The Coaxial Cable

How Cable TV Operators Use the Laplacian Operator

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This article illustrates the interplay between topology and analysis in the context of an everyday example. We describe how topology governs the transmission of TV signals through a cable. This example helps us to understand some of the mathematical ideas contained in an earlier article by V Pati in *Resonance*. A number of steps are left as exercises so that the reader can actively participate in the exposition.

Cable TV is fairly common these days. The TV set receives signals through a cable that is plugged into it. The next time you get a chance, unplug the cable from the TV or the wall socket and examine it. You will find an outer cylindrical metal tube and an inner wire (see *Figure 1*) separated by a dielectric medium. The function of the cable is to transport electromagnetic energy from one of its ends to the other. Let us now try to understand the relation between the structure of the cable and its function.

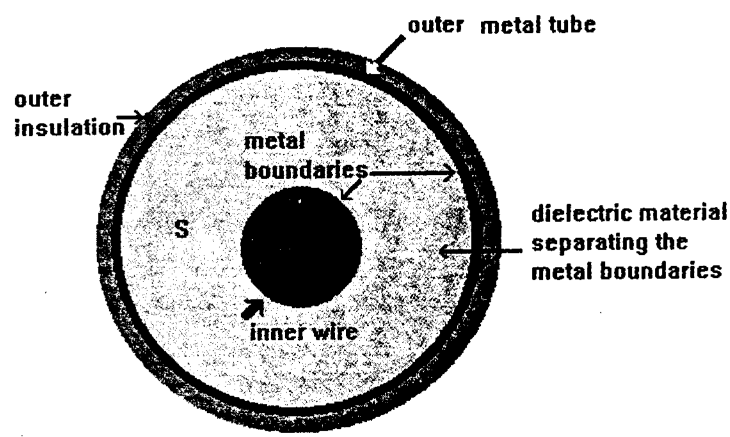

Figure 1 A cross section of the cable. S is the region in which the fields propagate.

The cable 'guides' the electromagnetic energy along its length – it is a *waveguide*. The outer tube confines the electromagnetic energy and prevents it from escaping. But what is the function of the inner wire? Is it possible to send the electromagnetic signal through the tube without an inner wire? We address these questions in this article and in the process of answering them, we will learn a bit about transmission lines and topology.

Maxwell's equations lead us to the transverse Laplace equation on the cross section of the cable.

Can one send electromagnetic energy through a hollow metal tube? Of course, one can! If you look through a metal pipe you see direct proof that electromagnetic energy does propagate through a tube. Light is an electromagnetic wave! Then why do cable TV operators put in an inner wire? To find the answer, we need to write down all the relevant equations and solve them. This looks like a problem in calculus, but, as you will see, we will end up discussing topology.

At this point it is best to get paper and a pencil to check some of the steps yourself. Topology and physics are not passive entertainment like Star TV! The equations we have to solve are the source-free Maxwell's equations inside the cable subject to boundary conditions at the metal surface. Let us assume the cable to be laid straight along the z axis. The cross section of the cable is the same for all t and z. We can therefore solve the Maxwell equations by separating variables. We assume a solution of the form $\vec{E}(x,y,z,t) = \vec{\mathcal{E}}(x,y)f(t,z)$, where $\vec{\mathcal{E}}(x,y)$ depends only on the transverse coordinates (x,y) and f depends only on (t,z). Plugging this form into the wave equation B(5)(see *Box 1*) for the electric field, we find that

i) $f(t,z)$ satisfies the equation

$$-\frac{\partial^2 f}{\partial t^2} + c^2\frac{\partial^2 f}{\partial z^2} = Df, \qquad (1)$$

ii) $\vec{\mathcal{E}}(x,y)$ satisfies the equation

$$-c^2\nabla_\perp^2\vec{\mathcal{E}} = D\vec{\mathcal{E}}, \qquad (2)$$

where D is a separation constant and ${\nabla^2}_\perp$ the two dimensional Laplacian $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. (A form similar to (1) is also assumed for the magnetic field, which will not be written here explicitly. We merely note that after the electric

Box 1 Source-free Maxwell's Equations.

Electromagnetism is governed by the set of four Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \qquad (B1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (B2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad (B3)$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{\epsilon_0}\frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}, \qquad (B4)$$

where μ_0 and ϵ_0 are constants characterizing the vacuum. ρ and $\vec{j}$ are the charge density and current density respectively. They act as *sources* for the $\vec{E}$ and $\vec{B}$ fields. In the absence of sources we set $\rho = 0$, $\vec{j} = 0$ and get the 'source-free' Maxwell equations. Throughout this article we have set the sources to zero in Maxwell's equations, since we work in the region between the metal boundaries, where there are neither charges nor currents.

