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Stars: Their Structure and Evolution

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1. Introduction

The subject of astrophysics began with the study of the stars. It may be recalled that the positivist philosophers who were so influential in European thinking had asserted that it was in the nature of things that one can never know what the stars are. And yet, with Fraunhofer's discovery of dark lines in the spectrum of the Sun and the stars, and their subsequent explanation in terms of atomic absorption lines, a major scientific revolution had occurred – a question that appeared meaningless within the premise of science had acquired a meaning. Lane, Kelvin and Helmholtz laid the foundations for the theory of stars towards the end of the 19th century. But the credit for constructing a remarkably successful theory of the stability and equilibrium of stars must go to Sir Arthur Eddington. His book **The Internal Constitution of the Stars** published in 1926 is undoubtedly one of the greatest masterpieces of the 20th century.

It is from this book that young Chandrasekhar learnt about the theory of the stars. The year was 1928, and he was an undergraduate student in the Presidency College in Madras. The newly discovered Compton Effect was much in the news, and was the subject of his first scientific publication entitled *Thermodynamics of the Compton Effect with reference to the Interior of the Stars* (Chandrasekhar 1928). He was 18 years old then. That same year he learnt about the discovery of the Fermi-Dirac statistics from Arnold Sommerfeld who happened to visit Madras. Straightaway he applied the new statistics to Compton scattering, and this paper was communicated to the Royal Society by R.H. Fowler (Chandrasekhar 1929). During the next ten years he made monumental contributions to the theory of stellar structure and stellar evolution. Much of it is summarized in his classic book entitled **An Introduction to the Study of Stellar Structure** published in 1939. Almost immediately after writing this book he decided to leave the field, and turned his attention to problems in Stellar Dynamics. This decision was primarily due to the fact that his epoch making discoveries, instead of being lauded, stirred up a great controversy largely on account of Eddington rejecting them. In this article we shall attempt to recall some of his pioneering contributions to the theory of the stars. This is undoubtedly a daunting task. Fortunately, Chandrasekhar himself had made a selection of his most important papers from this period for inclusion in Volume I of his *Selected Papers*. We shall choose a few of them, explain their content, and trace their impact on subsequent developments.

2. The theory of white dwarfs

Chandrasekhar's first significant papers were devoted to the theory of white dwarfs. Following Chandrasekhar's own style of writing, we shall digress for a moment to enable one to place his contributions in proper perspective.

The discovery of white dwarfs, such as the companion of Sirius, with mean densities of the order of 10^5 – 10^6 g cm^{-3}, appeared to spell trouble for the enormously successful theory due to Eddington. As Eddington himself put it in his book:

> I do not see how a star which has once got into this compressed state is ever going to get out of it. ... It would seem that the star will be in an awkward predicament when its supply of subatomic energy fails.

The difficulty may be explained as follows. The electrostatic energy E_v per unit volume of an assembly of completely ionized atoms (with nuclear charge Z) is given by

$$E_v = 1.32 \times 10^{11} Z^2 \rho^{4/3}, \tag{1}$$

where ρ is the mass density. The kinetic energy E_{kin} per unit volume of the free particles (under the assumption that it is a *perfect gas*) is given by

$$\begin{aligned} E_{\text{kin}} &= \frac{3}{2}\frac{k}{\mu m_H}\rho T \\ &= \frac{1.24 \times 10^8}{\mu}\rho T, \end{aligned} \tag{2}$$

where μ is the mean molecular weight. If the external pressure (in this case gravitational pressure) were removed, the matter can expand and return to its original state of normal atoms only if

$$E_{\text{kin}} > E_v, \tag{3}$$

or if

$$\rho < \left(0.94 \times 10^{-3} \frac{T}{\mu Z^2}\right)^3. \tag{4}$$

The point underlying Eddington's remarks is that this inequality is violated under the conditions that obtain in white dwarfs. This paradox was resolved in 1926 by Fowler in one of the most prescient papers in the astronomical literature. Fowler argued that at high densities electrons will be highly degenerate and therefore their kinetic energy and pressure should be calculated not according to Boyle's law, but according to the newly discovered quantum statistics. According to the Fermi-Dirac statistics the pressure and kientic energy of a highly degenerate electron gas are given by

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m} n^{5/3}, \tag{5}$$

and

$$E_{\text{kin}} = \frac{3}{40}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m}n^{5/3}$$
$$= 1.39 \times 10^{13}\,(\rho/\mu)^{5/3}, \tag{6}$$

where n and ρ are the number density and mass density, and μ is the mean-molecular weight. Fowler's point was that the inequality (3) will be easily satisfied if one uses the quantum statistical expression for the kinetic energy instead of its classical value.

Chandrasekhar became aware of Fowler's seminal paper in 1929, and immediately applied the theory of polytropic gas spheres to the new equation of state and derived the mass-radius relation for completely degenerate white dwarf configurations. This historic paper was communicated to the *Philosophical Magazine* by Fowler. To recall briefly, the Fermi-Dirac pressure of a degenerate electron gas (at absolute zero of temperature) is given by a polytropic equation of state

$$P = K\rho^{5/3}. \tag{7}$$

This corresponds to a polytropic index $n = 3/2$ (where $5/3 = 1 + \frac{1}{n}$). Using the theory of equilibrium states of polytropic gas spheres, Chandrasekhar obtained the following relations between the masses and radii of white dwarfs, as well as their mean density:

$$\frac{M}{M_\odot} = \frac{2.14 \times 10^{28}}{\mu^5} \cdot \frac{1}{R^3} \tag{8}$$

$$\rho = 2.162 \times 10^6 \left(\frac{M}{M_\odot}\right)^2. \tag{9}$$

To quote from this early paper (Chandrasekhar, 1931a):

(i) the radius of a white dwarf is inversely proportional to the cube root of the mass,

(ii) the density is proportional to the square of the mass,

(iii) the central density would be six times the mean density.

Thus, according to this theory *all* stars, regardless of their mass, will end their lives peacefully as white dwarfs.

THE LIMITING MASS: Soon after completing this work Chandrasekhar set out to Cambridge to continue his research under the guidance of R.H. Fowler. During the voyage he began to worry about the effects of Special Relativity on the conclusions he had just arrived at. He concluded that at electron densities $> 6 \times 10^{29}$ cm^{-3} one must use a modified form for the pressure of an electron gas. In the extreme relativistic case (when the rest mass of the electrons can be neglected) the pressure is given by

$$\begin{aligned} P &= \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} \cdot hc \cdot n^{4/3} \\ &= K_2\rho^{4/3}. \end{aligned} \tag{10}$$

Once again applying the theory of polytropic gas spheres (this time for an equation of state with a polytropic index $n = 3$) he derived the relation

$$\left(\frac{GM}{M'}\right)^2 = \frac{(4K_2)^3}{4\pi G}, \tag{11}$$

where $M' = 2.7176$. With an assumed value of 2.5 for the mean molecular weight (the "canonical value" in 1930) this yielded

$$M = 1.822 \times 10^{33}\ \mathrm{g} \approx 0.91 M_\odot. \tag{12}$$

More generally

$$M = 0.197\left[\left(\frac{hc}{G}\right)^{3/2} \cdot \frac{1}{m_H^2}\right]\frac{1}{\mu_e^2} = 5.76\ \mu_e^{-2} M_\odot, \tag{13}$$

where μ_e is the mean molecular weight per electron. Thus a fully degenerate star, in the extreme relativistic limit, has a *unique mass* ! Chandrasekhar concluded that this must represent the *maximum mass of an ideal white dwarf* (Chandrasekhar, 1931b).

