\sigma \in \Sigma$, smoothly over each coordinate patch on Σ. Then any point m_σ on the fibre $\Pi^{-1}(\sigma)$ can uniquely be written as $m_\sigma = gp_\sigma$ where $g \in G$ and so the fibre can be identified with the group G. $(\mathcal{M}, \Sigma, \Pi, G)$ then forms a principal fibre bundle[11] with structure group G. Tangent vectors at m orthogonal to ξ are called horizontal and these form a 'horizontal subspace' of the tangent space at m. Let C be a curve in $\Sigma(\sigma(u) \in \Sigma$, $0 \leq u \leq 1)$ and $m(0) \in \Pi^{-1}(\sigma(0)) \subset \mathcal{M}$ an initial point on the fibre over $\sigma(0)$. There is a unique curve $\tilde{C}(m(u) \in \Pi^{-1}(\sigma(u)) \subset \mathcal{M})$ in $\mathcal{M}$, called the 'horizontal lift' of C, whose tangent vector is horizontal. This is precisely the definition of a connection[12] on a fibre bundle. The connection 1-form A and the curvature 2-form F are then naturally regarded as forms on Σ. In general, the lift of a closed curve with base point $\sigma(0) = \sigma(1)$, is open $(m(0) \neq m(1))$, which means the connection has curvature. $m(0)$ and $m(1)$ lie on the same fibre but differ in their time coordinate by $\Phi(C)$.

In physical terms, $\mathcal{M}$ is space-time, Σ is space and Π assigns each event to its spatial location. A stationary observer at $\sigma \in \Sigma$ would regard an event $m' \in \Pi^{-1}(\sigma)$, at a neighbouring spatial location σ', as simultaneous with $m \in \Pi^{-1}(\sigma)$, if the infinitesimal curve joining m to m' is horizontal. This notion of simultaneity can be operationally realised by flashing light signals[9]. Stationary observers along any open curve in space can thus synchronize their clocks. But this cannot be done if the curve is closed.

5. NUT SPACETIME AS A GRAVITATIONAL MONOPOLE

Given the detailed similarity between rotation and magnetic fields, we ask if (in a theory of pure gravity) there is an exact solution of Einstein's field equations that represents a gravitational monopole. The field of a Dirac monopole at rest at the origin is stationary and spherically symmetric. So we look for a stationary, spherically symmetric vacuum solution of Einstein's field equations with the rotation field of a monopole. These symmetries imply that there exist Killing vectors ξ and R_a $(a = 1, 2, 3)$ satisfying the algebra

$$[\xi, R_a] = 0, \quad [R_a, R_b] = \varepsilon_{abc} R_c. \tag{14}$$

It is convenient to project the problem (and its symmetries) down to the three-dimensional space Σ, using a formalism due to Geroch[13]. The fields on Σ are ψ, A_i and γ_{ij}. Einstein's equations can be projected down to Σ as equations for these[14]. The topology of Σ is taken to be $\mathbf{R}^3 - \{0\}$, where $\{0\}$ represents the location of the monopole (and the centre of symmetry). The rotational Killing vectors, R_a on $\mathcal{M}$ can be projected down to Σ, preserving their algebra. From the original spherical symmetry of the space-time metric on $\mathcal{M}$, it follows that the tensors γ_{ij}, $F = \mathrm{d}A$ and ψ on Σ are spherically symmetric. Using this symmetry, we introduce coordinates (r, θ, ϕ) on Σ, where the radial coordinate r is chosen so that the area of a sphere in Σ is $4\pi r^2$ and θ and ϕ are the usual polar coordinates on the sphere. The spatial line element is then constrained to be

$$(\mathrm{d}s^2)_\Sigma = a(r)\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{15}$$

and ψ must depend on r alone. The general form for F allowed by spherical symmetry is

$$F = g(r)\sin\theta,\ \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{16}$$

But since F is closed, $g(r)$ must be a constant, g, which represents the strength of the monopole. A 'vector potential' that leads to this 'field strength' is

$$A = g(1 - \cos\theta)\,\mathrm{d}\phi. \tag{17}$$

Solving the field equations for ψ and a leads uniquely to

$$\psi(r) = \left(1 - \frac{g^2}{2r^2}\right) - \frac{2M}{r}\left(1 - \frac{g^2}{4r^2}\right)^{1/2},$$
$$a^{-1}(r) = \left(1 - \frac{g^2}{2r^2}\right)\left(1 - \frac{g^2}{4r^2}\right) - \frac{2M}{r}\left(1 - \frac{g^2}{4r^2}\right)^{1/2}, \tag{18}$$

where M is an integration constant representing the mass of the source. For $g = 0$, we recover the well-known Schwarzschild spacetime, which corresponds to the Coulomb field in electromagnetism. The solution with $M = 0, g \neq 0$ represents a pure gravitational monopole. In general the metric with (M, g) describes a gravitational dyon, where g relates to M as magnetic charge to electric charge. This solution is known in the relativity literature as the NUT metric[15] and has a coordinate singularity at $\theta = \pi$, which is the string singularity of the Dirac[16] monopole. It can be moved around by 'gauge transformations', and eliminated entirely by covering Σ with two coordinate patches.

