

## Impossibility of a continuous phase convention for polarised light

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**Abstract.** The possibility of extending the well known Poincare sphere representation of polarised light to include phase is considered. Any attempt to define the zero of phase for each vibration represented on the Poincare sphere runs into discontinuities at at least one point. These are shown to be inevitable using a topological argument.

**Keywords.** Poincare sphere; polarised light.

### 1. Introduction

The state of polarisation of a completely polarised beam of light can be characterised by the ratio  $b/a$  of the minor and major axes of the ellipse, the sense with which the ellipse is traversed by the electric vector, and the angle  $\lambda$  made by the major axis with some fixed line (figure 1a). It is convenient to replace the axial ratio and sense by an angle  $\omega$  running from  $-\pi/4$  to  $+\pi/4$ , so chosen that  $|\tan \omega| = b/a$ . Positive and negative values of  $\omega$  are used to distinguish the two possible senses of rotation for elliptic vibrations. The coordinates  $2\lambda$  and  $2\omega$  run from 0 to  $2\pi$  and  $-\pi/2$  to  $+\pi/2$  respectively and are used as longitude and latitude to represent the state of polarisation on the Poincare sphere, reviewed by Ramachandran and Ramaseshan (1961). As shown in figure 1b, the poles of this sphere correspond to the two opposite circular vibrations and the equator to linear vibrations with varying azimuth.

The problem considered in this paper is whether one can define an initial phase for each of these states of vibration, chosen in such a way that there is no discontinuity anywhere on the Poincare sphere. For example, one might decide to define the zero

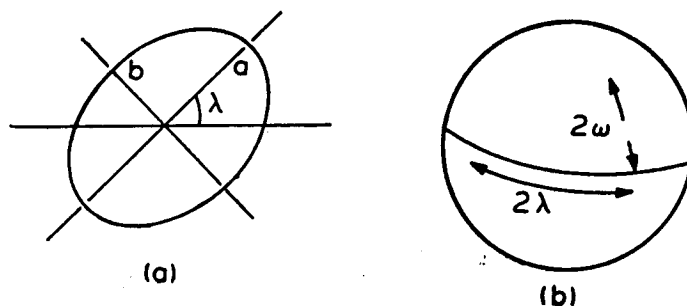


Figure 1. a. A general elliptic vibration showing the definitions of azimuth and ellipticity. b. The Poincare sphere representation of a general elliptic vibration.

of phase at one of the extremities of the major axis of each elliptic vibration. Figure 2a represents the result on a projection in which  $2\lambda$  and  $2\omega$  are used as cartesian coordinates. One sees that this attempt runs into two types of discontinuities. First of all, the zero of phase for the circular vibration at the pole of the sphere is indeterminate and indeed, we get a different answer for each meridian along which we approach the pole. Secondly, on traversing a small circle at a fixed latitude, we find that the dot representing the zero of phase has shifted to the opposite end of the major axis. A second attempt at establishing a phase convention over the whole of the Poincare sphere is shown in figure 2b. Here there is no discontinuity at the north pole or along any meridian but there is a clear discontinuity at the south pole.

Considering a few such examples leads to the belief that a continuous convention for the phase zero is impossible. A more precise statement and proof of this result form the rest of this paper.

## 2. A faithful three dimensional geometric representation of polarisation and phase

If we consider the collection of all states of (in general elliptic) vibration and all possible phases for each, we clearly need three numbers for a full specification, since the phase adds one degree of freedom to the already two dimensional surface of the Poin-