By taking the curl of (B3) and using (B4) and (B1) we get (remember that $\rho = 0$ and $\vec{j} = 0$)

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0, \qquad (B5)$$

where $c^2 = \frac{1}{\mu_0 \epsilon_0}$. This is the *wave equation* for the field $\vec{E}$. Similarly, by taking the curl of (B4) and using (B3) and (B2) we get

$$\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0, \qquad (B6)$$

the *wave equation* for the magnetic field. Apart from satisfying Maxwell's equations, which are differential equations, the fields also have to satisfy boundary conditions: at the metal boundaries, the tangential component of the electric field must vanish:

$$\vec{t}.\vec{E} = 0. \qquad (B7)$$

Here $\vec{t}$ is any vector tangential to the boundary surface. Further, the normal component of the magnetic field must also vanish at the metal boundaries:

$$\vec{n}.\vec{B} = 0, \qquad (B8)$$

where $\vec{n}$ is the vector normal to the boundary.

field is known, the magnetic field is determined from the Maxwell's equations.)

Equation (1) is solved by $f(t,z) = \cos(kz - \omega t)$ where ω and k are constants satisfying the relation

$$\omega^2 = D + c^2k^2. \tag{3}$$

ω is the angular frequency of the wave and k is the wave number, which is related to the wavelength by $\lambda = 2\pi/k$. $\cos(kz-\omega t)$ represents a solution propagating in the positive z direction with phase velocity $v_{ph} = \omega/k$ which depends on the wavelength for $D \neq 0$. Thus in this case, there is *dispersion* of the TV signal. This is analogous to dispersion of light in optical media, where the refractive index (and hence the velocity of propagation) depends on the wavelength.

Equation (3) relates the (angular) frequency ω of waves being transmitted through the cable and their wave number k and is called the *dispersion relation.* Let us now assume that the transmission is dispersionless (see *Box 2*). In other words, we require that pulse shapes travel undistorted through the cable. This implies $\omega = ck$ (or equivalently, $D = 0$) so that waves of all frequencies travel at the speed of light, c.

The possibility of dispersionless wave propagation through a metal tube depends on the topology of the cross-section of the tube.

From (2) and (3) the condition for dispersionless propagation is that the field $\vec{\mathcal{E}}(x,y)$ must satisfy the transverse Laplace equation

$$\nabla_\perp{}^2\vec{\mathcal{E}} = 0 \tag{4}$$

subject to the condition (B7). The latter implies that

$$\vec{t}.\vec{\mathcal{E}} = 0 \tag{5}$$

at the metal boundary for every vector $\vec{t}$ tangent to the boundary. Now solve (4) with the boundary condition (5).

Let us start with the z-component of (4). Note the identity

$$\int_S \vec{\nabla}_\perp\mathcal{E}_z.\vec{\nabla}_\perp\mathcal{E}_z dxdy = \int_S \vec{\nabla}_\perp.(\mathcal{E}_z\vec{\nabla}_\perp\mathcal{E}_z)dxdy - \int_S \mathcal{E}_z\nabla_\perp{}^2\mathcal{E}_z dxdy \tag{6}$$

Box 2: Dispersion and Damping.

You may wonder why you can see light through a hollow tube but cannot detect a TV signal along such a tube. For a signal to propagate through a hollow tube the frequency of the signal has to be higher than a threshold frequency ω_0 set by the dimensions of the tube. (*Exercise:* Use (6) to show that $D \geq 0$ and can therefore be written as ${\omega_0}^2$. Use dimensional arguments to estimate the lowest positive value of ω_0) Consider the dispersion relation $k^2 = \omega^2 - \omega_0^2$ (equation (3)). Notice that for $\omega > \omega_0$, k is real and we get a propagating wave ($\sim \cos(kz - \omega t)$). On the other hand, for $\omega < \omega_0$, k is purely imaginary and the wave is damped and it decays exponentially along the length of the cable. In the case of light, the angular frequency ω is about 10^{15}rad/s. (It is more common to give the frequency $\nu = \omega/2\pi$ measured in Hz.) Thus the frequency of a light signal is clearly greater than the cut-off frequency $\nu_0 \sim 10^{11}$Hz associated with a cable 1mm thick. This explains why light can be seen through a hollow tube. In contrast, a TV signal has a typical frequency $\nu \sim 10^7$Hz, which is much smaller than the cut-off frequency ν_0 of a cable. So a TV signal cannot be transmitted through a 1mm thick hollow tube. Similarly, light cannot pass through a thin hollow metal tube (say 1000 Angstrom in diameter or less).