The Killing vectors corresponding to the spherical symmetry of the NUT metric are

$$R_1 = g \cos\phi \tan\frac{\theta}{2}\frac{\partial}{\partial t} + \sin\phi\frac{\partial}{\partial\theta} + \cos\phi \cot\theta\frac{\partial}{\partial\phi},$$
$$R_2 = g \sin\phi \tan\frac{\theta}{2}\frac{\partial}{\partial t} - \cos\phi\frac{\partial}{\partial\theta} + \sin\phi \cot\theta\frac{\partial}{\partial\phi}, \tag{19}$$
$$R_3 = g\frac{\partial}{\partial t} - \frac{\partial}{\partial\phi}.$$

For the motion of a test particle in the 'monopole' background, the conserved angular momentum of the particle is given by

$$L_X \equiv R^{\mu}_1 p_\mu = [\sin\phi(p_\theta - EA_\theta) + \cos\phi \cot\theta(p_\phi - EA_\phi)] + gE\sin\theta\cos\phi,$$
$$L_Y \equiv R^{\mu}_2 p_\mu = [-\cos\phi(p_\theta - EA_\theta) + \sin\phi \cot\theta(p_\phi - EA_\phi)] + gE\sin\theta\sin\phi$$
$$L_Z \equiv R^{\mu}_3 p_\mu = [-(p_\phi - EA_\phi)] + gE\cos\theta. \tag{20}$$

While the 'gauge-invariant' expressions in brackets correspond to the usual $\bar{r} x m \bar{v}$ angular momentum, there is an extra piece

$$gE(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{21}$$

like the term $eg\hat{r}$ familiar from magnetic monopoles. By noticing that

$$\sin\theta\cos\phi\, L_X + \sin\theta\sin\phi\, L_Y + \cos\theta\, L_Z = gE, \tag{22}$$

one sees the motion is restricted to a cone with axis along the angular momentum vector, exactly as in the electromagnetic case. Thus the analogy with electromagnetism is complete (table 1).

The NUT solution was first discovered by Newman *et al*[15] and many of its properties elucidated by Misner[17]. It has many features that defy physical intuition[18] like closed time-like curves and the absence of global space-like hyperfaces. Its monopole interpretation was first realized by Demianski and Newman[2]. Other interpretations have been attempted[19] but these are not as appealing. A particularly clear discussion of the NUT solution as a gravitational dyon is due to Dowker[3]. The 'interior' NUT is perfectly acceptable as a homogeneous and anisotropic cosmological solution and first discussed by Taub[20]. Our treatment which goes beyond the linearized theory leads uniquely to the NUT solution as a gravitational monopole in general relativity.

Table 1 *Comparison of the electromagnetic and gravitational cases*

Electromagnetism	Gravity
Vector potential	$A_i = g_{oi}/g_{oo}$
Gauge transformations	Resetting of clocks
Parallel transport along a curve	Synchronizing clocks along a spatial curve
Magnetic field	Rotation field, F_{ij}
Electric charge	Mass-Energy
Coupling constant e	Total conserved Energy (E)
Bohm-Aharonov effect	Sagnac effect
Coulomb solution	Schwarzschild solution
Magnetic charge	Dual mass, g
Dirac monopole	NUT ($M = 0$)
Dyon	NUT ($M, g \neq 0$)

6. GRAVITATIONAL MONOPOLES IN GENERAL

The NUT solution regarded as a magnetic monopole suffers from the unphysical feature that it has closed timelike curves. This malady is not peculiar to the NUT solution but afflicts gravitational monopoles in general. A gravitational monopole is defined as an object with a nonzero gravimagnetic flux integral $\int_s F$ over a two surface S in space. We do not require that the object be spherically symmetric or a solution of Einstein's field equations. We will see below that this necessarily leads to the existence of closed timelike curves.