At this stage Chandrasekhar was not aware of the work of W.Anderson (1929) and E.C.Stoner (1929, 1930) who had independently investigated this problem. Fowler drew his attention to these papers upon his arrival in Cambridge. Stoner's approach was more heuristic. In his 1929 paper Stoner had derived the *limiting* density for white dwarfs using the following argument: The number of electrons with momenta within a definite range cannot exceed a certain maximum. Any increase in the density involves an increase of energy. In the limiting case, at absolute zero, the star can contract until the decrease in gravitational energy becomes insufficient to balance the increase of kinetic energy of electrons. The limiting density corresponds to the value of n when

$$\frac{d}{dn}\left(E_{\text{kin}} + E_{\text{grav}}\right) = 0. \tag{14}$$

This yielded a value of density approximately twice the *average* density calculated by Chandrasekhar using the more rigorous approach of hydrostatic equilibrium.

Historically, Anderson (1929) was the first to appreciate that at high densities special relativistic effects would have to be taken into account. In particular, he isolated the fundamental result that as the density increases the mass will approach a "limiting mass". Stoner (1930) improved upon this result (but confining himself to his earlier framework of considering the energetics of homogeneous spheres) and derived a limiting mass of 2.19×10^{33} g. Curiously, neither Anderson nor Stoner pursued this further!

The credit for elucidating the significance of the limiting mass must go solely to Chandrasekhar. Faced with the skepticism of R.H.Fowler and E.A.Milne, he puzzled over this intriguing result. But by October of that year (1930) it became clear to him that what was happening was that the relation $R \propto M^{-1/3}$ given by the nonrelativistic theory was modified by the inclusion of relativistic effects in the following way. Consider an approximation in which a white dwarf consists of a nonrelativistically degenerate 'envelope' (in which the pressure $\propto \rho^{5/3}$), and a 'core' (in which the pressure $\propto \rho^{\frac{4}{3}}$). . In this approximation Chandrasekhar showed (1931c) that "the completely relativistic model, considered as a limit of this composite series is a point mass with $\rho_c = \infty$!".

Armed with this insight Chandrasekhar proceeded to work out a complete theory which allowed for the effects of special relativity in an exact manner. For this he wrote the equation of state in a parametric form

$$P = Af(x), \quad \rho = n\mu_e m_H = Bx^3, \tag{15}$$

where

$$A = \frac{\pi m^4 c^5}{3h^3}, \quad B = \frac{8\pi m^3 c^3 \mu_e m_H}{3h^3}, \tag{16}$$

and

$$f(x) = x\left(x^2+1\right)^{1/2}\left(2x^2-3\right) + 3\sinh^{-1} x. \tag{17}$$

The above expression for the pressure approximates the relation $P = K_1\rho^{5/3}$ for low electron densities, and $P = K_2\rho^{4/3}$ in the ultrarelativistic limit. A detailed consideration of equilibrium configurations built on the above (exact) equation of state led to a mass-radius relation shown in fig. 1. While the exact relation (full line) approximates the relation obtained (in the nonrelativistic approximation) for $M \to 0$, it predicted that the *radius tends to zero for* $M \to M_{\text{limit}}$. In other words, *finite degenerate equilibrium configurations exist only for* $M < M_{\text{limit}}$. Given the chemical composition (or equivalently the mean molecular weight per electron μ_e) this limiting mass is uniquely determined by a combination of fundamental constants

$$\begin{aligned} M_{\text{limit}} &= 0.197\left[\left(\frac{hc}{G}\right)^{3/2} \cdot \frac{1}{m_H^2}\right]\frac{1}{\mu_e^2} \\ &= 5.76\ \mu_e^{-2}\ M_\odot. \end{aligned} \tag{18}$$

This mass has rightly come to be known as the **Chandrasekhar limit**, and plays a central role in the theory of relativistic stars, and we shall return to it presently.

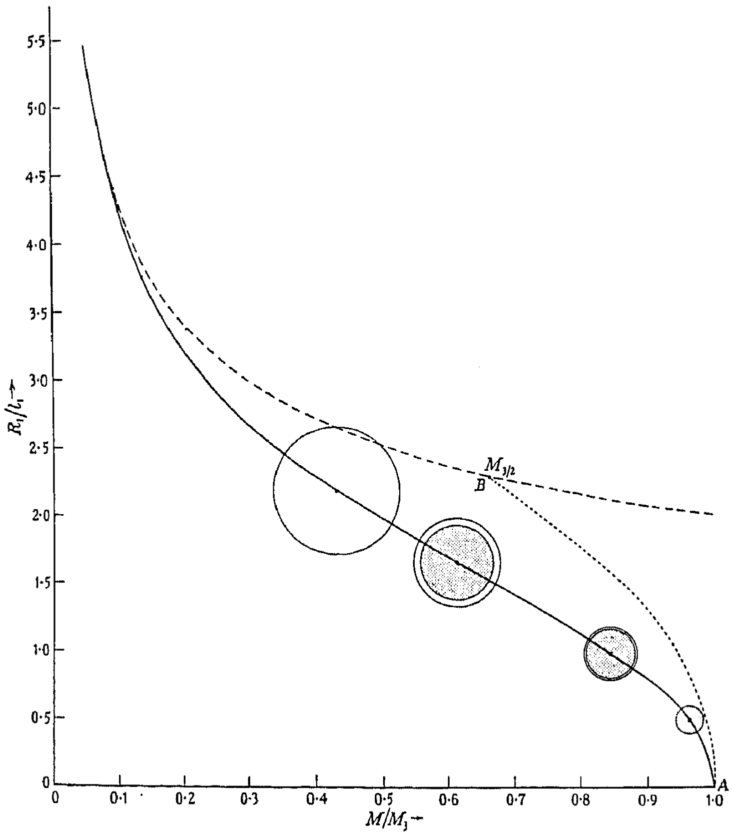

Figure 1. The solid line represents the exact mass-radius relation for completely degenerate configurations. The mass, along the abscissa, is measured in units of the limiting mass (denoted by M_3) and the radius, along the ordinate, is measured in the unit $l_1 = 7.72 \times 10^8 \ \mu_e^{-1}$ cm. The dashed curve represents the relation that follows from the equation of state given in eq. (7); at the point B along this curve the fermi momentum p_F of the electrons at the centre of the configuration is exactly equal to mc. Along the exact curve, at the point where a full circle (with no shaded part) is drawn, p_F (at the centre) is again equal to mc; the shaded part of the other circles represent the regions in these configurations where the electrons may be considered to be relativistic ($p_F >> mc$). (Reproduced from Chandrasekhar 1935).

3. Why are the stars as they are?

After completing his investigations of white dwarfs Chandrasekhar turned to a detailed investigation of the internal constitution of stars. He extended and developed three basic methods. They are

I. The method of integral theorems

II. The method of homologous transformations

III. The method of stellar envelopes.

He published the results in a series of very detailed papers, and subsequently distilled and summarized them in his classic monograph **An Introduction to the Study of Stellar Structure**. Here we shall single out one of the integral theorems because it serves to illustrate his approach to the subject as a whole. The method of integral theorems consists in finding *inequalities* for quantities like the central pressure, mean pressure, the potential energy, the mean value of gravity, etc.. The particular theorem we shall refer to concerns the *central pressure* in a star.

But first let us digress a little. As already remarked, Eddington's *standard model* of the stars was enormously successful. Perhaps most importantly it provided an explanation for the observed masses of stars. It is an extraordinary fact that the overwhelming majority of stars have masses close to that of the Sun: stars with masses very much less than, or very much more than, the mass of the Sun are relatively infrequent. Why is this so? Eddington posed this question to himself and answered it in his parable of a physicist on a cloud bound planet (**The Internal Constitution of the Stars**):

> "The outward flowing radiation may be compared to a wind flowing through the star and helping to distend it against gravity. The formulae to be developed later enable us to calculate what proportion of the weight of the material is borne by this wind, the remainder being supported by the gas pressure. To a first approximation the proportion is the same at all parts of the star. It does not depend on the density nor on the opacity of the star. It depends only on the mass and molecular weight. Moreover, the physical constants employed in the calculation have all been measured in the laboratory, and no astronomical data are required. We can imagine a physicist on a clound-bound planet who has never heard tell of the stars calculating the ratio of radiation pressure to gas pressure for a series of globes of gas of various sizes, starting, say, with a globe of mass 10 g, then 100 g, 1000 g and so on, so that his nth globe contains 10^ng. Table I shows the more interesting part of his results."