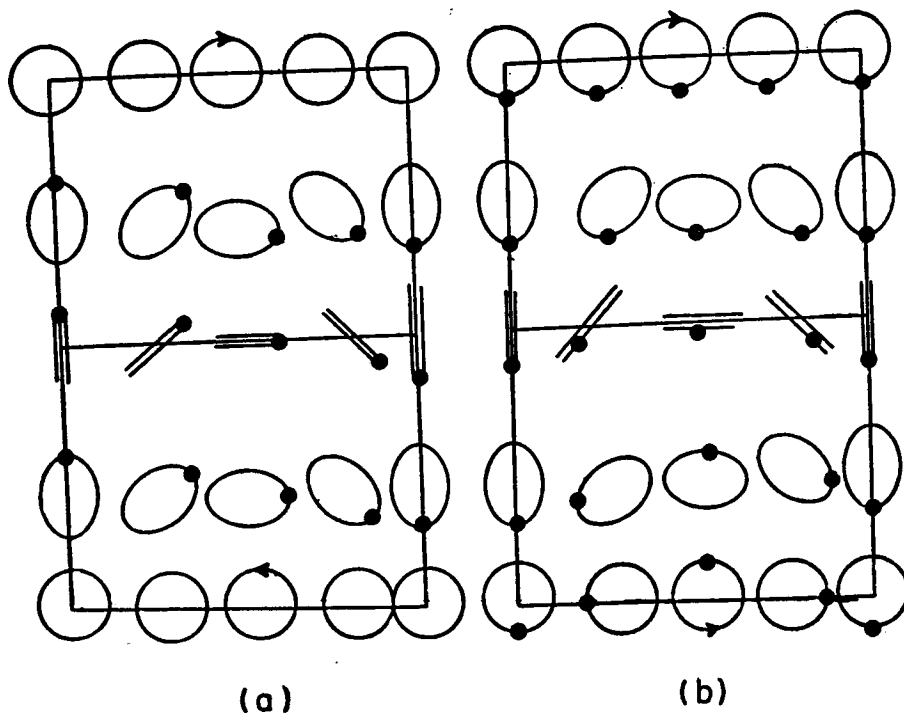


Figure 2. a. An attempt to define a phase convention on the Poincare sphere.  $2\lambda$  and  $2\omega$  are plotted as rectangular coordinates. The dot represents the point chosen as the zero of phase. Linear polarisation is represented by a double line to distinguish between the two directions of travel at the same point. b. A second attempt, based on resolving a given state into opposite circular components and using the phase zero of one of them.

care sphere. For example, we could represent the elliptic vibration at one instant of time by giving its  $x$  and  $y$  coordinates.

$$x = r \cos \theta \cos (\phi_1 + \omega t), \tag{1a}$$

$$y = r \sin \theta \cos (\phi_2 + \omega t). \tag{1b}$$

Here  $r$  determines the intensity of the light which is not of interest here.  $\theta$  (range 0 to  $\pi/2$ ) determines the relative strength of the linear vibrations along  $y$  and  $x$  which combine to give the elliptic vibration.  $\phi_1 - \phi_2$  determines the relative phase of the two linear vibrations along  $x$  and  $y$  and hence the state of polarisation of the resultant.  $\phi_1$  itself can be regarded as the phase, varying over 0 to  $2\pi$ .

It is tempting to represent  $\theta$ ,  $\phi_1$  and  $\phi_2$  as rectangular coordinates as in figure 3. Every point in the region shown corresponds to a state of polarisation or phase and every such state is represented. Values of  $\theta$  beyond the range 0 to  $\pi/2$  are not needed as we can change the sign of (1a) or (1b) by adding  $\pi$  to  $\phi_1$  or  $\phi_2$ . Lines of constant  $\phi_1 - \phi_2$  are drawn in the figure. Along one such line only the phase varies, and each line corresponds to a given state of polarisation, i.e. a point on the Poincare sphere. Really each line is closed because the values 0 and  $2\pi$  for either  $\phi_1$  or  $\phi_2$  should be regarded as equivalent. This is done by identifying pairs of points on opposite vertical faces of the rectangular solid in figure 3. A further property of this representation is that for  $\theta=0$ , we have a linear vibration along  $x$ ;  $\phi_1$  is the phase while  $\phi_2$  is then irrelevant and thus all points with different  $\phi_2$ 's must be identified, as seen from (1b). Similarly, for  $\theta=\pi/2$ , all points with different coordinates  $\phi_1$  are identified while  $\phi_2$  represents the phase. After we make these identifications, the three dimensional region in figure 3 is a faithful representation of all distinct phase and polarisation states, with neighbouring states going into neighbouring points of this region. Here neighbouring states are defined as those for which the corresponding values of  $x$  and  $y$ , the two components of the electric field defined in (1a) and (1b), differ only slightly for all values of  $t$ .

The problem of defining the initial phase of each state on the Poincare sphere can now be recast. In the three dimensional region of figure 3, draw a continuous surface

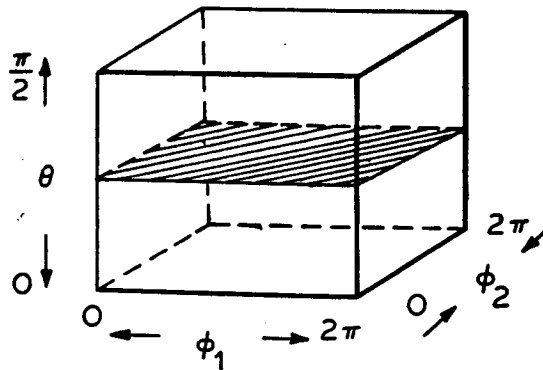


Figure 3. A three-dimensional space for representing polarisation and phase. Opposite points on the vertical faces whose  $\phi_1$  or  $\phi_2$  coordinates differ by  $2\pi$  are to be identified. On the bottom face,  $\theta=0$ , all points with different  $\phi_2$ 's along a line  $\phi_1$ =constant are identified. On the top face,  $\theta=\pi/2$ , all points with different  $\phi_1$ 's are to be identified. A family of lines  $\phi_1 - \phi_2$ =constant is shown in the interior of the region.

