Freezing transition in the barrier crossing rate of a diffusing particle: supplemental materials for

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We give principal details of the calculations and simulations described in the main text of the Letter.

I. THE DETAILS OF COMPUTING $\tilde{\psi}(x,s)$ AND THE SURVIVAL PROBABILITY FOR THE POTENTIAL $U(x) = \alpha|x|$

In the presence of an absorbing barrier at $x = a$, the quantum potential is given by

$$V(x) = \begin{cases} \dfrac{\alpha^2}{4D} - \alpha\delta(x) & \text{for } x < a, \\ \infty & \text{for } x \geq a. \end{cases} \tag{1}$$

Therefore, the Schrödinger equation (in imaginary time) in the in the Laplace space, $\tilde{\psi}(x,s) = \int_0^\infty \psi(x,t)\, e^{-st}\, dt$, reads

$$D\tilde{\psi}''(x,s) - [\alpha^2/(4D) - \alpha\delta(x) + s]\tilde{\psi}(x,s) = -\delta(x), \quad \text{for } \ x \leq a, \tag{2}$$

with boundary conditions $\tilde{\psi}(x \to -\infty, s) = 0$ and $\tilde{\psi}(x = a, s) = 0$.

We solve Eq. (2) separately for $x < 0$ and $0 < x < a$, and then match the solutions at $x = 0$, where the wave function is continuous, but its first derivatives undergoes a jump due to the δ-function at $x = 0$. In each of these regions Eq. (2) reads $D\tilde{\psi}''(x,s) - [\alpha^2/(4D) + s]\tilde{\psi}(x,s) = 0$. Therefore, the solutions can be written as

$$\tilde{\psi}(x,s) = \begin{cases} A_1 e^{px/(2D)} + B_1 e^{-px/(2D)} & \text{for } x < 0, \\ A_2 e^{px/(2D)} + B_2 e^{-px/(2D)} & \text{for } 0 < x < a, \end{cases} \tag{3}$$

with $p = \sqrt{\alpha^2 + 4Ds}$ and A_1, B_1, A_2 and B_2 are four unknown constants to be fixed from the boundary and the matching conditions.

The boundary condition $\tilde{\psi}(x \to -\infty, s) = 0$ implies $B_1 = 0$. On the other hand the absorbing boundary condition $\tilde{\psi}(x = a, s) = 0$ yields $A_2 = -B_2 e^{-pa/D}$. Therefore, the solutions read

$$\tilde{\psi}(x,s) = \begin{cases} A_1 e^{px/(2D)} & \text{for } x < 0, \\ B_2 \left[1 - e^{-p(a-x)/D}\right] e^{-px/(2D)} & \text{for } 0 < x < a. \end{cases} \tag{4}$$

The two remaining constants A_1 and B_2 can be determined by matching the solutions at $x = 0$. Integrating Eq. (2) in an infinitesimal region across $x = 0$ gives the continuity of the solution $\tilde{\psi}(0^+,s) = \tilde{\psi}(0^-,s) \equiv \tilde{\psi}(0,s)$ and the discontinuity of the derivatives $D\left[\tilde{\psi}'(0^+,s) - \tilde{\psi}'(0^-,s)\right] + \alpha\tilde{\psi}(0,s) = -1$. The continuity of the solution implies $A_1 = B_2\left(1 - e^{-pa/D}\right)$. Finally, the discontinuity of the solution gives $B_2 = 1/A(p)$ where

$$A(p) = p - \alpha\left(1 - e^{-pa/D}\right), \tag{5}$$

with $p = \sqrt{\alpha^2 + 4Ds}$. Therefore,

$$\tilde{\psi}(x,s) = \begin{cases} \frac{1}{A(p)} \left[1 - e^{-pa/D}\right] e^{px/(2D)} & \text{for } x \leq 0, \\ \frac{1}{A(p)} \left[1 - e^{-p(a-x)/D}\right] e^{-px/(2D)} & \text{for } 0 \leq x \leq a, \end{cases} \tag{6}$$

as mentioned in the main text.

We can obtain $\psi(x,t)$ by using the Bromwich integral

$$\psi(x,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\psi}(x,s)\, e^{st}\, ds, \tag{7}$$

where c is a real number such that all the singularities of $\tilde{P}(x,s)$ are on the left of the vertical contour $\mathrm{Re}(s) = c$, in the complex-s plane. The most dominant large t behavior comes from the singularity closest to the contour, the second dominant contribution comes from the next singularity and so on.

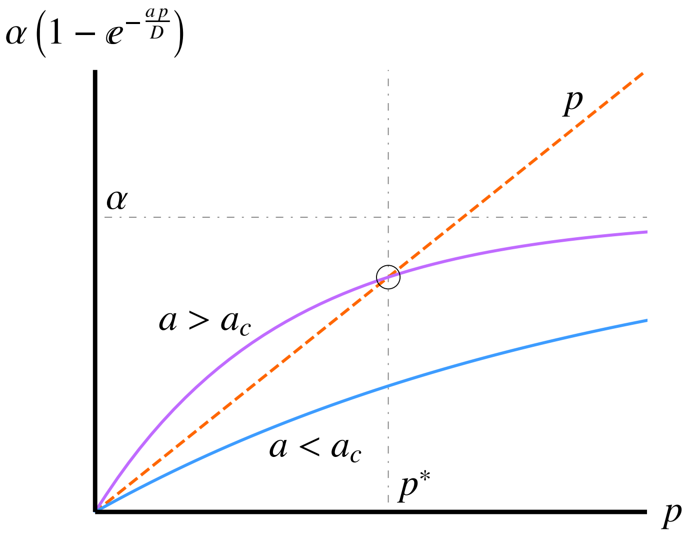

FIG. 1: Graphical representation of the non-trivial (non-zero) solution p^* of the transcedental equation $p = \alpha(1-e^{-pa/D})$, which exists only for $a > a_c = D/\alpha$. For $a \to \infty$, $p^* = \alpha$, whereas for $a \to a_c^+$, we have $p^* \to 0$.

Let us first consider the limit $a \to \infty$, i.e., when there is no absorbing barrier. In this case, we have from (5), $A(p) = p - \alpha$, and hence, $\tilde{\psi}(x,s)$ has a pole at $s = 0$, i.e., $p = \alpha$. There is also a branch-point (and not a pole) at $s = -\alpha^2/(4D)$, i.e., $p = 0$. Therefore, the leading order behavior, that corresponds to the stationary state at $t \to \infty$, comes from the contribution from the pole at $s = 0$ and the approach to the stationary state comes from the branch-point at $s = -\alpha^2/(4D)$. Evaluating the residue at $s = 0$ gives

$$\psi(x,t\to\infty) = \frac{\alpha}{2D}\exp\left(-\frac{\alpha|x|}{2D}\right). \tag{8}$$

Consequently, using $P(x,t) = e^{-\alpha|x|/(2D)}\psi(x,t)$, we get the stationary distribution

$$P(x,t\to\infty) \equiv p_{\rm ss}(x) = \frac{\alpha}{2D}\exp\left(-\frac{\alpha|x|}{D}\right). \tag{9}$$

The contribution from the branch-point gives the approach to the stationary state as

$$\psi(x,t) - \psi(x,t\to\infty) = Q(x,t)\exp\left(-\frac{\alpha^2}{4D}t\right) \tag{10}$$

where $Q(x,t)$ is obtained from the integral around the branch-cut from $-\infty$ to $-\alpha^2/(4D)$, and is given by

$$Q(x,t) = \frac{1}{\pi}\int_0^\infty dr\,\frac{e^{-rt}}{\alpha^2+4Dr}\left[\sqrt{4Dr}\cos\left(\frac{|x|\sqrt{r}}{\sqrt{D}}\right) - \alpha\sin\left(\frac{|x|\sqrt{r}}{\sqrt{D}}\right)\right]. \tag{11}$$

The large t behaviors, for arbitrary x, can be computed by expanding $[\alpha^2+4Dr]^{-1}$ about $r = 0$ and carrying out the above integral term by term. The leading order behavior is given by

$$Q(x,t) = \frac{\sqrt{D}\,e^{-x^2/(4Dt)}}{\sqrt{\pi}\,\alpha^2 t^{3/2}}\left(1 - \frac{x^2}{2Dt} - \frac{\alpha|x|}{2D}\right) + O(t^{-5/2}). \tag{12}$$

In terms of the energy spectrum of the quantum Hamiltonian $\mathscr{H}$ given in the Schrödinger equation in the main text, there is a single bound state corresponding to energy $E_0 = 0$, and continuum spectrum of scattering states with energy $E \geq \alpha^2/(4D)$. Hence, the gap is given by $\Delta = \alpha^2/(4D) - E_0 = \alpha^2/(4D)$. The continuum energy spectrum from $\alpha^2/(4D)$ to ∞, manifests as the branch-cut from $-\alpha^2/(4D)$ to $-\infty$ in the Laplace transform $\tilde{\psi}(x,s)$.

