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N-term pairwise-correlation inequalities, steering, and joint measurability

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Chained inequalities involving pairwise correlations of qubit observables in the equatorial plane are constructed based on the positivity of a sequence of moment matrices. When a jointly measurable set of positive-operator-valued measures (POVMs) is employed in the first measurement of every pair of sequential measurements, the chained pairwise correlations do not violate the classical bound imposed by the moment matrix positivity. We find that incompatibility of the set of POVMs employed in first measurements is only necessary, but not sufficient, in general, for the violation of the inequality. On the other hand, there exists a one-to-one equivalence between the degree of incompatibility (which quantifies the joint measurability) of the equatorial qubit POVMs and the optimal violation of a nonlocal steering inequality, proposed by Jones and Wiseman [S. J. Jones and H. M. Wiseman, Phys. Rev. A **84**, 012110 (2011)]. To this end, we construct a local analog of this steering inequality in a single-qubit system and show that its violation is a mere reflection of measurement incompatibility of equatorial qubit POVMs, employed in first measurements in the sequential unsharp-sharp scheme.

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I. INTRODUCTION

Conceptual foundations of quantum theory deviate drastically from the classical world view. The prominent counterintuitive features pointing towards the quantum-classical divide are a subject of incessant debate ever since the birth of quantum theory. Pioneering works by Bell [1], Kochen and Specker [2], and Leggett and Garg [3] are significant in bringing forth the perplexing features arising within the quantum scenario, in terms of correlation inequalities, constrained to obey classical bounds. Violation of the inequalities sheds light on the the nonexistence of a joint probability distribution for the measurement outcomes of all the associated observables [4–6].

In fact, noncommutativity of the observables forbids assignment of joint sharp realities to their outcomes in projective-valued (PV) measurements. Subsequently, it is not possible to envisage a bona fide joint probability distribution for the outcomes of PV measurements of noncommuting observables. However, the generalized measurement framework [7] goes beyond the conventional PV measurement scenario, where positive-operator-valued measures (POVMs) are employed. Joint measurability (or compatibility) of a set of POVMs is possible even when they do not commute. To declare that a set of POVMs is jointly measurable there should exist a global POVM, the measurement statistics of which enables one to retrieve that of the set of compatible POVMs. Within the purview of generalized measurements, it is possible to assign fuzzy joint realities (and in turn a valid joint probability distribution) to the statistical outcomes of noncommuting observables when the corresponding POVMs are compatible.

In recent years there has been a surge of research activity dedicated to exploring the notion of measurement incompatibility and its connection with counterintuitive quantum notions such as nonlocality, contextuality, and nonmacrorealism [8–23]. In particular, it is known that measurement incompatibility plays a key role in bringing to the surface the violations of the so-called no-go theorems in the quantum world. Wolf *et al.* [11] proved that a set of two incompatible dichotomic POVMs is necessary and sufficient to violate the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [24]. However, this result may not hold, in general, for Bell nonlocality tests where more than two incompatible POVMs with any number of outcomes are employed, i.e., it is possible to identify a set of nonjointly measurable POVMs, which fail to reveal Bell-type nonlocality, in general [25]. Interestingly, there exists a one-to-one equivalence [16–18] between measurement incompatibility and quantum steerability (i.e., Alice's ability to nonlocally alter Bob's states by performing local measurements on her part of the quantum state [26]). More specifically, a set of fuzzy POVMs is said to be incompatible if and only if it can be used to show steering in a quantum state.

From the point of view of an entirely different mathematical perspective, the classical moment problem [22,27–31] addresses the issue of the existence of a probability distribution corresponding to a given a sequence of statistical moments. Essentially, the classical moment problem points out that a given sequence of real numbers qualifies to be the moment sequence of a legitimate probability distribution if and only if the corresponding moment matrix is positive. In other words, the existence of a valid joint probability distribution, consistent with the given sequence of moments, necessitates positivity of the associated moment matrix. A moment matrix constructed in terms of pairwise correlations of observables in the quantum scenario is not necessarily positive [22,30,31] and thus one witnesses violation of Bell, Leggett-Garg,

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and noncontextual inequalities (which can be realized to be the positivity constraints on the eigenvalues of the moment matrix). In turn, violation of these inequalities points towards the nonexistence of joint probabilities corresponding to the measurements of all the observables employed. Moments extracted from measurements of a set of POVMs result in a positive moment matrix if the degree of incompatibility is restricted to lie within the range specified by the compatibility of the set of POVMs employed [22].

In this paper we construct N-term chained correlation inequalities involving pairwise correlations of N dichotomic random variables based on the positivity of a sequence of 4×4 moment matrices. The bound on the linear combination of pairwise correlations (recognized through the positivity of moment matrices) ensures the existence of joint probabilities for the statistical outcomes. When the dichotomic classical random variables are replaced by qubit observables, one witnesses a violation of the chained correlation inequalities [32,33]. The maximum violation of the inequalities in the quantum scenario (i.e., the corresponding Tsirelson-like bound) has been established in Refs. [32,33]. The dichotomic observables, which result in the maximum quantum violation, correspond to the qubit observables in a plane. Here we investigate the degree of incompatibility necessary for the joint measurability of the equatorial plane noisy qubit POVMs (i.e., a mixture of qubit observables in the equatorial plane and the identity matrix). Based on this, we determine that the chained inequalities are always satisfied when the equatorial noisy qubit POVMs, employed in first measurements of sequential pairwise measurements, are all jointly measurable; however, incompatible POVMs are in general not sufficient for violation of the chained inequalities for $N > 3$. On the other hand, we show that there is a one-to-one correspondence between the joint measurability of a set of equatorial plane noisy qubit POVMs and the optimal violation of the linear nonlocal steering inequality proposed by Jones and Wiseman [34]. This leads us towards the construction of a local analog of this steering inequality in a single-qubit system, the violation of which gives evidence for the nonjoint measurability of the set of equatorial plane qubit POVMs, employed in the first of every sequential pair of measurements.

We organize the contents of the paper as follows. In Sec. II we outline the notion of compatible POVMs. As a specific case, we discuss the compatibility of qubit POVMs in the equatorial plane of the Bloch sphere and obtain the necessary condition for the unsharpness parameter quantifying the degree of incompatibility. Section III is devoted to (i) the formulation of chained N-term pairwise correlation inequalities constructed from the positivity of moment matrices and (ii) the optimal violation of the inequalities in the quantum scenario when qubit observables in the equatorial plane are employed and the connection between the degree of incompatibility of POVMs, used in the first of every sequential pair measurements, and the strength of violation of the correlation inequalities. In Sec. IV we show that the steering inequality proposed by Jones and Wiseman [34] has a one-to-one correspondence with the joint measurability of equatorial qubit POVMs. A local analog of this steering inequality for a single-qubit system, involving N settings of sequential unsharp-sharp pairwise correlations, is constructed. Section V contains a summary of our results and concluding remarks.

II. JOINT MEASURABILITY OF POVMS

In the conventional quantum framework, measurements are described in terms of the spectral projection operators of the corresponding self-adjoint observables. Joint measurability of two commuting observables is ensured because the results of a single PV measurement are comprised of those of both observables. However, noncommuting observables are declared as incompatible under the regime of PV measurements. The introduction of POVMs by Ludwig [35] and subsequent investigations of their applicability [36] led to a mathematically rigorous generalization of measurement theory. It is the notion of compatibility (the notion of compatibility of a set of POVMs will be defined in the following), rather than commutativity, that gains importance so as to recognize if a given set of POVMs is jointly measurable or not [37].

