The geometric phase and ray space isometries

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Abstract. We study the behaviour of the geometric phase under isometries of the ray space. This leads to a better understanding of a theorem first proved by Wigner: isometries of the ray space can always be realised as projections of unitary or anti-unitary transformations on the Hilbert space. We suggest that the construction involved in Wigner's proof is best viewed as an use of the Pancharatnam connection to 'lift' a ray space isometry to the Hilbert space.

1. Introduction

The states of a quantum system are in one-to-one correspondence with rays in Hilbert space. The 'overlap' between rays is a measure of the distance between them and can be directly measured in the laboratory as a transition probability. A symmetry [1] of a quantum system maps the ray space onto itself preserving distances – i.e, it is a ray space isometry. Since it is inconvenient to work directly on the ray space (defined as an equivalence class of states in Hilbert space), most quantum mechanical calculations are carried out in Hilbert space. Wigner [2] proved that any ray space isometry can be realised on the Hilbert space of quantum mechanics by either a unitary or antiunitary transformation. This theorem underlies much of the study of symmetry in quantum mechanics. A complete and elementary account of Wigner's proof of his theorem is given by Bargmann [3]. More abstract and axiomatic accounts exist [4]. Our purpose here, is to use geometric phase ideas [5], which have recently been of interest, to shed light on Bargmann's exposition of Wigner's theorem. This work follows on an observation by Mukunda and Simon [6] regarding the relation between Bargmann's paper and the geometric phase. This paper is structured as follows. We first review some well-known facts about the geometric phase to set this paper in context and then fix our notation in § 2. In § 3, we study the behaviour of the Pancharatnam excess phase under ray space isometries. In § 4 we state Wigner's theorem and discuss its significance. In § 5 we use the result of § 3 to prove Wigner's theorem by an explicit construction. Section 6 is a concluding discussion. A fine point from § 3 is relegated to an appendix.

Berry [7] noticed many years ago that standard treatments of the adiabatic theorem in quantum mechanics had overlooked an important phenomenon: when a quantum system in a slowly changing environment returns to its original ray, the state of the system picks

up an extra phase of geometric origin above and beyond the phase that one naively expects on dynamical grounds. Berry's phase attracted wide attention because of its essentially geometric character. The phase depends only on the path traversed by the system in ray space and not on its rate of traversal. It is a measureable and gauge invariant quantity, independent of phase conventions. An important paper by Barry Simon [8] shows that Berry's phase is a consequence of the curvature of the natural connection on a line bundle over the ray space. Berry's original observation was made in the context of the adiabatic theorem of quantum mechanics. Aharonov and Anandan [9] showed how one could see Berry's phase even in nonadiabatic situations. The key input here was to identify the dynamical phase as the time integral of the expectation value of the Hamiltonian. When this dynamical phase is removed, the geometrical picture described by Simon [8] applies. Although one starts with the Schrödinger equation, after one identifies and removes the dynamical phase, the resulting parallel transport law is *purely* kinematic and depends only on the geometry of the Hilbert space. For this reason, Berry's phase is also known as the 'geometric phase'.

It was pointed out by Ramaseshan and Nityananda [10] that Berry's phase had been anticipated by Pancharatnam [11] in his studies of the interference of polarized light in the fifties. They showed that Berry's phase for a two state system was a special case of Pancharatnam's general study of interference of polarized light. Pancharatnam had given a physically motivated criterion for comparing the phases of two beams of polarized light. He went on to notice that this criterion was not integrable. Two beams A and B in phase with a third beam C are not in phase with each other. The phase difference between A and B is equal to half the solid angle subtended by the triangle (ABC) on the Poincaré sphere [12, 13]. In the limit that the discrete points approach a continuous curve, the Pancharatnam phase reduces to Berry's phase for a two state system.

