Of Connections and Fields – I

Chern's Mathematical Ideas in Physics

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Joseph Samuel is a theoretical physicist and by natural inclination a classical mechanic. Over the years he has strayed into other fields like optics, general relativity and very recently DNA elasticity. A unifying theme in his work is differential geometry and topology in physics. He keeps moderately fit by raising and lowering indices and relaxes by playing semiclassical guitar.

We describe some instances of the appearance of Chern's mathematical ideas in physics. By means of simple examples, we bring out the geometric and topological ideas which have found application in describing the physical world. These applications range from magnetic monopoles in electrodynamics to instantons in quantum chromodynamics to the geometric phase of quantum mechanics. The first part of this article is elementary and addressed to a general reader. The second part is somewhat more demanding and is addressed to advanced students of mathematics and physics.

Topological Invariance

Take a loop of string and lay it out flat on a table so it doesn't cross itself. Keeping your finger pointed along the string, move it all the way around the loop at unit speed. Note that the direction of your finger describes a full circle. Writing θ for the angle between your finger and the x axis, (x and y axes chosen conveniently along perpendicular edges of the table) we find that the 'angular velocity' $\Omega = d\theta/dt$ of the tangent vector (the unit tangent vector to the string is $\hat{t} = (\cos\theta, \sin\theta)$) integrates to 2π. We now have the remarkable relation:

$$1/2\pi \int_0^L \Omega dt = 1/2\pi \int_0^{2\pi} d\theta = 1, \qquad (1)$$

where L is the length of the string. What is so remarkable about this formula? If one wiggles the string, the value of Ω at points of the string will change. However,

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the integral (1) is unchanged. It is a 'topological invariant', unaffected by small deformations. The angular velocity $d\theta/dt$ of the tangent vector is just the curvature of the string. Remember, we traverse the string at unit speed so, if s is the arc length parameter, $ds/dt = 1$ and $\Box = d\theta/ds = d\theta/dt$. Physically, $\Box$ is the transverse force necessary to keep on track a bug of unit mass traversing the loop at unit speed. From its definition $\Box = d\theta/ds$, we see that $\Box$ is a geometrical quantity, described by a real number. However, its integral $1/2\pi \int \Box ds$ is an integer and a topological quantity, the number of times the tangent vector $\hat{t}$ winds around the circle of directions in the plane. The general flavour of the simple equation (1) is that the integral of a certain geometric quantity over the loop is a topological quantity.

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This is a simple example of a class of mathematical results discovered by the mathematicians Chern, Weil and Pontryagin. These results are deep ones involving global differential geometry and topology. They have had applications in physics, in such diverse fields as elementary particle physics and the quantum hall effect. The purpose of this article (and the next) is to bring out the main impact of this body of mathematical work in the physical sciences. It is impractical within the space of these articles to precisely define the mathematical objects that we deal with (manifolds, connections, bundles). We are concerned less with completeness and rigour than with enticing the reader into a beautiful subject. A full appreciation of even this popular article would need some familiarity with modern differential geometry and the standard model of elementary particle physics[1]. The hurried reader is advised to read this article impressionistically and not worry too much about the technical details. A less hurried reader may be motivated to try the suggested reading given at the end, to pick up the necessary background to appreciate the mathematical ideas described here and how they are

[1] Rohini M Godbole and Sunil Mukhi, Nobel for a Minus Sign, *Resonance*, Vol.10, No.2, pp.33-51, 2005.

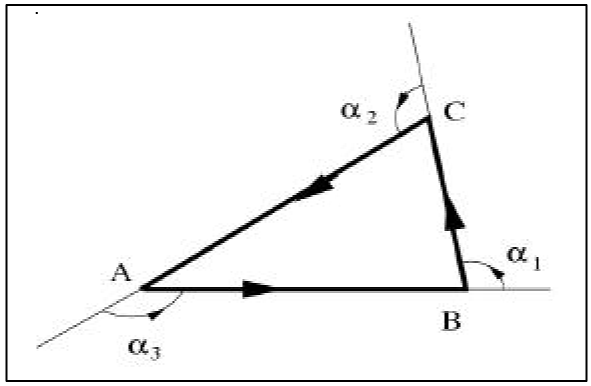

Figure 1.

used in physics. While reading this article, you may want to keep pencil and paper handy so you can check some of the formulae. Also required are string, a pair of scissors, glue (or some less messy substitute like cellotape or a stapler) and a ball. Almost any soft ball will do, like a tennis or basket ball. If you are not the sporting type, try and steal a ball from a child you know. Using your hands to play with models will make the subject come alive as no mathematical formulae can.