We now turn to finite a. As we bring the absorbing wall from ∞ to a finite value a, the location of the rightmost pole in Eq. (6) shifts from $s^*(\infty) = 0$ to a negative value at $s = s^*(a)$ (with $s^*(a) < 0$) that varies continuously with a. In the quantum problem, this corresponds to the fact that the ground state, while still remains a bound state, its energy $E_0(a) = -s^*(a)$ increases continuously with decreasing a. On the other hand, the position of the branch-point of Eq. (6) remains fixed at $s=-\alpha^2/(4D)$, i.e., at $p=0$, irrespective of a. Therefore, the continuum spectrum for the scattering states is always from $\alpha^2/(4D)$ to ∞. Clearly, as a decreases, at some critical value $a=a_c$, the gap, $\Delta(a)=\alpha^2/(4D)-E_0(a)$, between the scattering band starting at $\alpha^2/(4D)$ and the ground state $E_0(a)$ must vanish, triggering a phase transition.

To locate this critical value a_c, we look for the rightmost pole of Eq. (6), i.e., the nonzero solution of the transcendental equation $A(p)=p-\alpha\left(1-e^{-pa/D}\right)=0$. In Fig. (1) we plot both $\alpha\left(1-e^{-pa/D}\right)$ (solid lines) and p (dashed line) vs p. The slope of $\alpha\left(1-e^{-pa/D}\right)$ at $p=0$ is given by $\alpha a/D$. The two curves p and $\alpha\left(1-e^{-pa/D}\right)$ will cross each other at a nonzero value $p*>0$ only for the slope $\alpha a/D>1$, i.e., $a>a_c=D/\alpha$. As $a\to a_c$ from above, the nonzero solution $p^*\to 0$, triggering the phase transition. In the quantum language, at $a=a_c$, the gap $\Delta(a)$ vanishes. The mechanism of this phase transition is thus very similar to the mean-field transition in ferromagnetic Ising model. For $a<a_c$, $A(p)=0$ has only a trivial solution $p=0$ (which however is not a pole as it cancels with the numerator of $\tilde{\psi}(x,s)$). For $a>a_c=D/\alpha$, the pole at $s=s^*<0$ on the real-s line is given by

$$s^*(a)=-\frac{1}{4D}\left[\alpha^2-p^{*2}(a)\right]. \tag{13}$$

Therefore, for $a>a_c$, the energy of the bound state is given by

$$E_0(a)=-s^*(a)=\frac{1}{4D}\left[\alpha^2-p^{*2}\right], \tag{14}$$

and the corresponding gap in the spectrum is given by

$$\Delta(a)\equiv\frac{\alpha^2}{4D}-E_0(a)=\frac{[p^*(a)]^2}{4D}. \tag{15}$$

The gap vanishes at $a\to a_c^+$, as $p^*\to 0$. There is no pole, and only the branch-point at $s=-E_1=-\alpha^2/(4D)$, for $a<a_c$ [see Fig. 1]. Therefore, the spectrum remains gapless for $a<a_c$.

For $a>a_c$, $\psi(x,t)$ is given by

$$\psi(x,t)=R(x)\,e^{-E_0(a)t}+Q(x,t)\,e^{-\alpha^2t/(4D)}, \tag{16}$$

where $R(x=\lim_{s\to s^*}(s-s^*)\tilde{\psi}(x,s)$ arises from the residue at s^* The prefactor $Q(x,t)$ of the sub-dominant branch-point contribution is from the contour integral around the branch-cut from $-\alpha^2/(4D)$ to $-\infty$, and the leading order time dependence is given by $Q(x,t)\sim t^{-3/2}$.

At $a=a_c$, there is no pole corresponding to $p=0$, and only a branch-point. In this case

$$\psi(x,t)=Q(x,t)\,e^{-\alpha^2t/(4D)}, \tag{17}$$

where the leading order time dependence of $Q(x,t)$ is given by $Q(x,t)\sim 1/\sqrt{t}$.

For $a<a_c$, again there is no pole and the contribution comes only from the branch-point. Therefore, $\psi(x,t)$, is still given by Eq. (17), however, with a different $Q(x,t)$. In particular, $Q(x,t)\sim t^{-3/2}$ for large t.

Survival/first-passage probability: The survival probability $S_a(t|0)$ with the starting position $x_0=0$, can be obtained by integrating $P(x,t)$ obtained above, over x fro $0-\infty$ to a. Hence the Laplace transform of the survival probability $\tilde{S}_a(s|0)=\int_0^\infty S_a(t|0)\,e^{-st}\,dt$ is given by

$$\tilde{S}_a(s|0)=\int_{-\infty}^{a}e^{-\alpha|x|/(2D)}\,\tilde{\psi}(x,s)\,dx, \tag{18}$$

where $\tilde{\psi}(x,s)$ is given by Eq. (6). Performing the integral yields

$$\tilde{S}_a(s|0)=\frac{1}{s}\left[1-\frac{p}{A(p)}\,e^{-(\alpha+p)a/(2D)}\right], \tag{19}$$

with $p=\sqrt{\alpha^2+4Ds}$ and $A(p)=p-\alpha\left(1-e^{-pa/D}\right)$. The first-passage time distribution is related to the survival probability by $F_a(t|x_0)=-\partial_t S_a(t|x_0)$. In the Laplace space, this relation reads $\tilde{F}_a(s|x_0)=1-s\tilde{S}_a(s|x_0)$. Hence, using Eq. (19), we get the Laplace transform of the first-passage distribution as

$$\tilde{F}_a(s|0)=\frac{p}{A(p)}\,e^{-(\alpha+p)a/(2D)}. \tag{20}$$

By inverting formally the Laplace transform (19), the survival probability can be expressed as a Bromwich integral

$$S_a(t|0) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{S}(x_0,s)\, e^{st}\, ds. \tag{21}$$

Let us first consider the trivial limit $a\to\infty$, where $\tilde{S}(x_0,s)=1/s$. The pole at $s=0$ gives $S_\infty(t|0)=1$, which means $\theta(\infty) := -\lim_{t\to\infty} t^{-1} S_\infty(t|0) = 0$.

Now as we bring the absorbing wall from ∞ to a finite value a, how does the exponent $\theta(a) := -\lim_{t\to\infty} t^{-1} S_a(t|0)$ change as a function of a? It can be checked from Eq. (19) that, $s=0$ is no longer a pole for any finite values of a. Similarly, although $p=0$ is a trivial solution of $A(p)=0$, it does not correspond to a pole as it cancels with p in the numerator. Nevertheless, $p=0$ is a branch-point.

For $a > a_c = D/\alpha$, there exists a pole at $s=s^*(a)<0$, on the real-s line, as argued above. Therefore, for $a>a_c$, the leading contribution at large times, comes from this pole, and the branch-point gives the subleading correction:

$$S_a(t|0) = R\exp\left(-\frac{\alpha^2 - p^{*2}}{4D}\,t\right) + Q(t)\,\exp\left(-\frac{\alpha^2}{4D}\,t\right), \tag{22}$$

where $R=\lim_{s\to s^*}(s-s^*)\tilde{S}_a(s|0)$ arises from the residue at s^* and is given by

$$R = \frac{2p^{*2}\exp\left[-\frac{a}{2a_c}\,e^{-p^* a/D}\right]}{\alpha(\alpha+p^*)\left[1-\frac{a}{a_c}\,e^{-p^* a/D}\right]}. \tag{23}$$

The prefactor $Q(t)$ of the branch-point contribution arises from the contour integral around the branch-cut from $-\infty$ to $-\alpha^2/(4D)$, and is given by

$$Q(t) = e^{-\alpha a/(2D)}\int_0^\infty dr\, e^{-rt}\,\frac{\sqrt{4Dr}}{\pi\left[r+\alpha^2/(4D)\right]}\,\mathrm{Re}\left[-\frac{e^{-ia\sqrt{r/D}}}{A(i\sqrt{4Dr})}\right], \tag{24}$$

where Re[.] is the real part of the function inside the square bracket. In Fig. 2 (a), we compare the leading behavior, i.e., the contribution from the pole, given by the first line of Eq. (22) with numerical simulation results and find very good agreement.

At $a=a_c$, to the leading order, $A(p)=p^2/(2\alpha)+O(p^3)$, as $p\to 0$. Therefore, as $a\to a_c^+$, the pole disappears and there is only a branch-point at $s=-\alpha^2/(4D)$, corresponding to $p=0$. Note that although the location of the branch-point is independent of a, the nature of the singularity for $a=a_c$ is different from that for $a\neq a_c$. At, $a=a_c$, following Eqs. (24), the leading order behavior for large t is given by

$$S_{a_c}(t|0) = Q(t)\,\exp\left(-\frac{\alpha^2}{4D}\,t\right), \tag{25}$$

where

$$Q(t) = e^{-\frac{1}{2}}\,\frac{4\sqrt{D}}{\alpha\sqrt{\pi t}} + O(t^{-3/2}), \tag{26}$$

Figure 2 (b) compares this leading behavior with numerical simulation and shows a very good agreement.

For $a<a_c$, there is no pole [see Fig. 1], and only a branch-point at $s=-\alpha^2/(4D)$. Therefore,

$$S_a(t|0) = Q(t)\,\exp\left(-\frac{\alpha^2}{4D}\,t\right), \tag{27}$$

where $Q(t)$ is given by Eq. (24). For $a\ll a_c$, the leading order behavior for large t can be evaluated as

$$Q(t) = \frac{e^{-\alpha a/(2D)}}{(1-a/a_c)^2}\,\frac{2a\sqrt{D}}{\alpha^2\sqrt{\pi}\,t^{3/2}} + O(t^{-5/2}), \tag{28}$$

Figures 2 (c) and (d) compare this result with numerical simulations, where $Q(t)$ evaluated exactly by performing numerical integration of Eq. (24) in (c), and the leading order behavior of $Q(t)$ given by Eq. (28) is used in (d). Naturally, the agreement is perfect in Fig. 2 (c) whereas in Fig. 2 (d), it becomes better as a moves away from a_c.

