Homfly, the work of Edward Witten and quantum groups

Ravi Kulkarni

Recent developments in mathematical physics have generated much excitement amongst workers in this field. Much of this stems from the work of Edward Witten of the Institute for Advanced Studies in Princeton. What is striking about these results is that it relates areas that appear entirely distinct from one another. On the one hand theoretical physicists study what they call 'exactly solvable models', on the other, algebraists have, for some time been studying rather esoteric objects called 'quantum groups' and surprisingly, the algebraists' work provides a complete solution to the physicists' problems! Even more dramatic was Witten's demonstration that the theory of knots (yes, ordinary everyday knots) is closely linked to quantum field theory. And to complete the circle, it is now becoming clear that quantum groups have close connections with knot theory! In this article I have attempted to explain some of the main ideas involved.

Guide for the reader

One of the main ingredients in this pot-pourri of ideas is the theory of knots (and something called Homfly . . . read on!). This theory is really very elementary and section two below provides a concise introduction to the subject. This section should be considered as being the major part of this review. A careful reading (of this section) will enable the reader to calculate the Homfly or the Jones polynomial of any knot. The reader is urged to work her way through the calculations—they are easy, fun to do and require nothing more than the ability to draw simple pictures and multiply polynomials. This section can be read independently of the other two (which need not be read at all).

The first section provides a quick look at Witten's basic discovery of a link between Chern–Simons field theory and the knot invariants. The description is mainly qualitative.

The third section is on quantum groups and presupposes some algebraic sophistication—not for the lay reader! This section shows why the theory of quantum groups (to be more precise, the representation theory of quasi-triangular Hopf algebras) is of relevance to statistical mechanics (in low dimensions).

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1. Chern–Simons field theory and knot invariants

The aim of this section is to give a qualitative description of Witten's discovery of a deep relationship between what physicists call Chern–Simons field theory and the theory of knots. First a quick look at CS field theory.

A field theory is prescribed once an action functional is chosen. For some time now physicists have been interested in a particular action functional (in three dimensions) which does not require the choice of a metric in the space. (The standard Yang–Mills action does require a metric.) The CS action functional is

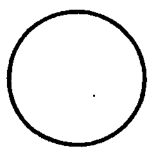
$$S=\int \mathrm{Tr}\left(A\wedge \mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\right),$$

where A is a connectior for a gauge group G. Such a theory is completely topological in nature—there are no dynamical degrees of freedom, but there are topological degrées of freedom and the theory therefore is not entirely vacuous. A natural choice of observables in this theory are the so called Wilson lines. Given an irreducible representation R of G, the Wilson line (or loop) is associated to any circle C in this space and is defined to be

$$W_R(C)=\mathrm{Tr}_R\, P\exp\int_c A\cdot \mathrm{d}x.$$

Observe that no metric is needed to define this bservable and therefore $W(C)$ should be a topological nvariant.

Knot theory makes an appearance because a circle C an be embedded in a three-dimensional space in a ariety of ways:



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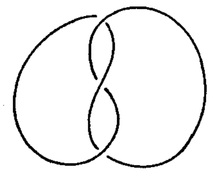


Also, one is interested in expectation values of products

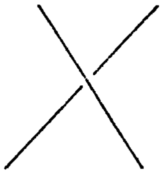
$$< W(L) > = < W(C_1) \ldots W(C_n) >,$$

where $C_1, \ldots C_n$ are circles in the space—which could be linked in complicated ways. Mathematicians have for a long time been interested in distinguishing one knot or link from another and have invented various link invariants. A link invariant is an object (a polynomial say) associated to a knot such that deforming the knot does not change this invariant (and thus the name). But the Wilson line $W(C)$ has precisely this property! Since $W(C)$ does not depend on a metric, deforming the loop (knot) should not change $W(C)$. It was suspected therefore that there should be a relation between the Wilson line expectation values and link invariants discovered by mathematicians. A prime choice was an invariant called the Jones polynomial. Following a suggestion by Atiyah, Edward Witten showed that the coefficients of the various terms in a perturbative expansion of $W(C)$ are related to the coefficients of the Jones polynomial for the link C. The Jones polynomial, which had hitherto seemed a totally mysterious object admitted an 'explanation' in terms of quantum field theory! This was the result that surprised mathematicians and physicists alike and led to Witten's sharing of the Fields medal for this year.