In ref. [14], Pancharatnam's ideas were carried over to the Hilbert space of quantum mechanics. The Pancharatnam criterion was used to compare the phases of states on any two non-orthogonal rays. One defines two states to be 'in phase' if their inner product is real and positive. This permits us to define the Pancharatnam lift: Given a discrete sequence of rays (successive rays not orthogonal), one can use the Pancharatnam connection to 'lift' the discrete set of rays to Hilbert space. This connection contains the natural connection as a special case and tends to it in the limit that the sequence of points becomes a continuous curve.

The importance of geodesics on the ray space of quantum mechanics was emphasized in ref. [14], which states and proves the geodesic rule: Pancharatnam's criterion is equivalent to parallel transport of the phase along the shortest geodesic in the ray space. Given three non-orthogonal rays, one finds that Pancharatnam's excess phase is the integral of a two form over a *geodesic* triangle in the ray space. This is the direct analogue of Pancharatnam's 'half the solid angle' result. This general framework permits an extension of Berry's work to nonunitary and noncyclic situations. Needless to say, this work also subsumes the unitary and cyclic situations as a special case. It is also observed in ref. [14] that Berry's phase appears in systems subject to quantum measurements. Analogue optical experiments demonstrating this effect are reported in refs [15, 16]. A review of the field and a collection of papers upto 1989 is contained in the book by Shapere and Wilczek [5]. A more recent and detailed treatment is given by Mukunda and Simon [6], who note the connection between Pancharatnam's excess phase and invariants

considered by Bargmann [3]. Here, we follow on this observation made in [6]. We show how Pancharatnam's connection can be used to better understand a construction due to Wigner. We show below how one can use the Pancharatnam connection to 'lift' a given ray space isometry to the Hilbert space.

2. Preliminaries

Let $\mathcal{H}$ be the Hilbert space of a quantum system and $\mathcal{N} := \mathcal{H} - \{0\}$ the space of normalizable states. We define [17] rays to be equivalence classes of normalizable states differing only by multiplication by a nonzero complex number. We define two elements $|\Psi_1\rangle$ and $|\Psi_2\rangle$ of $\mathcal{N}$ to be equivalent ($|\Psi_1\rangle \sim |\Psi_2\rangle$) if $|\Psi_1\rangle = \alpha|\Psi_2\rangle$, where $\alpha \in \mathbb{C}, \alpha \neq 0$. The ray space is defined as the quotient of $\mathcal{N}$ by the equivalence relation

$$\mathcal{R} = \mathcal{N}/\sim .$$

Elements of both $\mathcal{H}$ and $\mathcal{N}$ will be written as kets $|\ \rangle$. The natural projection

$$\Pi : \mathcal{N} \to \mathcal{R}$$

maps each normalizable state $|\Psi\rangle$ to the ray Ψ on which it lies. We define the *overlap* between two rays Ψ_1 and Ψ_2 as follows:

$$|\Psi_1 \cdot \Psi_2|^2 := \frac{\langle \Psi_1|\Psi_2\rangle\langle \Psi_2|\Psi_1\rangle}{\langle \Psi_1|\Psi_1\rangle\langle \Psi_2|\Psi_2\rangle}.$$

By Schwartz inequality, $|\Psi_1 \cdot \Psi_2| \leqslant 1$ and $|\Psi_1 \cdot \Psi_2| = 1$ if and only if $\Psi_1 = \Psi_2$. We define the distance $\delta(\Psi_1, \Psi_2)$ between the rays Ψ_1 and Ψ_2 by

$$|\Psi_1 \cdot \Psi_2| = \cos(\delta/2),$$

where δ lies between zero and π. Note that $\delta(\Psi_1, \Psi_2) = 0$ if and only if $\Psi_1 = \Psi_2$.

Let $\{\gamma(\lambda), 0 \leqslant \lambda \leqslant 1\}$ be a curve in $\mathcal{R}$ and $|\gamma(0)\rangle$ a vector on the ray $\gamma(0)$. We define the 'horizontal lift' of $\gamma(\lambda)$ as the unique curve $|\gamma(\lambda)\rangle$ starting from $|\gamma(0)\rangle$ which satisfies $\Pi(|\gamma(\lambda)\rangle) = \gamma(\lambda)$ and

$$\left\langle \gamma(\lambda) \left| \frac{\mathrm{d}\gamma}{\mathrm{d}\lambda} \right\rangle \right. = 0. \tag{1}$$

Equation (1) gives us a rule (mathematically a connection) for comparing vectors on neighbouring rays.