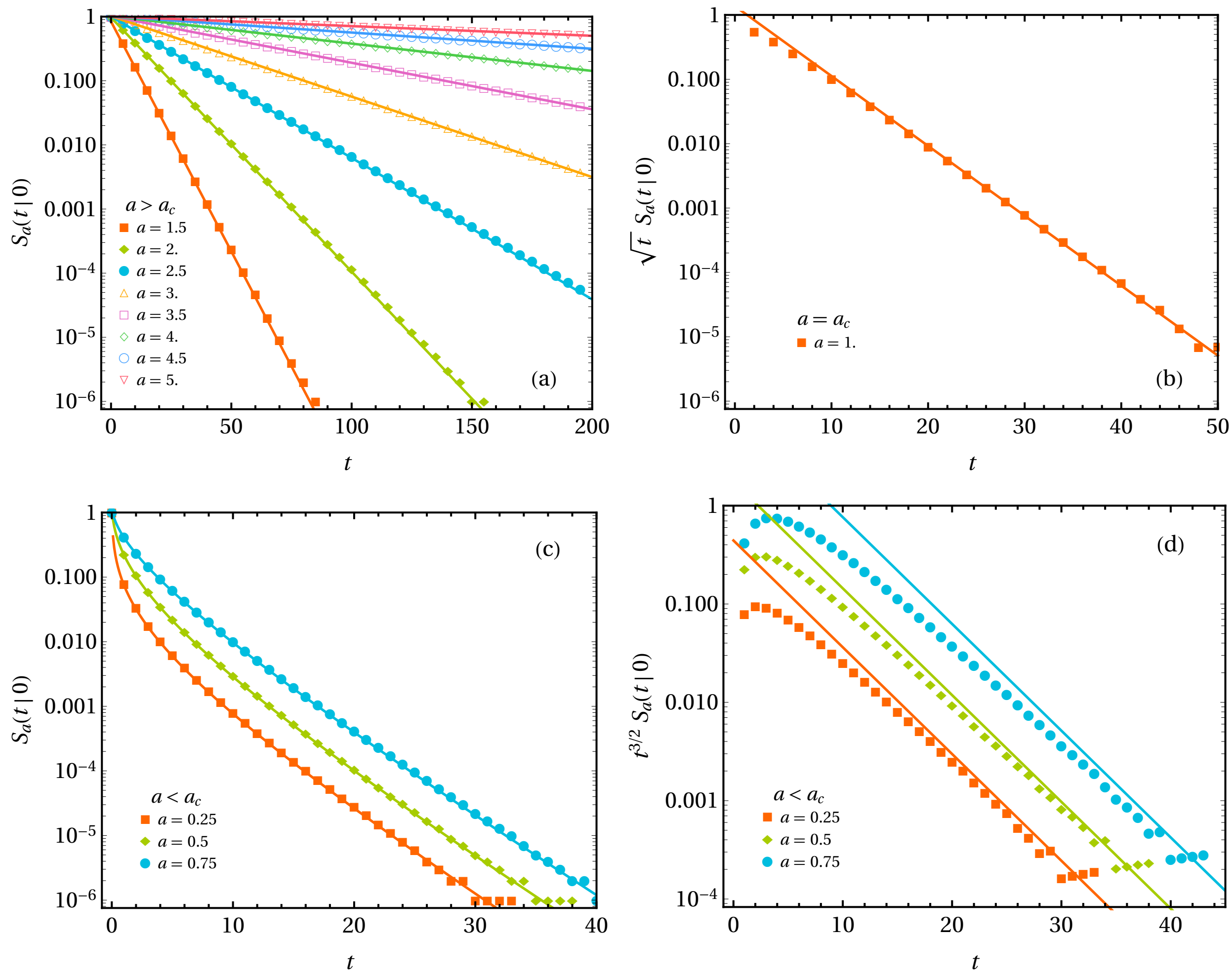

FIG. 2: Survival probability of a Brownian particle in a potential $U(x) = \alpha|x|$, starting at the position $x_0 = 0$ and in the presence of an absorbing barrier at $x = a > 0$. We set the diffusion coefficient $D = 1$ and $\alpha = 1$, so that $a_c = D/\alpha = 1$. The points are from numerical simulations whereas the solid lines are theoretical results. In (a) $a > a_c$, the theoretical lines plot the dominant behavior given by the first line of Eq. (22). In (b), $a = a_c$, the theoretical line is the leading behavior from Eq. (25). In (c) and (d), $a < a_c$, the theoretical lines plot Eq. (27) with $Q(t)$ obtained by numerical integration of Eq. (24) in (c) and using Eq. (28) in (d).

To summarize, the leading asymptotic of $S_a(t|0)$ is given by

$$S_a(t|0) \sim \exp\left[-\theta(a)t\right], \tag{29}$$

where

$$\theta(a) = \begin{cases} \dfrac{1}{4D}\left[\alpha^2 - p^{*2}(a)\right] & \text{for } a > a_c \\ \dfrac{\alpha^2}{4D} & \text{for } a < a_c \end{cases} \tag{30}$$

in which $0 < p^*(a) < \alpha$ solves the transcendental equation $p = \alpha\left(1 - e^{-pa/D}\right)$ [see Fig. 1]. Note that while $\theta(a)$ varies as a function of a for $a > a_c$, it remains at the constant value $\alpha^2/(4D)$ for $a < a_c$. Near $a = a_c^+$, we have $p^* = 2D(a - a_c)/a_c^2 + O\left[(a - a_c)^2\right]$. Therefore, as a approaches a_c from above,

$$\theta(a) = \frac{\alpha^2}{4D} - \frac{D}{a_c^4}(a - a_c)^2 + O\left[(a - a_c)^3\right]. \tag{31}$$

Link to the interlace theorem: We now show how our results are consistent with a powerful interlace theorem derived in Refs. [2, 3]. Let us first consider an unconstrained motion of particle (no absorbing barrier or equivalently $a \to \infty$ limit) diffus-

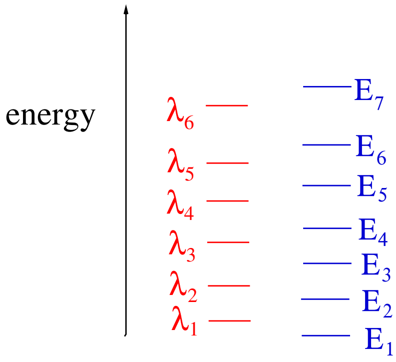

FIG. 3: For a sufficiently confining potential in the unconstrained system (with the absorbing barrier $a \to \infty$), the discrete energy eigenvalues $\{E_1 < E_2 < E_3 \ldots\}$ of the associated Schrödinger equation are shown schematically on the right (blue). As the barrier location $a \geq 0$ decreases, the new ordered eigenvalues (first-passage eigenvalues) $\{\lambda_1(a) < \lambda_2(a) < \lambda_3(a) \ldots\}$ are shown (schematic) on the left (red). The sets $\{E_i\}$ and $\{\lambda_i(a)\}$ interlace, i.e., between any two consecutive E_i's, there is one and only one λ_i, as shown schematically in the figure.

ing in a sufficiently confining potential $U(x)$, such that its quantum counterpart $V(x) = [U'(x)]^2/4D - U''(x)/2$ has a discrete spectrum say $\{E_1 < E_2 < E_3 < \ldots\}$. These energy eigenvalues of the Schrödinger equation in the quantum potential $V(x)$ are identical to the relaxation eigenvalues of the unconstrained Fokker-Planck equation, $\partial_t P - D\partial_x^2 P - \partial_x[U'(x)P] = 0$ on the full real line. Now, imagine that we bring the absorbing barrier from $a = \infty$ to a finite value $a \geq 0$, i.e., the barrier always stays to the right of the initial position of the particle which we set to be $x = 0$. If the barrier is to the left of the particle, one can reverse x and it would be a similar analysis. In this paper, we therefore restrict ourselves to $a \geq 0$. The presence of the absorbing barrier at $a \geq 0$ changes the spectrum of the Fokker-Planck operator. The new ordered eigenvalues $\{\lambda_1(a) < \lambda_2(a) < \lambda_3(a) \ldots\}$ obviously depend on a. Note that the lowest eigenvalue $\lambda_1(a) \equiv \theta(a)$ by definition. Hence as a decreases towards 0, the whole spectrum $\{\lambda_i(a)\}$ moves upwards. The unconstrained spectrum $\{E_i\}$ of course is independent of a. According to the interlace theorem, for any given a, the $\{\lambda_i\}$'s are interlaced with $\{E_i\}$'s, i.e., $E_1 \leq \lambda_1(a) \leq E_2 \leq \lambda_2(a) \leq E_3 \leq \cdots$, —between any pair of consecutive E_i's, there is one and only one λ_i (see Fig. (3)). As a decreases, the whole spectrum $\{\lambda_i(a)\}$ slides upwards, but always respects the interlacing rule for any $a \geq 0$. In this case, the lowest eigenvalue $\lambda_1(a) = \theta(a)$ monotonically increases with decreasing a as $a \to 0$. As mentioned above, this occurs for sufficiently confining potentials, e.g., when $U(x) \sim |x|^\gamma$ as $x \to -\infty$ with $\gamma > 1$.