Table I

No. of globe	Radiation pressure	Gas pressure	No. of globe	Radiation pressure	Gas pressure
32	0.0016	0.9984	36	0.951	0.049
33	0.106	0.894	37	0.984	0.016
34	0.570	0.430	38	0.9951	0.0049
35	0.850	0.150	39	0.9984	0.0016

> "The rest of the table would consist mainly of long strings of 9's and 0's. Just for the particular range of mass about the 33rd to 35th globes the table becomes interesting, and then lapses back into 9's and 0's again. Regarded as a tussle between matter and either (gas pressure and radiation pressure) the contest is overwhelmingly one-sided except

between numbers 33–35, where we may expect something to happen."

"What *happens* is the stars."

"We draw aside the veil of cloud beneath which our physicist has been working and let him look up at the sky. There he will find a thousand million globes of gas nearly all of mass between his 33rd and 35th globe – that is to say, between $\frac{1}{2}$ and 50 times the Sun's mass. The lightest known star is about 3×10^{32} g and the heaviest about 2×10^{35} g. The majority are between 10^{33} and 10^{34} g, where the serious challenge of radiation pressure to compete with gas pressure is beginning."

But why is the relative extent to which radiation pressure provides support against gravity a relevant factor to the "happening" of stars?! A more rational argument due to Chandrasekhar is the following. To quote him, "Domains of natural phenomena are often circumscribed by well-defined scales, and theories concerning them are successful only to the extent that these scales emerge naturally in them. Thus, to the question *'why are the atoms as they are?'* the answer *'because the Bohr-radius* – $h^2/\left(4\pi^2 m_e e^2\right) \sim 0.5 \times 10^{-8}$ cm – *provides a correct measure of their dimensions'* is apposite. In a similar vein, we may ask 'why are the stars as they are?', intending by such a question to seek the basic reason why modern theories of stellar structure and stellar evolution prevail."

The answer may be found along the following lines. According to one of the integral theorems proved by Chandrasekhar (1936, a; 1936, b) the pressure P_c at the centre of a star of mass M, in hydrostatic equilibrium and in which the density $\rho(r)$ at any point r does not exceed the mean density $\bar{\rho}(r)$ interior to that point r, must satisfy the inequality

$$\frac{1}{2}G\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}\bar{\rho}^{\frac{4}{3}}\ M^{\frac{2}{3}} \leq P_c \leq \frac{1}{2}G\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}\rho_c^{\frac{4}{3}}\ M^{\frac{2}{3}}, \tag{19}$$

where $\bar{\rho}$ denotes the mean density of the star and ρ_c its density at the centre. The meaning of this inequality is the following: the pressure at the centre of a star must be intermediate between those at the centres of two configurations of *uniform* density, one at a density equal to the mean density $\bar{\rho}$ and the other at a density equal to the density ρ_c at the centre. If this inequality is violated, then there must be regions in the star where adverse density gradients prevail, and this will lead to instabilities. Thus, *satisfying this inequality is a necessary condition for the stable existence of a star.*

Before proceeding further one must eliminate the explicit temperature dependence of the central pressure P_c. This is readily done as follows. Let us introduce the fraction β defined by

$$P = p_{\rm gas} + p_{\rm rad}$$

$$= \frac{1}{\beta}p_{\text{gas}} = \frac{1}{1-\beta}p_{\text{rad}}$$
$$= \frac{1}{\beta}\frac{\rho kT}{\mu m_H} = \frac{1}{1-\beta}\frac{1}{3}aT^4. \tag{20}$$

We may eliminate T from these relations and express the total pressure in terms of ρ and β. Thus

$$P = \left[\left(\frac{k}{\mu m_H}\right)^4 \cdot \frac{3}{a} \cdot \frac{1-\beta}{\beta^4}\right]^{\frac{1}{3}} \rho^{\frac{4}{3}} \equiv C(\beta)\rho^{\frac{4}{3}}. \tag{21}$$

Let us use this expression for the total pressure in the right-hand part of the inequality (22), as a necessary condition for the existence of a stable star,

$$\left[\left(\frac{k}{\mu m_H}\right)^4 \cdot \frac{3}{a} \cdot \frac{1-\beta_c}{\beta_c^4}\right]^{\frac{1}{3}} \leq \left(\frac{\pi}{6}\right)^{\frac{1}{3}} GM^{\frac{2}{3}}. \tag{22}$$

Substituting for Stefan's constant a and simplifying one gets

$$\mu^2 M \left(\frac{\beta_c^4}{1-\beta_c}\right)^{\frac{1}{2}} \geq 0.19 \left[\left(\frac{hc}{G}\right)^{\frac{3}{2}} \cdot \frac{1}{m_H^2}\right]. \tag{23}$$

We observe that the above inequality has isolated the following combination of fundamental constants of the dimensions of a mass:

$$\left(\frac{hc}{G}\right)^{\frac{3}{2}} \frac{1}{m_H^2} \simeq 29.2 M_\odot, \tag{24}$$

and that this mass is of *stellar magnitude* (Chandrasekhar 1937). This inequality also provdes an *upper limit* to $(1-\beta_c)$ for a star of a given mass:

$$1 - \beta_c \leq 1 - \beta_\star, \tag{25}$$

where $(1-\beta_\star)$ is uniquely determined by the mass M of the star and the mean molecular weight, μ, by the quartic equation

$$\mu^2 M = 5.48 \left[\frac{1-\beta_\star}{\beta_\star^4}\right]^{\frac{1}{2}} M_\odot. \tag{26}$$

From this equation it follows that for a star of solar mass(and $\mu = 1$) *the radiation pressure at the centre cannot exceed 3% of the total pressure*. (It also follows that this fraction increases with increasing mass – a point to which we shall return later.) What do we conclude from the foregoing calculation? "*We conclude that to the extent eq. (26) is at the base of the equilibrium of actual stars, to that extent the combination of natural constants* $(hc/G)^{\frac{3}{2}}\,(1/m_H^2)$, *providing a mass of proper magnitude for the measurement of stellar masses, is at the base of a physical theory of stellar structure.*" (Chandrasekhar 1984).

4. On the evolution of the main sequence of stars

By the end of the 1930s many of the major questions concerning the structure of the main-sequence stars were settled, and the attention shifted to the evolution of the stars away from the main-sequence. In their seminal papers Weizsäcker (1938) and Bethe (1939) had established that the source of energy radiated by the main-sequence stars is the transformation of hydrogen into helium through the 'C-N-O cycle'. Gamow was among the first to formulate a picture of stellar evolution on the basis of the Bethe-Weizsäcker theory (1939 a, b). His model was based on three assumptions: (i) the stars evolve gradually through a sequence of *equilibrium configurations*, (ii) successive equilibrium configurations are homologous; and (iii) nuclear reactions continue to take place till all the hydrogen in the star is exhausted.

