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Accretion onto a Kerr black hole in the presence of a dipole magnetic field

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Abstract. We analyze the accretion of charged matter onto a rotating black hole immersed in an aligned dipolar magnetic field. We specialize to motion in the equatorial plane and calculate the 'Keplerian' angular momentum distribution, the marginally stable and marginally bound orbits, and the efficiency of mass-to-energy conversion as functions of the angular momentum of the black hole and of the product of the dipole moment and the charge of the infalling matter. Although the detailed results are quite different from those previously obtained in the case of an uniform magnetic field, the astrophysically relevant results are very similar; when hydrodynamical accretion is considered, these effects of the magnetic field are always very small. But for test particles the efficiency can be significantly increased for limited ranges of the parameters.

Keywords. Black holes; accretion; magnetic fields.

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1. Introduction

Accretion onto supermassive black holes has now become the most widely accepted basic model for the powerhouse of active galactic nuclei (for recent reviews see Begelman *et al* 1984; Wiita 1985). Geometrically thin accretion disks have been investigated for over a decade and are certainly present in catacylsmic variables and other astronomical systems (*e.g.* Pringle 1981). But the realization that thick disks, supported either by radiation (*e.g.* Paczynski and Wiita 1980; Jaroszynski *et al* (1980) or by ion pressure (Rees *et al* 1982) have a geometry suitable for producing the jets that are ubiquitous in quasars and radio galaxies has led to a great deal of interest in this topic (*e.g.* Narayan *et al* 1983; Abramowicz *et al* 1984). The vast majority of work on accretion has neglected magnetic fields.

Magnetic fields must, however, be present in the plasma accreted by a black hole, and the magnetic flux is likely to be concentrated as the material is compressed and sheared before it is swallowed by the central object (*e.g.* Bisnovatyi-Kogan 1979). Several models have incorporated magnetized thin accretion disks in dynamo mechanisms (*e.g.* Lovelace 1976) or have used magnetic fields to tap the rotational energy of a black hole (*e.g.* Blandford and Znajek 1977; Phinney 1983). Detailed general relativistic formalisms are available for analyzing electromagnetic fields in strongly curved spacetimes (Macdonald and Thorne 1982, Thorne and Macdonald 1982), and fully relativistic charged disks have also been examined (Prasanna and Chakraborty 1981). All of these somewhat disparate theories are of interest and ought to be investigated further, but each makes certain simplifying assumptions, and no realistic, self-consistent models have yet emerged.

This paper does not actually contain such a realistic model; rather, it involves a different approach to some aspects of the effects of magnetic fields on accretion that was begun by Dadhich and Wiita (1982, hereafter DW) for Schwarzschild black holes and continued by Wiita *et al* (1983, hereafter WVSI) for Kerr black holes. The key questions investigated in these papers were how the presence of a uniform aligned magnetic field changed both the location of the inner edges of accretion disks and the efficiency of mass-to-energy conversion of the matter falling into the black hole. Standard thin accretion disks are assumed to terminate at the last stable circular orbit, r_{ms} ($= 6m$ for a Schwarzschild black hole; we use units where $G = c = 1$). For such disks the efficiency is given by the binding energy per unit mass at that point, b_{ms} ($= 0{\cdot}572$ in that case). As the angular momentum, a, of the black hole increases, $r_{ms} \to r_+$, the outer event horizon, and b_{ms} increases, approaching $\sim 0{\cdot}42$ as $a/m \to 1$ (Thorne 1974). However, fluid rings around black holes (*e.g.* Abramowicz *et al* 1978) or thick accretion disks (*e.g.* Lynden-Bell 1978), can have inner edges that come closer to the event horizon; the matter can approach the marginally bound orbit, r_{mb} ($= 4m$ in the Schwarzschild metric), where the binding energy vanishes. Thus thick accretion disks have lower mass-to-energy conversion efficiencies than do thin disks, but they can radiate at above the Eddington luminosity and can more easily accelerate and collimate jets (Paczyński and Wiita 1980).

The basic conclusions drawn from DW and WVSI were that realistic hydrodynamic accretion disks could only be very slightly affected by the presence of uniform magnetic fields, but that test particles could be very much affected, with conversion efficiencies rising the most around slowly rotating black holes. One significant problem with these previous papers was that an unorthodox definition of binding energy was necessary, since the unphysical assumption of a uniform field extending to infinity implied that the potential diverged at large distances. Of course in any realistic solution the field must decrease rapidly enough to remove any singularity, and since most of the interesting effects involve the regions quite close to the event horizons, WVSI argued that a proper normalization should yield reasonable results.

In this paper we shall consider the case of a rotating black hole of mass m immersed in a dipole-like magnetic field; in this case no unorthodox definitions or renormalizations are necessary. It will be assumed that even the maximal field strength obeys the condition $|Bm| \ll 1$, so that the mass-energy of the field is small compared to that of the black hole. Physical constraints on the magnetic field strengths allowed in stable disks guarantee this condition (see §3 and DW). Prasanna and Vishveshwara (1978; hereafter PV) analyzed the trajectories of charged particles with fixed angular momenta in equatorial orbits around Kerr black holes surrounded by both uniform and dipolar magnetic fields. Just as WVSI used the effective potential formalism for uniform fields, we shall use its dipole form to numerically solve for the 'Keplerian' angular momentum distribution, r_{ms}, r_{mb}, and the efficiency at r_{ms} (b_{ms}). The necessary equations for the potential and the angular momentum distribution, which surprisingly do not turn out to be much more complex than those for the uniform field case, are developed in §2.