In the case $\gamma = 1$, e.g., for $U(x) = \alpha|x|$ as $x \to -\infty$, the Schrödinger equation in the full space (i.e., when the barrier location $a \to \infty$) has a single discrete eigenvalue (the lowest one) and a band of continuous eigenvalues, separated from the ground state by a finite gap $\alpha^2/(4D)$, as shown in Fig. 2 in the main text. As a decreases, the continuous part of the spectrum does not change, but the lowest discrete eigenvalue increases (see Fig. 2 in the main text). Since the continuous part of the spectrum is independent of a, clearly the discrete eigenvalue can not increase beyond the gap $\alpha^2/(4D)$. Had it done so, it will go over the lowest eigenvalue of the continuous sector and clearly violate the interlacing rule mentioned above. Hence $\theta(a)$ freezes when it hits the gap value $\alpha^2/(4D)$, at a certain critical value $a_c = D/\alpha$ as shown in our paper. Thus this freezing transition and the mechanism behind it is completely consistent with the interlacing theorem, applied to the discrete part of the spectrum in the critical case $\gamma = 1$ till the onset of the freezing transition.

II. THE BACKWARD FOKKER-PLANCK EQUATION FOR THE SURVIVAL PROBABILITY IN THE LAPLACE SPACE

As mentioned in the main text, the backward Fokker-Planck equation satisfied by the survival probability, with the starting position x_0, reads

$$\frac{\partial S_a}{\partial t} = D\frac{\partial^2 S_a}{\partial x_0^2} - U'(x_0)\frac{\partial S_a}{\partial x_0} \tag{32}$$

with the initial condition $S_a(0|x_0) = 1$ and the boundary conditions, $S_a(t|x_0 \to -\infty) = 1$ and $S_a(t|x_0 = a) = 0$. The Laplace transform $\tilde{S}_a(s|x_0) = \int_0^\infty S_a(t|x_0)\, e^{-st}\, dt$ then satisfies

$$D\tilde{S}_a''(s|x_0) - U'(x_0)\tilde{S}_a'(s|x_0) - s\tilde{S}_a(s|x_0) = -1, \tag{33}$$

with the boundary conditions

$$\tilde{S}_a(s|x_0 \to -\infty) = 1/s \quad \text{and} \quad \tilde{S}_a(s|x_0 = a) = 0. \tag{34}$$

Setting

$$\tilde{S}_a(s|x_0) = \frac{1}{s} + \tilde{q}(x_0), \tag{35}$$

$\tilde{q}(x_0)$ satisfies the homogeneous differential equation

$$D\tilde{q}''(x_0) - U'(x_0)\tilde{q}'(x_0) - s\tilde{q}(x_0) = 0, \tag{36}$$

with the boundary conditions $\tilde{q}(x_0 \to -\infty) = 0$ and $\tilde{q}(x_0 = a) = -1/s$.

III. THE DETAILS OF COMPUTING THE SURVIVAL PROBABILITY FOR A POTENTIAL THAT GROWS FASTER THAN $|x|$ AS $x \to -\infty$

The example of the potential considered in the main text is

$$U(x) = \begin{cases} \frac{1}{2}\mu x^2 & \text{for } x < -b, \\ \alpha|x| & \text{for } -b < x < a, \end{cases} \tag{37}$$

where $b > 0$ and for the sake of continuity of the potential at $x = -b$, we set $\mu = 2\alpha/b$.

We have to solve Eq. (36) separately in the three regions, (i) $x_0 < -b$, (ii) $-b < x_0 < 0$, and (iii) $0 < x_0 < a$ and then match the solutions for the continuity of the solutions as well as the derivatives at both $x_0 = -b$ and $x_0 = 0$. For $x_0 < -b$, Eq. (36) reads

$$D\tilde{q}''(x_0) - \mu x_0\tilde{q}'(x_0) - s\tilde{q}(x_0) = 0. \tag{38}$$

Now substituting

$$\tilde{q}(x_0) = e^{\mu x_0^2/(4D)}\, w(x_0\sqrt{\mu/D}), \tag{39}$$

above gives the differential equation

$$w''\big(x_0\sqrt{\mu/D}\big) + \left[-\frac{s}{\mu} + \frac{1}{2} - \frac{1}{4}\big(x_0\sqrt{\mu/D}\big)^2\right] w\big(x_0\sqrt{\mu/D}\big) = 0, \tag{40}$$

whose general solution can be expressed in terms of the two linearly independent parabolic cylinder functions $D_{-s/\mu}\big(x_0\sqrt{\mu/D}\big)$ and $D_{-s/\mu}\big(-x_0\sqrt{\mu/D}\big)$. Noting that $D_{-s/\mu}\big(x_0\sqrt{\mu/D}\big) \to \infty$ whereas $D_{-s/\mu}\big(-x_0\sqrt{\mu/D}\big) \to 0$ as $x_0 \to -\infty$, the solution of Eq. (38) that tends to zero as $x_0 \to -\infty$, is given by

$$\tilde{q}(x_0) = A_1\, e^{\mu x_0^2/(4D)} D_{-\frac{s}{\mu}}\left(-x_0\sqrt{\frac{\mu}{D}}\right), \tag{41}$$

where the constant A_1 to be determined.

For $-b < x_0 < 0$, Eq. (36) reads

$$D\tilde{q}''(x_0) + \alpha\tilde{q}'(x_0) - s\tilde{q}(x_0) = 0. \tag{42}$$

The general solution is given by

$$\tilde{q}(x_0) = A_2\, e^{(p-\alpha)x_0/(2D)} + B_2 e^{-(p+\alpha)x_0/(2D)}, \tag{43}$$

with $p = \sqrt{\alpha^2 + 4Ds}$ and the constants A_2 and B_2 to be determined.

Finally, for $0 < x_0 < a$, Eq. (36) reads

$$D\tilde{q}''(x_0) - \alpha\tilde{q}'(x_0) - s\tilde{q}(x_0) = 0, \tag{44}$$

whose general solution is given by

$$\tilde{q}(x_0) = A_3\, e^{(\alpha+p)x_0/(2D)} + B_3 e^{(\alpha-p)x_0/(2D)} \tag{45}$$

with the constant A_3 and B_3 to be determined.

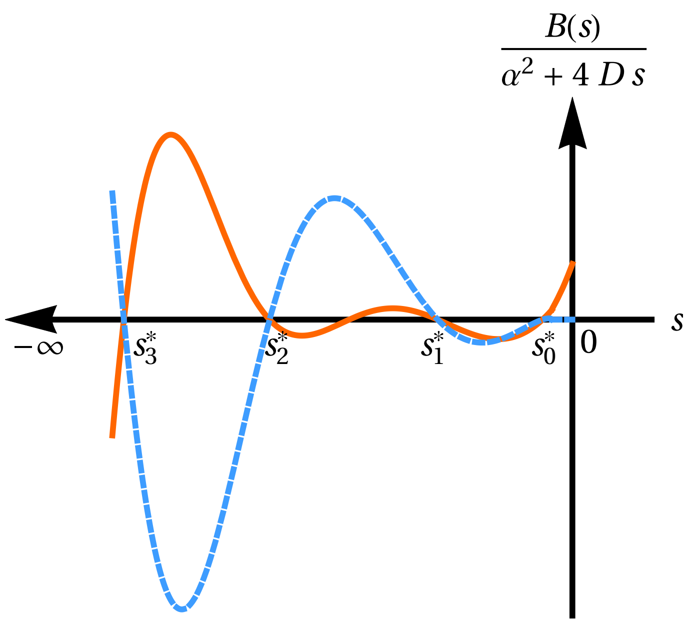

FIG. 4: $B(s)/(\alpha^2+4Ds)$ as a function s for $s \le 0$, for certain parameter values $\alpha = D = \mu = 1$ and $b = 2$. The red solid line and the blue dashed line plot respectively the real part and the imaginary part of the function. The zeros of the function given by the points where both real and imaginary part become zero.