intersecting each line of constant  $\phi_1 - \phi_2$  once and only once. This point of intersection is the phase zero. Because of the numerous (four!) identification rules, it is not convenient to solve the problem using the representation in figure 3. In the next section, an alternative representation in which the identification rules are naturally taken into account is given.

### 3. Representation of polarisation and phase by two toroids and impossibility proof

Two coordinates  $\phi_1$  and  $\phi_2$ , each running from 0 to  $2\pi$  with 0 and  $2\pi$  identified are most naturally represented on the surface of a torus. Here  $\phi_1$  increases from 0 to  $2\pi$  along the 'horizontal' circle  $C_1$  while  $\phi_2$  increases from 0 to  $2\pi$  along a vertical circle  $C_2$  (Figure 4a). Since we have a third coordinate  $\theta$ , it is natural to consider a nested family of tori labelled by the parameter  $\theta$ . Further, when  $\theta=0$ , we want points labelled by different values of  $\phi_2$  to coalesce. This is achieved by shrinking the torus to a circular ring when  $\theta=0$ , as shown in figure 4a by using the 'core' of the solid toroid for  $\theta=0$ . Only the fourth identification rule, viz. that for  $\theta=\pi/2$  all points with different  $\phi_1$ 's must coalesce, cannot easily be represented with a single toroid. We overcome this by using a single toroid to represent all values of  $\theta$  from 0 to  $\pi/4$ , say. A second toroid is used to represent values of  $\theta$  from  $\pi/4$  to  $\pi/2$ . In this we interchange the roles of  $\phi_1$  and  $\phi_2$  so that  $\phi_2$  now varies as we move along the 'length' of the torus on a 'horizontal' circle, while  $\phi_1$  varies as we traverse a 'vertical' circular section. Figure 4b should make the choice of coordinates in the second toroid clear. The advantage is now that when  $\theta=\pi/2$ , all points with different  $\phi_1$ 's coalesce as required. Finally, our representation consists of two solid toroids, one each for  $\theta \leq \pi/4$  and  $\theta \geq \pi/4$ . The surfaces of the two toroids are made to correspond by identifying 'horizontal' circles on one with 'vertical' ones on the other. Such a correspondence between the surface of one torus and another is continuous and quite permissible. Topologists recognise that the distinction between the two types of circles is not intrinsic to the torus regarded as a surface, but only appears when we embed it in a three-dimensional space.

In this representation by two toroids, the lines of constant  $\phi_1 - \phi_2$  for a given value of  $\theta$  not equal to 0 or  $\pi/2$  form closed curves, winding on the surface of a torus such that the coordinates  $\phi_1$  and  $\phi_2$  both increase by  $2\pi$  on traversing the curve once. Each line corresponds to a state of polarisation and each point on it to a given phase, as

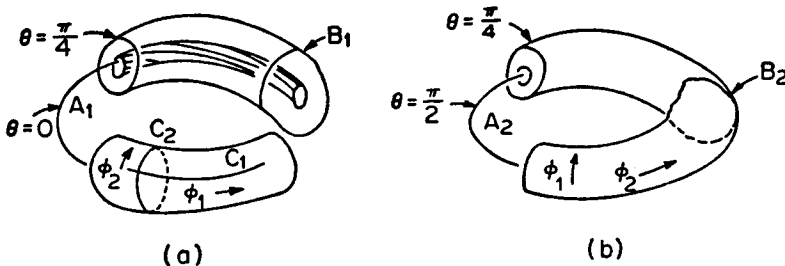


Figure 4. An alternative representation for polarisation and phase. a. A toroid representing all states between  $\theta=0$  and  $\theta=\pi/4$ . It has been cut away to show the special line  $\theta=0$  and the trajectories  $\phi_1 - \phi_2 = \text{constant}$  on a torus with a nonzero value of  $\theta$ . b. The toroid representing  $\theta=\pi/4$  to  $\pi/2$ . Note the interchange of the  $\phi_1$  and  $\phi_2$  axes with respect to a.

explained earlier. The two circles  $A_1(\theta=0, \phi_1$  varying from 0 to  $2\pi$ ) and  $A_2(\theta=\pi/2, \phi_2$  varying from 0 to  $2\pi$ ) represent vibrations along  $x$  and  $y$  respectively, and lie at the 'cores' of the two toroids. We require a continuous surface intersecting each member of this family of lines once and once only.