The results for r_{mb}, r_{ms} and b_{ms} are presented in §3, a discussion of these results and a comparison with previous work comprise §4.

2. Effective potential and angular momentum distribution

In the usual Boyer-Lindquist coordinates the line element for the Kerr metric is given by

$$ds^2 = -(1-2mr\Sigma^{-1})dt^2 - 4mra\Sigma^{-1}\sin^2\theta dt\, d\phi + \Sigma\Delta^{-1}dr^2 + \Sigma\, d\theta^2 + F\Sigma^{-1}\sin^2\theta\, d\phi^2, \tag{1}$$

where $\Sigma = (r^2 + a^2\cos^2\theta)$, $\Delta = (r^2 + a^2 - 2mr)$, and $F = (r^2+a^2)^2 - \Delta a^2 \sin^2\theta$. Employing Teukolsky's (1973) perturbation equations in the Newman-Penrose formalism, the solution for a stationary axisymmetric electromagnetic field in the above background may be obtained (Chitre and Vishveshwara 1975; Petterson 1975). For a dipole magnetic field, with dipole moment μ parallel to the rotation axis of the black hole, the non-vanishing components of the vector potential are (PV)

$$A_t = -3a\mu(2\Sigma\zeta^2)^{-1}\{[r(r-m) + (a^2 - mr)\cos^2\theta] + (2\zeta)^{-1}\ln[(r-m+\zeta)/(r-m-\zeta)] - (r - m\cos^2\theta)\}, \tag{2a}$$

and

$$A_\phi = -3\mu\sin^2\theta(4\Sigma\zeta^2)^{-1}\{(r-m)a^2\cos^2\theta + r(r^2+mr+2a^2) - [r(r^3 - 2ma^2 + a^2 r) + \Delta a^2\cos^2\theta](2\zeta)^{-1} \ln[(r-m+\zeta)/(r-m-\zeta)]\}, \tag{2b}$$

where $\zeta = (m^2 - a^2)^{1/2}$.

We note that this is a solution which is regular for large **r** but diverges on the horizon. There exists an independent dipole solution whose behaviour is just the opposite to the above. In a realistic problem the two are matched at the source such as a current loop around the black hole (*e.g.* Chitre and Vishveshwara 1975; Dhurandhar and Dadhich 1984). In the present treatment we have chosen the first solution throughout the region of interest for the sake of simplicity. This is tantamount to having the source closer to the black hole than the least distance we may have to probe within the accretion disk.

If **e** is the charge and **M** the rest mass of a particle orbiting the black hole, the canonical angular momentum and energy, which are given by

$$U_\phi + eA_\phi = l, \qquad U_t + eA_t = -E, \tag{3}$$

are constants of the motion since the fields have **t** and $\boldsymbol{\phi}$ Killing vectors. Here U^i denotes the particle's proper velocity and all quantities are normalized by dividing by **M**. In terms of the dimensionless variables

$$\rho = r/m, \quad \sigma = s/m, \quad \alpha = a/m, \quad L = l/m, \quad \tau = t/m, \quad \bar{A}_\phi = A_\phi/m, \tag{4}$$

the radial velocity for motion confined in the equatorial plane is given by

$$(U^\rho)^2 = (d\rho/d\sigma)^2 = \rho^{-3}\{[\rho(\rho^2+\alpha^2)+2\alpha^2](E+A_\tau)^2 - 4\alpha(E+A_\tau)(L-\bar{A}_\phi) - (\rho-2)(L-\bar{A}_\phi)^2 - \rho\Delta\}. \tag{5}$$

This specialization to the equatorial plane is reasonable since the minimum values for r_{mb} and r_{ms}, and thus the determinants of the efficiency of a disk, are in the plane where $\theta = \pi/2$.

The effective potential for radial motion is obtained by solving for the turning points of the orbit. PV show that setting $U^{\rho} = 0$ yields

$$V = -A_{\tau} + K/R, \tag{6}$$

where

$$K = 2\alpha(L - \bar{A}_{\phi}) + \Delta^{1/2}[\rho^2(L - \bar{A}_{\phi})^2 + \rho R]^{1/2}, \tag{7a}$$

$$R = \rho^3 + \rho\alpha^2 + 2\alpha^2, \tag{7b}$$

and in these units

$$\Delta = \rho^2 - 2\rho + \alpha^2. \tag{7c}$$

In terms of the dimensionless variables the potentials become

$$A_{\tau} = -2c\alpha[(1 - \rho^{-1})\Delta_1 - 2\beta\rho^{-1}], \tag{8a}$$

$$\bar{A}_{\phi} = -c\{2\beta(1 + \rho + 2\alpha^2\rho^{-1}) - [\rho^2 + \alpha^2(1 - 2\rho^{-1})]\Delta_1\}, \tag{8b}$$

where

$$\beta = (1 - \alpha^2)^{1/2}, \quad c = (3/8)\lambda\beta^{-3}, \quad \Delta_1 = \ln[(\rho - 1 + \beta)/(\rho - 1 - \beta)], \tag{8c}$$

and the important parameter characterizing the product of the charge of the external matter and the dipole moment of the magnetic field is

$$\lambda = e\mu/m^2. \tag{8d}$$

Note that we can now write

$$K = 2\alpha(L - \bar{A}_{\phi}) + \rho\Delta^{1/2}(L^2 - XL + Y)^{1/2}, \tag{9a}$$

with

$$X = 2\bar{A}_{\phi} \quad \text{and} \quad Y = \bar{A}_{\phi}^2 + R/\rho. \tag{9b}$$