Let S be any closed 2-surface in Σ enclosing the monopole at the origin. We regard S as a family of loops C_s, $0 \leq s \leq 1$, all with the same base point σ on $S(\sigma_s(0) = \sigma_s(1) = \sigma)$. The family starts from the trivial loop $C_0(\sigma_1(u) = \sigma)$ at σ, goes around S and ends at the trivial loop $C_1(\sigma_1(u) = \sigma)$ at σ. The lift of C_s is a curve $\tilde{C}_s$ in $\mathscr{M}$ with end points $m_s(0)$, $m_s(1) \in \Pi^{-1}(\sigma)$ separated in t by $\Phi(C_s)$. As s goes from 0 to 1, $\Phi(C_s)$ goes from 0 to $4\pi g$. But C_1 is a single point and so its lift must also be a single point. Hence we are forced to identify points on the fibre $\Pi^{-1}(\sigma)$ separated in t by $4\pi g$. This implies that the time-coordinate is periodic and that the fibres $\Pi^{-1}(\sigma)$, $\sigma \in \Sigma$ have topology S^1 rather than $\mathbf{R}$ and the structure group must be $u(1)$ rather than $\mathbf{R}$. Note that the argument is independent of the field equations and so applies to any stationary metric with a non-zero $\int_s F$. In physical language, each C_s can be regarded as a Sagnac tube. As s goes from 0 to 1, the Sagnac shift goes from 0 to $(2v_0)(4\pi g)$. But C_1 is an infinitesimal tube and so must not register any Sagnac shift. The only way out is to identify points on each fibre as discussed above.

The existence of a gravitational monopole implies that time is a periodic coordinate i.e. that the integral curves of ξ have topology S^1 rather than $\mathbf{R}$. This means that the space-time has closed timelike curves. Indeed there are closed timelike curves through any event in the space-time, no matter how far removed from the monopole. This leads to logical paradoxes if one allows for observers with free will. This makes gravitational monopoles unacceptable as they violate causality.

The above argument also shows that gravimagnetic charge must be quantized. If T is the coordinate length of the fibre, the monopole charge must come in multiples of $T/4\pi$:

$$g = nT/4\pi. \tag{23}$$

This condition is derived in a purely classical framework. In the quantum-mechanical context a time periodicity of T would imply energy quantization for test particles in units of $\varepsilon = 2\pi\hbar/T$. Equation (23) then reads

$$gE = n\hbar/2.$$

This is analogous to the Dirac quantization condition for magnetic monopoles.

7. CONCLUSION

We have presented a simple and physically-motivated discussion of gravitational monopoles using an exact analogy between rotation in stationary space-time and magnetic fields. The essential ingredient in this discussion was a decomposition of space-time into space and time, using timelike Killing vector. This is entirely in the spirit of Kaluza-Klein and leads to the identification of the conserved energy E as a coupling constant. Rotation is viewed as a gauge field which couples to E. This approach leads to an exact and remarkably complete analogy between the gravitational and electromagnetic situations. For instance, the geodesics of NUT space lie on a cone and the expression for the particle angular momentum has an extra contribution.

We have also presented a general argument to show that closed timelike curves are inevitable if gravitational monopoles exist. Our framework gives a simple way of thinking about gravitational monopoles and removes confusions which might otherwise remain. In a recent letter Zee[21] raises a number of issues which can easily be settled in our framework. The first is the question whether it is the total energy or the rest mass which is quantized in the field of a gravitational monopole. Our analysis shows quite clearly that it is the conserved energy of the test particle which serves as the gravimagnetic coupling constant. Hence it is the conserved energy which is quantized. A related question is whether the photon energy will be quantized. Our approach which does not rely on the post-Newtonian approximation applies to photons as well and shows again that the conserved frequency of the photon is quantized. The other question relates to the magnitude of the basic quantum of energy. Quite independent of the subtleties of gravitational monopoles or the NUT solution within the framework of conventional quantum mechanics, if energy is quantized in units of ε then time must be periodic. For, the wave function of any isolated system can then be expanded in terms of stationary states

$$\psi(x, t) = \sum_{\alpha} U_{\alpha}(x) \exp\left(-\frac{i\varepsilon n_{\alpha} t}{\hbar} \right),$$

which are all periodic with period $T = 2\pi\hbar/\varepsilon$. This means that the wave function too is periodic and events repeat after a time T. If energy quantization is invoked in the world we live in, T would need to be of cosmological scale, so as to escape detection. If ε were as large as Zee suggests (10^{-23} eV), events would have repeated every thirteen years or so, which would have been noticed.

25 April 1986

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