Chandrasekhar turned his attention to this problem around 1940 and two of his papers from that period (Henrich & Chandrasekhar 1941; Schönberg & Chandrasekhar 1942) turned out to be landmark papers. In this section we shall briefly summarize the main conclusions of these two papers, and also attempt to place them in perspective against our present understanding of stellar evolution. Both these papers are devoted to a discussion of stellar models with *isothermal cores*. At the end of hydrogen burning, the star is left with an inert helium core surrounded by a hydrogen-rich envelope. Hydrogen contines to burn in a *shell* at the bottom of the envelope. The helium core must be nearly isothermal, and hence the attempt to construct stellar models with isothermal cores and radiative envelopes. Henrich and Chandrasekhar considered the case in which the value of the mean molecular weight, μ, was the same in both regions. In the subsequent paper, Schönberg and Chandrasekhar discussed the more general (and more relevant) case of different molecular weights μ_e and μ_c, for the envelope and the core, respectively. The basic approach was to require that at the interface the values of the pressure, temperature, and mass of the core should be identical:

$$P(r_i)_{\text{core}} = P(r_i)_e; \quad T(r_i)_{\text{core}} = T(r_i)_e; \quad M(r_i)_{\text{core}} = M(r_i)_e, \tag{27}$$

where P, T and M denote the total pressure, the temperature, and the mass within the radius, respectively. The subscript i indicates that the values refer to the interface and the subscript e indicates that the quantities correspond to the envelope solutions of the equilibrium equations. The above conditions are the only ones to be satisfied, and so one gets a family of configurations for any given set of M, R and L. To fit the core and envelope solutions at the interface they introduced the *homology invariants* U and V to describe the isothermal core,

$$U = \frac{4\pi r^3 \rho_{\text{core}}}{M(r)}, \quad V = \frac{\mu_c m_H}{k} \cdot \frac{GM(r)}{rT(r)}. \tag{28}$$

The equations of fit in the new variables are

$$U_i = 4\pi \left[\frac{r^3 \rho(r_i)_e}{M(r_i)_e}\right] \frac{\mu_c}{\mu_e}, \quad V_i = \left[\frac{\mu_e m_H}{k} \cdot \frac{GM(r_i)_e}{r_i T(r_i)_e}\right] \frac{\mu_c}{\mu_e} \tag{29}$$

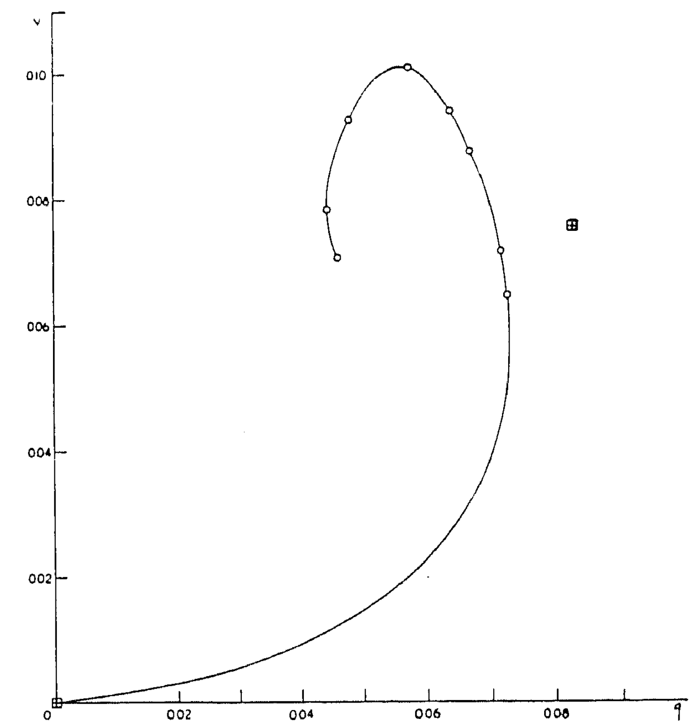

Figure 2. Plotted along the abscissa is the radius of the isothermal core as a fraction of the stellar radius $\left(q = \frac{r_i}{R}\right)$, and along the ordinate is the mass of the core as a fraction of the stellar mass $\left(\nu = M_c/M\right)$. The circles refer to models with isothermal cores, and the squares refer to models with convective core. Starting from its minimum value, ν increases rapidly as q grows, reaches its absolute maximum, and starts spiraling. The absolute maximum is the Schönberg–Chandrasekhar limiting mass (from Schönberg & Chandrasekhar 1942).

The method employed in these papers – which then became very widely used in the literature – was to "fit" the core and envelope solutions in the *U–V* plane by finding their *intersection*. Since the quantities q and ν,

$$q = \frac{r_i}{R} \quad \text{and} \quad \nu = \frac{M(r_i)}{M} \tag{30}$$

are homology invariants, they could be used to label the different configurations corresponding to the same stellar mass and the same central temperature. Using this technique Chandrasekhar and his colleagues were able to derive several properties of models with isothermal cores and radiative envelopes. Their most important conclusion – as borne out by subsequent developments – was the following: *There are no equilibrium configurations with the isothermal cores having masses exceeding a critical mass. This upper limit is a decreasing function of* μ_c/μ_e. In the case of equal molecular weights the upper limit for the ratio $\nu = M_c/M$ is ~ 0.35. Schönberg and Chandrasekhar, who considered the more general case, estimated an upper limit for $M_c/M \sim 0.1$. The dependence of the fractional core mass on the fractional core radius is reproduced in fig. 2. As may be seen, there are no

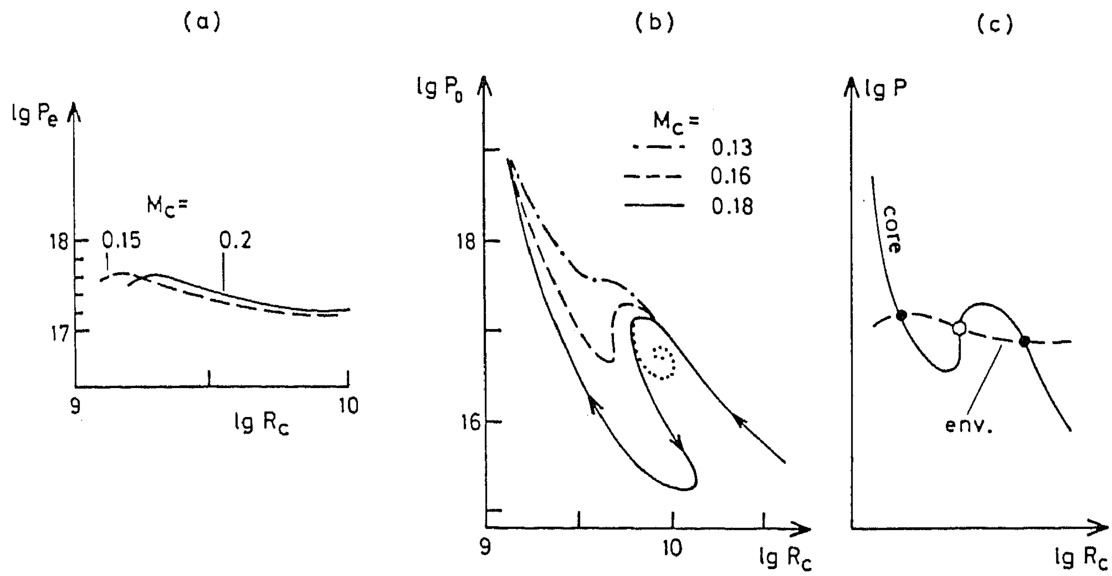

Figure 3. Models with isothermal cores and radiative envelopes. (a) The pressure P_e at the bottom of the envelope (in dyn cm^{-2}) plotted against the core radius R_c for a $2M_\odot$ star. The two curves correspond to two values of the core mass (in $M_\odot$). Thus the envelope solution for the interface is nearly independent of M_c. (b) The pressure P_0 at the surface of the core for different core masses (in $M_\odot$). The arrows along the solid curve indicate the direction of increasing central pressure. The curve spirals around a certain value (the dotted curve) if degeneracy of the core is neglected. The inclusion of degeneracy unwinds the spiral, and it rises once again for increasing central pressure. (c) The intersection of the core and envelope solutions. The filled circles represent stable solutions, and the open circle represents an unstable solution (from computations by Roth (1973) reproduced from Kippenhahn & Weigert 1990).

equilibrium configurations with cores containing less than 0.065 or more than 0.1 of the stellar mass. The lower limit is due to the appearance of convective instability at the interface, while the upper one is due to the impossibility of fitting a core to an envelope. Starting from its minimum value, ν increases rapidly as q grows, reaches a maximum, and starts spiralling around a certain value (not shown in the figure reproduced here but discussed in the text).