In this way the entire potential V is determined as a function of ρ for specified values of the parameters α, λ and L, as discussed in PV for the case of large λ and L. We now seek the angular momentum distribution which is characterized by a balance between rotational, gravitational and electromagnetic forces, but which neglects radiation losses. This distribution is the analog of that of circular Keplerian orbits in the Newtonian case, and will agree very closely with the angular momentum at the inner edge of an accretion disk. This function $L_K(\rho; \alpha, \lambda)$ is found by setting $\mathrm{d}V/\mathrm{d}\rho = 0$. After a large amount of algebra involving equations (6)–(9) we find

$$\begin{aligned} V' = \mathrm{d}V/\mathrm{d}\rho = T_1 + T_2(L^2 - XL + Y)^{1/2} + T_3 L \\ + T_4(L^2 - WL + Z)(L^2 - XL + Y)^{-1/2}, \end{aligned} \tag{10}$$

where the coefficients of L are functions solely of ρ, as long as α and λ are taken as parameters. The explicit forms of these coefficients and auxiliary functions are found in

(7b), (7c), (8c) and the following:

$$
\begin{aligned}
&Q = 3\rho^2+\alpha^2, \qquad X = -2S_3, \qquad Y = S_3^2+R/\rho,\\
&W = -2S_3-\rho S_4, \quad Z = S_3^2+R/(2\rho)+Q/2+\rho S_3 S_4,\\
&S_1 = 2\alpha c[(1-1/\rho)\Delta_1-2\beta/\rho],\\
&S_2 = (2\alpha c\rho^{-2})[\Delta_1-2\rho\beta\Delta^{-1}(1-\alpha^2\rho^{-1})],\\
&S_3 = c[2\beta(1+\rho+2\alpha^2\rho^{-1})-\Delta_1(\rho^2+\alpha^2-2\alpha^2\rho^{-1})],\\
&S_4 = 2c[2\beta\Delta^{-1}(\rho^2-\rho+\alpha^2\rho^{-1}-\alpha^4\rho^{-2})-\Delta_1(\rho+\alpha^2\rho^{-2})],\\
&T_1 = S_2+(2\alpha R^{-1})S_4-(2\alpha Q R^{-2})S_3,\\
&T_2 = \rho(\rho-1)R^{-1}\Delta^{-1/2}-\rho Q\Delta^{1/2}R^{-2},\\
&T_3 = -2\alpha Q R^{-2}, \qquad T_4 = \Delta^{1/2}R^{-1}.
\end{aligned}
\tag{11}
$$

If we now set $V' = 0$, we convert (10) into an explicit quartic equation for L, which actually has the same form as for the uniform field case (WVSI, equation 12).

$$
\begin{aligned}
&L^4[(T_2+T_4)^2-T_3^2]\\
&\quad +L^3[T_3^2X-2T_1T_3-2(T_2+T_4)(T_2X+T_4W)]\\
&\quad +L^2[(T_2X+T_4W)^2+2(T_2+T_4)(T_2Y+T_4Z)\\
&\qquad\qquad -(T_3^2Y+T_1^2-2T_1T_3X)]\\
&\quad +L[T_1^2X-2T_1T_3Y-2(T_2X+T_4W)(T_2Y+T_4Z)]\\
&\quad +[(T_2Y+T_4Z)^2-T_1^2Y]=0
\end{aligned}
\tag{12}
$$

It turns out that the terms T_2, T_3 and T_4 look the same as for the uniform field case, but that T_1 is different, as are the explicit expressions for X, Y, W, and Z. Thus we expect that the same numerical techniques that yielded the desired results concerning the marginally bound and marginally stable orbits when uniform fields were considered will also suffice in this case. This is nearly true, but the much greater complexity of the coefficients in the current situation meant that the numerical evaluations were significantly more difficult to accomplish. Equation (12) must be solved numerically and the smallest real positive root corresponding to each value of $\rho > \rho_+$ $(= 1+[1-\alpha^2]^{1/2}$, the event horizon) is chosen as $L_K(\rho;\alpha,\lambda)$.

Once the Keplerian angular momentum distribution is found we can return to (9a) and (8) to find the potential, $V(\rho;\ L_K(\rho))$. Since we are interested in cases with small values of r_{mb} and r_{ms} so as to provide high efficiency and narrow funnels, we only consider co-rotating particles or disks, *i.e.*, we always assume $L\alpha > 0$. We note that in the limit of $\rho \to \infty$,

$$
V \to 1-2/\rho+\alpha^2/\rho^2 \to 1. \tag{13}
$$

This means that we can define the binding energy in the standard fashion as just $b = 1 - V$, and it approaches 0 smoothly, unlike for uniform fields (WVSI).