The Pancharatnam connection which we now describe is a more general notion that permits a comparison of vectors on *any* two non-orthogonal rays. Let $|A\rangle$ and $|B\rangle$ be two non-orthogonal vectors. We define them as being 'in phase' if the inner product $\langle A|B\rangle$ is real and positive. Given $|A\rangle$ and a ray B, there is an unique $|B\rangle$ which is *in phase* with $|A\rangle$ and has the same size ($\langle B|B\rangle = \langle A|A\rangle$). We refer to $|B\rangle$ as the Pancharatnam lift of B (with $|A\rangle$ as reference).

Given three pairwise non-orthogonal rays A, B, C, one can define the quantity

$$\Delta_{\mathrm{ABC}} = \frac{\langle A|B\rangle\langle B|C\rangle\langle C|A\rangle}{\langle A|A\rangle\langle B|B\rangle\langle C|C\rangle}, \tag{2}$$

where, $|A\rangle, |B\rangle$ and $|C\rangle$ are representative elements from the corresponding rays. Δ_{ABC} depends only on the rays A, B, C and not on the representatives. We will sometimes abbreviate Δ_{ABC} to Δ. The phase β of the complex number $\Delta = \rho \exp(i\beta)$ is the Pancharatnam excess phase, which is well defined (modulo 2π) if $\rho \neq 0$.

An *isometry* of the ray space is a map

$$\mathbf{T} : \mathcal{R} \to \mathcal{R} \tag{3}$$

which preserves distances. Writing

$$\Psi' = \mathbf{T}\,\Psi, \tag{4}$$

$\mathbf{T}$ is an isometry if

$$|\mathbf{A}' \cdot \mathbf{B}'| = |\mathbf{A} \cdot \mathbf{B}|. \tag{5}$$

Under isometries the rays A, B, C go to A', B', C' and Δ_{ABC} goes to $\Delta_{A'B'C'}$, which we will abbreviate to Δ'.

3. Isometries and the Pancharatnam phase

We now study the transformation of Δ under ray space isometries. Let A, B, C be three distinct pairwise non-orthogonal rays. Let us choose unit representatives $|A\rangle, |B\rangle, |C\rangle$ from these rays. Further, let us choose the phases of these representatives so that $|B\rangle$ is in phase with $|A\rangle$ (their inner product $\langle A|B\rangle$ is real and positive)

$$\langle A|B\rangle = \cos c/2 \tag{6}$$

and $|C\rangle$ is in phase with $|A\rangle$

$$\langle C|A\rangle = \cos b/2. \tag{7}$$

This of course means that $|C\rangle$ is not (in general) in phase with $|B\rangle$. In fact,

$$\langle B|C\rangle = \cos a/2 \exp(i\beta). \tag{8}$$

a, b and c above are the distances (lengths of the shortest geodesics in $\mathcal{R}$) between the rays (A, B, C). (a, b, c) are the sides of the geodesic triangle with vertices (A, B, C) and take values strictly between 0 and π.

Let

$$|\mu_B\rangle = |B\rangle - \cos(c/2)|A\rangle \tag{9}$$

be the component of $|B\rangle$ orthogonal to $|A\rangle$. Since $\langle \mu_B|\mu_B\rangle = \sin^2(c/2)$, we define the unit vector

$$|\hat{\mu}_B\rangle = |\mu_B\rangle / \sin(c/2). \tag{10}$$

Using $|A\rangle$ and $|\hat{\mu}_B\rangle$ as an orthonormal basis in the $|A\rangle - |B\rangle$ plane, one sees (on the Poincaré sphere) that the horizontal curve $\{|\gamma_B(\lambda)\rangle, 0 \leqslant \lambda \leqslant 1\}$, joining $|A\rangle$ to $|B\rangle$ $(|\gamma_B(0)\rangle = |A\rangle, |\gamma_B(1)\rangle = |B\rangle)$

$$|\gamma_B(\lambda)\rangle = \cos(\lambda c/2)|A\rangle + \sin(\lambda c/2)|\hat{\mu}_B\rangle \tag{11}$$

projects down to the shortest geodesic $\gamma_B(\lambda)$ connecting **A** and **B** ($\gamma_B(0) = \mathbf{A}$, $\gamma_B(1) = \mathbf{B}$).