A POVM is a set $\mathbb{E}_x = \{E_x(a) = M_x^\dagger(a)M_x(a)\}$ comprising positive self-adjoint operators $0 \leqslant E_x(a) \leqslant \mathbb{1}$, satisfying $\sum_a E_x(a) = \sum_a M_x^\dagger(a)M_x(a) = \mathbb{1}$, where a denotes the outcome of measurement and $\mathbb{1}$ is the identity operator. Under measurement $\{M_x(a)\}$ a quantum system, prepared in the state ρ, undergoes a positive trace-preserving generalized Lüder transformation, i.e.,

$$\rho \mapsto \sum_a M_x(a)\rho M_x^\dagger(a), \tag{1}$$

and an outcome a occurs with probability $p(a|x) = \mathrm{Tr}[\rho M_x^\dagger(a)M_x(a)] = \mathrm{Tr}[\rho E_x(a)]$. Results of PV measurements can be retrieved as a special case, when the POVM $\{E_x(a)\}$ consists of complete orthogonal projectors.

A finite collection $\{\mathbb{E}_{x_1},\mathbb{E}_{x_2},\ldots,\mathbb{E}_{x_N}\}$ of N POVMs is said to be jointly measurable (or compatible) if there exists a grand POVM $\mathbb{G} = \{G(\lambda);\ 0 \leqslant G(\lambda) \leqslant \mathbb{1},\ \sum_\lambda G(\lambda) = \mathbb{1}\}$, with outcomes denoted by a collective index $\lambda \equiv \{a_1,a_2,\ldots,a_N\}$, such that the individual POVMs $\mathbb{E}_{x_i}$ can be expressed as its marginals [10]

$$E_{x_k}(a_k) = \sum_{\substack{a_1,a_2,\ldots,a_{k-1},\\ a_{k+1},\ldots,a_N}} G(\lambda = \{a_1,a_2,\ldots,a_N\}) \tag{2}$$

for all $k = 1,2,\ldots,N$. From now on, we denote the collective index $\lambda = \{a_1,a_2,\ldots,a_N\}$ characterizing measurement outcomes of the global POVM $\mathbb{G}$ by $\mathbf{a} = (a_1,a_2,\ldots,a_N)$ for brevity.

When a measurement of the global POVM $\mathbb{G} \equiv \{G(\mathbf{a})\}$ is carried out in an arbitrary quantum state ρ, an outcome $\mathbf{a}$ occurs with probability $\mathrm{Tr}[\rho G(\mathbf{a})] = p(\mathbf{a})$. Then the corresponding results $[p(a_k|x_k),a_k]$ (viz., the outcomes a_k and the probabilities $p(a_k|x_k) = \mathrm{Tr}[\rho E_{x_k}(a_k)]$) for all the compatible POVMs $\mathbb{E}_{x_k}$ can be deduced by postprocessing the collective measurement data [37] $(p(\mathbf{a}),\{\mathbf{a}\})$ of the global POVM $\mathbb{G}$:

$$p(a_k|x_k) = \sum_{\substack{a_1,a_2,\ldots,a_{k-1},\\ a_{k+1},\ldots,a_N}} p(\mathbf{a}). \tag{3}$$

A set of POVMs $\{\mathbb{E}_{x_k}\}$, $k = 1,2,\ldots,N$, is declared to be compatible if and only if it consists of marginals of a global POVM $\mathbb{G}$ [as expressed in (2)].

A. Example of noisy qubit POVMs

Consider a pair of qubit observables $\sigma_x = \sum_{a_x=\pm 1} a_x \Pi_x(a_x)$ and $\sigma_z = \sum_{a_z=\pm 1} a_z \Pi_z(a_z)$. Sharp PV measurements of these self-adjoint observables σ_x and σ_z are incorporated in terms of their spectral projectors

$$\begin{aligned}\Pi_x(a_x) &= \tfrac{1}{2}(\mathbb{1} + a_x\sigma_x),\\ \Pi_z(a_z) &= \tfrac{1}{2}(\mathbb{1} + a_z\sigma_z).\end{aligned} \tag{4}$$

Within the conventional framework of PV measurements, the noncommuting qubit observables σ_x and σ_z are not jointly measurable. However, it is possible to consider a particular choice of jointly measurable noisy qubit POVMs $\mathbb{E}_x = \{E_x(a_x)\}$ and $\mathbb{E}_z = \{E_x(a_x)\}$ by mixing white noise with the respective projection operators, i.e.,

$$\begin{aligned}E_x(a_x) &= \eta\, \Pi_x(a_x) + (1-\eta)\tfrac{\mathbb{1}}{2} = \tfrac{1}{2}(\mathbb{1} + \eta a_x\sigma_x),\\ E_z(a_z) &= \eta\, \Pi_z(a_z) + (1-\eta)\tfrac{\mathbb{1}}{2} = \tfrac{1}{2}(\mathbb{1} + \eta a_z\sigma_z),\end{aligned} \tag{5}$$

where $0 \leqslant \eta \leqslant 1$ denotes the unsharpness parameter. When $\eta = 1$, the noisy qubit POVMs (5) reduce to their corresponding sharp PV counterparts. Throughout this paper we will be focusing on the joint measurability (compatibility) of noisy qubit observables of the form given by (5).

The dichotomic POVMs $\mathbb{E}_x$ and $\mathbb{E}_z$ are jointly measurable if there exists a four-outcome global POVM $\mathbb{G} = \{G(a_x,a_z);\ a_x = \pm 1, a_z = \pm 1\}$ such that

$$\begin{aligned}&\sum_{a_z=\pm 1} G(a_x,a_z) = E_x(a_x),\\ &\sum_{a_x=\pm 1} G(a_x,a_z) = E_z(a_z),\\ &\sum_{a_x,a_z=\pm 1} G(a_x,a_z) = \mathbb{1}, \quad G(a_x,a_z) \geqslant 0.\end{aligned} \tag{6}$$

It has been shown [7,10] that the POVMs $\mathbb{E}_x$ and $\mathbb{E}_z$ are jointly measurable in the range $0 \leqslant \eta \leqslant 1/\sqrt{2}$, i.e., it is possible to construct a global POVM $\mathbb{G}$ comprised of the elements

$$G(a_x,a_z) = \tfrac{1}{4}(\mathbb{1} + \eta a_x\sigma_x + \eta a_z\sigma_z), \quad 0 \leqslant \eta \leqslant 1/\sqrt{2}, \tag{7}$$

which obey (6). Similarly, triplewise joint measurements of the qubit observables σ_x, σ_y, and σ_z could be envisaged by considering the fuzzy POVMs $\mathbb{E}_x$, $\mathbb{E}_y$, and $\mathbb{E}_z$, elements of which are given, respectively, by

$$\begin{aligned}E_x(a_x) &= \tfrac{1}{2}(\mathbb{1} + \eta a_x\sigma_x), \quad a_x = \pm 1\\ E_y(a_y) &= \tfrac{1}{2}(\mathbb{1} + \eta a_y\sigma_y), \quad a_y = \pm 1\\ E_{\sigma_z}(z) &= \tfrac{1}{2}(\mathbb{1} + \eta a_z\sigma_z), \quad a_z = \pm 1\end{aligned}$$

in the range $0 \leqslant \eta \leqslant 1/\sqrt{3}$ of the unsharpness parameter [10,13].

In general, the necessary condition on the unsharpness parameter such that the qubit POVMs $\{E_{x_k}(a_k = \pm 1) = \frac{1}{2}[\mathbb{1} + \eta a_k \vec{\sigma}\cdot\hat{n}_k],\ k = 1,2,\ldots,N\}$ are jointly measurable is derived in Refs. [13,38]:

$$\eta \leqslant \frac{1}{N}\max_{\mathbf{a}} |\vec{m}_{\mathbf{a}}|, \tag{8}$$

where $\vec{m}_{\mathbf{a}}$ is defined by

$$\vec{m}_{\mathbf{a}} = \sum_{k=1}^{N} \hat{n}_k a_k, \quad a_k = \pm 1. \tag{9}$$

The maximization is carried out over all 2^N outcomes [39] $\{\mathbf{a} = (a_1 = \pm 1, a_2 = \pm 1, \ldots, a_N = \pm 1)\}$. A sufficient condition places the following constraint on the unsharpness parameter (derived in Ref. [13]):

$$\eta \leqslant \frac{2^N}{\sum_{\mathbf{a}} |\vec{m}_{\mathbf{a}}|}. \tag{10}$$

In Table I we list the optimal value $\eta_{\rm opt}$ of the unsharpness parameter [evaluated using (8) and (10)], below which the qubit POVMs $\{E_{x_k}(a_k) = \frac{1}{2}(I + \eta a_k \vec{\sigma}\cdot\hat{n}_k);\ a_k = \pm 1\}$ are jointly measurable, in the specific cases [40] of $\hat{n}_k$, $k = 1,2,3$, constituting (i) orthogonal axes and (ii) trine axes [13,38], i.e., three coplanar unit vectors with $\hat{n}_k\cdot\hat{n}_l = -\frac{1}{2}$, $k \neq l = 1,2,3$.