This upper limit to the mass of the isothermal core has come to be known as the **Schönberg–Chandrasekhar limit**. This limit is certainly exceeded by helium cores left behind after central hydrogen burning in stars of the upper main sequence. What, then, is the significance of this limit? This became clear only after detailed numerical results (which also included the degeneracy of the core) became available. In view of the fundamental significance of this result, we shall summarize our present understanding of this problem.

For illustration we show the results of detailed numerical integration by Roth (1973) which has been reproduced from the excellent modern textbook on the subject by Kippenhahn and Weigert (1990). Fig. 3(a) shows the pressure at the bottom of the envelope as a function of the core radius for two assumed values for the core mass. As would be expected, the pressure is insensitive to the core mass. Fig. 3(b) shows the behaviour of the pressure at the surface of the isothermal core for several values of the core mass. Consider first the solid curve which is for $M_c = 0.18M_\odot$ and $T_0 = 2.24 \times 10^7$ K. If the degeneracy of the core is neglected then with increasing central pressure this curve would rise from the lower right hand part of the diagram and spiral in as indicated by the *dotted* curve. The inclusion of degeneracy "unwinds" this spiral and it begins to rise again (degeneracy becomes more and more important with increasing central pressure). As may be seen, the minimum is less and less pronounced for smaller core masses. Next let us consider the intersection of the core and envelope solutions. For very small values of the core mass the core solution is monotonic and there is only one intersection. When the mass of the core increases to a critical value – the Schönberg–Chandrasekhar limit – the core solution develops a pronounced maximum and if the envelope solution passes between the maximum and minimum there will be three intersections: the one with the largest R_c corresponds to the core described by an ideal gas, the one with intermediate R_c corresponds to partial degeneracy, and the one with the smallest R_c corresponds to large degeneracy of the core.

The emerging picture is best summarized by fig. 4 in which linear series of complete equilibrium solutions are shown for stars with four different masses (in units of $M_\odot$). R_c is the radius and $M_c = q_c M$ is the mass of the isothermal core. The curves for the more massive stars have three branches; the solid sections represent thermally stable branches and the dashed section represents unstable models. On the upper branch the cores are nondegenerate, but they are strongly degenerate in the lower branch. *The turning point with the larger value of core mass* (q_{SC}) *defines the Schönberg–Chandrasekhar limit.* As we go to smaller stellar masses the two turning points approach each other, and at $M \sim 1.4M_\odot$ they merge and disappear.

Let us conclude this discussion of the Schönberg–Chandrasekhar limit with a brief discussion of its implication for the evolution of the stars in the upper main-sequence. To be specific, let us consider a 3 solar mass star and follow its evolution with the aid of fig. 5(a). When the mass of the isothermal helium core is still relatively small the star will settle into an equilibrium state represented by the upper branch. This corresponds to a dwarf star close to the main sequence (as shown in fig. 5b). As the mass of the helium core grows due to hydrogen burning in the shell the star will "move" along this upper branch maintaining equilibrium. This proceeds continuously till the core mass reaches the Schönberg–Chandrasekhar limit. When the core mass exceeds this critical value the only equilibrium models are in the lower branch and the core will have to contract discontinuously. This contraction of the core will be accompanied by an expansion of the star and the star will move rapidly in the HR diagram from the main-sequence to the region of the Hayashi line. This central conclusion, namely that the core will contract and the star will transform itself into a red-giant, is borne out by detailed evolutionary calculation such as the one shown in fig. 6. The core contraction (and the expansion of the

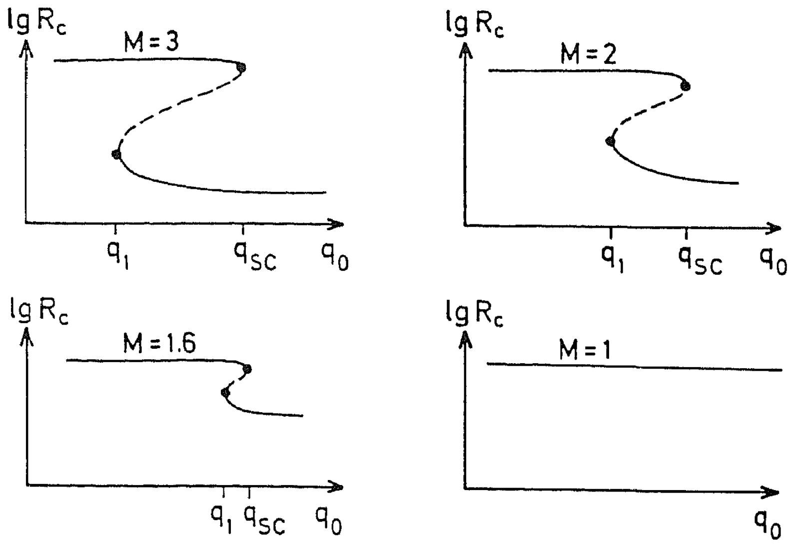

Figure 4. Linear series of complete equilibrium solutions for four different stellar masses M (in $M_\odot$) having isothermal core of mass $M_c = q_0 M$. Each solution is characterized by its core radius R_c and its relative core mass q_0. Branches with thermally stable solutions are shown by solid lines, and the unstable branches are shown as dashed lines. The turning point $q_0 = q_{SC}$ defines the Schönberg–Chandrasekhar limit. (After Roth 1973; from Kippenhahn and Weigert 1990).

star) occurs roughly on the Kelvin-Helmholtz timescale of the core ($\sim 3 \times 10^6$ yr for a $5M_\odot$ star). In contemporary literature this consequence of the Schönberg–Chandrasekhar limit is taken as the explanation for the existence of the well known *Hertzsprung gap* in the HR-diagram, a region between the main-sequence and the red giants where there is a paucity of observed stars.

Such a sudden contraction of the core may also have implications for dramatic mass loss from the upper main-sequence stars. The discovery of white dwarfs in some open clusters with main-sequence turn off mass $\sim 6M_\odot$ certainly implies such a mass loss. For otherwise, according to our current understanding, the degenerate carbon cores of stars with mass $\geq 6M_\odot$ will eventually grow to $\sim 1.4M_\odot$ and ignite, resulting in a detonation of the star as a Type I supernova. Whether the Schönberg–Chandrasekhar limit has anything to do with dramatic mass loss from the stars of the upper main-sequence or not, we would like to conclude this section by quoting from the last paragraph of this fundamental paper:

> "It therefore appears difficult to escape the conclusion that beyond this point the star must evolve through nonequilibrium configurations. It is difficult to visualize what form these nonequilibrium transformations will take; but, whatever their precise nature, they must depend critically on whether the mass of the star is greater or less than the upper limit M_3 $(= 5.7\mu^{-2}M_\odot)$ to the mass of degenerate configurations. For masses less than M_3 the nonequilibrium transformations need not take

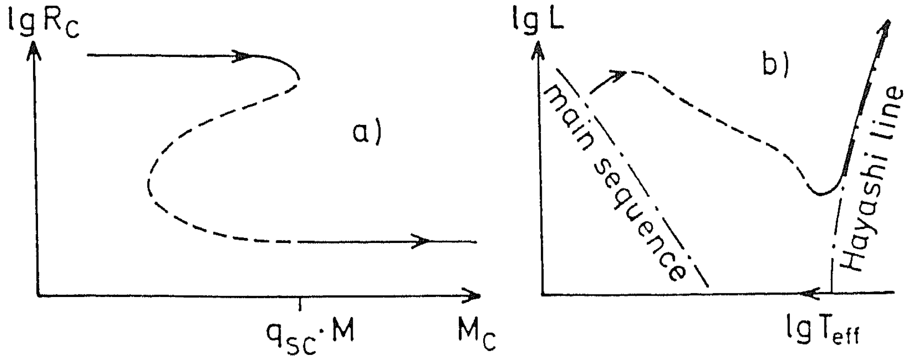

Figure 5. (a) Linear series of equilibrium solutions for a 3 solar mass star. With increasing core mass the model shifts along the solid lines as indicated by the arrows. When the mass of the core grows to $q_{SC}M$ – *the Schönberg–Chandrasekhar critical mass* – the core will have to contract discontinuously. This results in an expansion of the star, and the star moves rapidly from near the main-sequence to the red giant branch in the HR diagram. The rapid transit of the star across the *Hertzsprung gap* is indicated by the dashed line in (b) (from Kippenhahn & Weigert 1990).