The next section is an elementary exposition of knot theory and describes what the Jones polynomial is. Actually it is just as easy to describe a more general invariant called the Homfly and we now proceed to this.

2. Homfly and the Jones polynomial

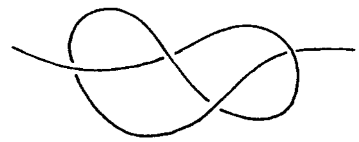
There is a basic convention followed in drawing pictures of knots on paper. The diagram below of two strings crossing



(2.1)

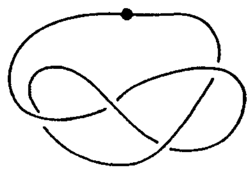
means that the string going from lower right to upper left passes *over* the other string. In short a broken line is an underpass, an unbroken line an overpass.

Ordinary everyday knots have the ends hanging loose as shown below



(2.2)

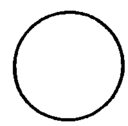
Since mathematicians do not like to leave any loose ends, they conventionally identify them:



(2.3)

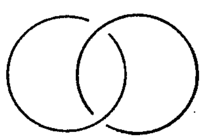
and all the knots we will talk about will have their loose ends identified thus, producing closed knots. With this convention, the unknot, that is an unknotted piece of string

becomes a circle:

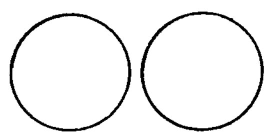


(2.4)

The advantage of this is obvious. An ordinary knot can be untied, but the 'knot' in (2.3) can never be made to look like (2.4) by any process [except cutting and rejoining of course, but this is not allowed!]. Actually we need to talk about objects more general than knots—they are called links. A knot is made from a single piece of string while links are constructed from several pieces (all loose ends being identified). The simplest link is:



or

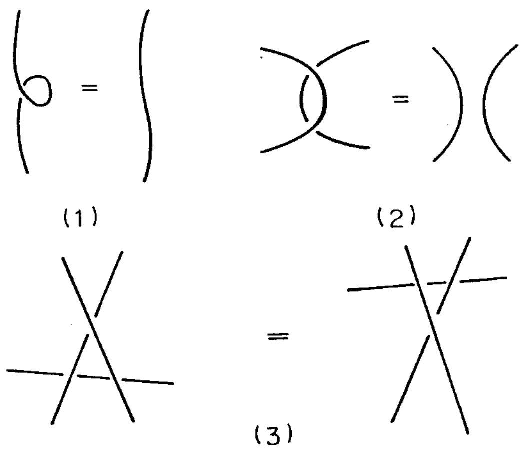


(2.5)

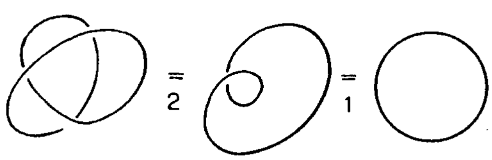
two unknots linked or unlinked. Each piece of string in a link is called a component of the link. A knot therefore is a single component link.

The Reidemeister moves

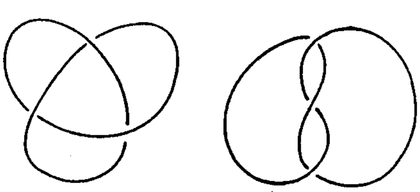
In untangling a knot or a link, one goes through a series of physical moves—trying to push one loop through another . . . we have all done this . . . . The mathematician Reidemeister showed that one really only needs three moves, these are called the Reidemeister moves. As the following pictures show they are obvious moves. In the sequel, the symbol = between two links will mean that one link can be physically deformed into the other.



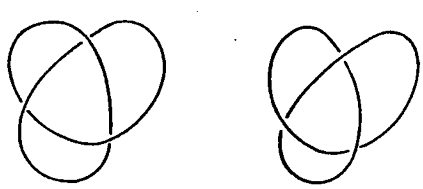
While it is obvious that these moves are valid, what Reidemeister did was to show that are *all* you need in trying to untangle a given link. Here is an example of the use of the Reidemeister moves.