TABLE I. Optimal value $\eta_{\rm opt}$ of the unsharpness parameter [evaluated using the necessary and sufficient conditions (8) and (10)], below which the joint measurability of the qubit POVMs $\{E_{x_k}(a_k) = \frac{1}{2}(\mathbb{1} + \eta a_k \vec{\sigma}\cdot\hat{n}_k)\}$ for different orientations $\hat{n}_k$ are compatible.

Number of POVMs	Orientation of $\hat{n}_k$	$\eta_{\rm opt}$
	Orthogonal axes	
$N = 3$	$\hat{n}_k\cdot\hat{n}_l = 0$, $k \neq l = 1,2,3$	$\frac{1}{\sqrt{3}}$
$N = 2$	$\hat{n}_1\cdot\hat{n}_2 = 0$	$\frac{1}{\sqrt{2}}$
	Trine axes	
$N = 3$	$\hat{n}_k\cdot\hat{n}_l = -\frac{1}{2}$, $k \neq l = 1,2,3$	$\frac{2}{3}$
$N = 2$	$\hat{n}_1\cdot\hat{n}_2 = -\frac{1}{2}$	0.732

B. Joint measurability of equatorial qubit observables

Here we consider joint measurability of N equatorial qubit observables $\sigma_{\theta_k} = \sigma_x\cos(\theta_k) + \sigma_y\sin(\theta_k)$, where $\theta_k = k\pi/N$ and $k = 1,2,\ldots,N$, which correspond geometrically to the points on the circumference of the circle in the equatorial half plane ($z = 0$) of the Bloch sphere, separated successively by an angle $\theta = \pi/N$. Consider equatorial qubit POVMs $\mathbb{E}_{\theta_k}$, elements of which are given by

$$E_{\theta_k}(a_k = \pm 1) = M^{\dagger}_{\theta_k}(a_k) M_{\theta_k}(a_k) = \tfrac{1}{2}(\mathbb{1} + \eta a_k \sigma_{\theta_k}). \tag{11}$$

When the set $\{\mathbb{E}_{\theta_k}\}$ of POVMs is compatible, there exists a global qubit POVM $\mathbb{G}$ comprised of 2^N elements

$$G(\mathbf{a}) = \frac{1}{2^N}\left(\mathbb{1} + \eta\sum_{k=1}^{N} a_k\sigma_{\theta_k}\right),$$

with $0 \leqslant \eta \leqslant \eta_{\rm opt}$. Using (8) and (10), we have evaluated the range of unsharpness parameter $0 \leqslant \eta \leqslant \eta_{\rm opt}$ such that the POVMs $\mathbb{E}_{\theta_k}$, $k = 1,2,\ldots,N$, are jointly measurable. Based on our computations of $\eta_{\rm opt}$, for small values of N, we recognize

TABLE II. Optimal value $\eta_{\rm opt}$ of the unsharpness parameter [see (12)] specifying the joint measurability of the equatorial qubit observables $\sigma_{\theta_k} = \sigma_x \cos(k\pi/N) + \sigma_y \sin(k\pi/N)$, $k = 1,2,\ldots,N$.

Number of POVMs	$\eta_{\rm opt}$
3	0.6666
4	0.6532
5	0.6472
6	0.6439
10	0.6392
20	0.6372
50	0.6367
100	0.6366

the following cutoff $\eta \leqslant \eta_{\rm opt}$ for any N:

$$\eta_{\rm opt} = \frac{1}{N}\sqrt{N + 2\sum_{k=1}^{[N/2]}(N-2k)\cos\left(\frac{k\pi}{N}\right)}. \tag{12}$$

The values of $\eta_{\rm opt}$ are listed in Table II. In the large-N limit, the degree of incompatibility (i.e., the cutoff value of the unsharpness parameter) approaches $\eta_{\rm opt} \to 0.6366$ and thus the POVMs associated with the set of all qubit observables $\sigma_\theta, 0 \leqslant \theta \leqslant \pi$, in the equatorial plane of the Bloch sphere are jointly measurable in the range $0 \leqslant \eta^{(\infty)}_{\rm opt} \leqslant 0.6366$.

Recently, Uola *et al.* [41] investigated the incompatibility of some noisy observables in finite-dimensional Hilbert spaces by developing a technique that they referred to as an adaptive strategy. In particular, they independently identified the following sufficient condition for the simultaneous measurements of qubit observables in a plane $\sigma_{\theta_k} = \sigma_x \cos(\theta_k) + \sigma_y \sin(\theta_k)$, where $\theta_k = k\pi/N$ and $k = 1,2,\ldots,N$, based on their approach [41]:

$$\eta \leqslant \frac{2}{N}\sum_{k=1}^{[N/2]}\cos\left(\frac{(2k-1)\pi}{2N}\right) = \frac{1}{N\sin(\pi/2N)}, \tag{13}$$

which too agrees perfectly with the optimal value (12) of the unsharpness parameter (see Table II).

III. CHAINED N-TERM CORRELATION INEQUALITIES AND JOINT MEASURABILITY

The local realistic framework places bounds on correlations between the outcomes of measurements, carried out by spatially separated parties, and Bell inequalities formulated in terms of these correlations get violated in the framework of quantum theory. On the other hand, quantum theory too places a strict limit on the strength of these correlations. The maximum violation of CHSH inequality [24], by nonlocal quantum correlations, is constrained by the Tsirelson bound [42] $2\sqrt{2}$. The CHSH inequality involves measurements of two pairs of dichotomic observables on a bipartite system [denoted by (A_1,A_2) and (B_1,B_2), which are local observables measured by Alice and Bob, respectively] and four correlation terms

$$\langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle \leqslant 2. \tag{14}$$

An interesting connection between joint measurability and violation of the CHSH inequalities, within the framework of quantum theory, has been revealed recently by Banik *et al.* [43]. In general, they showed that, in a no-signaling probabilistic theory, the maximum strength of violation of the inequality (14) by any pair of (A_1^η, A_2^η) of quantum dichotomic observables [unsharp counterparts of (A_1,A_2)] is essentially determined by the optimal degree of incompatibility $\eta_{\rm opt}$, which, in quantum theory, is identified as $\frac{1}{\sqrt{2}}$, i.e.,

$$\langle A_1^\eta B_1\rangle + \langle A_1^\eta B_2\rangle + \langle A_2^\eta B_1\rangle - \langle A_2^\eta B_2\rangle \leqslant \frac{2}{\eta_{\rm opt}} = 2\sqrt{2}. \tag{15}$$

In other words, the degree of incompatibility $\eta_{\rm opt} = \frac{1}{\sqrt{2}}$ of measurements in the quantum framework is shown to place limitations on the maximum strength of violations of the four-term CHSH inequality, by retrieving the quantum Tsirelson bound $2\sqrt{2}$.

Does this connection between the degree of incompatibility and the Tsirelson-like bound (maximum strength of violation) hold in general when more than two incompatible measurements are involved? We explore this question through N-term correlation inequalities, which we formulate from positivity of a sequence of moment matrices.