The tangent vector to the curve $|\gamma_B(\lambda)\rangle$ at $\lambda = 0$ is

$$|\dot{\gamma}_B(0)\rangle = (c/2)|\hat{\mu}_B\rangle. \tag{12}$$

Similarly $\{|\gamma_C(\lambda)\rangle, 0 \leqslant \lambda \leqslant 1\}$ defined as

$$|\gamma_C(\lambda)\rangle = \cos(\lambda b/2)|A\rangle + \sin(\lambda b/2)|\hat{\mu}_C\rangle \tag{13}$$

is the horizontal lift of the shortest geodesic connecting **A** with **C**. In (13) $|\hat{\mu}_C\rangle$ is the normalized vector $|\hat{\mu}_C\rangle = |\mu_C\rangle / \sin(b/2)$ where $|\mu_C\rangle = |C\rangle - \cos(b/2)|A\rangle$. The tangent vector to the curve $|\gamma_C(\lambda)\rangle$ at $\lambda = 0$ is

$$|\dot{\gamma}_C(0)\rangle = b/2|\hat{\mu}_C\rangle. \tag{14}$$

The angle A between the geodesics $\gamma_B(\lambda)$ and $\gamma_C(\lambda)$ at **A** is given by

$$\cos(A) = \frac{\Re(\langle\dot{\gamma}_B|\dot{\gamma}_C\rangle)}{(\langle\dot{\gamma}_B|\dot{\gamma}_B\rangle\langle\dot{\gamma}_C|\dot{\gamma}_C\rangle)^{1/2}}, \tag{15}$$

where $\Re(\alpha)$ means the real part of its α. This is easily worked out as

$$\cos(A) = \frac{\cos(a/2)\cos(\beta) - \cos(b/2)\cos(c/2)}{\sin(c/2)\sin(b/2)}. \tag{16}$$

This gives us the formula

$$\cos(\beta) = \frac{\cos(A)\sin(c/2)\sin(b/2) + \cos(b/2)\cos(c/2)}{\cos(a/2)} \tag{17}$$

for the cosine of the Pancharatnam phase. The right hand side of this equation contains only the sides a, b, c and (one of) the angles of the geodesic triangle connecting the rays **A**, **B**, **C**. All these quantities are manifestly invariant under isometries of the ray space. It follows that $\cos(\beta)$ is also an isometry invariant. Since $\rho = |\Delta|$ is clearly isometry invariant, it follows that $\Re(\Delta)$ is isometry invariant and hence

$$\Delta' = \chi(\Delta) \tag{18}$$

where $\chi(\alpha) = \alpha$ or $\chi(\alpha) = \bar{\alpha}$. Equation (18) is valid for *all* triplets of rays (including orthogonal ones, for which it becomes trivial). Since the map **T** is continuous, the function χ must be the *same* all over the ray space (see appendix) and can be determined [3] from **T** [18].

4. Statement of the theorem

We address the following problem. Given a ray space isometry **T**, construct a map $T : \mathcal{N} \to \mathcal{N}$ so that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{N} & \overset{T}{\longrightarrow} & \mathcal{N} \\
\Pi \downarrow & & \downarrow \Pi \\
\mathcal{R} & \overset{\mathbf{T}}{\longrightarrow} & \mathcal{R}
\end{array}
$$