After determining the five constants by using the four matching conditions at $x_0 = -b$ and $x_0 = 0$, and the boundary condition at $x_0 = a$, for the starting position $x_0 = 0$ (taking for simplicity), we get

$$\tilde{S}_a(s|0) = \frac{1}{s}\left[1 - \tilde{F}_a(s|0)\right] \tag{46}$$

where the Laplace transform of the first-passage time distribution is given by

$$\tilde{F}_a(s|0) = \frac{p}{B(s)}\, e^{-(\alpha+p)a/(2D)}\, \chi(s), \tag{47}$$

with $p = \sqrt{\alpha^2 + 4Ds}$ and

$$B(s) = \left[p^2\left(e^{\frac{pb}{D}} + e^{-\frac{pa}{D}}\right) + 3\alpha^2\left(1 - e^{-\frac{pa}{D}}\right)\left(e^{\frac{pb}{D}} - 1\right) - \alpha p\left(4e^{\frac{pb}{D}} + 1 - e^{\frac{pb}{D}}e^{-\frac{pa}{D}} - 4e^{-\frac{pa}{D}}\right)\right] D_{-\frac{s}{\mu}}\left(b\sqrt{\frac{\mu}{D}}\right)$$
$$- 2D\left[\alpha\left(1 - e^{-\frac{ap}{D}}\right)\left(e^{\frac{bp}{D}} - 1\right) + p\left(e^{-\frac{pa}{D}} - e^{\frac{pb}{D}}\right)\right]\sqrt{\frac{\mu}{D}}\, D_{-\frac{s}{\mu}+1}\left(b\sqrt{\frac{\mu}{D}}\right), \tag{48}$$

$$\chi(s) = \left[p + 3\alpha + (p - 3\alpha)e^{\frac{bp}{D}}\right] D_{-\frac{s}{\mu}}\left(b\sqrt{\frac{\mu}{D}}\right) + 2D\left(e^{\frac{pb}{D}} - 1\right)\sqrt{\frac{\mu}{D}}\, D_{-\frac{s}{\mu}+1}\left(b\sqrt{\frac{\mu}{D}}\right). \tag{49}$$

After substituting, $s = (p^2 - \alpha^2)/(4D)$, it is easy to check that $\tilde{F}_a(s(p)|0) \equiv \tilde{G}_a(p)$ is a symmetric function with respect to p, indicating that series expansion of $\tilde{G}_a(p)$ around $p = 0$ contains only even powers of p. Therefore, $p = 0$, and equivalently $s = -\alpha^2/(4D)$, is not a branch-point. Similarly, it is easy to check that $s = -\alpha^2/(4D)$ is not a pole of $\tilde{F}_a(s|0)$, i.e., $p = 0$ is not a pole of $\tilde{G}_a(p)$.

Although $s = -\alpha^2/(4D)$ is a zero of $B(s)$, it cancels with the numerator as both the numerator and the denominator tend to zero as $\alpha^2 + 4Ds$ near $s = -\alpha^2/(4D)$. Therefore, to find the poles of $\tilde{S}_a(s|0)$ one should look at the zeros of $B(s)/(\alpha^2 + 4Ds)$. This function has infinite number of zeros on the negative s axis [see Fig. 4], denoted by $-\infty < \cdots < s_2^*(a) < s_1^*(a) < s_0^*(a) < 0$. Therefore, inverting the Laplace transform, the survival probability is given by

$$S_a(t|0) = \sum_{i=0}^{\infty} R_i(a)\, e^{s_i^*(a)t}, \tag{50}$$

where

$$R_i(a) = \lim_{s\to s_i^*}(s - s_i^*)\,\tilde{S}_a(s|0) = \frac{\sqrt{\alpha^2 + 4Ds_i^*(a)}\;\chi(s_i^*(a))}{\left[-s_i^*(a)\right] B'(s_i^*(a))}\,\exp\left(-\frac{a}{2D}\left[\alpha + \sqrt{\alpha^2 + 4Ds_i^*(a)}\right]\right). \tag{51}$$

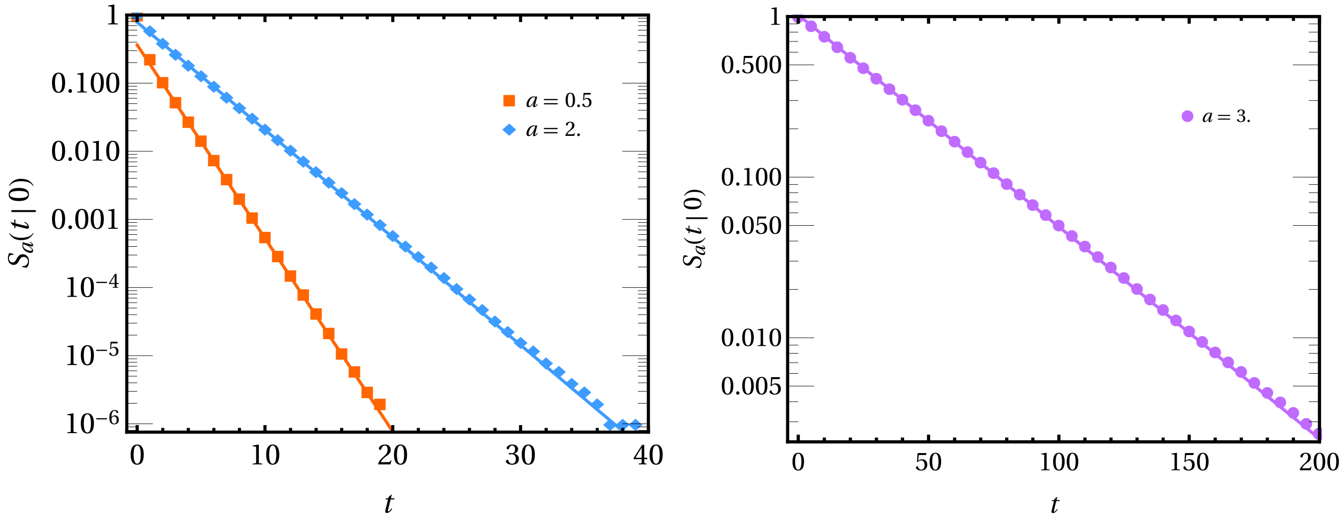

FIG. 5: Survival probability of a Brownian particle in a potential given by Eq. (37), starting at the position $x_0 = 0$ and in the presence of an absorbing barrier at $x = a > 0$, for $a = 0.5$ and 2.0 on the left panel and $a = 3.0$ om the right panel. We set $D = \alpha = \mu = 1$ and $b = 2$. The points are from numerical simulations whereas the solid lines plot the leading order theoretical expression given by Eq. (52).

Evidently, the leading behavior of the survival probability at large times is given by

$$S_a(t|0) = R_0(a)\, e^{-\theta(a)\,t} + O(e^{s_1^*(a)t}), \tag{52}$$

where $\theta(a) = -s_0^*(a)$. We compare this expression with numerical simulation in Fig. 5 and find very good agreement.

IV. THE DETAILS OF COMPUTING THE SURVIVAL PROBABILITY FOR A POTENTIAL THAT GROWS SLOWER THAN $|x|$ AS $x \to -\infty$

The example of the potential considered in the main text is

$$U(x) = \begin{cases} c\ln(-x/\lambda) & \text{for } x < -b, \\ \alpha|x| & \text{for } -b < x < a, \end{cases} \tag{53}$$

where $\lambda > 0$ sets a length scale, $b > 0$, and for the sake of continuity of the potential at $x = -b$, we set $c\ln(b/\lambda) = \alpha b$.

We have to solve Eq. (36) separately in the three regions, (i) $x_0 < -b$, (ii) $-b < x_0 < 0$, and (iii) $0 < x_0 < a$ and then match the solutions for the continuity of the solutions as well as the derivatives at both $x_0 = -b$ and $x_0 = 0$. In the region $x < -b$, Eq. (36) reads

$$D\tilde{q}''(x_0) - (c/x_0)\,\tilde{q}'(x_0) - s\tilde{q}(x_0) = 0. \tag{54}$$

The solution of this equation, that tends to zero as $x \to -\infty$, is given by

$$\tilde{q}(x_0) = A_1|x_0|^{\nu} K_\nu\left(|x_0|\sqrt{s/D}\right), \tag{55}$$

where A_1 is a constant to be determined, $\nu = (c/D+1)/2$, and $K_\nu(z)$ is the modified Bessel function of the second kind.

In the region $-b < x_0 < 0$, Eq. (36) reads

$$D\tilde{q}''(x_0) + \alpha\tilde{q}'(x_0) - s\tilde{q}(x_0) = 0. \tag{56}$$

The general solution is given by

$$\tilde{q}(x_0) = A_2\, e^{(p-\alpha)x_0/(2D)} + B_2 e^{-(p+\alpha)x_0/(2D)}, \tag{57}$$

with $p = \sqrt{\alpha^2 + 4Ds}$ and the constants A_2 and B_2 to be determined.

Finally, for $0 < x_0 < a$, Eq. (36) reads

$$D\tilde{q}''(x_0) - \alpha\tilde{q}'(x_0) - s\tilde{q}(x_0) = 0, \tag{58}$$

whose general solution is given by

$$\tilde{q}(x_0) = A_3\, e^{(\alpha+p)x_0/(2D)} + B_3 e^{(\alpha-p)x_0/(2D)} \tag{59}$$

with the constant A_3 and B_3 to be determined.