Consider N classical random variables $X_k, k = 1,2,\ldots,N$, with outcomes $a_k = \pm 1$. Let $\xi_k^T = (1, a_1a_k, a_ka_{k+1}, a_1a_{k+1})$, $k = 2,3,\ldots,N-1$, denote row vectors. We construct a sequence of 4×4 moment matrices $M_k = \langle \xi_k\xi_k^T\rangle$, expressed explicitly as

$$M_k = \begin{pmatrix} 1 & \langle X_1X_k\rangle & \langle X_kX_{k+1}\rangle & \langle X_1X_{k+1}\rangle \\ \langle X_1X_k\rangle & 1 & \langle X_1X_{k+1}\rangle & \langle X_kX_{k+1}\rangle \\ \langle X_kX_{k+1}\rangle & \langle X_1X_{k+1}\rangle & 1 & \langle X_1X_k\rangle \\ \langle X_1X_{k+1}\rangle & \langle X_kX_{k+1}\rangle & \langle X_1X_k\rangle & 1 \end{pmatrix}, \tag{16}$$

where $\langle X_kX_l\rangle$, $k \neq l$, denote pairwise correlations of the variables X_k and X_l (here $\langle\cdot\rangle$ denotes the expectation value).

In the classical probability setting, the moment matrix is, by construction, real symmetric and positive semidefinite. The eigenvalues $\lambda_i^{(k)}$, $i = 1,2,3,4$, of the moment matrix are given by

$$\begin{aligned} \lambda_1^{(k)} &= 1 + \langle X_1X_k\rangle - \langle X_kX_{k+1}\rangle - \langle X_1X_{k+1}\rangle, \\ \lambda_2^{(k)} &= 1 - \langle X_1X_k\rangle + \langle X_kX_{k+1}\rangle - \langle X_1X_{k+1}\rangle, \\ \lambda_3^{(k)} &= 1 - \langle X_1X_k\rangle - \langle X_kX_{k+1}\rangle + \langle X_1X_{k+1}\rangle, \\ \lambda_4^{(k)} &= 1 + \langle X_1X_k\rangle + \langle X_kX_{k+1}\rangle + \langle X_1X_{k+1}\rangle. \end{aligned} \tag{17}$$

Replacing classical random variables X_k by quantum dichotomic observables $\mathbf{X}_k = \vec{\sigma}\cdot\hat{n}_k$, $k = 1,2,\ldots,N$, with eigenvalues ± 1 and the classical probability distribution by a density matrix, the moment matrix positivity results in linear constraints on pairwise correlations of the observables measured sequentially.

Based on the positivity of a sequence of $N-1$ moment matrices $M_2, M_3, \ldots, M_{N-1}$ one obtains the inequalities $\sum_{k=2,3,\ldots,N-1}\lambda_i^{(k)} \geqslant 0$ for the sum of the eigenvalues [see (17)], which correspond to the following chained inequalities involving pairwise correlations:

$$\sum_{k=2}^{N-1}\langle \mathbf{X}_k\mathbf{X}_{k+1}\rangle + \langle \mathbf{X}_1\mathbf{X}_N\rangle - \langle \mathbf{X}_1\mathbf{X}_2\rangle \leqslant N-2, \quad (18)$$

$$2\sum_{k=2}^{N}[\langle \mathbf{X}_1\mathbf{X}_k\rangle - \langle \mathbf{X}_k\mathbf{X}_{k+1}\rangle] + \langle \mathbf{X}_1\mathbf{X}_N\rangle \leqslant N-2, \quad (19)$$

$$\sum_{k=1}^{N-1}\langle \mathbf{X}_k\mathbf{X}_{k+1}\rangle - \langle \mathbf{X}_1\mathbf{X}_N\rangle \leqslant N-2, \quad (20)$$

$$\sum_{k=2}^{N-1}\langle \mathbf{X}_k\mathbf{X}_{k+1}\rangle + 2\sum_{k=2}^{N-2}\langle \mathbf{X}_1\mathbf{X}_{k+1}\rangle + \langle \mathbf{X}_1\mathbf{X}_N\rangle \leqslant N-2. \quad (21)$$

Violation of these inequalities implies at least one of the moment matrices $M^{(k)}$ is not positive, which in turn highlights the nonexistence of a valid joint probability distribution for the outcomes of all the observables employed. However, it may be realized that by employing unsharp measurements of the observables, within their joint measurability region, one can retrieve positivity of the sequence of moment matrices and consequently the chained inequalities (18)–(21) are satisfied.

In particular, (20) is analogous to the N-term temporal correlation inequality investigated by Budroni *et al.* [33]. The pairwise correlations $\langle \mathbf{X}_k\mathbf{X}_{k+l}\rangle$ arise from the sequential measurements of the observables $\mathbf{X}_k$ and $\mathbf{X}_{k+l}$ in a single quantum system. Such inequalities involving sequential pairwise correlations of observables in a single quantum system [in contrast to correlations of the outcomes of local measurements at different ends of a spatially separated bipartite system as in (14)] have been well explored to highlight quantum contextuality [44] and nonmacrorealism [3,5,45].

Budroni *et al.* [33] computed the maximal achievable value (Tsirelson-like bound) of the left-hand side of the chained N-term temporal correlation inequality (20) and obtained

$$\mathcal{S}_N^Q = \sum_{k=1}^{N-1}\langle \mathbf{X}_k\mathbf{X}_{k+1}\rangle_{\rm seq} - \langle \mathbf{X}_1\mathbf{X}_N\rangle_{\rm seq} \leqslant N\cos\left(\frac{\pi}{N}\right). \quad (22)$$

The classical bound $N-2$ on the chained N-term inequality (20) can get violated in the quantum framework and a maximum value of $N\cos(\frac{\pi}{N})$ could be achieved by choosing sequential measurements of appropriate observables. In particular, when a single qubit is prepared in a maximally mixed state $\rho = \mathbb{1}/2$, sequential PV measurements of the observables $\sigma_{\theta_k} = \sigma_x\cos(\theta_k) + \sigma_y\sin(\theta_k)$, where $\theta_k = k\pi/N$ and $k = 1,2,\ldots,N$, lead to pairwise correlations

$$\langle \mathbf{X}_k\mathbf{X}_{k+l}\rangle_{\rm seq} = \langle \sigma_{\theta_k}\sigma_{\theta_{k+l}}\rangle_{\rm seq} = \cos(\theta_{k+l}-\theta_k) = \cos\left(\frac{l\pi}{N}\right). \quad (23)$$

Substituting (23) in (22), we obtain the quantum Tsirelsen-like bound $\mathcal{S}_N^Q = N\cos(\frac{\pi}{N})$.

It is pertinent to point out that the observables $\{\sigma_{\theta_k} = \sigma_x\cos(\theta_k) + \sigma_y\sin(\theta_k), \theta_k = k\pi/N, k = 1,2,\ldots,N\}$ need not, in general, be associated with any particular time evolution; they are considered to be any ordered set of observables. Moreover, the pairs of sequential measurements are performed in independent statistical trials, i.e., the input state in every first measurement of the pair is $\rho = \mathbb{1}/2$.

A. Degree of incompatibility and violation of the chained correlation inequality (20)

It is seen that the average pairwise correlations $\langle \mathbf{X}_k\mathbf{X}_{k+l}\rangle_{\rm seq}$ of qubit observables $\mathbf{X}_k \equiv \sigma_{\theta_k}$, $k = 1,2,\ldots,N$, evaluated based on the results of sequential sharp PV measurements, lead to maximal violation of the chained correlation inequality (20). Instead of sharp PV measurements of the observables, we consider here an alternate sequential measurement scheme. We separate the set of observables $\{\mathbf{X}_k \equiv \sigma_{\theta_k}, k = 1,2,\ldots,N\}$ of first measurements of every sequential pair. We ask if the chained inequality (20) is violated, when measurement of first observables of every pair correlation $\langle \mathbf{X}_k\mathbf{X}_{k+l}\rangle_{\rm seq}$ is done using noisy POVMs, while sharp PV measurements are employed for second observables in the sequence. Interestingly, we identify that the chained inequality (20) is not violated, whenever a compatible set of POVMs $\{\mathbb{E}_{\theta_k}, k = 1,2,\ldots,N\}$ [see (11)] is employed to carry out measurements of first observables of every sequential pair, irrespective of the fact that second measurements are all sharp (and hence incompatible). In other words, incompatibility of the set of POVMs employed in carrying out first measurements in the sequential scheme is sufficient to witness violation of the chained inequality (20). We now proceed to describe the sequential measurement scheme explicitly in the following.