To continue with the elementary exposition, now consider a special case. Pull the loop of string taut in the shape of a triangle with vertices ABC. The tangent vector $\hat{t}$ (and so θ) remains constant along the lines AB, BC and CA but swings abruptly through the exterior angles $\alpha_1, \alpha_2, \alpha_3$ at the vertices A, B and C respectively (*Figure* 1). Adding up the total rotation of the tangent vector we find that

$$2\pi = (\pi - A) + (\pi - B) + (\pi - C), \qquad (2)$$

where we write A, B, C for the interior angles at the vertices A, B, C respectively. From (2)) follows the equation $A + B + C = \pi$ that you learned at your mother's knee: the sum of the interior angles of a plane triangle is two right angles! Elementary, yes, but it has in it the germ of deeper ideas to come. The reader is invited to puzzle over the following questions. What happens if the string is laid on the table so that it crosses itself as for example, a figure of eight? What happens if the string is laid not on a flat table but on the curved surface of a globe? How does one define the total rotation of the tangent vector? Is it still a topological quantity? What happens if you suppose that the loop is infinitesimal so that the globe can be approximated by a plane? What happens if you expand the loop (say it is elastic) and shrink it back to a small loop on the other side of the globe? A little reflection will show that these simple questions can be pursued in very interesting directions. Let us now return to our main theme: the connection between local and global quantities in mathematics and physics.

The Gaussian curvature is zero for cylinders and planes, positive for spheres and eggshells, and negative for saddles and some parts of flower vases.

Gauss–Bonnet

Consider a sphere $\mathcal{S}$ (you have one near you, the surface of the ball) of radius r embedded in three dimensional space. The curvature of $\mathcal{S}$ (the Gaussian curvature which is also called the intrinsic curvature) is given by $1/r^2$ and we have the easily checked result

$$\frac{1}{4\pi}\int_{\mathcal{S}} \square dA = 1, \tag{3}$$

where $dA = r^2 \sin\theta d\theta d\phi$ in standard polar co-ordinates. Now deform the ball by squeezing it. The Gauss-Bonnet theorem states that (3) still holds true. One can understand this result as follows. At every point on the sphere $\mathcal{S}$ with local co-ordinates (x^1, x^2), there is a unit normal $\hat{n}(x^1, x^2)$ which points out of the sphere. If (x^1, x^2) are varied over a small region of (signed) area dA, the unit normal varies over a signed area $dA_{\hat{n}}$ on the sphere of directions. The Gaussian curvature is in fact a measure of this variation and can be expressed as

$$\square = \frac{dA_{\hat{n}}}{dA}. \tag{4}$$

(The reader who is not familiar with Gaussian curvature is invited to check for herself, using (4), that the

Gaussian curvature is zero for cylinders and planes[2], positive for spheres and eggshells, and negative for saddles and some parts of flower vases.) Thus, the Gaussian curvature is a real number and a geometric quantity describing the local curving of the sphere. However, its integral over the sphere is given by

$$\frac{1}{4\pi}\int_{S} \Box dA = \frac{1}{4\pi}\int_{S} \frac{dA_{\hat{n}}}{dA} dA = \frac{1}{4\pi}\int_{S} dA_{\hat{n}} = 1, \qquad (5)$$

since the unit normal 'winds once around' the sphere of normals when the sphere is traversed once. Just as in the case of the loop of string, we find that although the integrand is local and geometrical, the integral is a topological quantity which being an integer, does not change under small variations. While deforming the ball one creates non-uniformities in the curvature. However, every decrease in the curvature is compensated by an increase somewhere else so that the integral remains unchanged.

[2] This is why a flat paper label can be stuck on the cylindrical surface of a beer bottle without crumpling! In contrast, try tracing Australia from a globe to a flat piece of tracing paper.

Fibre Bundles and Connections

The Chern–Weil theorem is a general mathematical result of a similar nature dealing with connections on fibre bundles. A fibre bundle is a space which is locally a product of two spaces. A simple example of a fibre bundle is a Möbius strip. To make your very own bundles, you can use a xerox copy of *Figure* 2. Cut along the dark black lines to get four strips. Taking a strip at a time, hold the two ends of each strip against the light and align the paper so that the square, triangle and circle overlay the square, triangle and circle at the other end. Make sure that the triangle is properly oriented. For the second strip (B) you will have to turn one of the ends around once to do this. Glue the ends in place. You are now the proud owner of four fibre bundles, labelled A,B,C and D! Take good care of them; they may be worth a bundle someday. Let us compare the first two, A and B a cylinder and Möbius strip respectively.