3. Accretion efficiency

Within the limitations discussed above, we are interested in investigating solutions for a two-parameter family. The black hole angular momentum parameter can range from α

$= 0$ (Schwarzschild case) through $\alpha = 1$ (extreme Kerr case). Numerical difficulties preclude solving (12) for those limiting cases, but we were able to obtain useful results for $0{\cdot}00001 \leqslant \alpha \leqslant 0{\cdot}99999999$ which appear to be very close to those expected for the bounding values.

It is more difficult to evaluate bounds on the parameter λ, which couples the charge (per unit mass) of the accreted material to the dipole strength of the magnetic field. We first examine the dipole moment; the conversion between the dimensionless variable μ/m^2 and physical units for the dipole moment is

$$\mu_p = (G^{3/2}M^2c^{-2})(\mu/m^2)$$
$$= 7{\cdot}58 \times 10^{34}(M/M_\odot)^2(\mu/m^2)\text{gauss cm}^3, \tag{14}$$

where M is the mass of the black hole in grams. A generous constraint on μ is found by demanding that the pressure due to the magnetic field does not exceed the maximal sum of the gas and radiation pressures in a fluid disk. As discussed by DW, the thick disk models of Wiita (1982) give us our best current estimates of the pressures and magnetic fields; for large ($> 10^6\, M_\odot$) black holes we must have

$$B < 1{\cdot}3 \times 10^8 \text{ gauss} \quad \text{or} \quad |Bm| < 5{\cdot}6 \times 10^{-12}(M/M_\odot). \tag{15}$$

When combined with (14) this translates into bounds on the dipole moment

$$\mu_p < 4{\cdot}1 \times 10^{23}(M/M_\odot)^3 \text{ gauss cm}^3,$$

or

$$|\mu/m^2| < 5{\cdot}4 \times 10^{-12}\,(M/M_\odot). \tag{16}$$

For self-consistency, these models demand $M < 10^7\, M_\odot$, so a good limit is

$$|\mu/m^2| < 10^{-4}. \tag{17}$$

Although higher mass black holes probably exist in active galactic nuclei (*e.g.* Wiita 1985), they would imply a slower increase in pressure, thus a small rise in the limit on μ. Realistically, we expect magnetic field strengths a factor of 10 to 1000 below these limits (WVSI).

While the values of **e** for test particles can be very large (for an isolated proton, $e = 1{\cdot}112 \times 10^{18}$), we certainly expect any bulk plasma that is being accreted to be electrically neutral over large scales. But, as discussed in DW, we anticipate that a plasma disk will be subject to forces that produce some charge separation in the disk's innermost regions. However, in geometrical units, this equivalent charge should be quite small, $|e| \lesssim 1$. Multiplying **e** by μ/m^2, we conclude that while for charged test particles $\lambda \gg 1$ is possible, for any hydrodynamic accretion flow we expect $\lambda < 10^{-4}$.

We proceeded to compute $L_K(\rho)$ and $V(\rho)$ for a large number of values of the parameter pair, with $10^{-3} < \alpha < 0{\cdot}99999999$ and $10^{-6} \leqslant \lambda \leqslant 10^3$. For $\lambda \leqslant 10^{-5}$ the results appeared to be completely insensitive to λ, while for $\lambda > 10^3$, no interesting new results were obtained, and the efficiency fell monotonically. The marginally bound orbit is found as the first radius for which $V = 1$ (*i.e.*, $b = 0$) as V decreases while ρ increases above ρ_+. The marginally stable orbit is found as the one where $\mathrm{d}L_K/\mathrm{d}\rho = 0$. Then the maximal efficiency is evaluated as $b(r_{ms}) = b_{ms}$. These three quantities are found numerically by following the approach of DW and WVSI.

But in the present circumstances things are more complicated. Since (12) is a quartic equation, four roots are found, and, while only two are usually real and positive, for many values of the variables, a third, and even fourth, seemingly valid solution was found. (For the uniform field case two roots were always complex and one was negative, so there was never any question as to the valid solution.) A careful analysis did allow for the discarding of the invalid roots, which typically required nearly radial orbits, with $L \approx 0$, or else produced negative energy states with $L < 0$ and $b > 1{\cdot}0$.

Our results are summarized in tables 1–5 and figures 1–3. Each table corresponds to a fixed α and the rows are defined by the values of λ in the first column. The second column gives r_{mb}, the third r_{ms}, the fourth $L_{ms} = L(r_{ms})$, and the final column, b_{ms}, all in dimensionless units. The only limiting cases that could be used to test our numerical results involved testing models as $\lambda \to 0$ or as $\rho \to \infty$. As the field vanished, our values for r_{ms}, r_{mb} and b_{ms} all nicely approached those computed by Thorne (1974) for accretion onto Kerr black holes without electromagnetic fields. Special attention is paid to $\alpha = 0{\cdot}9978$, since this is the canonical value achieved by a black hole "spun-up" by accretion (*e.g.* Thorne 1974). However, it has been argued that larger values are possible if thick accretion disks are involved (Abramowicz and Lasota 1980).