or algebraically,

$$\Pi(T(\Psi))) = \mathbf{T}(\Pi(|\Psi\rangle)). \tag{19}$$

T is called the 'lift' of $\mathbf{T}$. Clearly, there are many such lifts T since, given $|\Psi\rangle$, we could pick as its image $|\Psi'\rangle$ an arbitrary point from the fibre above $\mathbf{T}(\Pi(|\Psi\rangle))$. We could, in fact turn this nonuniqueness to advantage and demand that the lift T has some nice properties. For instance, we could demand that T be continuous. We *will* assume below that T is continuous but even this restriction allows much residual freedom. For example, if $\mathbf{T}$ is the identity map, for each continuous, nonzero complex function f on $\mathcal{N}$, T_f defined by $T_f(|\Psi\rangle) = f|\Psi\rangle$ is a continuous lift. Clearly, we can do much better and demand that T has some *more* nice properties. The conditions we impose should be as strong as we can demand (so that the lift has desirable properties and is reasonably unique) and yet weak enough that a lift exists. Continuity of T is clearly too weak. We are free to impose more conditions on the lift T. Wigner's theorem does just that. Wigner showed that one can find a continuous lift which preserves intensities (**W1** below) as well as superpositions (**W2** below).

Wigner's theorem. There exists a lift T of $\mathbf{T}$ which

W1 satisfies $\langle\Psi'|\Psi'\rangle = \langle\Psi|\Psi\rangle$
W2 when extended to $\mathcal{H}$ by $T|0\rangle = |0\rangle$ satisfies

$$T(|A\rangle + |B\rangle) = T(|A\rangle + |B\rangle).$$

The lift is unique up to an overall phase [18].

The content of Wigner's theorem is that *all* ray space isometries (i.e all maps $\mathbf{T}$ which satisfy (5)) can be realised by maps on $\mathcal{H}$ satisfying (**W1,W2**). No other isometries exist and nothing is lost by restricting attention to maps T which satisfy (**W1,W2**). We prove Wigner's theorem below by explicitly constructing the map T.

5. Wigner's construction

Let $|e\rangle$ be any fixed vector in $\mathcal{N}$, e its ray and e' the image of e under $\mathbf{T}$. Let us arbitrarily pick $|e'\rangle$ from e' satisfying $\langle e'|e'\rangle = \langle e|e\rangle$ and define $T|e\rangle$ to be $|e'\rangle$. $|e'\rangle$ is arbitrary up to a phase. This is the only arbitrariness in the entire construction which follows. Let $\mathcal{P} = \{|\Psi\rangle \in \mathcal{H}|\langle e|\Psi\rangle = 0\}$ be the set of elements in $\mathcal{H}$ orthogonal to $|e\rangle$. And let $\mathcal{P}^c$ be its complement – the set of elements in $\mathcal{H}$ which are *not* orthogonal to $|e\rangle$. We now define the action of T on all elements of $\mathcal{P}^c$ using the Pancharatnam lift. Let $|\Psi\rangle \in \mathcal{P}^c$ be such an element. From (5), it follows that $|(\Psi', e')|$ is not zero. We map $|\Psi\rangle$ to the unique element $|\Psi'\rangle \in \Psi'$ which satisfies (20, 21) below.

$$\langle\Psi'|\Psi'\rangle = \langle\Psi|\Psi\rangle \tag{20}$$

determines the amplitude of $|\Psi'\rangle$. Since $|\langle e'|\Psi'\rangle| = |\langle e|\Psi\rangle|$, we can choose the phase of $|\Psi'\rangle$ to satisfy

$$\langle e'|\Psi'\rangle = \chi(\langle e|\Psi\rangle). \tag{21}$$

It follows from (18) rewritten here as

$$\frac{\langle e'|A'\rangle\langle A'|B'\rangle\langle B'|e'\rangle}{\langle e'|e'\rangle\langle A'|A'\rangle\langle B'|B'\rangle} = \chi\left(\frac{\langle e|A\rangle\langle A|B\rangle\langle B|e\rangle}{\langle e|e\rangle\langle A|A\rangle\langle B|B\rangle}\right) \tag{22}$$

that if $|A\rangle$ and $|B\rangle$ are any two vectors in $\mathcal{P}^c$, $|A'\rangle$ and $|B'\rangle$ defined as in (20, 21) above satisfy