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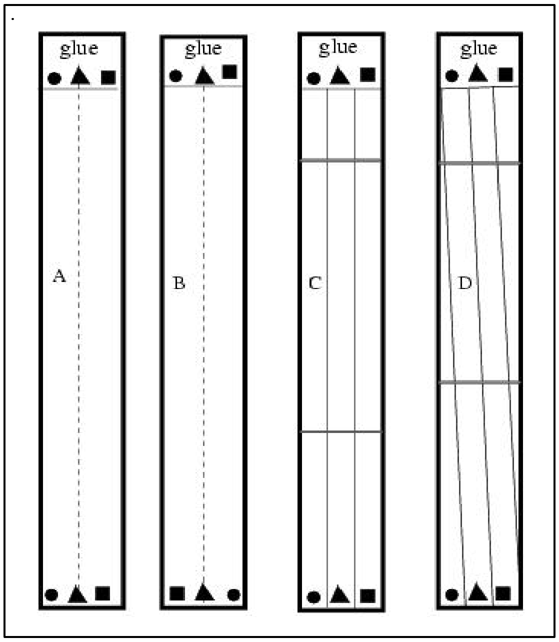

Figure 2.

Locally, the Möbius strip B is indistinguishable from the cylindrical strip A (one without the half twist), but it is globally different. A bug walking along a small part of the dotted line cannot tell the difference, but we know that the two are quite different spaces. For example, the Möbius strip has only one side (try colouring it) as opposed to two sides for a cylindrical strip. If you cut A and B all the way around along the dotted line, you will find that A falls into two components, but B does not! (What happens if you repeat this operation on B?) The Möbius strip is a twisted product of a circle S^1 and a line. The circle is called the base of the fibre bundle and the line the fibre. More generally, a fibre bundle consists of $(\mathcal{E}, \mathcal{B}, \mathcal{F}, \Pi)$, where $\mathcal{E}$ is the total space and Π is a projection from $\mathcal{E}$ to the base space $\mathcal{B}$ and $\mathcal{F}$ is the fibre. We require that any point $b \in \mathcal{B}$ has a neighbourhood $\mathcal{U}_b$ so that $\Pi^{-1}(\mathcal{U}_b)$ looks like $\mathcal{U}_b \times \mathcal{F}$. This is the requirement that the space is locally a product space. It may happen that globally $\mathcal{E} = \mathcal{B} \times \mathcal{F}$, in which case, we say that the bundle is trivial. More generally, the bundle is only locally a product not globally.

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The mathematical language of fibre bundles turns out to be just right for describing a class of physical theories called gauge theories. Gauge theories have turned out to be extremely successful in describing the interactions between elementary particles. The most familiar example of such a theory is electromagnetic theory, that you may have studied. Mathematically, gauge theories (more precisely classical gauge theories) are connections on fibre bundles. A connection gives a rule for 'parallel transport' or moving between fibres. See the accompanying article in this issue by Siddhartha Gadgil for more on this. You can visualise a connection by using the paper strip models C and D, which are fibre bundles with a connection on it. The fibres are shown as short grey lines across the strip. The rule for horizontal motion in each of these bundles is: move parallel to the dark thin lines. With the bundle C you find that going all the way around the strip brings you back to the same point on the fibre: the connection is integrable. With the bundle D you come back to a *different* point on the same fibre. This connection is not integrable. Integrable connections are flat and have vanishing curvature. Locally integrable connections have vanishing curvature, but there may be global obstructions to integrability as in the case of the Möbius strip. Can you put an integrable connection on a Möbius strip? Try it out and see.

Magnetic Monopoles

A magnetic monopole is an object from which magnetic field lines emerge (just as electric field lines emerge from an electrically charged particle such as an electron). Magnetic monopoles have never been seen experimentally.

We now turn to a simple example of Chern's ideas in electromagnetic theory: the quantisation condition for the charge of magnetic monopoles. A magnetic monopole is an object from which magnetic field lines emerge (just as electric field lines emerge from an electrically charged particle such as an electron). Magnetic monopoles have never been seen experimentally, but have fascinated physicists over many generations. J J Thomson, P A M Dirac, M N Saha have studied monopoles in years past and more recently, G 'tHooft and A M Polyakov have made seminal contributions to the subject. Dirac showed that if even a single monopole exists, the electric charges of all partices must be multiples of a certain basic unit of charge $q = n(\hbar/2g)$ or

$$qg = \frac{n\hbar}{2}, \tag{6}$$

where g is the magnetic charge of the monopole and $\hbar$ is Planck's constant. Even though monopoles have not been seen, *quantisation of electric charge is a fact of Nature*. This lends support to the idea that magnetic monopoles *may* exist. In fact, the standard model of the weak nuclear interactions predicts monopoles of the kind studied by 'tHooft and Polyakov.