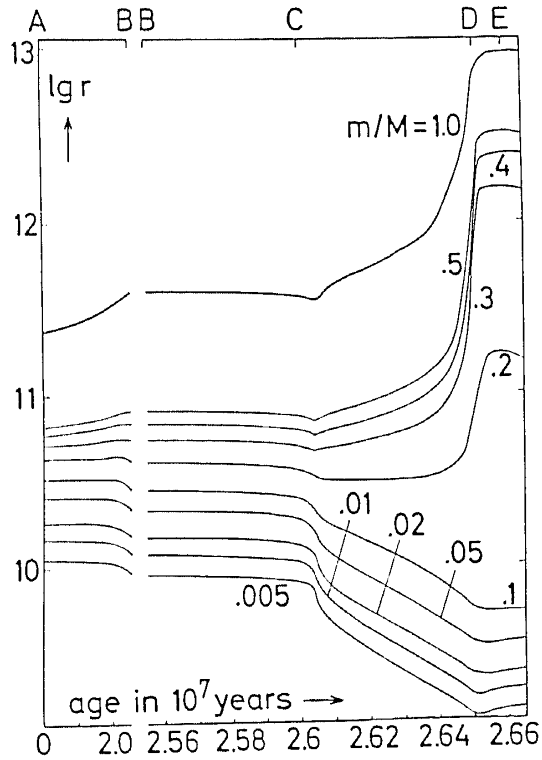

Figure 6. The radial variations of different mass shells characterized by their m/M values in the post-main-sequence phase of a $7M_\odot$ star. The rapid contraction of the core and the expansion of the envelope is clearly seen (from Hofmeister *et al.* 1964).

particularly violent forms, as finite degenerate white-dwarf states exist for these stars. However, when $M > M_3$, the star must eject the excess mass first, before it can evolve through a sequence of composite models consisting of degenerate cores and gaseous envelopes toward the completely degenerate state. *Our present conclusions tend to confirm a suggestion made by one of us (S.C.) on different occasions that the supernova phenomenon may result from the inability of a star of mass greater than M_3 to settle down to the final state of complete degeneracy without getting rid of the excess mass.*"

5. The fate of massive stars

Let us now return to the early 1930s once again. The discovery of the limiting mass for white dwarfs in which the gravitational pressure is balanced by the degeneracy pressure of the electrons was to play a central role in astrophysics in the second half of this century. Equally fundamental and far reaching in its significance was a paper Chandrasekhar published in 1932 entitled *Some Remarks on the State of Matter in the Interior of Stars.*

Although the astronomical community was indifferent to the isolation of the limiting mass of white dwarfs, and its high priests hostile to the idea, Chandrasekhar took the result seriously enough and attempted to relate it to the life history of gaseous stars with masses greater than the critical mass. The first question to be resolved was the condition under which a star, initially gaseous, can develop a degenerate core. This question could be answered by comparing the electron pressure as given by the classical perfect gas equation of state with the expression for the degeneracy pressure (see fig. 7). Since the former is a function of both ρ and T, while the latter is independent of temperature, Chandrasekhar expressed the classical pressure in terms of ρ and β_e where the fraction of β_e is defined through the relation

$$P_{\text{tot}} = \frac{1}{\beta_e}\left(\frac{k}{\mu_e m_H}\rho T\right) = \frac{1}{1-\beta_e}\left(\frac{1}{3}aT^4\right). \tag{31}$$

One gets

$$p_e = \left[\left(\frac{k}{\mu_e m_H}\right)^4 \cdot \frac{3}{a} \cdot \frac{1-\beta_e}{\beta_e}\right]^{\frac{1}{3}} \rho^{\frac{4}{3}}. \tag{32}$$

If degeneracy sets in at all in the core it will do so under conditions in which special relativistic corrections would be important. Hence the above expression should be compared with the *relativistic degeneracy pressure*

$$p_{\text{deg}} = K_2 \rho^{\frac{4}{3}}, \quad K_2 = \frac{1}{8}\left(\frac{3}{\pi}\right)^{\frac{1}{3}} \frac{hc}{(\mu_e m_H)^{\frac{4}{3}}}. \tag{33}$$

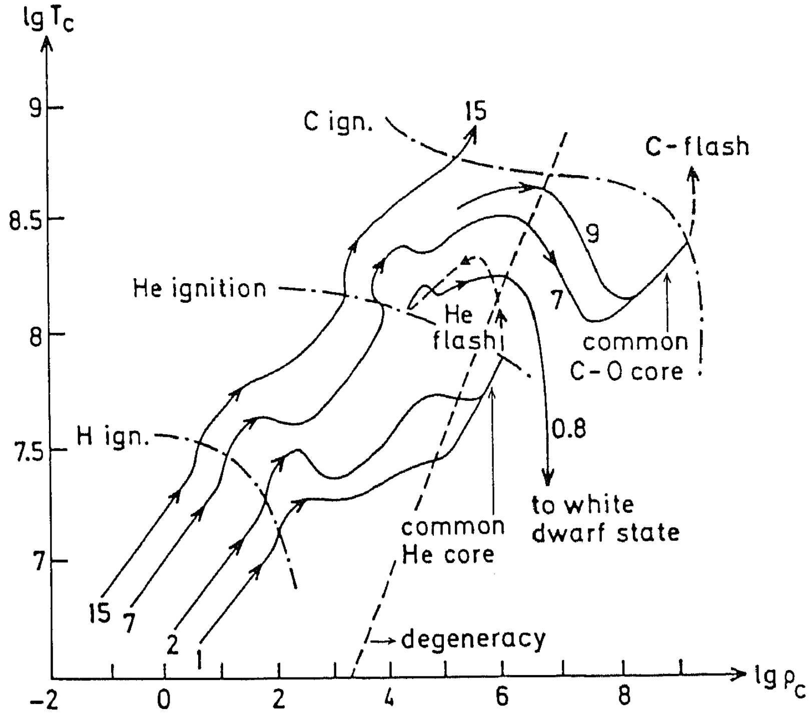

Figure 7. This plot of $\log p$ *vs.* $\log \rho$ illustrates the onset of degeneracy for increasing density at constant β. The straight line ABK represents the equation of state $p = K_1 \rho^{\frac{5}{3}}$ and BC the equation of state $p = K_2 \rho^{\frac{4}{3}}$. ABC gives roughly the equation of state of a degenerate gas. DE represents the classical equation of state (eq. (32) with $\mu = 2$, and $\beta = 0.98$). It intersects the degenerate equation of state (AB, C) at E. Thus for a star with $\beta = 0.98$ there are two surfaces of demarcation: a perfect gas envelope, a degenerate zone EB, and then a relativistically degenerate zone. If $\beta = 0.9079$, then GB represents the perfect gas equation of state and the degenerate zone reduces to a single layer, and the relativistically degenerate zone is described equally well by the perfect gas equation. Now if $\beta < 0.9079$ perfect gas equation of state has no intersections with ABC and this means that however high the density may become the temperature rises sufficiently rapidly to prevent the matter from becoming degenerate (from Chandrasekhar 1932).