After determining the five constants by using the four matching conditions at $x_0 = -b$ and $x_0 = 0$, and the boundary condition at $x_0 = a$, for the starting position $x_0 = 0$ (taking for simplicity), we get

$$\tilde{S}_a(s|0) = \frac{1}{s}\left[1 - \tilde{F}_a(s|0)\right] \tag{60}$$

where the Laplace transform of the first-passage time distribution is given by

$$\tilde{F}_a(s|0) = \frac{p}{B(s)}\, e^{-(\alpha+p)a/(2D)}\, \chi(s), \tag{61}$$

with $p = \sqrt{\alpha^2 + 4Ds}$ and

$$\begin{aligned} B(s) = &\left[p^2 e^{pb/D} - \left(\alpha - (\alpha+p)e^{-pa/D}\right)\left(\alpha\left(e^{pb/D} - 1\right) + p\right)\right] K_\nu\left(b\sqrt{\frac{s}{D}}\right) \\ &+ 2D\left[p\left(e^{pb/D} - e^{-pa/D}\right) - \alpha\left(1 - e^{-pa/D}\right)\left(e^{pb/D} - 1\right)\right]\sqrt{\frac{s}{D}}\, K_{\nu-1}\left(b\sqrt{\frac{s}{D}}\right), \end{aligned} \tag{62}$$

$$\chi(s) = \left[p - \alpha + (\alpha+p)e^{pb/D}\right] K_\nu\left(b\sqrt{\frac{s}{D}}\right) + 2D\left(e^{pb/D} - 1\right)\sqrt{\frac{s}{D}}\, K_{\nu-1}\left(b\sqrt{\frac{s}{D}}\right). \tag{63}$$

Note that in the limit $a \to \infty$, we get $\tilde{F}_a(s|0) \to 0$, and consequently, $\tilde{S}_a(s|0) \to 1/s$. This gives, $\lim_{a\to\infty} S(t|0) = 1$ at all times, as expected. For any finite a, after substituting, $s = (p^2 - \alpha^2)/(4D)$, it is easy to check that $\tilde{F}_a(s|0)$ is a symmetric function with respect to p, indicating that series expansion of $\tilde{F}_a$ around $p = 0$ contains only even powers of p. Therefore, $p = 0$, and equivalently $s = -\alpha^2/(4D)$, is not a branch-point. For several sets of representative values of the parameters, we have numerically checked that $\tilde{F}_a(s|0)$ does not have any pole, indicating a non-exponential late time behaviour for $F_a(t|0)$ as well as for $S_a(t|0)$. Anticipating such non-exponential behaviour to show up as branch-point singularities, we analyse $\tilde{F}_a(s|0)$ near $s = 0$. In our case, we choose the potential $U(x)$ to be confining so that there is an equilibrium state (for $a \to \infty$) given by the Boltzmann distribution $P(x, t \to \infty) \propto e^{-U(x)/D}$ that is integrable, i.e., $\int e^{-U(x)/D}\, dx$ is finite. This requires $c/D > 1$, and consequently, $\nu = (1 + c/D)/2 > 1$.

Using the leading behaviour of the modified Bessel function of the second kind near $s = 0$, for a given non-integer $\nu \in (n, n+1)$ with $n \geq 1$ being an integer, we get

$$K_\nu\left(b\sqrt{\frac{s}{D}}\right) = s^{-\nu/2}\left[\sum_{m=0}^{n} \alpha_m\, s^m + \alpha_\nu\, s^\nu + O\left(s^{n+1}\right)\right], \tag{64}$$

$$\sqrt{\frac{s}{D}}\, K_{\nu-1}\left(b\sqrt{\frac{s}{D}}\right) = s^{-\nu/2}\left[\sum_{m=1}^{n} \beta_m\, s^m + \beta_\nu\, s^\nu + O\left(s^{n+1}\right)\right], \tag{65}$$

where the coefficients in the above series can be obtained explicitly. In particular,

$$\alpha_0 = 2^{\nu-1} b^{-\nu} D^{\nu/2}\Gamma(\nu), \quad \alpha_\nu = 2^{-\nu-1} b^\nu D^{-\frac{\nu}{2}}\Gamma(-\nu), \tag{66}$$

$$\beta_1 = 2^{\nu-2} b^{1-\nu} D^{\frac{\nu}{2}-1}\Gamma(\nu-1), \quad \text{and} \quad \beta_\nu = 2^{-\nu} b^{\nu-1} D^{-\frac{\nu}{2}}\Gamma(1-\nu). \tag{67}$$

The remaining factors in both $B(s)$ and $\chi(s)$ are analytic about $s = 0$. Therefore, for a given (finite and fixed) value of a, $\tilde{F}_a(s|0)$ can be expressed as

$$\tilde{F}_a(s|0) = \frac{\left[\sum_{j=0}^\infty a_j s^j\right]\left[\sum_{m=0}^n \alpha_m\, s^m + \alpha_\nu\, s^\nu + O\left(s^{n+1}\right)\right] + \left[\sum_{j=0}^\infty b_j s^j\right]\left[\sum_{m=1}^n \beta_m\, s^m + \beta_\nu\, s^\nu + O\left(s^{n+1}\right)\right]}{\left[\sum_{j=0}^\infty c_j s^j\right]\left[\sum_{m=0}^n \alpha_m\, s^m + \alpha_\nu\, s^\nu + O\left(s^{n+1}\right)\right] + \left[\sum_{j=0}^\infty d_j s^j\right]\left[\sum_{m=1}^n \beta_m\, s^m + \beta_\nu\, s^\nu + O\left(s^{n+1}\right)\right]}, \tag{68}$$

where the coefficients $\{a_j, b_j, c_j, d_j\}$ can be computed. In particular, the first few coefficients are given by

$$a_0 = c_0 = 2\alpha^2 e^{-\frac{\alpha a}{D}} e^{\frac{\alpha b}{D}}, \quad a_1 = 2e^{-\frac{a\alpha}{D}}\left[e^{\frac{\alpha b}{D}}(-a\alpha+2\alpha b+3D)+D\right], \quad b_0 = 2\alpha D e^{-\frac{a\alpha}{D}}\left(e^{\frac{\alpha b}{D}}-1\right), \tag{69}$$

$$c_1 = e^{-\frac{a\alpha}{D}}\left[2D\left(2e^{\frac{\alpha(a+b)}{D}}-e^{\frac{a\alpha}{D}}+e^{\frac{\alpha b}{D}}+2\right)-4\alpha(a-b)e^{\frac{\alpha b}{D}}\right], \quad d_0 = 2\alpha D e^{-\frac{a\alpha}{D}}\left(e^{\frac{a\alpha}{D}}+e^{\frac{\alpha b}{D}}-2\right). \tag{70}$$

Equation (68) can be expanded in a series as

$$\tilde{F}_a(s|0) = \sum_{m=0}^{n}\frac{(-1)^m}{m!}T_{a,m}(0)\,s^m + \nu\,\Gamma(-\nu)\,a_\nu\,s^\nu + o(s^\nu), \tag{71}$$

where $T_{a,m}(0) = \int_0^\infty t^m F_a(t|0)\,dt$ is the m-th moment of the first-passage time distribution. In particular, we have $T_{a,0}(0) = a_0/c_0 = 1$, as expected from normalization of the first-passage time distribution. The mean first-passage time is given by

$$T_a(0) \equiv T_{a,1}(0) = \frac{1}{a_0\alpha_0}\left[\alpha_0(c_1-a_1)+\beta_1(d_0-b_0)\right], \tag{72}$$

which, after explicit evaluation, agrees with Eq. (88) obtained from the general expression Eq. (85).

The coefficient of the most dominant non-analytic term is given by

$$\nu\,\Gamma(-\nu)\,a_\nu = \frac{\beta_\nu(b_0-d_0)}{a_0\alpha_0}, \tag{73}$$

which after evaluation (while using $c\ln(b/\lambda) = \alpha b$) gives

$$a_\nu = \left(\frac{2D}{\alpha\lambda}\right)\left(e^{\frac{\alpha a}{D}}-1\right)\frac{1}{\Gamma(\nu)}\left(\frac{\lambda^2}{4D}\right)^{\nu}. \tag{74}$$

Using the relation

$$\int_0^\infty dz\left[e^{-z}-\sum_{m=0}^{n}\frac{(-z)^m}{m!}\right]z^{-(\nu+1)} = \Gamma(-\nu) \quad \text{for } n<\nu<n+1, \tag{75}$$

we find that the late time behavior of the first-passage time distribution is given by the power-law distribution

$$F_a(t|0) = \nu\,a_\nu\,t^{-(\nu+1)} + o(t^{-(\nu+1)}). \tag{76}$$

Consequently, the survival probability has the asymptotic power-law decay

$$S_a(t|0) = \int_t^\infty F_a(t'|0)\,dt' = a_\nu\,t^{-\nu} + o(t^{-\nu}). \tag{77}$$

For integer values of $\nu = n$, the Bessel's functions have $\ln s$ singularities, which in turn gives rise to $\ln s$ singularities in $\tilde{F}_a(s|0)$. Consequently, the first-passage time distribution and survival probability have power-low decays accompanied by $\ln t$ corrections.