Of course you may begin with a genuine knot and then no amount of deformation will ever unknot it! So, a central problem of knot theory is to be able to say when a given knot diagram represents the unknot, or whether two given knots can be deformed into one another. Can you deform these two knots into one another?



An even more interesting example is provided by these two knots

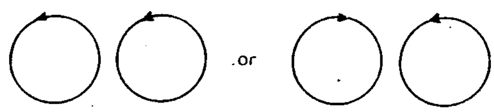


(2.6)

It appears as if they are the same knot, but they are not! Check that no deformation will change one to the other! These are called the right-handed and left-handed trefoil knots and they are mirror images of each other. Some technical terminology: What we have loosely been refering to as deformation—any process using the three Reidemeister moves—is called ambient isotopy by knot theorists. Thus the two knots above are not *ambient isotopic* to one another.

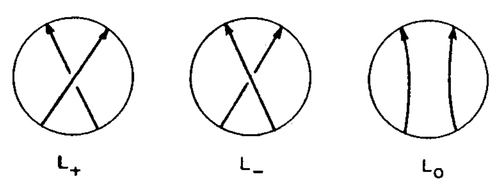
Knot theorists have solved these problems of distinguishing one knot from the other by associating a certain polynomial (called the Homfly) with each knot. This Homfly has the property that if the polynomials of two knots are different, then the knots are distinct and cannot be deformed into each other!

We need a technicality before we go any further. Each link comes with an orientation—this is a choice of direction along each component of the link, marked by an arrow. Thus the second link in (2.5) for instance can have two orientations



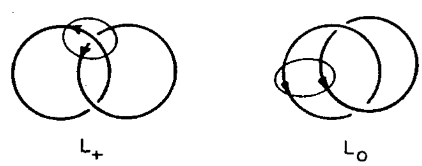
and as oriented links, these must be considered distinct.

In order to understand what Homfly is, we must first understand the skein relations for a link. Every link can be given one of three names L_+, L_- or L_0. This is done as follows. Given a link, draw a little circle over the knot-diagram such that inside this little circle there are only two strings with one of the following crossings inside. (The little circle is marked by a thin line.)



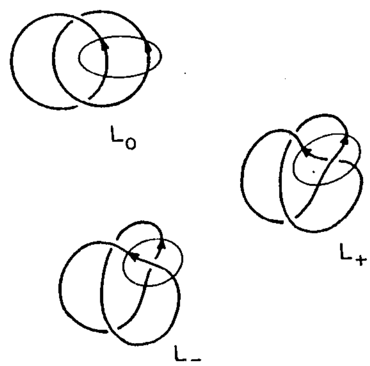
(2.7)

(Remember, each string comes with an orientation.) The link gets its name from one of these pictures. The name of the link therefore depends on where one chooses the little circle. As an example, here is a link named in two different ways.



(2.8)

But now, given any link we can construct two other links said to be skein-related to the first link. Name the first link using the procedure just outlined. Now draw two other links which are the same everywhere except inside the little circle and such that the strings inside the little circle are of the type L_+, L_- or L_0. If the given link is called L_0, the two other links are L_+ and L_-. An example to illustrate:



(2.9)

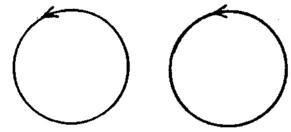
The reader will recognize L_+ as the trefoil and L_- as the unknot, and is urged to construct her own examples.

And we are now ready to understand what the Homfly is. The Homfly of a link is a polynomial in two variables t and z. However it is the kind of polynomial which can have negative powers of t and z (in other words it is a Laurent polynomial). Here is an example: $t^3z^2-2t^{-2}z^{-1}+t^{-1}z$. More important it has the property that its coefficients are always integers (so we are talking about an element in $\mathbb{Z}[t, t^{-1}, z, z^{-1}]$). Given a link, name it and construct the two skein-related links. We therefore have three links named L_+, L_- and L_0. The Homfly polynomial $P[L]$ of the first link L is defined by the following conditions:

(1) $tP[L_+]-t^{-1}P[L]=zP[L_1]$
(2) $P[\text{unknot}]=1$
(3) The Homfly is invariant under ambient isotopy.