Consider N noisy qubit observables $\mathbb{E}_{\theta_k}$ with elements $\{E_{\theta_k}(a_k = \pm 1) = M^\dagger_{\theta_k}(a_k)M_{\theta_k}(a_k)\}$ given by (11). From our discussion in Sec. II B, it is seen that there exists a global qubit POVM $\mathbb{G}$, when η lies in the range $0 \leqslant \eta \leqslant \eta_{\rm opt}$ [see (12) for the values of the parameter $\eta_{\rm opt}$], such that the POVMs $\mathbb{E}_{\theta_k}$, $k = 1,2,\ldots,N$, are all jointly measurable.

As before, we consider the initial state of the qubit to be $\rho = \mathbb{1}/2$, a maximally mixed state. Carrying out an unsharp measurement $M_{\theta_k}(a_k)$, yielding an outcome a_k, the initial state gets transformed to

$$\rho \to \rho_{a_k} = \frac{M_{\theta_k}(a_k)\rho M^\dagger_{\theta_k}(a_k)}{p(a_k|\theta_k)} = \frac{1}{2}(\mathbb{1} + \eta a_k\sigma_{\theta_k}), \quad (24)$$

where we have defined ${\rm Tr}[\rho M^\dagger_{\theta_k}(a_k)M_{\theta_k}(a_k)] = {\rm Tr}[\rho E_{\theta_k}(a_k)] = p(a_k|\theta_k)$. Following this with a second PV measurement of $\sigma_{\theta_{k+l}}$ on the state ρ_{a_k} results in the pairwise correlations

$$\begin{aligned}\langle \mathbf{X}_k^{(\eta)}\mathbf{X}_{k+l}\rangle_{\rm seq} &= \sum_{a_k} p(a_k|\theta_k){\rm Tr}\big[\rho_{a_k}\sigma_{\theta_{k+l}}\big] = \eta\cos(\theta_{k+l}-\theta_k)\\ &= \eta\cos(\pi l/N). \end{aligned} \quad (25)$$

So the left-hand side of chained correlation inequality (20) assumes the value

$$\begin{aligned}\mathcal{S}_N^Q(\eta) &= \sum_{k=1}^{N-1}\langle \mathbf{X}_k^{(\eta)}\mathbf{X}_{k+1}\rangle_{\rm seq} - \langle \mathbf{X}_1^{(\eta)}\mathbf{X}_N\rangle_{\rm seq}\\ &= \eta N\cos\left(\frac{\pi}{N}\right), \end{aligned} \quad (26)$$

when pairwise unsharp-sharp measurements of equatorial qubit observables are carried out. Within the joint

TABLE III. Maximum attainable value $\mathcal{S}_N^Q(\eta_{\rm opt}) = \eta_{\rm opt} N \cos(\frac{\pi}{N})$ of the left-hand side of the N-term temporal correlation inequality (22) when the qubit POVMs employed are jointly measurable [see (27)].

No. of POVMs employed	Classical bound $N-2$	Quantum bound $N\cos(\frac{\pi}{N})$	Maximum achievable value $\mathcal{S}_N^Q(\eta_{\rm opt})$
3	1	1.5	1
4	2	2.83	1.85
5	3	4.05	2.62
6	4	5.20	3.35
10	8	9.51	6.08
20	18	19.75	12.59
50	48	49.90	31.77
100	98	99.95	63.62

measurability domain of the set $\{\mathbb{E}_{\theta_k}, k = 1,2,\ldots,N\}$ of first unsharp measurements in this sequential scheme, the sum of pairwise correlations obeys

$$\mathcal{S}_N^Q(\eta) = \sum_{k=2}^{N-1} \langle X_k^{(\eta)} \mathbf{X}_{k+1}\rangle_{\rm seq} - \langle X_1^{(\eta)} \mathbf{X}_N\rangle_{\rm seq} \leqslant \eta_{\rm opt} N \cos\left(\frac{\pi}{N}\right). \tag{27}$$

Using the optimal values $\eta_{\rm opt}$ specifying the degree of incompatibility of the equatorial qubit observables [see (12) and the values listed in Table II], we evaluated the maximum value $\mathcal{S}_N^Q(\eta_{\rm opt}) = \eta_{\rm opt} N \cos(\frac{\pi}{N})$ attainable by the left-hand side of the inequality (27) for different values of N; these values are listed together with the corresponding classical and quantum bounds in Table III. It is evident that as the number of measurements N increases, the quantum Tsirelson-like bound approaches the algebraic maximum value N, while the maximum achievable value of (27) approaches $\mathcal{S}_N^Q(\eta_{\rm opt}) \to 0.6366N$. More specifically, the classical bound is always satisfied when the first measurements in the sequential scheme are carried out by compatible POVMs. However, unlike the situation in the CHSH Bell inequality [11,43,46] (14), the maximum achievable value $\mathcal{S}_N^Q(\eta_{\rm opt})$ is not identically equal to the classical bound of $N-2$, except in the case of $N=3$ [22]. So it is evident that the incompatible set $\{\mathbb{E}_{\theta_k};\, \eta > \eta_{\rm opt}, k = 1,2,\ldots,N\}$ of POVMs is necessary but not sufficient to violate the chained N-term correlation inequality (20). Is it possible to find a steering protocol for which incompatibility of equatorial qubit measurements is both necessary and sufficient? In the next section we discuss a linear steering inequality involving equatorial qubit observables [34] and unravel how violation of the inequality gets intertwined with measurement incompatibility.

IV. LINEAR STEERING INEQUALITY AND JOINT MEASURABILITY

Quantum steering (introduced by Schrödinger [47]) has gained much impetus in recent years. Reid [48] proposed an experimentally testable steering criterion, which revealed that, apart from Bell-type nonlocality, steering is yet another distinct manifestation of Einstein-Podolsky-Rosen (EPR) nonlocality in spatially separated composite quantum systems. A conceptually clear formalism of EPR steering [in terms of local hidden state (LHS) model] has been formulated by Wiseman *et al.* [49]. They elucidated that steering constitutes a different kind of nonlocality, which lies between entanglement and Bell-type nonlocality. Several steering inequalities, suitable for the experimental demonstration of this form of EPR spooky action at a distance, have been derived in Ref. [26]. Moreover, it has been realized that putting steering phenomena to experimental test is much easier compared to demonstrations of Bell-type nonlocality [49,50]. Interestingly, the steering framework is useful to investigate the joint measurability problem and vice versa [16–22]. In this section we discuss a linear steering inequality, derived by Jones and Wiseman [34], where measurements of equatorial qubit observables are employed. We show that this steering inequality exhibits a striking equivalence with the joint measurability of the equatorial qubit observables, discussed in Sec. II.

Suppose Alice prepares a bipartite quantum state ρ_{AB} and sends a subsystem to Bob. If the state is entangled and Alice chooses suitable local measurements on her part of the state, she can affect Bob's quantum state remotely. How would Bob convince himself that his state is indeed steered by Alice's local measurements? In order to verify that his (conditional) states are steered, Bob asks Alice to perform local measurements of the observables $\mathbf{X}_k = \sum_{a_k} a_k \Pi_{x_k}(a_k)$ on her part of the state and communicate the outcomes a_k in each experimental trial. If Bob's conditional reduced states (unnormalized) $\varrho^B_{a_k|x_k} = {\rm Tr}_A[\Pi_{x_k}(a_k) \otimes \mathbb{1}_B \rho_{AB}]$ admit a LHS decomposition [49], viz., $\varrho_{a_k|x_k} = \sum_\lambda p(\lambda) p(a_k|x_k,\lambda)\rho^B_\lambda$ [where $0 \leqslant p(\lambda) \leqslant 1$, with $\sum_\lambda p(\lambda) = 1$, and $0 \leqslant p(a_k|x_k,\lambda) \leqslant 1$, with $\sum_{a_k} p(a_k|x_k,\lambda) = 1$; $(p_\lambda, \rho^B_\lambda)$ denote Bob's LHS ensemble], then Bob can declare that Alice is not able to steer his state through local measurements at her end. In addition to entanglement being a necessary (but not sufficient [49]) ingredient, incompatibility of Alice's local measurements too plays a crucial role to reveal steering [16–18]. In the following section we unfold the intrinsic link between steering and measurement compatibility in a specific two-qubit protocol.