$$\langle A'|B'\rangle = \chi(\langle A|B\rangle). \tag{23}$$

Note that this lift preserves superpositions. For if $|\Psi\rangle = |A\rangle + |B\rangle$, (all $|\ \rangle$s in $\mathcal{P}^c$), a simple calculation shows that the norm of

$$|\phi'\rangle = |\Psi'\rangle - (|A'\rangle + |B'\rangle)$$

vanishes. It follows that

$$|\Psi'\rangle = |A'\rangle + |B'\rangle. \tag{24}$$

Actually, more is true. If $|A\rangle + |B\rangle = |C\rangle + |D\rangle$ (all $|\ \rangle$s in $\mathcal{P}^c$), we find that $|A'\rangle + |B'\rangle = |C'\rangle + |D'\rangle$. The proof as before, is to just compute the norm of the difference of both sides and use (23). Note that the sum $|A\rangle + |B\rangle$ need not be in $\mathcal{P}^c$. We can therefore define the action of T on elements of $\mathcal{P}$ by superposition. Any element $|\Phi\rangle \in \mathcal{P}$ can be written as sums of elements in $\mathcal{P}^c$. For example

$$|\Phi\rangle = (|\Phi\rangle - |e\rangle) + |e\rangle. \tag{25}$$

In fact there are many ways to express $|\Phi\rangle$ as sums of elements of $\mathcal{P}^c$. It does not matter which of these ways one chooses and that the extension of T to $\mathcal{P}$ is well defined. We have thus defined T on all of $\mathcal{H}$ satisfying (**W1,W2**).

6. Conclusion

The key new observation of [6] which led to the present work is that the quantity Δ which has recently been of interest in the context of the Pancharatnam phase is exactly what was used by Bargmann to discriminate between unitary and anti-unitary transformations. Bargmann remarks [3] that one can determine the function $\chi(\alpha)$ merely from a knowledge of the map **T** (for dim($\mathcal{H}$) > 1 [18]). One starts with Δ, which is defined on the ray space. Using **T**, one determines Δ' and from $\Delta' = \chi(\Delta)$, one can determine χ.

The main difference between our exposition and ref. [3] is that Bargmann deduces (18) as a *corollary*, after constructing a lift of **T**. We reverse the order and, using geometric phase ideas, first prove (18) as a geometric identity on the ray space. This result is then used as an *input* for constructing the lift and showing that it does have the desired properties (**W1,W2**). This leads to a considerably simplified and elementary exposition of Wigner's theorem based on ideas from the geometric phase.

We have derived a formula (17) expressing the cosine of the Pancharatnam excess phase in terms of isometry invariants. This leads to two distinct possibilities for the transformation of the Pancharatnam phase under isometries: it is either preserved or reversed. The lift T is accordingly unitary or anti-unitary. Note that the Pancharatnam phase β itself is *not* an isometry invariant, but only its cosine. The non-invariance of β is

precisely what Bargmann uses to distinguish between unitary and anti-unitary transformations.

It is interesting to note that trigonometry in ray space is qualitatively different from plane or spherical trigonometry. In ray space, the sides of a triangle (a, b, c) do not determine its angles (A, B, C). To see this, it is enough to consider a 3 (complex) dimensional Hilbert space $\mathcal{H}$ (since three rays are involved). A triangle in $\mathcal{R}$ is determined by 3 distinct rays in $\mathcal{R}$. Since $\mathcal{R} = C\mathbb{P}^2$ is 4 (real) dimensional, the set of triangles is 12 (real) dimensional. The isometry group of $C\mathbb{P}^2$ is 8 (real) dimensional and acts freely on triangles. It follows that a triangle in the ray space has 4 independent isometry invariants. We chose to express (17) $\cos(\beta)$ in terms of the four independent variables (a, b, c, A). One could equally choose any four of these six variables.

For simplicity, we assumed that a symmetry maps the ray space $\mathcal{R}$ to itself. More generally, one can have maps between different ray spaces. Such a situation arises if there is more than one superselection sector in the theory. An example of such a mapping is charge conjugation, which maps different charge superselection sectors to each other. Our analysis is easily adapted to mappings between different superposition sectors.