As Chandrasekhar pointed out, *in this connection it will have to be remembered that considerations of relativity do not affect the equation of state of a perfect gas. $p = NkT$ is true independent of relativity* ! Thus, if

$$\left[\left(\frac{k}{\mu_e m_H}\right)^4 \cdot \frac{3}{a} \cdot \frac{1-\beta_e}{\beta_e}\right]^{\frac{1}{3}} > K_2, \tag{34}$$

then the pressure p_e given by the classical equation of state will be greater than the degeneracy pressure, *not only for the prescribed ρ and T, but for all ρ and T having the same β_e*. Inserting the value for Stefan's constant a, the above inequality reduces to

$$\frac{960}{\pi^4} \cdot \frac{1-\beta_e}{\beta_e} > 1 \tag{35}$$

or, equivalently

$$1 - \beta_e > 0.0921 = 1 - \beta_\omega \text{ (say).} \tag{36}$$

Thus, *the criterion for a star to develop degeneracy is that the radiation pressure be less than* 9.2% *of the total pressure.* This exact result is of singular importance in all the contemporary schemes of stellar evolution.

It is an important result of the *standard model* due to Eddington that radiation pressure must play a more dominant role as the mass of a star increases. Chandrasekhar proceeded to calculate the mass of a star (in the standard model) in which radiation pressure is precisely equal to 9.2% of the total pressure. It may be recalled that in the standard model the fraction β is assumed to be constant throughout the star. Under this assumption, stars are polytropes of index 3 since the total pressure can be written as

$$P = \left[\left(\frac{k}{\mu m_H}\right)^4 \cdot \frac{3}{a} \cdot \frac{1-\beta}{\beta^4}\right]^{\frac{1}{3}} \rho^{\frac{4}{3}} = C(\beta)\rho^{\frac{4}{3}}. \tag{37}$$

For such a star the mass is uniquely determined by the constant of proportionality in the polytropic equation of state:

$$M = 4\pi\left(\frac{C(\beta)}{\pi G}\right)^{\frac{3}{2}} \times 6.89. \tag{38}$$

For $\beta = \beta_\omega = 0.908$ this gives

$$M = 6.65 \;\; \mu^{-2} M_\odot. \tag{39}$$

Thus, in the standard model, stars with masses exceeding $6.65\mu^{-2}M_\odot$ will have radiation pressure that will exceed 9.2% of the total pressure, and consequently they cannot, during the course of their evolution, develop degeneracy in their interiors, and, accordingly, an eventual white-dwarf state is impossible for them without a substantial ejection of mass. Although this remarkable conclusion was soundly rejected by Eddington and Milne – two of the most distinguished and influential astrophysicists of that time – Chandrasekhar himself was so convinced of his result that he asserted with supreme confidence:

> *"For all stars of mass greater than* 6.6 $\mu^{-2}M_\odot$, *the perfect gas equation of state does not break down, however high the density may become, and the matter does not become degenerate. An appeal to Fermi-Dirac statistics to avoid the central singularity cannot be made."*

Although convinced of it, Chandrasekhar was nevertheless uneasy about the above conclusion. Since infinite density cannot be entertained, and since no other equation of state was available at that stage, he invoked the assumption that there must exist a *maximum density* $\rho_{\max}$ which matter is capable of. Accordingly he constructed models with gaseous envelopes and homogeneous cores at the maximum density of matter (at the density of nuclear matter). But he concluded this remarkable paper on a cautious note:

> "Great progress in the analysis of stellar structure is not possible before we can answer the following fundamental question: *Given an enclosure containing electrons and atomic nuclei (total charge zero) what happens if we go on compressing the material indefinitely?*".

It should be remarked that in 1932 (when the paper discussed above was published) he had not yet convinced himself of the significance of the maximum mass of white dwarfs. However, after working out the exact theory of white dwarfs, he concluded (Chandrasekhar 1934a):

> "Finally, it is necessary to emphasize one major result of the whole investigation, namely, that it must be taken as well established that the life-history of a star of small mass must be essentially different from the life-history of a star of large mass. For a star of small mass the natural white-dwarf stage is an initial step towards complete extinction. A star of large mass cannot pass into the white-dwarf stage, and one is left speculating on other possibilities."

Uncharacteristically, he even speculated (Chandrasekhar 1934b):

> "It is conceivable, for instance, that at very high critical density the atomic nuclei come so near one another that the nature of the interaction might suddenly change and be followed subsequently by a sharp alteration in the equation of state in the sense of giving a maximum density of which matter is capable. However, we are now entering a region of pure speculation, and it is best to conclude the discussion at this stage."

But in a paper published around the same time Baade and Zwicky (1934) were less cautious:

> "With all reserve we advance the view that supernovae represent the transitions from ordinary stars into neutron stars, which in their final stages consist of extremely closely packed neutrons."

In a prescient paper published in 1939 Oppenheimer and Volkoff pointed out that as one approached the limiting mass of white dwarfs along the sequence of the completely degenerate configurations, the central density will become high enough for the electrons at the fermi level to combine with the protons to form neutrons. Thus beyond a critical density neutrons will be the more stable particles. Oppenheimer and Volkoff studied the mass-radius relation of neutron stars with the aid of general relativistic equation for hydrostatic equilibrium. Based on this work Chandrasekhar (1939a) concluded the following regarding the fate of stars more massive than the limiting mass of white dwarfs, but less massive than 6.65 $\mu^{-2}M_{\odot}$ (in the standard model):

> "If the degenerate cores attain sufficiently high densities (as is possible for these stars) the protons and electrons will combine to form neutrons. This would cause a sudden diminution of pressure resulting in the collapse of the star onto a neutron core giving rise to an enormous liberation of gravitational energy. This may be the origin of the supernova phenomenon."

These, then, were the predictions made nearly sixty years ago. Have they been confirmed by observations, as well as detailed computations? Yes! As may be seen in fig. 8, a star of, say, $15M_{\odot}$ does not develop degeneracy in the core during successive stages of nuclear burning. The fusion reactions proceed in a controlled fashion till an iron core forms. And degeneracy finally sets in in the iron core for the following reason. In stars with masses $\geq 10M_{\odot}$, radiation pressure always remains in excess of 9.2% of the total pressure (i.e., $1-\beta_e > 1-\beta_\omega$). However, when the carbon core finally ignites there will be a copious emission of neutrinos resulting in a cooling of the core, thus lowering $(1-\beta_e)$; but it will still be in excess of $(1-\beta_\omega)$. The increase in density and temperature will eventually result in neon ignition. The resultant neutrino emission will further lower $(1-\beta_e)$. It is important to appreciate that the emission of neutrinos occurs selectively in the central region. This succession of nuclear ignitions followed by a lowering of $(1-\beta_e)$ will continue till $(1-\beta_e)$ becomes *less* than $(1-\beta_\omega)$, and a *relativistically degenerate iron core will form.* The mass of this core will quickly grow to $\sim 1.4M_{\odot}$. Soon instability of some sort will set in resulting in the collapse of the core.

Let us examine this more closely. In the framework of the Newtonian theory of gravitation for the spherical core to be stable against radial perturbations the pressure-averaged value of the adiabatic index Γ must be greater than $\frac{4}{3}$. That is

$$\bar{\Gamma} = \int_0^M \Gamma(r)\, P(r)\, dM(r) \div \int_0^M P(r)\, dM(r) > \frac{4}{3}, \tag{40}$$

where

$$\Gamma = \frac{\Delta P/P}{\Delta\rho/\rho}. \tag{41}$$

If $\bar{\Gamma} < 4/3$, *dynamical instability* of a global character will set in with a characteristic timescale given by the sound travel time across the core. Fig. 9 schematically shows the behaviour of Γ as a function of density. As the density approaches $\sim 10^9$ g cm^{-3}, Γ will tend to 4/3 and matter will become marginally stable. When the density reaches $\sim 10^{10}$ g cm^{-3} inverse β-decay will set in resulting in the neutronization of matter. The decrease in the number density of electrons will cause a diminution of the pressure resulting in the collapse of the core. Initially the number