A. Linear steering inequality for a two-qubit system

Consider a qubit observable [34]

$$\mathbf{S}_{\rm plane} = \frac{1}{\pi}\int_0^\pi d\theta\, \alpha_\theta \sigma_\theta, \tag{28}$$

where $\sigma_\theta = \sigma_x \cos(\theta) + \sigma_y \sin(\theta)$ denotes an equatorial qubit observable and $-1 \leqslant \alpha_\theta \leqslant 1$. The expectation value of the observable $\mathbf{S}_{\rm plane}$ is upper bounded by

$$\langle \mathbf{S}_{\rm plane}\rangle \leqslant \frac{1}{\pi}\int_0^\pi d\theta \langle\sigma_\theta\rangle = \frac{1}{\pi}\int_0^\pi d\theta[\langle\sigma_x\rangle\cos(\theta) + \langle\sigma_y\rangle\sin(\theta)] = \frac{2}{\pi}\langle\sigma_y\rangle \Rightarrow \langle \mathbf{S}_{\rm plane}\rangle \leqslant \frac{2}{\pi}. \tag{29}$$

Suppose Alice and Bob share a two-qubit state ρ_{AB}; Bob asks Alice to perform measurements of σ^A_θ and communicate the outcome $a_\theta = \pm 1$ of her measurements. After Alice's measurements, Bob will be left with an ensemble $\{p(a_\theta|\theta), \rho^B_{a_\theta}\}$ where $\rho^B_{a_\theta|\theta} = {\rm Tr}_A[\Pi_\theta(a_\theta) \otimes \mathbb{1}_B \rho_{AB}]/p(a_\theta|\theta)$;

$p(a_\theta|\theta) = \mathrm{Tr}[\Pi_\theta(a_\theta)\otimes \mathbb{1}_B\rho_{AB}]$ denotes Bob's conditional states [here $\{\Pi_\theta(a_\theta=\pm1)\}$ denote PV measurements of the observable σ_θ]. At his end, Bob would then measure the observable σ_θ^B. Suppose he gets an outcome $b_\theta=\pm1$ with probability $p(b_\theta|a_\theta;\theta) = \mathrm{Tr}[\Pi_\theta(b_\theta)\rho^B_{a_\theta|\theta}]$. He evaluates the conditional expectation value of the observable σ_θ^B, based on the statistical data he obtains, as follows:

$$\langle\sigma_\theta^B\rangle_{a_\theta} = \sum_{b_\theta=\pm1} b_\theta\, p(b_\theta|a_\theta;\theta). \tag{30}$$

If the conditional probabilities $p(b_\theta|a_\theta;\theta)$ originate from a LHS model, i.e., if

$$\begin{aligned} p(b_\theta|a_\theta;\theta) &= \sum_\lambda p(\lambda)p(a_\theta|\theta,\lambda)\mathrm{Tr}[\Pi_{\theta_B}(b_\theta)\rho^B_\lambda] \\ &= \sum_\lambda p(\lambda)p(a_\theta|\theta,\lambda)\langle\Pi_\theta(b_\theta)\rangle_\lambda \end{aligned} \tag{31}$$

[where we have defined $\sum_{b_\theta=\pm1} b_\theta\langle\Pi_\theta(b_\theta)\rangle_\lambda = \langle\sigma^B_\theta\rangle_\lambda$], one gets the conditional expectation value in the LHS model as follows:

$$\begin{aligned} \langle\sigma_\theta^B\rangle_{a_\theta|\theta} &= \sum_\lambda p(\lambda)p(a_\theta|\theta,\lambda)\left\{\sum_{b_\theta=\pm1} b_\theta\langle\Pi_\theta(b_\theta)\rangle_\lambda\right\} \\ &= \sum_\lambda p(\lambda)p(a_\theta|\theta,\lambda)\langle\sigma_\theta\rangle_\lambda. \end{aligned} \tag{32}$$

Whenever the LHS model holds, the inequality

$$\frac{1}{\pi}\int_0^\pi d\theta\,\alpha_\theta\langle\sigma_\theta^B\rangle_{a_\theta|\theta} \leqslant \frac{2}{\pi} \tag{33}$$

is obeyed for any $-1\leqslant\alpha_\theta\leqslant1$ in the LHS framework. Now, defining $\sum_{a_\theta=\pm1} a_\theta p(a_\theta|\theta)\langle\sigma^B_\theta\rangle_{a_\theta|\theta} = \langle\sigma^A_\theta\sigma^B_\theta\rangle$, one obtains the linear steering inequality [34]

$$\frac{1}{\pi}\int_0^\pi d\theta\langle\sigma^A_\theta\sigma^B_\theta\rangle \leqslant \frac{2}{\pi}. \tag{34}$$

Violation of the inequality (34) in any bipartite quantum state ρ_{AB} demonstrates nonlocal EPR steering phenomena (more specifically, violation implies falsification of the LHS model, which confirms that Alice can indeed steer Bob's state remotely via her local measurements).

Note that implementing an infinite number of measurements (i.e., measurement of σ^B_θ by Bob conditioned by the outcomes of Alice's measurement of σ^A_θ, in the entire equatorial half plane $0\leqslant\theta\leqslant\pi$) is a tough task in a realistic experimental scenario. So it would be suitable to consider a finite setting of N evenly spaced equatorial measurements of σ_{θ_k} (such that the successive angular separation is given by π/N, i.e., $\theta_{k+1}-\theta_k=\pi/N$) by Bob, conditioned by the ±1 valued outcomes a_k of Alice's measurements $\sigma^A_{\theta_k}$. This leads to the following linear steering inequality in the finite setting [34]:

$$\frac{1}{N}\sum_{k=1}^N\langle\sigma^A_{\theta_k}\sigma^B_{\theta_k}\rangle \leqslant f(N), \tag{35}$$

where

$$f(N) = \frac{1}{N}\left[\left|\sin\left(\frac{N\pi}{2}\right)\right| + 2\sum_{k=1}^{[N/2]}\sin\left((2k-1)\frac{\pi}{2N}\right)\right] \tag{36}$$

corresponds to the maximum eigenvalue of the observable $\frac{1}{N}\sum_{k=1}^N\sigma_{\theta_k}$.

One obtains $f(2)=1/\sqrt{2}$, $f(3)\approx0.6666$, $f(4)=0.6533$, and $f(10)\approx0.6392$ for smaller values of N. [Note that there is a striking match between the degree of incompatibility $\eta_{\rm opt}$ listed in Table II and the upper bound $f(N)$ of the inequality (35).] The factor $f(N)\to2/\pi\approx0.6366$ in the limit $N\to\infty$. We discuss the violation of the steering inequality (35) when Alice and Bob share a maximally entangled two-qubit state.

B. Violation of the linear steering inequality by a two-qubit maximally entangled state

Let Alice and Bob share a maximally entangled Bell state $|\psi^-\rangle=(1/\sqrt{2})[|0_A,1_B\rangle-|1_A,0_B\rangle]$. Alice performs a PV measurement $\{\Pi_{\theta_k}(a_k)=\frac{1}{2}(\mathbb{1}+a_k\sigma_{\theta_k})\}$ of one of the equatorial qubit observables σ_{θ_k}, which results in an outcome $a_k=\pm1$, leaving Bob's conditional state in the form

$$\rho^B_{a_k|\theta_k} = \mathrm{Tr}_A[\Pi_{\theta_k}(a_k)\otimes\mathbb{1}_B|\psi^-\rangle\langle\psi^-|]/p(a_k|\theta_k) = \Pi_{\theta_k}(a_k). \tag{37}$$

[Alice's outcomes $a_k=\pm1$ are totally random and occur with probability $p(a_k|\theta_k)=1/2$ for any measurement setting θ_k.]