Figure 1 shows the effective potential in the equatorial plane as a function of radius (in a log-linear plot) for six pairs of (α, λ). Figure 2 gives the Keplerian angular momentum per mass for the same cases (on a log-log graph), with the minima corresponding to r_{ms} marked on the curves. In making comparisons between these curves it is important to note that $r_+(\alpha = 0{\cdot}5) = 1{\cdot}866025$, $r_+(0{\cdot}9978) = 1{\cdot}066296$ and $r_+(0{\cdot}99999999) = 1{\cdot}000141$; for the tabulated α's not represented on these figures, $r_+(0{\cdot}1) = 1{\cdot}994987$ and $r_+(0{\cdot}9999) = 1{\cdot}014142$. The general trends illustrated by these figures and abstracted from the tables include: for a fixed α, increasing λ up to a point causes r_{mb}, r_{ms} and L_{ms} all to increase, while b_{ms} declines. These smooth variations occur throughout the explored parameter space for $\alpha = 0{\cdot}1$ (save for r_{mb}, which does evince the jump to lower values at one point), and correspond in general to cases where only one minima in $L_K(\rho)$ and in $V(L_K)$ are found outside the event horizon.

However, for all other examined values of α, a second valid solution to (12) minimizes L_K for a specific range of λ's. These solutions show two clear minima in V and L_K, with values of r_{mb} quite close to r_+. The extreme closeness when $(\alpha, \lambda) = (0{\cdot}99999999, 0{\cdot}001)$ means that the curves for that case are hard to plot on the same scale as the other five

Table 1. Accretion parameters as functions of λ for $\alpha = 0{\cdot}1$.

λ	ρ_{mb}	ρ_{ms}	L_{ms}	b_{ms}
0·00001	3·7974	5·6877	3·3671	0·0606
0·0001	3·7974	5·6877	3·3672	0·0606
0·001	3·7981	5·6877	3·3683	0·0606
0·01	3·8043	5·6878	3·3716	0·0605
0·0315	3·8193	5·6879	3·3813	0·0601
0·1	3·8666	5·8210	3·4115	0·0591
0·315	4·0121	6·0210	3·5019	0·0561
1·0	4·4500	6·8272	3·7529	0·0490
3·15	5·6425	9·0002	4·3401	0·0367
10·0	2·0241	14·248	5·4934	0·0230
31·5	2·3150	25·395	7·4317	0·0126
100·0	3·1500	50·368	10·492	0·0063
315·0	4·8088	104·85	15·090	0·0031
1000·0	7·8715	224·74	21·980	0·0014

Table 2. Accretion parameters as functions of λ for $\alpha = 0{\cdot}5$.

λ	ρ_{mb}	ρ_{ms}	L_{ms}	b_{ms}
0·00001	2·9142	4·2254	2·9029	0·0821
0·001	2·9150	4·2259	2·9035	0·0821
0·01	2·9222	4·2490	2·9093	0·0818
0·1	2·9930	4.3899	2·9652	0·0788
0·316	3·1584	4·7111	3·0879	0·0728
1·0	3·6402	5·6100	3·4035	0·0601
1·35	3·8670	5·9746	3·5378	0·0556
1·37	1·8688	1·9750	1·9233	0·5403
1·40	1·8692	1·9846	1·8138	0·5388
1·60	1·8720	2·0445	1·2045	0·5142
1·80	1·8738	2·0973	0·7087	0·4729
2·00	1·8759	2·2271	1·6834	0·4603
2·30	1·8794	2·2864	1·8962	0·4307
2·60	1·8834	2·3251	2·0647	0·4088
2·90	1·8879	2·3849	2·3109	0·3852
3·00	1·8895	7·6641	4·0398	0·0427
4·0	2·0611	8·5482	4·2793	0·0381
6·0	2·2190	10·219	4·6808	0·0319
10·0	2·5117	12·946	5·3041	0·0248
31·5	3·6742	24·217	7·2884	0·0132
100·0	5·8586	49·155	10·373	0·0065
315·0	9·7120	102·41	14·984	0·0031
1000·0	16·638	219·51	21·881	0·0015

Table 3. Accretion parameters as functions of λ for $\alpha = 0{\cdot}9978$.

λ	ρ_{mb}	ρ_{ms}	L_{ms}	b_{ms}
0·00001	1·0961	1·2505	1·4001	0·3180
0.001	1·1023	1·2701	1·4204	0·3108
0·01	1·1501	1·3999	1·5353	0·2737
0·0251	1·2090	1·5358	1·6431	0·2439
0·0282	1·0752	1·0951	0·9013	0·5908
0·04	1·0778	1·1089	0·7660	0·6497
0·1	1·0873	1·1395	0·5050	0·7458
0·2	1·0977	1·1756	0·3450	0·7739
0·225	1·0998	1·1840	0·3178	0·7747
0·3	1·1054	1·2006	0·2472	0·7716
0·631	1·1240	1·2645	0·0158	0·7175
0·708	1·1277	3·2923	2·6343	0·1012
1·0	1·1404	3·7963	2·8286	0·0880
2·24	1·8489	5·3373	3·3961	0·0612
5·0	2·3264	7·9268	4·1655	0·0407
10·0	2·9875	11·430	5·0448	0·0277
31·5	4·8763	22·680	7·0972	0·0140
100·0	8·0623	46·853	10·218	0·0067
315·0	13·548	98·619	14·847	0·0032
1000·0	23·358	216·00	21·753	0·0015

Table 4. Accretion parameters as functions of λ for $\alpha = 0{\cdot}9999$.