V. THE MEAN FIRST-PASSAGE TIME

The mean first-passage time $T_a(x_0)$ to a position a, starting with the position $x_0 < a$ can be computed exactly for arbitrary confining potential $U(x)$ [1]. Here, for convenience, we reproduce this proof. The mean first-passage time is defined by

$$T_a(x_0) = \int_0^\infty t\,F_a(t|x_0)\,dt. \tag{78}$$

Using $F_a(t|x_0) = -\partial_t S_a(t|x_0)$ in the above integral and then integrating by parts using the boundary conditions $S_a(t=0|x_0) = 1$ and $S_a(t\to\infty|x_0) = 0$, we get

$$T_a(x_0) = \int_0^\infty S_a(t|x_0)\,dt \equiv \tilde{S}_a(s=0|x_0). \tag{79}$$

Therefore, setting $s=0$ in Eq. (19), we get the differential equation

$$D\frac{d^2T_a}{dx_0^2}-U'(x_0)\frac{dT_a}{dx_0}=-1. \tag{80}$$

Using $W(x_0)=T_a'(x_0)$, we get

$$\frac{dW}{dx_0}-\frac{U'(x_0)}{D}W(x_0)=-\frac{1}{D}. \tag{81}$$

Multiplying both sides of the above equation by $e^{-U(x_0)/D}$ we get

$$\frac{d}{dx_0}\left[W(x_0)\,e^{-U(x_0)/D}\right]=-\frac{1}{D}\,e^{-U(x_0)/D}, \tag{82}$$

which can be integrated to

$$W(x_0)e^{-U(x_0)/D}=-\frac{1}{D}\int_{-\infty}^{x_0}e^{-U(z)/D}\,dz, \tag{83}$$

where we have used the boundary condition $\lim_{x_0\to-\infty}W(x_0)e^{-U(x_0)/D}=0$. Therefore,

$$\frac{dT_a}{dx_0}=-\frac{1}{D}e^{U(x_0)/D}\int_{-\infty}^{x_0}e^{-U(z)/D}\,dz. \tag{84}$$

Integrating the above equation from x_0 to a and using the boundary condition $T_a(x_0=a)=0$, we get

$$T_a(x_0)=\frac{1}{D}\int_{x_0}^{a}dy\,e^{U(y)/D}\int_{-\infty}^{y}e^{-U(z)/D}\,dz. \tag{85}$$

For $U(x)=\alpha|x|$, after performing the integrals (for $x_0=0$) we get,

$$T_a(0)=\frac{2D}{\alpha^2}\left(e^{\alpha a/D}-1\right)-\frac{a}{\alpha}. \tag{86}$$

For the potential given by Eq. (37), we get

$$T_a(0)=\frac{D}{\alpha^2}\left(e^{\alpha a/D}-1\right)\left[2-e^{-\alpha b/D}+\sqrt{\frac{\pi}{2}}\,\frac{\alpha}{\sqrt{\mu D}}\,\mathrm{erfc}\left(b\sqrt{\frac{\mu}{2D}}\right)\right]-\frac{a}{\alpha},\quad \text{with } \mu b=2\alpha. \tag{87}$$

On the other hand, for the potential given by Eq. (53), we get

$$T_a(0)=\frac{D}{\alpha^2}\left(e^{\alpha a/D}-1\right)\left[2-e^{-\alpha b/D}+\frac{\alpha b\,e^{-\alpha b/D}}{2D(\nu-1)}\right]-\frac{a}{\alpha}. \tag{88}$$

Note that, by taking the limit $b\to\infty$ in both Eq. (87) and Eq. (88), one gets back Eq. (86), as required.

VI. SURVIVAL PROBABILITY FOR A POTENTIAL GIVEN BY EQ. (89)

In the main text, we argued that the freezing transition of the decay rate $\theta(a)$ is robust, i.e, it occurs for any confining potential $U(x)$ that behaves asymptotically $U(x)\sim|x|$ as $x\to-\infty$. The actual value of a_c depends on the details of the potential $U(x)$, but the existence of the freezing transition does not depend on the details of $U(x)$ in the bulk as long as $U(x)\sim|x|$ when $x\to-\infty$. This argument was based on a general mapping to a quantum problem and using the properties of the Schrödinger equation in one dimension. We showed that the freezing transition coincides with the vanishing of the gap between the ground state (which is a bound state) and the continuum of scattering states when $a\to a_c$ from above. We provided an exactly solvable example of $U(x)$ in Eqs. (7a) and (7b) of the main text. In this appendix, we provide another example of $U(x)$ which differs from the preceding example in the bulk, but still behaves asymptotically $U(x)\sim|x|$ when $x\to-\infty$. We show below, analytically and numerically, that the freezing transition again occurs, supporting our claim of the robustness of this transition. We chose the following potential

$$U(x)=\begin{cases}\alpha|x| & \text{for } x<-b\\ \frac{1}{2}\mu x^2 & \text{for } x>-b\end{cases} \tag{89}$$

where $b > 0$ and $\mu = 2\alpha/b$ for continuity.

As before, the Laplace transform of the survival probability can be found by solving Eq. (19) piecewise for $x_0 < -b$ and $x_0 > -b$ and then matching the solutions at $x_0 = -b$ as well as using the boundary conditions at $x_0 = a$ and $x_0 \to -\infty$. Since we have already solved Eq. (19) for both linear and quadratic potentials in the examples above, we skip the details here. Moreover, for simplicity, we set $\alpha = 1$, $D = 1$, and $b = 1$. We find

$$\tilde{S}_a(s|0) = \frac{1}{s}\left[1 - \tilde{F}_a(s|0)\right] \tag{90}$$

where the Laplace transform of the first-passage time distribution is given by

$$\tilde{F}_a(s|0) = \frac{\sqrt{\pi} e^{-\frac{a^2}{2}} 2^{-\frac{s}{4}}}{\Gamma\left(\frac{s+2}{4}\right)} \frac{\chi(s)}{B(s)}, \tag{91}$$

where

$$\chi(s) = 2\sqrt{2} D_{1-\frac{s}{2}}\left(-\sqrt{2}\right) + 2\sqrt{2} D_{1-\frac{s}{2}}\left(\sqrt{2}\right) + \left(\sqrt{4s+1}+3\right)\left[D_{-\frac{s}{2}}\left(-\sqrt{2}\right) - D_{-\frac{s}{2}}\left(\sqrt{2}\right)\right], \tag{92}$$

$$\begin{aligned} B(s) = & \left[2\sqrt{2} D_{1-\frac{s}{2}}\left(\sqrt{2}\right) - \left(\sqrt{4s+1}+3\right) D_{-\frac{s}{2}}\left(\sqrt{2}\right)\right] D_{-\frac{s}{2}}\left(\sqrt{2}a\right) \\ & + 2\sqrt{2} D_{1-\frac{s}{2}}\left(-\sqrt{2}\right) D_{-\frac{s}{2}}\left(-\sqrt{2}a\right) + \left(\sqrt{4s+1}+3\right) D_{-\frac{s}{2}}\left(-\sqrt{2}\right) D_{-\frac{s}{2}}\left(-\sqrt{2}a\right). \end{aligned} \tag{93}$$

By analyzing $\tilde{F}_a(s|0)$ we find that $s = -1/4$ is a branch-point. Moreover, for $a > a_c \approx 1.06$, $\tilde{F}_a(s|0)$ has a pole at $s^*(a) \in (-1/4, 0)$. Therefore, as in Sec. I, here also the survival probability behaves as $S_a(t|0) \sim e^{-\theta(a)t}$ with $\theta(a) = -s^*(a)$, for $a > a_c$ and $S_a(t|0) \sim t^{-3/2} e^{-\theta(a)t}$ with $\theta(a) = 1/4$, independent of a, for $a < a_c$.

We verify the theoretical predication using numerical simulation. In the presence of an absorbing barrier at $x = a$, using the Langevin dynamics, we compute the survival probability of a Brownian particle starting from the origin, for various values of $a > 0$. The results are shown in Fig. 6, where we find that for larger values of a, the survival probability behaves as $S_a(t|0) \sim e^{-\theta(a)t}$ with a monotonically decreasing $\theta(a)$ as a function of a [see Fig. 6 (b)]. On the other hand, for smaller values of a, the survival probability behaves as $S_a(t|0) \sim t^{-3/2} e^{-\theta(a)t}$ with $\theta(a) = 1/4$, independent of a [see Fig. 6 (a)]. The numerically estimated values of $\theta(a)$ together with the theoretically calculated values are shown in Fig. 6 (c), which again shows the freezing transition similar to the one reported in the main text of the Letter.

VII. SURVIVAL PROBABILITY FOR A POTENTIAL GIVEN BY EQ. (94)

In the main text, using a mapping to a quantum problem, we argued that if the confining potential $U(x)$ increases slower than $|x|$ as $x \to -\infty$, then $\theta(a) = 0$ for all a, indicating a slower than exponential decay with time, of the first-passage/survival probability. We have analytically demonstrated that for $U(x) \sim \ln(-x)$ as $x \to -\infty$, both the first-passage and the survival probability exhibit power law decay with time [see Sec. IV]. Here we consider another case

$$U(x) = \begin{cases} 2c\sqrt{-x} & \text{for } x < -b \\ \alpha|x| & \text{for } x > -b \end{cases} \tag{94}$$

where $b > 0$, and for the sake of continuity of the potential at $x = -b$, we set $c = \alpha\sqrt{b}/2$. Using numerical simulation we show [see Fig. 7] that the survival probability again has a slower than an exponential decay, namely, $S_a(t|0) \sim e^{-r\sqrt{t}}$.

[1] H. Risken, *The Fokker Planck Equation* (Springer, Berlin, New York, 1984).
[2] D. Hartich and A. Godec, New J. Phys. **20**, 112002 (2018).
[3] D. Hartich and A. Godec, J. Stat. Mech. 024002 (2019).