Bob then performs sharp measurements $\{\Pi_{\theta_k}(b_k)\}$ on his state and computes the conditional average value of the observable $\sigma^B_{\theta_k}$ to obtain

$$\langle\sigma^B_{\theta_k}\rangle_{a_k|\theta_k} = \sum_{b_k=\pm1} b_k\mathrm{Tr}[\Pi_{\theta_k}(a_k)\Pi_{\theta_k}(b_k)] = a_k. \tag{38}$$

Further, evaluating the average of $\langle\sigma^B_{\theta_k}\rangle_{a_k|\theta_k}$ together with Alice's outcomes a_k, one obtains

$$\langle\sigma^A_{\theta_k}\sigma^B_{\theta_k}\rangle = \sum_{a_k=\pm1} a_k p(a_k)\langle\sigma^B_{\theta_k}\rangle_{a_k|\theta_k} = 1. \tag{39}$$

Thus, the left-hand side of the linear steering inequality (35) may be readily evaluated and it is given by $\frac{1}{N}\sum_{k=1}^N\langle\sigma^A_{\theta_k}\sigma^B_{\theta_k}\rangle=1$, which is clearly larger than the upper bound $f(N)$ of the steering inequality [note that $f(N)$ varies from its largest $f(2)\approx0.7071$ for $N=2$ measurement settings to its limiting value $f(\infty)=0.6366$ when $N\to\infty$]. In the next section we show that the violation of the steering inequality reduces to an inequality $\eta>\eta_{\rm opt}$ (i.e., the unsharpness parameter η of Alice's local equatorial qubit POVMs exceeds the cutoff value $\eta_{\rm opt}$ specifying their compatibility), which in turn implies that the set of Alice's measurements is incompatible.

It is pertinent to point out a modification of the finite setting linear steering inequality (35), a violation of which has been tested experimentally [51]: Including a single nonequatorial measurement of σ_z by Bob, the linear steering inequality (35), constructed for a finite set of equatorial observables, gets modified into a nonlinear steering inequality [34], the violation of which is shown to be more feasible for experimental detection than that of its linear counterpart [34]. In an ingenious experimental setup [51] where a single photon is split into two ports by a beam splitter, it has been rigorously demonstrated that a set of six different equatorial measurements in one port (i.e., Alice's end) can indeed steer the state of the photon in the other port (Bob's end).

C. Joint measurability condition from linear steering inequality

Now we proceed to discuss the implications of joint measurability on the linear steering inequality (35). If Alice performs an unsharp measurement of one of the equatorial qubit POVMs $\mathbb{E}_{\theta_k} = \{E_{\theta_k}(a_k) = \frac{1}{2}(\mathbb{1} + \eta a_k \sigma_{\theta_k})\}$ with an outcome $a_k = \pm 1$, Bob is left with the conditional state

$$\begin{aligned}\rho^B_{a_k|\theta_k} &= \mathrm{Tr}_A\{[E_{\theta_k}(a_k) \otimes \mathbb{1}_B]|\psi^-\rangle\langle\psi^-|\}/p(a_k|\theta_k)\\ &= \tfrac{1}{4}(\mathbb{1} - \eta a_k \sigma_{\theta_k})/p(a_k|\theta_k) = \bar{E}_{\theta_k}(a_k), \end{aligned} \tag{40}$$

the probability of Alice's obtaining the outcome a_k being $p(a_k|\theta_k) = 1/2$. Here we have denoted the spin-flipped version of the POVM $\{E_{\theta_k}(a_k) = (\mathbb{1} + \eta a_k \sigma_{\theta_k})/2\}$ by $\{\bar{E}_{\theta_k}(a_k) = (\mathbb{1} - \eta a_k \sigma_{\theta_k})/2\}$. Following Alice's measurement, Bob carries out sharp measurements $\{\bar{\Pi}_{\theta_k}(b_k) = (\mathbb{1} - b_k \sigma_{\theta_k})/2\}$ on his state and computes the conditional average value of the observable $\sigma^B_{\theta_k}$ to obtain

$$\begin{aligned}\langle\sigma^B_{\theta_k}\rangle_{a_k|\theta_k} &= \sum_{b_k=\pm1} b_k \mathrm{Tr}\big[\rho^B_{a_k|\theta_k} \bar{\Pi}_{\theta_k}(b_k)\big]\\ &= \sum_{b_k=\pm1} b_k \mathrm{Tr}[\bar{E}_{\theta_k}(a_k)(b_k)] = \eta a_k. \end{aligned} \tag{41}$$

Averaging the conditional expectation value $\langle\sigma^B_{\theta_k}\rangle_{a_k|\theta_k}$ with Alice's outcomes a_k, we obtain

$$\langle\sigma^A_{\theta_k}\sigma^B_{\theta_k}\rangle = \sum_{a_k=\pm1} a_k p(a_k|\theta_k)\langle\sigma^B_{\theta_k}\rangle_{a_k|\theta_k} = \eta. \tag{42}$$

Thus the finite setting linear steering inequality (35) reduces to

$$\eta \leqslant f(N). \tag{43}$$

This reduces to the joint measurability condition $\eta \leqslant \eta_{\rm opt}$ for Alice's local unsharp measurements, as one can identify the striking agreement between the degree of incompatibility $\eta_{\rm opt}$ [given by (12) and listed in Table II] and the upper bound $f(N)$ [given in (36)] of the finite setting linear steering inequality (35). This is a clear example of the intrinsic connection (established in Refs. [16–18]) between steering and measurement incompatibility. Moreover, the equivalence between the degree of incompatibility [as given in (12)] and the linear steering inequality in the finite setting [see (43)] highlights the relation between a local quantum feature, i.e., nonjoint measurability, and a nonlocal one, viz., steerability. Would it be possible to demonstrate measurement incompatibility without employing a nonlocal resource (i.e., an entangled state)? In this direction, it is pertinent to point out that timelike analogs of steering have been formulated recently [19,52] and there has been ongoing research interest in developing resource theories of measurement incompatibility and nonlocal steerability [20,21,23]. This leads us to formulate (in the next section) a local analog of the linear steering inequality (35) in a single-qubit system, the violation of which implies incompatibility of the qubit POVMs employed in first measurements of the sequential pair.

D. Local analog of the linear steering inequality

As has been discussed in previous sections, the expectation value of the qubit observable $\mathbf{S}_{\rm plane} = (1/\pi)\int_0^\pi d\theta\, \alpha_\theta \sigma_\theta$, $-1 \leqslant \alpha_\theta \leqslant 1$, is bounded by $2/\pi$ [see (29)]. This bound is not obeyed, in general, if the expectation value of the observable $\langle\sigma_\theta\rangle$ is replaced by its conditional expectation value $\langle\sigma_\theta\rangle_{a_\theta|\theta}$, evaluated in a sequential measurement, with the first measurement resulting in an outcome a_θ. In particular, in the setting where a finite number of pairwise sequential measurements of the equatorial qubit observable σ_{θ_k}, with the same angle θ_k, are carried out [53], the analog of the steering inequality (35)

$$\frac{1}{N}\langle\sigma^{(1)}_{\theta_k}\sigma^{(2)}_{\theta_k}\rangle \leqslant f(N) \tag{44}$$

could get violated in the single-qubit system. Here we have defined $\langle\sigma^{(1)}_{\theta_k}\sigma^{(2)}_{\theta_k}\rangle = \sum_{a_k} a_k p(a_k|\theta_k)\langle\sigma^{(2)}_{\theta_k}\rangle_{a_k|\theta_k}$; $\langle\sigma^{(2)}_{\theta_k}\rangle_{a_k|\theta_k} = \sum_{b_k} b_k p(b_k|a_k,\theta_k)$ defines the conditional expectation value of σ_{θ_k}, given that the first measurement has resulted in an outcome a_k with probability $p(a_k|\theta_k)$. Now we proceed to identify explicitly that violation of the inequality (44) is merely a consequence of measurement incompatibility.