For purposes of exposition, we imposed the condition that the waves must propagate without dispersion. In practice, wave guides with dispersion are also used for transmitting signals. One chooses a narrow band of frequencies so that the dispersion is *negligible* over the band used.

We have also described region S inside the cable as 'metal free space'. This region is not empty but there is some solid dielectric material separating the two metal boundaries. This affects our analysis only to the extent that c is not the speed of light in vacuum but in the dielectric medium.

To avoid pathologies of a mathematical nature, we assume that the metal boundaries do not touch each other and that there are only a finite number of inner wires, and for a TV cable only a single inner wire.

where $\vec{\nabla}_\perp$ is the two dimensional gradient $\vec{\nabla}_\perp = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. The region of integration in (6) is S, the (cross section of the) metal-free space (the light grey region in the figure) inside the cable along with its boundary. The boundary of S consists of the inner surface of the outer metal tube and the outer surfaces of any inner wires present there. (In the figure we have shown a single central wire). The second term on the RHS of (6) vanishes by (4). The first term can be expressed (*Exercise:* supply the argument) as a line integral over the boundary of the cross section; this vanishes too, because the boundary condition (5) implies that $\mathcal{E}_z$ vanishes on the boundary. It follows that

$$\int_S \vec{\nabla}_\perp \mathcal{E}_z . \vec{\nabla}_\perp \mathcal{E}_z dxdy = \int_S (\vec{\nabla}_\perp \mathcal{E}_z)^2 dxdy = 0. \qquad (7)$$

Since the integrand is positive, $\vec{\nabla}_\perp \mathcal{E}_z = 0$. Thus $\mathcal{E}_z$ is zero (since it is zero at the boundary) everywhere in S.

Maxwell's equation (B1) now reads

$$\partial_x \mathcal{E}_x + \partial_y \mathcal{E}_y = 0. \qquad (8)$$

Let $\alpha = \partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x$ and let I the integral of α^2 over S be

$$I = \int (\alpha)^2 dxdy.$$

By rearranging the integral (*Exercise:* supply the argument) as before, and using (8), we conclude that $I = 0$. Since the integrand is positive,

$$\alpha = \partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x = 0. \qquad (9)$$

(9) would certainly hold if $\vec{\mathcal{E}}$ were the gradient of some scalar function. But can we write $\vec{\mathcal{E}}$ as a gradient? Fix a point p_0 in S and define a potential $\phi(p, \gamma)$ by

$$\phi(p, \gamma) = {}_\gamma\!\int_{p_0}^{p} \vec{\mathcal{E}} . d\vec{l}, \qquad (10)$$

where p is any point in S and the integral is over a curve γ joining p_0 to p. Consider the line integral of $\vec{\mathcal{E}}$ over any closed curve. Given any closed curve γ_0 in S, we can find a two dimensional surface whose boundary consists of γ_0 and possibly additional metallic components. Use of (5) and (9) shows that the integral vanishes:

$$\oint_{\gamma_0} \vec{\mathcal{E}} . d\vec{l} = 0. \qquad (11)$$

(*Exercise*: Show that $\phi(p, \gamma)$ is independent of γ as a result of (11)). Thus we can write $\phi(p)$ instead of $\phi(p, \gamma)$ and it follows from (10) that $\vec{\mathcal{E}}$ is a gradient: $\vec{\mathcal{E}} = \vec{\nabla}_\perp \phi$ (see

De Rham's[1] theorem in the article by V Pati in Suggested Reading). Now (8) reads

$$\vec{\nabla}_{\perp}^{2}\phi = 0, \quad (12)$$

and the boundary condition (5) tells us that

$$\vec{t}.\vec{\nabla}_{\perp}\phi(x,y) = 0, \quad (13)$$

i.e. the tangential component of the gradient of ϕ must be zero on the metallic boundary of the cross-section.

The analysis so far is general and encompasses all possiblities for the cross section of the tube. Let us now consider the following cases:

(a) *A hollow cylindrical tube:* In this case there is only one metallic boundary and the condition (13) implies that ϕ is a constant ($= \phi_1$, say) all over the boundary. Consider the function $\phi - \phi_1$ (which satisfies Laplace's equation (12) and vanishes on the boundary) and show that ϕ is constant all over S (this is left as an exercise). Hence the electric field vanishes in this region. There are therefore *no nontrivial solutions* to (4) satisfying the requisite boundary conditions.

(b) *A hollow tube with some other cross section:* One might wonder if changing the cross section of the tube would help. A glance at the argument given above shows that it does not. All we used in that argument was that the metal boundary of the cable consisted of a single piece. The use of neither a rectangular nor an irregular cross-section helps. It doesn't help to change the *geometry* of S. But, as we shall see later, it does help to change the *topology* of S.