To mathematicians, the ray space is a Kähler manifold [19, 20], with three interlinked structures: a metric, a symplectic structure and a complex structure. Any two of these determine the third. Physically, the metric represents transition probabilities and the symplectic 2-form is the curvature of the natural connection that emerges from Berry's phase [21]. Isometries of $\mathcal{R}$ preserve the metric, but may reverse the symplectic structure. Then the complex structure is also reversed.

We feel that this paper provides an interesting application of the Pancharatnam connection. Note that the Pancharatnam connection has been used in an essential way. The natural connection only permits a comparison of neighbouring rays and therefore could be used only in the tangent space around $|e\rangle$. The global nature of the Pancharatnam connection allows us to define a lift of T for (almost) all rays at once. The gaps are then filled in by superposition.

Acknowledgement

Appendix

In this appendix we show that continuity implies that χ is the same all over ray space. There is a subtlety here stemming from the fact that there are regions in ray space where $\Im(\Delta)$, the imaginary part of Δ vanishes and the two possibilities for χ coincide. Let us fix rays A, B and consider Δ as a function of ray C. Let us define $\mathcal{R}^+$ as the set of points of $\mathcal{R}$ where $\Im(\Delta) > 0$ and similarly $\mathcal{R}^-$ is the set where $\Im(\Delta) < 0$. We first show that $\mathcal{R}^-$ is path connected. Let C and $\tilde{C}$ be two rays in $\mathcal{R}^+$. Let us choose a representative vector $|C\rangle$ and decompose it into components $|C^{\parallel}\rangle$ in the $|A\rangle - |B\rangle$ plane and $|C^{\perp}\rangle$ orthogonal to it. By continuously decreasing the orthogonal component of $|C\rangle$ to zero, one can deform $|C\rangle$ to $|C^{\parallel}\rangle$ in the $|A\rangle - |B\rangle$ plane. In the expression (2) for Δ, $|C^{\perp}\rangle$ does

not contribute to the numerator and the denominator is real and positive. It follows that the sign of $\Im(\Delta)$ does not change in the process of decreasing the orthogonal component of $|C\rangle$ and so the deformation is entirely within $\mathcal{R}^+$. Likewise $|\tilde{C}\rangle$ can also be deformed within $\mathcal{R}^+$ to $|\tilde{C}^{\parallel}\rangle$ in the $|A\rangle - |B\rangle$ plane. The resulting kets are now in the two dimensional subspace spanned by $|A\rangle$ and $|B\rangle$ and we can now visualize the situation on the Poincaré sphere. Let C be the great circle through the points A and B on the Poincaré sphere. C divides the sphere into two hemispheres. $\Im(\Delta)$ vanishes only for points belonging to C, and $\Im(\Delta)$ is strictly positive on one hemisphere and strictly negative on the other. Since C and $\tilde{C}$ belong to $\mathcal{R}^+$, the rays $C^{\parallel}$ and $\tilde{C}^{\parallel}$ corresponding to the vectors $|C^{\parallel}\rangle$, $|\tilde{C}^{\parallel}\rangle$ lie in the same hemisphere. They can therefore be deformed into each other without passing through the equator. Throughout this deformation, $\Im(\Delta)$ is positive and it follows that $\mathcal{R}^+$ is connected. (An identical argument shows that $\mathcal{R}^-$ is connected.)

Since $\mathcal{R}^+$ is connected, continuity of T implies that χ must be the same all over $\mathcal{R}^+$. Likewise, χ must be the same all over $\mathcal{R}^-$. If χ were to differ between $\mathcal{R}^+$ and $\mathcal{R}^-$, both $\mathcal{R}^+$ and $\mathcal{R}^-$ would be mapped to the same component ($\mathcal{R}^+$ or $\mathcal{R}^-$). This contradicts the fact that the map T is onto. Therefore χ must be the same all over $\mathcal{R}$.

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