Consider sequential measurements of equatorial qubit observables σ_{θ_k} in a single-qubit state $\rho = \frac{1}{2}\mathbb{1}$. Suppose an unsharp measurement $\{E_{\theta_k}(a_k) = (1/2)[\mathbb{1} + \eta a_k \sigma_{\theta_k}]\}$ results in an outcome a_k, with probability $p(a_k|\theta_k) = \mathrm{Tr}[\rho E_{\theta_k}(a_k)] = 1/2$. Correspondingly, the state undergoes a transformation

$$\rho \to \rho_{a_k|\theta_k} = E_{\theta_k}(a_k) \tag{45}$$

after the first measurement. Following this with another sharp PV measurement $\{\Pi_{\theta_k}(b_k) = (1/2)[\mathbb{1} + b_k \sigma_{\theta_k}]\}$, the resulting postmeasured state takes the form

$$\rho_{b_k|a_k;\theta_k} = [\Pi_{\theta_k}(b_k)\rho_{a_k|\theta_k}\Pi_{\theta_k}(b_k)]/p(b_k|a_k), \tag{46}$$

where

$$p(b_k|a_k) = \mathrm{Tr}[\rho_{a_k|\theta_k}\Pi_{\theta_k}(b_k)] = \tfrac{1}{2}[1 + \eta a_k b_k] \tag{47}$$

is the conditional probability of obtaining the outcome b_k in the second measurement. The conditional expectation value of the observable σ_{θ_k} in the second measurement is then evaluated to obtain

$$\langle\sigma^{(2)}_{\theta_k}\rangle_{a_k|\theta_k} = \sum_{b_k} b_k p(b_k|a_k,\theta_k) = \eta a_k. \tag{48}$$

The average value $\langle\sigma^{(1)}_{\theta_k}\sigma^{(2)}_{\theta_k}\rangle$, evaluated using the statistical data of the first measurement, results in

$$\langle\sigma^{(1)}_{\theta_k}\sigma^{(2)}_{\theta_k}\rangle = \sum_{a_k=\pm1} a_k p(a_k|\theta_k)\langle\sigma^{(2)}_{\theta_k}\rangle_{a_k|\theta_k} = \eta. \tag{49}$$

Thus, the inequality (35) reduces to $\eta \leqslant f(N)$ when N pairs of unsharp-sharp measurements are carried out sequentially in a single-qubit system. Clearly, the inequality is violated when only sharp PV measurements (with $\eta = 1$) are carried out. On the other hand, the inequality is always obeyed when the set $\{E_{\theta_k}(a_k), k = 1,2,\dots,N\}$ of all POVMs, employed in the first measurements of every sequential pair measurements, is jointly measurable. In other words, we have shown that violation of the local analog of the steering inequality (44) in a single-qubit system is a consequence of incompatibility of measurements of the qubit POVMs employed in first measurements of the sequential scheme.

V. CONCLUSION

Discerning the intrinsic connection between quantum nonlocality and measurement incompatibility is significant in that it leads to conceptual clarity in understanding different manifestations of nonclassicality. An interesting result by Banik *et al.* [43] revealed that the degree of measurement incompatibility (quantifying joint measurability of two dichotomic observables) places restrictions on the maximum strength of violation of the CHSH Bell inequality. A natural question then is whether such a quantitative connection exists in general, when more than two measurement settings are involved. In this paper we have explored the connection between the maximum achievable bound (Tsirelson-like quantum bound) on the violation of N-term pairwise correlation inequality [33] and the degree of measurement incompatibility of N dichotomic qubit POVMs, employed in carrying out the first measurement of sequential pairs measurements. To this end, we have constructed N-term chained correlation inequalities based on the positivity of a sequence of 4×4 moment matrices in the classical probability setting. Replacing the classical dichotomic random variables by qubit observables and the classical probability distribution by a quantum state, we obtain the analog of chained N-term correlation inequalities in the quantum scenario; in general the correlations do not obey the classical bound, resulting in the violation of the inequalities. The maximum achievable quantum bound (Tsirelson-like bound) on one of these chained inequalities, involving pairwise correlations of statistical outcomes of dichotomic observables measured sequentially in a single quantum system, is known [33] and the dichotomic observables, which result in the maximum quantum violation of the inequality, correspond to qubit observables, having equal successive angular separations of π/N in a plane. We have shown in this work that the N-term chained inequality (20) is always obeyed when the set of all POVMs employed in first measurements of every pairwise correlation term is compatible. However, measurement incompatibility of equatorial qubit POVMs serves, in general, as a necessary condition. For $N > 3$, incompatibility is not sufficient to result in violation of (20). To be specific, a tight relation between the degree of incompatibility and the maximum strength of quantum violation of the correlation inequality holds mainly in two special cases. (i) Measurements of a pair of dichotomic observables on one part of a bipartite quantum system are considered. In this case, the degree of incompatibility $\eta_{\rm opt} = 1/\sqrt{2}$ (for the pair of dichotomic observables to be jointly measurable) places an upper bound $2/\eta_{\rm opt} = 2\sqrt{2}$ on the maximum achievable quantum bound of the CHSH Bell inequality [43]. (ii) In a three-term correlation inequality (20), with a classical upper bound 1, sequential pairwise measurements of $N = 3$ dichotomic observables were carried out in a single-qubit system prepared initially in a maximally mixed state. The inequality is known to be violated maximally (the quantum upper bound being $3/2$) when the three dichotomic observables correspond to qubit orientations, forming a trine axis (three axes with equal successive angular separations of $\pi/3$ in a plane). In this case, the degree of measurement incompatibility of the three POVMs is given by $\eta_{\rm opt} = 2/3$. When these POVMs are used in first measurements of the sequential pair measurements, the degree of incompatibility places restrictions on the maximum achievable quantum bound [22], i.e., $1/\eta_{\rm opt} = 3/2$. In view of the focus of recent research on the equivalence between joint measurability and nonlocal steering [16–18], we have explored a linear steering inequality, introduced by Jones and Wiseman [34], which involves measurements of N equatorial plane qubit POVMs. We have shown that this indeed reveals a striking connection between the optimal violation of the N-term steering inequality and the degree of incompatibility of equatorial qubit POVMs.

Within the perspective of our study, it appears natural to ask if one can devise a local test (by carrying out a set of sequential measurements on a single quantum system) to infer information about measurement incompatibility, rather than employing a nonlocal steering protocol (which requires an entangled state)? We have addressed this question (by restricting the discussion to the specific example pertaining to N equatorial qubit observables) and have shown that a local analog of the linear steering inequality of Ref. [34] can be formulated in a single quantum system, involving a linear combination of pairwise conditional correlations, resulting from N sequentially ordered unsharp-sharp pairwise measurements (performed in independent statistical trials for each pair, with the input state for every first measurement being $\rho = \mathbb{1}/2$) of equatorial qubit observables. Violation of this local steering inequality is shown to be a reflection of measurement incompatibility of POVMs employed in the first of the sequential pairwise measurements.

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[40] In the generalized framework, one finds that pairwise joint measurability of POVMs does not imply their triplewise compatibility, even in the case of two-dimensional Hilbert space of qubits, which is not the case in the conventional PV measurement scenario [10,13,38]. Due to the existence of pairwise (but not triplewise) jointly measurable qubit POVMs, violations of a state-dependent inequality, which highlights Kochen-Specker measurement contextuality, by a quantum system in the two-dimensional Hilbert space has been explored recently [13,38]. This extended notion of contextuality gains importance due to the fact that the Kochen-Specker noncontextuality [2] could only be refuted in quantum systems belonging to a Hilbert space dimension $d \geqslant 3$, when one is confined to the conventional PV measurement framework.
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