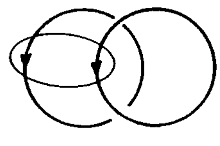
The reader should notice the following fact: If the given link is called L_+, we need to know $P[L_-]$ and $P[L_0]$ in order to calculate $P[L_+]$! But one can follow a recursive procedure to simplify a given link. At this point the reader may wonder exactly what it is that this polynomial signifies. I should make it clear that the polynomial is just a representative of the link in the sense I spoke about earlier: if the polynomials of two links are different then the links are different. This is characteristic of the method of topology: in order to distinguish between geometric objects, one associates certain algebraic objects to it (vector spaces, groups, etc.). If the algebraic objects thus associated are different, then so are the geometrical objects.

But now let us see how to calculate the Homfly polynomial of a link—let us try the trefoil knot. We begin with a simpler calculation however. Let us first find the Homfly of two unlinked unknots.



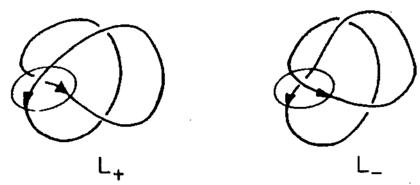
(2.10)

By the second Redemeister move, this is the same as



(2.11)

and with the choice above, this link is L_0. The skein-related links are



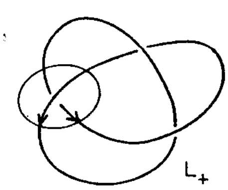
and are both unknots. The Homfly relation (1) now tells us what $P[L_0]$ is

$$tP[L_+]-t^{-1}P[L_-]=zP[L_0].$$

Therefore $t-t^{-1}=zP[L_0]$ and $P[L_0]=tz^{-1}-t^{-1}z^{-1}$. (2.12)

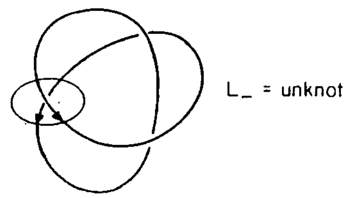
Notice that we used all the three defining relations of the Homfly polynomial: invariance under ambient isotopy in going from (2.9) to (2.10) and the Homfly relations (1) and (2) in obtaining (2.12). We now know the Homfly polynomial of the link (2.10).

Now let us try the trefoil.

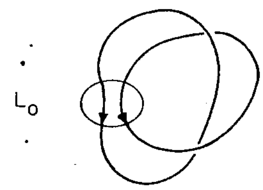


(2.13)

The two skein-related links are



(2.14)



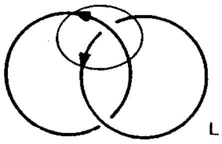
(2.15)

So if we know the Homfly of the link (2.15) we know that of the trefoil because

$$tP[L_+]-t^{-1}P[L_-]=zP[L_0] \qquad (2.16)$$

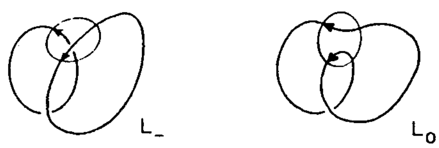
$$\therefore \quad tP[\text{trefoil}]-t^{-1}=z\cdot P[\text{the link 2.15}].$$

So let us find the Homfly of the link (2.15). Calling it L_0 would clearly be useless because that would give the trefoil as L_+. But there is an alternative:



(2.15a)

The two-skein related knots now are



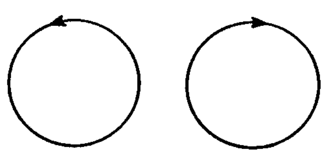
(2.17)

L will be recognized as the link (2.9) whose Homfly we know: equation (2.12)! And we are through:

$$P[\text{link 2.15}]=P[\text{link 2.15a}]$$
$$=(1/t)\,[zP[\text{unknot}]+t^{-1}P[\text{link 2.9}]]$$
$$=(1/t)[z+t^{-1}((t-t^{-1})/z)]. \qquad (2.18)$$

Putting this into (2.16) gives the Homfly of the trefoil.