λ	ρ_{mb}	ρ_{ms}	L_{ms}	b_{ms}
0·00001	1·0204	1·0801	1·2424	0·3813
0·0001	1·0229	1·0929	1·2558	0·3731
0·001	1·0405	1·1568	1·3211	0·3480
0·002	1·0537	1·1985	1·3594	0·3331
0·005	1·0658	1·2790	1·4284	0·3080
0·006	1·0683	1·2874	1·4449	0·3024
0·007	1·0200	1·0318	0·4525	0·7802
0·01	1·0211	1·0374	0·3922	0·8086
0·02	1·0240	1·0485	0·3061	0·8477
0·0315	1·0263	1·0512	0·2569	0·8652
0·08	1·0320	1·0735	0·1650	0·8867
0·0925	1·0331	1·0763	0·1512	0·8874
0·10	1·0337	1·0763	0·1444	0·8871
0·20	1·0396	1·0985	0·0685	0·8748
0·315	1·0444	1·1179	0·0032	0·8496
0·50	1·0503	2·9068	2·4586	0·1157
1·0	1·0623	3·6686	2·8265	0·0881
2·5	1·9008	5·5343	3·4865	0·0581
6·0	2·4741	8·7537	4·3738	0·0369
10·0	2·9892	11·578	5·0436	0·0277
31·5	4·8810	22·628	7·0966	0·0140
100·0	8·0707	47·223	10·218	0·0067
315·0	13·549	98·606	14·847	0·0032
1000·0	23·360	215·97	21·754	0·0015

cases. Such solutions allow for smaller values of r_{ms} and L_{ms}, but also for significantly larger values of b_{ms}. Single valid minima exist for (0·99999999, 1), but this case illustrates a situation where deviations from smooth curves indicate the second solution is having some effect.

Figure 3 shows the values for the efficiency for a thin accretion disk (or for test particles) which were found as functions of λ for all five tabulated values of α. These valid solutions only intrude over very limited ranges of the magnetic strength; these ranges decrease in absolute size (even if they grow in logarithmic length) as the angular momentum of the black hole rises. An objection might be raised towards our taking the inner solution with smaller r_{mb}, r_{ms} and higher b_{ms} since it does not continuously join onto the limiting solution for b_{ms} valid at both large and small λ. However, as seen in figure 1, the barrier usually presented by the intervening maximum is quite limited in both height and extent. In some cases the intervening maximum in V does not rise above unity at all. Thus, in any environment where viscosity or other interactions that could change the energy and angular momentum of the particles or fluid are present, these barriers could be overcome, thus justifying the choice of the second smooth solution and our claim for possible increased conversion efficiency.

The value of λ for which the maximum binding energy is achieved also decreases as α rises, from 1·37 for $\alpha = 0{\cdot}5$ to 0·009 for $\alpha = 0{\cdot}99999999$. In general, this rise in efficiency occurs only under conditions corresponding to charged test particles, and not to fluids. Only for values of α extremely, and probably unrealistically, close to unity do values for

Table 5. Accretion parameters as functions of λ for $\alpha = 0.99999999$.

λ	ρ_{mb}	ρ_{ms}	L_{ms}	b_{ms}
0·000001	1·0011	1·0139	1·1706	0·4148
0·00000315	1·0019	1·0200	1·1780	0·4112
0·00001	1·0034	1·0299	1·1889	0·4059
0·00002	1·0053	1·0380	1·1977	0·4017
0·0000315	1·0004	1·0014	0·1351	0·9325
0·00006	1·0005	1·0015	0·0971	0·9515
0·0001	1·0006	1·0019	0·0879	0·9561
0·000315	1·0008	1·0025	0·0556	0·9721
0·001	1·0010	1·0035	0·0375	0·9808
0·00315	1·0013	1·0050	0·0250	0·9860
0·008	1·0017	1·0069	0·0163	0·9879
0·009	1·0017	1·0075	0·0152	0·9880
0·01	1·0018	1·0078	0·0143	0·9880
0·02	1·0022	1·0123	0·0100	0·9851
0·05	1·0027	1·0213	0·0073	0·9745
0·09	1·0031	1·0383	0·0008	0·9552
0·095	1·0031	1·8561	1·8857	0·1910
0·10	1·0032	1·8984	1·8980	0·1887
0·315	1·1761	2·4928	2·2624	0·1358
1·0	1·5758	3·8057	2·8255	0·0882
3·15	2·0228	6·1431	3·6934	0·0517
10·0	2·9886	10·964	5·0496	0·0274
31·5	4·8767	21·320	7·0640	0·0157
100·0	7·9726	38·680	10·169	0·0099
315·0	13·773	87·424	14·813	0·0034

λ that might correspond to hydromagnetic flow also give rise to the possibility of boosted accretion efficiency. To quantify the increases in the efficiency, we can define

$$\Delta b(\alpha) = b_{\max}(\alpha, \lambda) - b(\alpha, 0), \; \delta b(\alpha) = b_{\max}(\alpha, \lambda)/b(\alpha, 0). \tag{18}$$