Let us assume that such a surface has been constructed. Studying the vicinity of its intersection with the circle  $A_1$ , we see that its intersection with a torus corresponding to some value of  $\theta$  slightly greater than zero is a closed curve which shrinks to a point as  $\theta \rightarrow 0$ . By continuity, this curve, for small  $\theta$ , is topologically equivalent to a circle of the 'vertical' family for which  $\phi_2$  increases by  $2\pi$  while  $\phi_1$  returns to its original value. It is useful here to recall that a general closed curve on the torus can be described by two integers  $n_1$  and  $n_2$  which give the multiples of  $2\pi$  by which  $\phi_1$  and  $\phi_2$  increase as we traverse it. Curves with different  $(n_1, n_2)$  pairs cannot be continuously deformed into one another as integers like  $n_1$  and  $n_2$  cannot change discontinuously.  $(n_1, n_2)$  and  $(-n_1, -n_2)$  are however equivalent descriptions of the same closed curve traversed in opposite senses. We thus conclude, by increasing  $\theta$  from 0 to  $\pi/4$ , that the surface under study intersects the outermost torus,  $\theta=\pi/4$ , in a curve  $B_1$  of the family (0, 1), which is topologically equivalent to a vertical circle. This is depicted in figure 4a.

Similarly, starting from an element of the surface meeting the circle  $A_2$  in the core of the second toroid, we conclude that the surface will meet the torus  $\theta=\pi/4$  in a curve  $B_2$  topologically equivalent to a 'vertical' circle. Remembering the different choice of the coordinates  $\phi_1, \phi_2$  in the second toroid, this means that the intersection is a curve  $B_2$  of the family (1, 0). But if we are to have a single continuous surface, then we should be able to deform the curve  $B_1$  to the curve  $B_2$ . Since these two curves, lying in the same torus have distinct 'quantum numbers' (0, 1) and (1, 0) the proof that such a continuous surface cannot be constructed is complete.

#### 4. Orbit interpretation

One can regard the ellipses traced out by the electric vector of a single monochromatic wave as the paths traced by a two-dimensional harmonic oscillator with equal restoring forces along the  $x$  and  $y$  directions. The Hamiltonian for such a system reads

$$H = p_1^2 + p_2^2 + q_1^2 + q_2^2, \tag{2}$$

with a suitable choice of units for the coordinates  $q_1, q_2$  and momenta  $p_1, p_2$ . Since we are not interested in the total energy, we can fix its value to be unity. The four parameters  $p_1, q_1, p_2, q_2$  can now be replaced by three polar angles  $\theta_1, \theta_2,$  and  $\phi$  which specify position on the (three dimensional) surface of a four dimensional unit sphere, using the relations

$$\begin{aligned} p_1 &= \cos \theta_1, & q_1 &= \sin \theta_1 \cos \theta_2, \\ p_2 &= \sin \theta_1 \sin \theta_2 \cos \phi, & q_2 &= \sin \theta_1 \sin \theta_2 \sin \phi. \end{aligned} \tag{3}$$

It can be verified that the orbits in this representation behave in the same way as in the alternative representation by two toroids, as of course they should.

## 5. Discussion

Bacry *et al* (1975) have remarked that one obtains the Poincare sphere representation starting from a two dimensional harmonic oscillator and 'projecting out' both the energy and phase variables as in § 4. They do not seem to have considered the structure of the three-dimensional manifold obtained by projecting out only the energy, which is the subject of the present paper.

When we consider the quantum description of polarisation for single photons, the Stokes parameters, and hence the Poincare sphere representation, remain valid (Fano 1954). However, the concept of phase is meaningless for single photon states, while it reappears for states with many (but not a definite number!) of photons. While this is qualitatively expressed by the number-phase uncertainty relation, a careful discussion is needed to define phase precisely (Carruthers and Nieto 1968). The entire discussion in the present paper is concerned with the choice of a coordinate system when one wishes to talk of all polarisations at once. Such a coordinate system is needed both for classical and quantum mechanics.

The question of a phase convention for polarised light arises naturally when one considers interference between two beams of different polarisation. In the context of radio astronomy, Morris *et al* (1964) have described the theory and application of an interferometer in which the two aerials respond to different, arbitrary, polarisations. The result of an observation with such an instrument is the amplitude and phase of the spatial coherence function (visibility function), and needs a convention for the phase difference between two vibrations of different polarisations. A study of this work motivated, though by no means compelled, the study of phase conventions and in particular of whether a global one could be found.

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