(c) *A tube with an annular cross-section:* Insertion of a central wire completely changes the picture. The region of interest is now an annulus and has *two* boundaries: an outer boundary and an inner boundary. (13) only implies that ϕ is constant over *each* piece of the boundary. While the potential can be a given constant ϕ_1 on the outer wire, it could be a *different* constant ϕ_2 on the inner wire. This possibility arises entirely *because the boundary of the annular region has two disconnected components* as opposed to one

[1] Readers may be amused to learn that apart from being a mathematician, de Rham was also an accomplished alpine mountaineer. In fact, in his obituary notice at his mountaineering club in Switzerland, they described him as being "also a mathematician"!

for the disc. Consequently there *are* nontrivial solutions to (12). It is quite easy to write down a solution for (12) namely,

$$\phi(x,y) = A log[r], \tag{14}$$

where A is a constant and r is the radial distance measured from the centre of the inner wire. The corresponding $\vec{\mathcal{E}}(x,y)$ is

$$\vec{\mathcal{E}} = A\hat{r}/r, \tag{15}$$

which points radially outward, just like the electric field of a line charge.

Cable TV provides an example where the existence of a solution to a differential equation in a region depends on the topology of the region.

Notice that the insertion of a central wire has qualitatively altered the situation. This is remarkable because the existence of a solution (or equivalently the possibility of transmission of a signal) does not depend on the thickness of the central wire, it merely depends on its presence! In mathematical terms, we were previously solving (12) on the disc. Now we solve (12) on the punctured disc. The latter problem has the solution (14) whereas the previous one does not. So cable TV provides an example where the existence of a solution to a *differential* equation in a region depends on the *topology* of that region. This is an example of how topology governs analysis.

It is now quite easy to write down the magnetic field from Maxwell's equations. Equation B (3) with a form for $\vec{\mathcal{B}}(x,y,z,t)$ similar to (1) tells us that $\mathcal{B}_z = 0, \mathcal{B}_x = -\mathcal{E}_y, \mathcal{B}_y = \mathcal{E}_x$. From (8), $\partial_x \mathcal{B}_y - \partial_y \mathcal{B}_x = 0$. So $\vec{\mathcal{B}}$ is a curl-free vector field. Nevertheless (*Exercise*) the line integral of $\vec{\mathcal{B}}$ along a closed curve encircling the inner wire is non-zero. Therefore $\vec{\mathcal{B}}$ *cannot* be written as the gradient of any function.

Many features of the cable TV problem we have discussed here are illustrations of the general mathematical ideas put forward in the article by Pati. (*Exercise:* Plot the vector field $\vec{\mathcal{B}}$ above and compare it with the 'whirlpool' vector field of Pati's article.) For example, our boundary conditions on ϕ (13) are satisfied by any constant function. But this is not the only solution. If the boundary is disconnected (that is, it is made of n distinct pieces), there are exactly n linearly independent functions on S which satisfy the boundary condition (13) and the differential equation

(4). (*Exercise*: Show that this is so). A function can be locally constant, yet take different values on each connected component. In the language of Pati's article, this is an example of cohomology. Similarly, a gradient is always curl-free, but curl-free vector fields need not be gradients, since closed line integrals of these fields may not vanish. The $\vec{\mathcal{E}}$ field *is* curl-free and also the gradient of a scalar function. On the other hand, the $\vec{\mathcal{B}}$ field is not a gradient although it is curl-free. The $\vec{\mathcal{E}}$ field represents a trivial (first) cohomology and the $\vec{\mathcal{B}}$ field a nontrivial one. The first cohomology measures the extent to which the local, differential equation (9) differs from the global, integral (11) one. Similarly, the zeroth cohomology measures the difference between the local, differential equation (12) and the global 'integrated' version $\phi(p_1) - \phi(p_2) = 0$ (for p_1 and p_2 on the boundary of S). The reader is invited to make a detailed comparison to understand both articles better.

Our aim in this article has been to show how a familiar realisation is often a valuable aid to understanding an abstract idea. There are numerous examples in physics which illustrate how topology affects analysis. Studying these could provide the reader with an (albeit back door) entry into topology and mathematics. We chose one specific example from classical electromagnetism, since this is a subject which is widely known. The reader is encouraged to seek others. After gaining experience with examples the reader will readily appreciate the value of the abstract ideas.

Acknowledgement: It is a pleasure to thank R Nityananda and V Pati for their help in writing this article.

Suggested Reading

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