We find that $\Delta b(0{\cdot}1) = 0$, $\delta b(0{\cdot}1) = 1$; $\Delta b(0{\cdot}5) = 6{\cdot}58$, $\delta b(0{\cdot}5) = 0{\cdot}458$; $\Delta b(0{\cdot}9978) = 2{\cdot}44$, $\delta b(0{\cdot}9978) = 0{\cdot}457$; $\Delta b(0{\cdot}9999) = 2{\cdot}32$, $\delta b(0{\cdot}9999) = 0{\cdot}505$; and $\Delta b(0{\cdot}99999999) = 2{\cdot}34$, while $\delta b(0{\cdot}99999999) = 0{\cdot}567$. In all cases the binding energy smoothly declines toward 0 at large ρ, matching the analytic result; as a matter of fact, by $\rho = 315$, the values were all within a few percent of each other. Considering the trends noted above, it is not absolutely impossible that at a larger λ non-zero Δb's for $\alpha = 0{\cdot}1$ could actually exist, or that the large b_{ms} solution would have dominated down to very small λ for α even greater than 0·99999999; however, neither of these possible outcomes would be of practical importance, and investigating them would have required an unwarranted large expenditure of computer time.

4. Conclusions

Our key result is that even the strongest plausible magnetic fields achievable in the vicinity of a black hole cannot affect the orbits of ordinary plasma to the extent

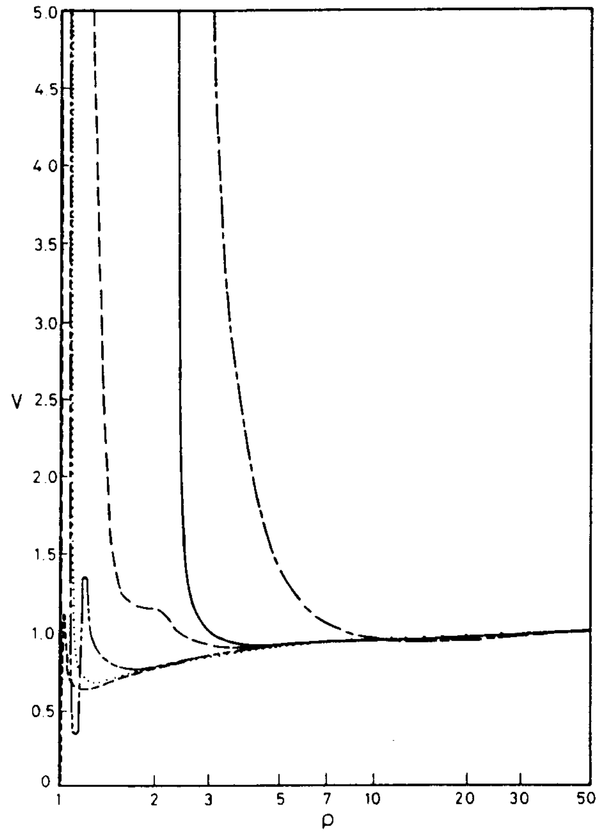

Figure 1. The effective potential *vs* distance for six choices of the parameter pair (α, λ). The different curves are labelled as follows: the solid curve, (0·5, 0·1); the dotted curve, (0·9978, 0·001); the double chain curve, (0·9978, 0·04); the single chain curve, (0·9978, 100); the short dashed curve, (0·99999999, 0·001); the long dashed curve, (0.99999999, 1). On this scale the short dashed curve is hard to read; for that case there is a minimum with $V = 0{\cdot}020$ at $\rho = 1{\cdot}0036$ and an intermediate maximum of $V = 1{\cdot}13$ at $\rho = 1{\cdot}030$.

necessary to materially change the efficiency at which that plasma is converted into energy as it is accreted. This also means that for an accreting plasma the decrease in r_{mb} is never sufficient to imply significantly narrower funnels (*e.g.* Paczynski and Wiita 1980). In view of the work of DW and WVSI, this is not at all surprising, but because the dipolar field configuration assumed in this paper is more physically likely than the uniform field assumed in that earlier work, we regard this confirmation as significant. It should be stressed that although the field configuration we have used is not a self-consistently locally generated one, the basic structure, with the field strength falling off with distance from the horizon should better mimic a full self-consistent configuration.

For charged test particles, our results agree in a general sense with those found earlier, but they are quantitatively very different. While we also find that for some values of α there are ranges of λ where the accretion efficiency is noticeably increased, here it is because a second minimum appears in the Keplerian angular momentum curve. When present, this effect depends rather weakly on α, with $\Delta b \approx 0{\cdot}5$ for all $\alpha \geqslant 0{\cdot}5$, but we found no effect for $\alpha = 0{\cdot}1$. This is directly opposite to what was found by WVSI: when