At this point the reader is urged to calculate the Homfly of the other trefoil (2.6), and to check that the two Homfly polynomials are not the same. By the property of ambient isotopy invariance of the Homfly polynomial this is a proof that the two trefoils cannot be deformed into each other. Also, as another exercise, try this link



Some properties of the Homfly must now be stated. First of all, what the knot is named is immaterial—the Homfly remains the same (provided you can calculate it! With a bad choice of name you can end up with more complicated skein-related knots). Finally the Homfly must be used with care: it is possible for two distinct (in the sense of ambient isotopy) links to have the same Homfly. Thus the Homfly polynomial can say with certainty when two links are distinct, but cannot guarantee that two links are ambient isotopic.

The Jones polynomial which is a little older than the Homfly is a one variable polynomial and is obtained from the Homfly simply by setting $z=t^{1/2}-t^{-1/2}$. The Homfly is more general because it can sometimes distinguish between knots when the Jones polynomial cannot. These then are the polynomial invariants which were shown to be relevant to quantum field theory by Witten. For another approach to the Jones polynomial, see the article by V. S. Sunder, page 1285 this issue.

Braids

Braids are more general objects than links and at this point, the reader is urged to read about them from the first few sections of Sunder's article on Vaughan Jones. The braid relations quoted there

$$g_i g_j = g_j g_i, \; i \neq j \pm 1$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$$

will make a surprising comeback in the next section! Braids are closely related to links and the reader is referred to Sunder's article for details.

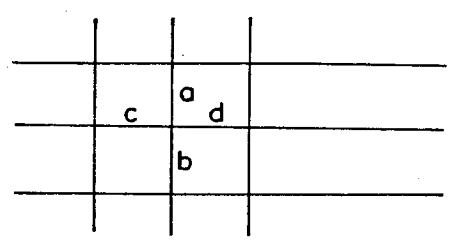
Before concluding, I should remind you that I have not shown you a proof of the existence of a polynomial satisfying the basic Homfly relations (1), (2) and (3); I have merely shown you how to compute it. Such a proof exists (the Homfly reference below) and I should end by telling you that the invariant Homfly is named after the six men who discovered it: Hoste, Ocneanu, Millett, Freyd, Lickorish and Yetter in 1985. A good reference for knot theory is Louis Kauffmann's book 'On Knots', *Annals of Mathematics Studies*, No. 115.

3. Quantum groups

This section is logically independent of the previous sections and is more technical in nature.

The motivation for the study of quantum groups comes from the study of the quantum inverse scattering problem and that of exactly solvable statistical models. Indeed, the two subjects are very closely related. There seem to be various approaches to the definition of quantum groups—in what follows I will provide a bird's eye view of the subject omitting details which can be found in the literature. We begin by looking at a typical statistical mechanical model in the plane.

Consider a two-dimensional square lattice where each link can be assigned different states.



(An early prototypes of this kind of model is the Ising model.) Let V denotes the vector space of states along the links.

Each vertex is assigned an energy $E(a,b,c,d)$ where a,b,c,d label the links determining the vertex and the total Hamiltonian is

$$\mathscr{H} = \sum_{\text{all vertices}} E(a,b,c,d).$$

The partition function $Z = \Sigma\, e^{-\beta H}$ and each vertex thus has a 'Boltzmann weight' of

$$R^{a\ b}_{c\ d} = \exp\,[\beta E\,(a,b,c,d)\,].$$

Observe that the upper indices label the vertical links and the lower label the horizontal links. This R-matrix as it is called, can be thought of as acting on $V \otimes V$ (vertical space $\otimes$ horizontal space) and plays a fundamental role in the theory. To see what this is, we need to define the monodromy matrix. First define $(t_{ab})_{cd} = R^{a\ b}_{c\ d}\cdot\, t_{ab}$ therefore acts on V. The monodromy matrix then is

$$T_{ab} := \sum_{a_i} t_{aa_1} \otimes t_{a_1a_2} \otimes \ \ldots \otimes t_{a_{n-1},b}.$$