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In the modern gauge theoretic approach towards electromagnetic theory, one views the magnetic fields (and electric fields) as analogous to curvature. The integral of the magnetic field over a closed surface $\mathcal{S}$ measures the total magnetic charge contained within that surface.

$$\frac{2q}{\hbar}\int_{\mathcal{S}} \vec{B}.d\vec{S} = n. \tag{7}$$

Thus we see by analogy that the general flavour of Dirac's quantisation condition (6) is the same as the Chern–Weil result: the integral of the curvature over a closed surface is an integer!

The Geometric Phase

One physical context where connections appear naturally in physics is the geometric phase. This is a vast topic, interesting in its own right. Let us briefly recall the main ideas in the context of the adiabatic theorem of quantum mechanics. A quantum system in a slowly changing environment displays a curious history dependent geometric effect: when the enviroment returns to its original state, the system also does, but for an additional phase. The phase is a complex number of modulus one and experimentally observable by quantum interference. This phenomenon is of geometric origin and is fundamentally due to the curvature of the ray space of quantum mechanics. (Ray space is the Hilbert space modulo multiplication by non-zero complex numbers.) The geometric phase provides us with a wealth of examples of globally non-trivial bundles, monopoles, instantons and more.

A quantum state $|\psi >$ is an element of a complex Hilbert space $\mathcal{H}$, (let's say finite dimensional $\mathcal{H} = \mathbb{C}^N$ to keep life simple) with the inner product $< \phi|\psi > = \Sigma_{i=1}^{N} \bar{\phi}_i \psi_i$. The time evolution of the system is determined by the Schrödinger equation

$$i\frac{d}{dt}|\psi > = \widehat{H}|\psi >, \tag{8}$$

where $\widehat{H}$ is an $N \times N$ Hermitian matrix. Let us suppose that the Hamiltonian $\widehat{H}$ depends on a set of parameters $\{x^1, x^2 ... x^m\}$ (which we write collectively as x, not to be confused with the ordinary spatial co-ordinate) describing the influence of the environment on the system. For x fixed, the equation

$$\widehat{H}(x)|\psi(x) > = E(x)|\psi(x) > \tag{9}$$

gives us the eigenstates of $\widehat{H}(x)$. From the Schrödinger equation, the eigenstates of $\widehat{H}$ evolve by a pure phase $\exp -iE(x)t|\psi(x) >$. This is the dynamical phase, which is easily accounted for. Consider a region of parameter space where the eigenvalues of $\widehat{H}$ are well separated from each other. Then $\tau = \hbar/(E - E')$ is small for any pair E, E' of eigenvalues. If the parameter x^i is slowly varied (over a time large compared to τ), a system which is initially in an eigenstate remains in an eigenstate of the instantaneous Hamiltonian. Further, since the energy levels do not cross, one can unambiguously keep track of the energy levels. For example, the eigenspace corresponding to the lowest energy level is a well defined notion. If the parameters x are varied in a cyclic fashion, one returns to the original ray. The eigenvalue equation(9) determines a ray in $\mathcal{H}$ for every x. This is a fibre bundle over the parameter space. This fibre bundle has a natural connection which can be deduced from the Schrödinger equation: move orthogonal to the fibres in the Hilbert space inner product.

$$< \psi | \frac{d}{dx} | \psi >= 0. \qquad (10)$$

The curvature of this connection is Berry's phase. Berry's phase has been seen experimentally in numerous situations. The simplest example is a spin half system which is described in *Box* 1. The connection occurring here is the same as that of the magnetic monopole.

Conclusion

We have come a long way from the loop of string we started with! The elementary results we used to motivate this exposition are only the tip of an iceberg. The basic idea is capable of considerable generalisation. Many of these ideas have application in modern physics. Such results find application in understanding gauge theories in many contexts, from Chern–Simons gauge theories of the quantum Hall effect to the non-Abelian gauge theories of the standard model of elementary particle physics.