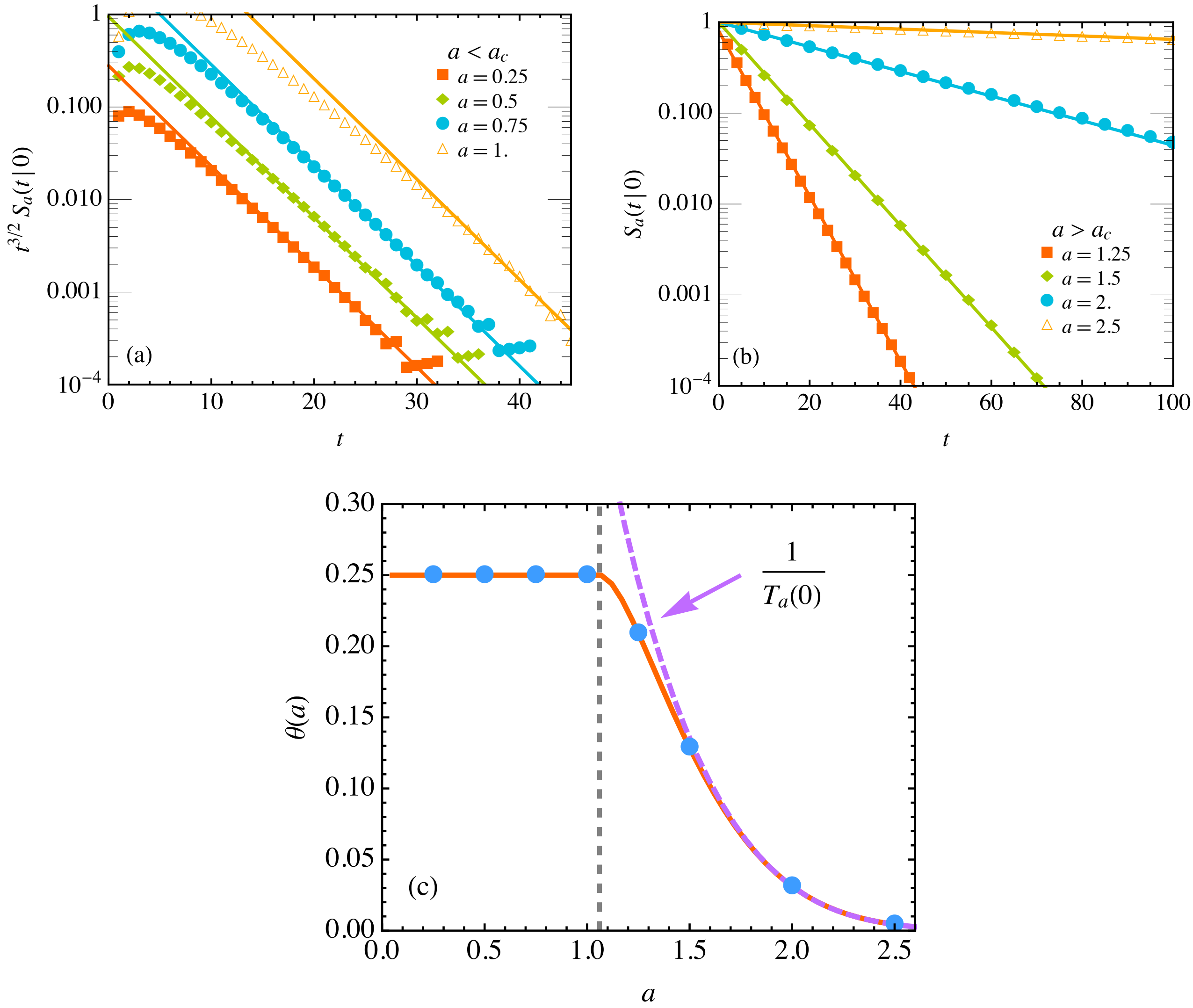

FIG. 6: (a) and (b): Survival probability of a Brownian particle in a potential given by Eq. (89) starting at the position $x_0 = 0$ and in the presence of an absorbing barrier at $x = a > 0$. We set $D = 1$, $\alpha = 1$, and $b = 1$. The points are from numerical simulations whereas the solid lines are fit to the exponential function $\propto e^{-\theta(a)t}$ where the prefactor to the exponential and the exponent $\theta(a)$ are chosen to match the simulation points. (c) The estimated values of $\theta(a)$ as a function of a are shown by points, together with the theoretically computed $\theta(a)$ by the solid (red) line. The dashed line shows the inverse of the mean first-passage time calculated exactly from Eq. (85). The vertical dashed line marks $a_c \approx 1.06$.

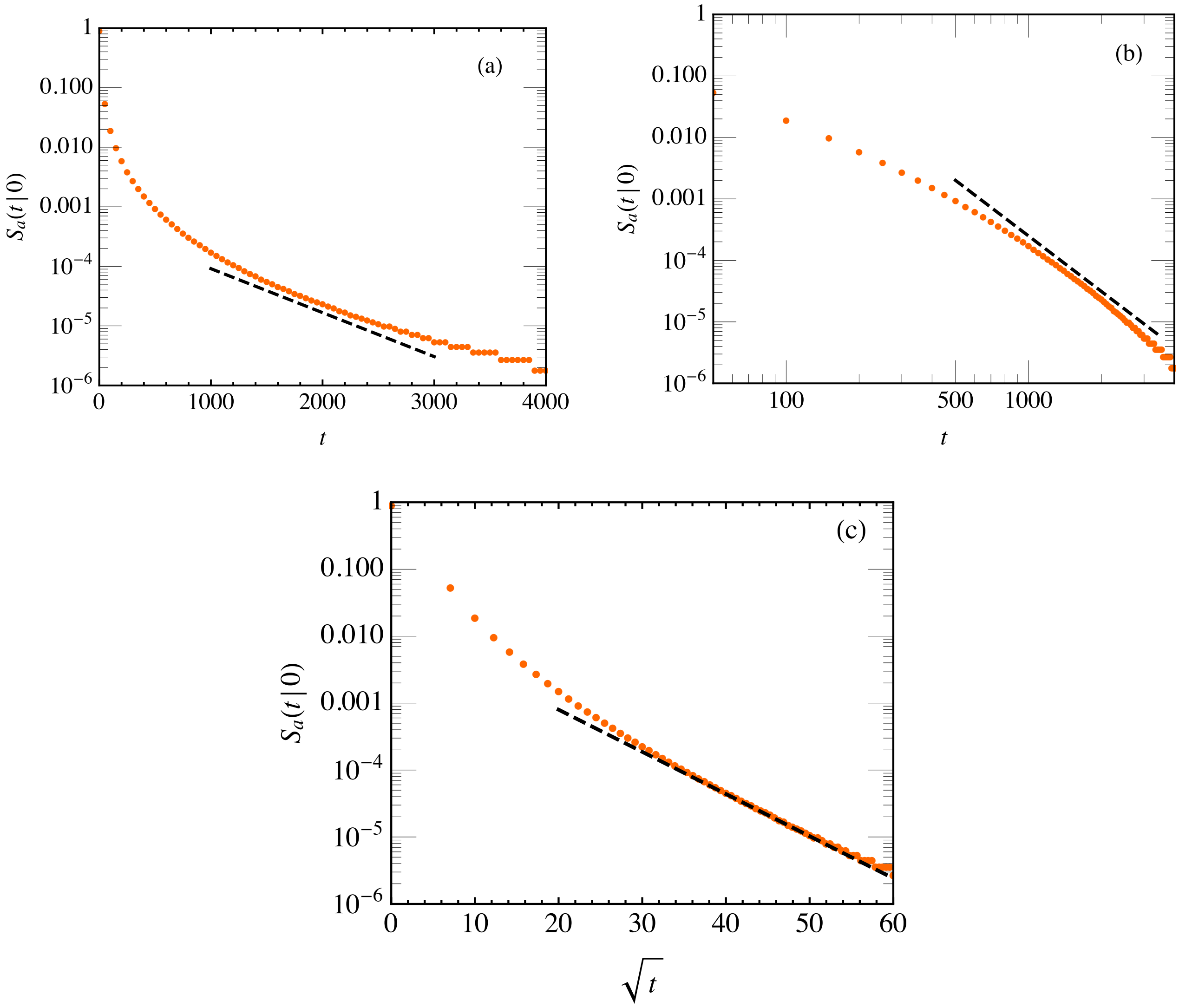

FIG. 7: Survival probability of a Brownian particle in a potential given by Eq. (94) starting at the position $x_0 = 0$ and in the presence of an absorbing barrier at $x = a > 0$. We set $D = 1$, $\alpha = 1$, $b = 1$, and $c = 1/2$. The points in (a)-(c) are the same numerical simulation data for the survival probability $S_a(t|0)$ presented in different scales whereas the dashed lines represent different functions. (a) The dashed line indicates that $S_a(t|0)$ has a slower than an exponential $\propto e^{-\theta t}$ decay, where θ and the proportionality constants are fitting parameters. (b) The dashed line indicates that $S_a(t|0)$ has a faster than a power-law $\propto t^{-3}$ decay. (c) Semi-log plot $S_a(t|0)$ as a function of $\sqrt{t}$. The dashed line demonstrates a stretched exponential decay $S_a(t|0) \sim e^{-r\sqrt{t}}$, where r is a fitting parameter.