The trace of the monodromy matrix, the so called row-to-row transfer matrix, is an important object because *its* trace is the partition function Z. So, diagonalizing the transfer matrices is equivalent to knowledge of the partition function. It turns out that this is possible if

$$R\cdot(T_1 \otimes T_2) = (T_2 \otimes T_1)\cdot R$$

where $T_1 := T \otimes 1$ and $T_2 := 1 \otimes T$. A consistency condition for the above equation is

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where $R_{ab} \in V \otimes V \otimes V$ and the indices a,b mean that R acts on the ath and bth factors of $V \otimes V \otimes V$. Either of the two equations above is referred to as the Yang–Baxter (YB) equation (or the quantum Yang–Baxter equation, the adjective quantum being used because precisely the same equation crops up in the quantum inverse scattering problem and to distinguish it from its linearized version which is called the classical Yang–Baxter equation). Thus an exactly solvable statistical mechanical model in the plane is determined once we find a matrix R satisfying the YB. One approach to the theory of quantum groups is through trying to formalize this structure—and the resulting structure is called a quasi-triangular Hopf algebra. The point is that if we know matrix representations of these Hopf algebras, then we will also have explicitly constructed R-matrices satisfying the YB. The theory of Hopf algebras therefore appears to be the natural language to study exactly solvable models in statistical mechanics and the quantum inverse scattering problem. Below, I give the definition of these algebras and provide examples.

A *Hopf algebra* is an algebra A with unit and three maps $\Delta: A \to A \otimes A$, $S: A \to A$ and $\varepsilon: A \to \mathbb{C}$ called respectively the comultiplication, the antipode and the counit. Δ and S are homomorphisms and ε is an anti-homomorphism. These maps are required to satisfy

$$(id \otimes \Delta)\,\Delta(a) = (\Delta \otimes id)\,\Delta(a)$$
$$(id \otimes S)\,\Delta(a) = (S \otimes id)\,\Delta(a) = \varepsilon(a)\,1$$
$$(\varepsilon \otimes id)\,\Delta(a) = (id \otimes \varepsilon)\,\Delta(a) = a.$$

(The reader will recognize that a Hopf algebra is nothing but a bialgebra equipped with an antipode map. It is best to think of these conditions in terms of diagrams.)

Example: Consider the group ring $k[G]$ of a finite group G. This has the structure of a Hopf algebra with $\Delta g = g \otimes g$, $\varepsilon g = 1$, $Sg = g^{-1}$ with $g \in G$ and extended by linearity to all of $K[G]$.

Example: The universal enveloping algebra U($\mathfrak{g}$) of a Lie algebra $\mathfrak{g}$ with

$$\Delta a = a \otimes 1 + 1 \otimes a,$$
$$\varepsilon a = 0,\ Sa = -a$$

for $a \in \mathfrak{g}$.

Now let $P: A \otimes A \to A \otimes A$ denote the permutation map $P(x \otimes y) = y \otimes x$. It is easy to check that if Δ is a comultiplication, then so is $P \circ \Delta$.

Finally, a quasi-triangular Hopf algebra is a Hopf algebra A, together with a matrix $R \in A \otimes A$ satisfying

$$P \circ \Delta(a) = R\Delta(a)\,R^{-1}$$

and

$$(id \otimes \Delta)\,R = R_{13}R_{12}$$
$$(\Delta \otimes id)\,R = R_{13}R_{23}$$
$$(S \otimes id)\,R = R^{-1}$$

where R_{ab} denotes the embedding of R in $A \otimes A \otimes A$, acting on the ath and bth factors. The relation to the exactly solvable models is now obvious—indeed the entire definition of quasi-triangularity was motivated by the physical problem.

The last set of conditions of R imply

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

the Yang–Baxter equation. Further, it is clear that if ρ is a representation of A then $(\rho \otimes \rho)\,R$ satisfies the YB. Thus classifying all representations of these Hopf algebras is equivalent to classifying all exactly solvable models! It has been shown that the classical Lie algebras lead naturally to examples of quasi-triangular Hopf algebras by a process of deformation—and these are the so-called quantum groups (so, a quantum group is not a group: it is a deformation of (the universal enveloping algebra of) a classical Lie algebra. Some authors also refer to a quasitriangular Hopf algebra as a quantum group).