While one may object that magnetic monopoles are 'science fiction' from the experimental viewpoint, the theoretical idea is very powerful and useful. The monopole configurations appear naturally in the geometric phase of a two state quantum system. With the right mathematical mappings, magnetic monopoles can be usefully employed to understand the elastic properties of DNA! (see Suggested Reading below). Even if magnetic monopoles are never directly seen in the laboratory, the theoretical ideas they bring with them are very deep and capable of application. Physics is a deeply intercon-

Suggested Reading

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Box 1. Monopoles, Möbius strips and Berry's Phase

Let us consider a very simple example in which a globally non-trivial connection (gauge field) appears: a spin 1/2 system in an external magnetic field. We regard the three components of the magnetic field as parameters $x^i; i = 1, 2, 3$; which can be varied. Throughout this box, the parameters x^i represent co-ordinates on the base space $\mathcal{B}$ which is the parameter space describing the influence of the environment, and not space-time as was the case in the previous examples. We write the Hamiltonian, a 2×2, Hermitian matrix as $H = x^i \sigma_i$, where σ_i are the three Pauli matrices. These matrices are traceless and Hermitian and satisfy $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$, the Clifford algebra. Since the eigenspaces of H are the same for x^i and λx^i (only the eigenvalue changes, not the eigenspace), it is enough to restrict attention to the unit sphere $S^2 = \{x^i \in \mathbb{R}^3 | \sum x^i x^i = 1\}$ in the parameter space. At each point of S^2, there is a two-dimensional, complex vector space $\mathbb{C}^2$ of spin states on which the Hamiltonian acts. Since H squares to unity, its eigenvalues are ± 1. The subspace of positive energy states $\{|v(x)> \in \mathcal{C}^2 | H(x)|v(x)> = |v(x)>\}$ defines a line bundle over S^2 (the Hopf bundle). To compute the Berry potential which arises when the parameters x^i are varied, note that $H(x) = h(x)\sigma_3 h^{-1}(x)$, where $h(x)$ is defined by

$$h(x) = \frac{1 + H\sigma_3}{\sqrt{2(1 + x^3)}};$$

at all points except the south pole, where $x^3 = -1$. If we pick a normalised positive energy state $|v^N>$ at the north pole, which satisfies $\sigma_3|v^N> = |v^N>$, then $|v(x)> = h(x)|v^N>$ is a normalised positive energy state all over the sphere (except the south pole). (Similar considerations also apply to the south polar patch, which excludes the north pole). The gauge potential describing the connection is $A = < v(x)d|v(x)> = < v^N|h^{-1}dh|v^N>$ and its field strength is $F = dA$. It is easily seen that this field strength describes a magnetic monopole situated at the origin of x^i, which has been excised from the parameter space. This magnetic monopole satisfies the Dirac quantisation condition with $n = 1$.

Consider the slice $x^2 = 0$. In this special case, we can write $H(\theta) = \cos\theta\,\sigma_3 + \sin\theta\,\sigma_1$. Or in the standard basis for Pauli matrices,

$$\hat{H}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

As θ goes from 0 to 2π, the real symmetric matrix $H(\theta)$ traverses a loop and returns to itself, but its eigenvector $(\cos\theta/2; \sin\theta/2)$ reverses sign! This is just the Möbius bundle you made with your hands.

nected subject and we can transport theoretical ideas from one context to another.

It is curious that the deep mathematical ideas of Chern–Weil theory which were discovered with a purely mathematical motivation actually turn out to be useful in physics [10]. Nature makes liberal use of geometric and topological objects. All the known interactions of Nature use the idea of connections on fibre bundles (gauge theories to physicists). This influx of mathematical ideas into physics is a good example of the symbiotic relation between the two disciplines. Mathematical ideas often find applications in physics and these applications provide a further stimulus for development of the subject.

The reader may wonder why there is so much mathematics in physics. The physics that we were taught at high school was of a more descriptive variety, with boring definitions to remember. The present state of the subject is considerably different and theoretical physicists need to have a fairly good base in mathematics to even get started. This is due to the natural evolution of the subject and the fact that both theory and experiments are getting more and more sophisticated. Throughout history, each generation of physicists has complained that the next generation is far too mathematical. (In the 1930s, physicists used to complain of the 'gruppenpest', the invasion of physics by group theory.) Physicists freely borrow state of the art mathematical language to formulate their theories. However, it is important to remember that theoretical physics is not mathematics. It uses mathematics to describe, understand and predict the behaviour of the physical world. The use of mathematics is not an end in itself, but a means to an end: a better understanding of Nature.

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