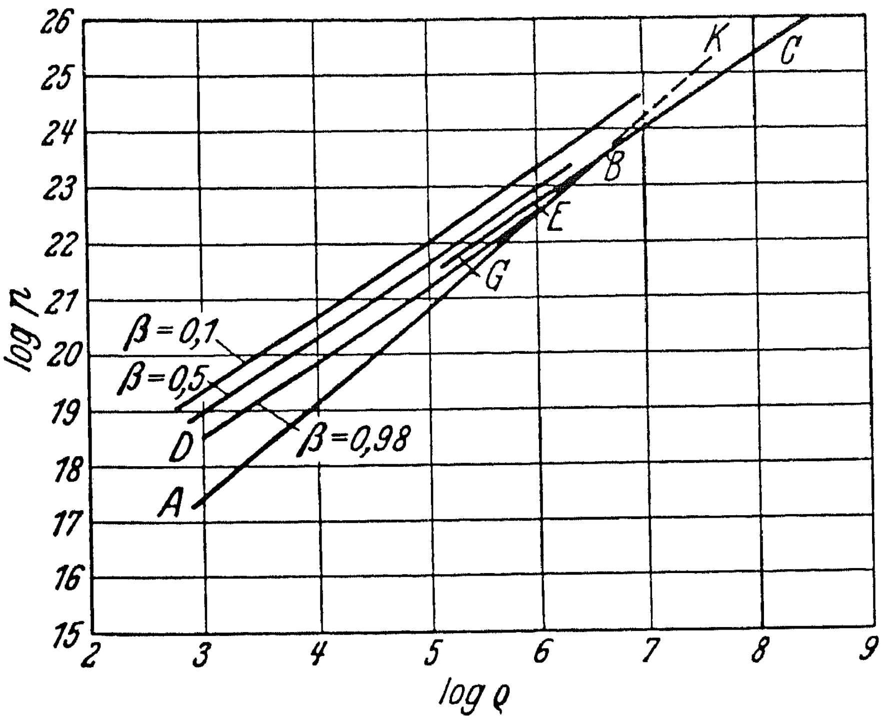

Figure 8. The evolution of the central temperature T_c (in K) and central density ρ_c (in g cm^{-3}) for stars of different masses. The conditions for the ignition of hydrogen, helium and carbon are indicated by dot-dashed-lines. In the region to the left of the sloping (dashed) straight line the core of the star is described by the classical perfect gas equation of state. As may be seen, the core of a $15M_\odot$ star remains nondegenerate through successive stages of nuclear ignition. This is because in such massive stars the radiation pressure exceeds 9.2% of the total pressure. Beyond the carbon burning phase the large neutrino luminosity lowers radiation pressure selectively in the central region, and eventually it becomes less than 9.2% of the total pressure. Consequently relativistic degeneracy sets in in the iron core (from Iben 1974).

of free neutrons will be small, and consequently they will not contribute significantly to the pressure. Eventually, when the density increases to $\sim 7 \times 10^{12}$ g cm^{-3} the degeneracy pressure of the free neutrons will become significant and the matter will become stable once again. The *mean-density* of the collapsed core (corresponding to $\bar{\Gamma} > 4/3$) will be $\sim 10^{14}$ g cm^{-3}. The resultant configuration will be a *neutron star* of mass $\sim 1.4M_\odot$.

As already mentioned, the above considerations were within the framework of the Newtonian theory of gravitation. As we saw, in this picture, neutronization of matter played a central role in the instability setting in. The situation is qualitatively different when one examines the same problem in the framework of the general theory of relativity. This discovery was made by Chandrasekhar in 1964 (and is discussed by J.Friedman in this volume). Chandrasekhar showed that in the post-Newtonian approximation to the general theory of relativity, the instability for radial

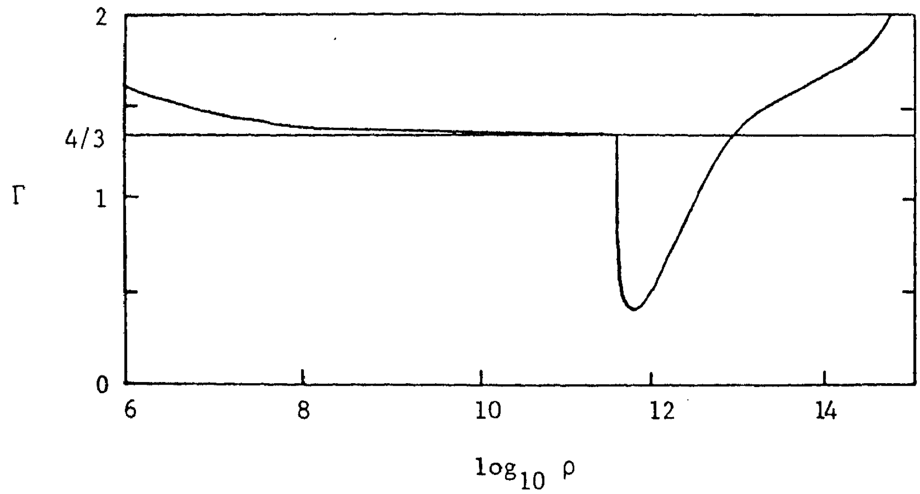

Figure 9. The adiabatic index as a function of the mass density. In Newtonian physics this index has to be greater than $4/3$ for stability. Inclusion of general relativistic effects increases the critical value to above $4/3$.

perturbations will set in for all stars with

$$R < \frac{K}{\Gamma - \frac{4}{3}} \cdot \frac{2GM}{c^2}, \tag{42}$$

where K is a constant. Chandrasekhar and Tooper (1964) applied this result for degenerate configurations near the limiting mass. Since the electrons in these highly relativistic configurations have velocities close to that of light, the effective value of Γ will be close to $4/3$. Thus *the post-Newtonian instability will set in for a mass slightly less than the limiting mass* because the modified stability criterion requires

$$\Gamma > \frac{4}{3} + K\frac{2GM}{Rc^2}. \tag{43}$$

The radius of the configuration when this global instability sets in will be $\sim 5 \times 10^3 R_S$ where $R_S = 2GM/c^2$.

6. Epilogue

The burst of neutrinos from the 1987a supernova in the Large Magellanic Cloud, the discovery of pulsars in the Crab Nebula and other supernova remnants, and the fact that the measured masses of neutron stars are almost equal to $1.4M_\odot$ are spectacular confirmations of the remarkable predictions made by Chandrasekhar in the 1930s.

Will all massive stars find peace as neutron stars? Since there is a maximum mass for neutron stars (in analogy with the Chandrasekhar mass limit for white dwarfs) the answer must be 'no'. Thus in sufficiently massive stars "*an appeal to the*

Fermi-Dirac statistics to avoid the central singularity cannot be made". Eddington clearly recognized the significance of this result. He thus stated (Eddington 1935):

> "The star apparently has to go on radiating and radiating and contracting and contracting until, I suppose, it gets down to a few kilometres radius when gravity becomes strong enough to hold the radiation and the star can at last find peace.".

But he denied the existence of an upper limit to mass of completely degenerate configurations (white dwarfs *and* neutron stars), and consequently rejected the above possibility. In a paper published in 1939 Oppenheimer and Snyder were unequivocal about the fate of sufficiently massive stars:

> "When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. This contraction will continue indefinitely till the radius of the star approaches asymptotically its gravitational radius. Light from the surface of the star will be progressively reddened and can escape over a progressively narrower range of angles till eventually the star tends to close itself off from any communication with a distant observer. Only its gravitational field persists."

In modern terminology, it will become a **black hole**.

Chandrasekhar began his research with a detailed study of white dwarfs. The results he obtained in the 1930s are at the base of much of relativistic astrophysics. In particular, his study led inescapably to the conclusion that sufficiently massive stars will ultimately find peace as black holes. He returned to this subject in 1964, and devoted the next thirty years to a detailed study of black holes and singularities.

Chandrasekhar credited Eddington with the founding of modern theoretical astrophysics, and creating the discipline of the structure, the constitution, and the evolution of the stars. The second half of this century will be remembered as the golden age of relativistic astrophysics. This is the era of Neutron Stars and Black Holes. Chandrasekhar said the first words on these, and went on to erect major pillars on which the superstructure of contemporary astrophysics rests.

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