Example: $U_q[sl(2)]$. This is the (non-commutative) algebra generated by $l, q^{H/2}$, X, X and $q^{-H/2}$. (The notation is standard: H is the generator of the Cartan subalgebra of $sl(2)$ and X_+ and X_- are the 'raising' and 'lowering' operators.) modulo the relations

$$q^{\pm H/2}q^{\mp H/2}=1$$
$$q^{H/2}X_{\pm}q^{-H/2}=q^{\pm 1}X_{\pm}$$
$$[X_+,X_-]=\frac{q^H-q^{-H}}{q-q^{-1}},$$

q is considered an indeterminate in the above algebra. Observe that as $q\to 1$. $U_q[sl(2)]$ reduces to $U[sl(2)]$. $U_q[sl(2)]$ as defined here is a quasitriangular Hopf algebra. The relevant maps are

$$\Delta q^{\pm H/2}=q^{\pm H/2}\otimes q^{\pm H/2}$$
$$\Delta X_{\pm}=X_{\pm}\otimes q^{H/2}+q^{-H/2}\otimes X_{\pm}$$
$$\varepsilon q^{\pm H/2}=1,\ \varepsilon X_{\pm}=0$$
$$SX_{\pm}=-q^{\pm 1}X_{\pm},\ Sq^{\pm H/2}=q^{\mp H/2}.$$

This shows that $U_q[sl(2)]$ is a Hopf algebra. Quasitriangularity is demonstrated by checking that the matrix given below is an R-matrix, i.e. it satisfies the YB.

$$R=q^{H\otimes H/2}\sum_{n=0}^{\infty}\left\{\frac{(1-q^{-2})^n}{[n]_q!}\left[q^{H/2}X_+\otimes q^{-H/2}X_-\right]^n q^{n(n-1)/2}\right\}.$$

We have used the 'quantum' notation:

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}},\ [n]_q!=[n]_q\ldots[1]_q.$$

This example of a solution to the YB was constructed by Drinfeld in 1985. (There are some topological niceties here which we have ignored.) For other examples and more information, the reader is referred to the article on Drinfeld by Vyjayanthi Chari and Dinesh Thakur (page 1297) appearing in this issue.

There are at least two other approaches to the subject of quantum groups. In the above example we were interested in explicitly determining a solution of YB. One can reverse this procedure and start with an R-matrix, i.e. a solution of the YB. The objective then would be to construct a quasitriangular Hopf algebra corresponding to this R-matrix. Thus, it is possible to construct Hopf algebras corresponding to the known exactly solvable models. When a Hopf algebra A is constructed this way, it frequently depends on a parameter q in such a way that as $q\to 1$, A reduces to the algebra of functions on a classical group. A is then referred to as a quantum group. The relation between this approach and the first is not yet clear for all the classical Lie algebras.

And finally there is the approach of Manin via noncommutative differential geometry. Let x and y denote coordinates in the plane, but subject to the relation $xy=q^{-1}yx$, i.e. x and y do not commute (q is a parameter). One can now look for linear transformations of the plane which preserve this relationship between the two coordinates. The resulting set of transformations has also been referred to as a quantum group. Again there is an R-matrix associated to this structure which satisfies the YB. As will be obvious from these remarks, much work is still being done in order to understand the interplay between models and the algebraic and geometric structure inherent in them.

We end by making contact with considerations of the previous section. Given an R-matrix define $\bar{R}=p\circ R$, p being the permutation map. If R acts on $V\otimes V$ define matrices $\bar{R}_i=1\otimes\ldots\otimes\bar{R}\otimes\ldots\otimes 1$, $\bar{R}$ sitting in the ith, $(i+1)$th slot. The YB then reduces to

$$\bar{R}_i\bar{R}_j=\bar{R}_j\bar{R}_i \text{ if } |i-j|>1$$
$$\bar{R}_{i+1}\bar{R}_i\bar{R}_{i+1}=\bar{R}_i\bar{R}_{i+1}\bar{R}_i$$

which the reader will recognize as being precisely the relations defining the braid group!

There are many more connections . . . Manin's approach seems to lead directly to the Jones polynomial, . . . there are applications in conformal field theory. . . . This plethora of ideas, all seemingly connected in some way, promises to lead us to deeper insight into our understanding of this world.

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