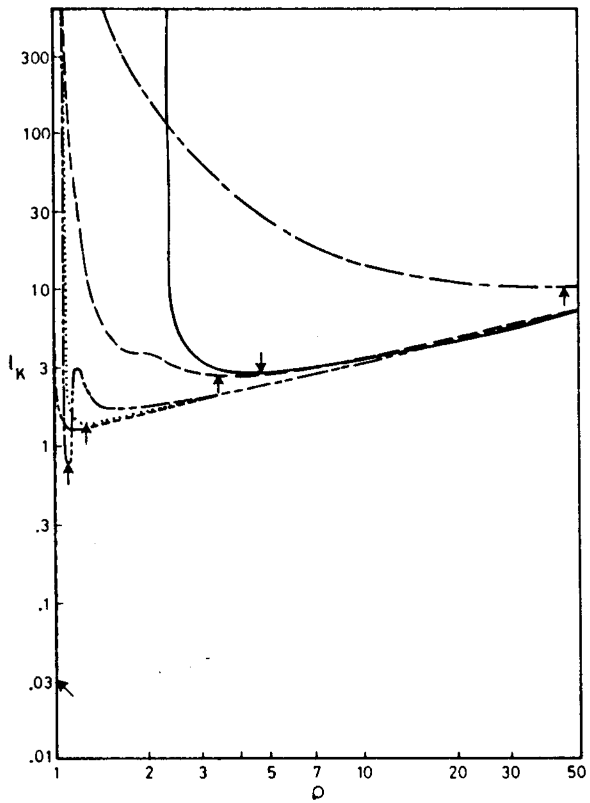

Figure 2. The Keplerian angular momentum (per unit mass) *vs* distance for the same six choices of the parameters. The curves are denoted as in figure 1. Here the values of L_{ms} are denoted by arrows. Again the case of (0·99999999, 0·001) is hard to read: it has a minimum at $L_K = 0{\cdot}0375$ at $\rho = 1{\cdot}0036$ and a secondary maximum of $L_K = 2{\cdot}32$ at $\rho = 1{\cdot}030$.

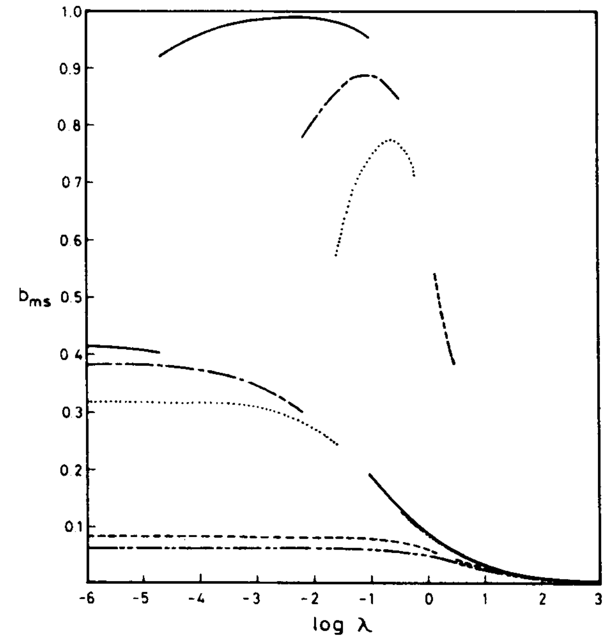

Figure 3. The binding energy *vs* λ for all five tabulated runs of α. The double chain curve corresponds to $\alpha = 0{\cdot}1$, the dashed to $\alpha = 0{\cdot}5$, the dotted to $\alpha = 0{\cdot}9978$, the single chained curve to $\alpha = 0{\cdot}9999$, and the solid curve to $\alpha = 0{\cdot}99999999$. Note the jumps to the second solution for all but the first case. Above $\lambda \approx 10$ all the curves nearly coincide.

uniform fields were assumed, the increase of efficiency (in both absolute and relative terms) decreased monotonically with α. We attribute this to the unphysical nature of the assumption of a constant magnetic field pervading all space. That assumption also produced configurations corresponding to rather narrow rings of material yielding the highest efficiencies. This is because the constant field eventually repelled particles, creating negative binding energies at large λ (unless $\alpha = 0$). But a dipole field falls off fast enough so as to allow smoothly vanishing binding energies at large λ, fulfilling the expectation of WVSI.

While we are quite confident in the negative results concerning hydrodynamic accretion, we ought to stress some of the simplifications that have entered into our calculations. We have specialized to the equatorial plane and have assumed a magnetic dipole moment aligned with the black hole's rotation axis. Other orientations could conceiveably, but not probably, alter the results in a major way. It would also be much more difficult to compute other such configurations, and we do not consider this task worthwhile.

Although the assumption of a dipole field configuration is clearly a simplification, the limits imposed in (17) were based on a maximum field strength, so our results should not vary in any important way if more complicated fields were to be considered. The analysis of locally generated self-consistent fields would be the best way of approaching this question, as these computations implicitly rely on the existence of a current source near the horizon; however, as stated above, we are confident that the inwardly growing fields analysed here would be rather typical of realistic self-generated fields, and are thus a good limiting case worthy of some attention. As opposed to the uniform fields examined in previous work, the dipole field has a satisfying limit at large distance, and does not require an unorthodox defintion of binding energy. The most serious omission in the current work, at least with regard to test particles, is our neglect of the radiation emitted by the infalling material. This might be worth further exploration as a question of principle, but should not be important for actual accretion disks.

Of course, many different accretion scenarios exist in which magnetism plays a dominating role (*e.g.* Lovelace 1976; Blandford and Znajek 1977; Blandford and Payne 1982; Rees *et al* 1982; Phinney 1983). Those effects, which can yield extremely high efficiencies in other ways, such as by extracting the rotational energy of the black hole, have not been considered here at all. Nonetheless, we have demonstrated that many straightforward effects of magnetism on accretion are negligible.

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