A REPORT

ON

GENERAL RELATIVITY

Submitted in fulfilment of

Summer Research Fellowship – 2013

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JUNE – JULY, 2013

ACKNOWLEDGEMENTS

I am very grateful to Indian Academy Of Sciences for giving me this golden opportunity to work under Dr.Sumati Surya as a Summer Research Fellow. I would like to thank Mr.G.Madhavan for his excellent cooperation and coordination.

I would like to thank Dr.Sumati Surya for giving me the project and for guiding me. I would also like to convey my thanks to the Post-doctoral fellows Sandipan Sen Gupta and Miguel Campiglia for patiently listening my doubts and clarifying them.

G. Chetan Krishna

July 2013

ABSTRACT

Einstein's relativistic theory of gravitation – general relativity – is about a century old. At core it is one of the most revolutionary ideas of modern science – the idea that gravity is the geometry of four-dimensional curved spacetime.

General relativity has been accurately tested in the solar system. It underlies our understanding of the universe on the largest distance scales and central to the explanation of various astrophysical phenomena.

This report just gives an introduction to general relativity and some of the phenomena explained by this theory like the formation of black holes, the cosmological red shift e.t.c

CONTENTS

1.INTRODUCTION

The study of bodies in motion and the cause for the motion is broadly called as "*Mechanics"*. There are two different interpretations of mechanics given each by Newton and Einstein namely "*Classical Mechanics*" and *"General Relativity".*

The Newtonian view of mechanics is popularly known as "*Classical Mechanics*".

The main assumptions of Classical Mechanics are:

1. Space and time as completely distinct.
2. The geometry of space is defined in Cartesian coordinates by the line element $$dS^2 = dx^2 + dy^2 + dz^2$$
3. Time is absolute.

The Newtonian mechanics fails to give accurate results when the particles move with velocities comparable to the speed of light i.e $v/c \to 1$. The Einstein's theory of "*Special Relativity*" comes to the rescue in this situation.

a).General Relativity:

The General Relativity is the more general version of special relativity which explains the motion of particles even from non-inertial frames. General relativity is the only theory with firm experimental verification which can explain "*Gravity*". It does not consider gravity as a force but considers it as geometry of space-time.

The space-time is a four dimensional unification of space and time. The space-time geometry is non-Euclidean and the presence of mass curves the space-time around it. The particles move in the straightest possible path in this curved space-time.

General Relativity is in excellent agreement with the all experimental results till date. It assumes that the speed of light is the speed limit and only massless particles can achieve this velocity. The particles with nonzero mass can never achieve the speed of light because the force needed to achieve this would become infinite.

It is in close connection with classical electrodynamics. Like the Maxwell's Equations explain classical electrodynamics the gravity can be explained by a set of ten equations called "*The Einstein Equation*". The solutions to the equation give the geometry around a star or a black hole. The Vacuum Einstein equation is the equation similar to Poisson's equation in classical electrodynamics. Schwarzschild geometry is one of the solutions of Vacuum Einstein equation.

This report deals with the concepts of Gravity as a result of curvature in the spacetime geometry caused by the presence of matter, the geodesics followed by the objects moving in space time and Schwarzschild geometry.

2.GRAVITY AS GEOMETRY

General relativity considers gravity as an effect of curvature of four dimensional spacetime and not as a phenomena arising from forces and fields.

a).The Equivalence Principle:

"No experiment can distinguish a uniform acceleration from a uniform gravitational field".

The above statement is called the "*Equivalence Principle*" given by Einstein.

For example let us assume two closed laboratories '*A*' and '*B*'. The laboratory *A* is on the earth and the laboratory *B* is in deep space accelerating with the same acceleration as that of the 'g' of earth. Now the observer in laboratory *A* drops ball and a feather and finds their acceleration to be same as 'g'. Now the observer in '*B*' repeats the same experiment and obtains the same result. So, the observer in laboratory '*B*' cannot decide whether he is in a gravitational field or is he accelerating.

From the above example we can infer that the laboratory should be in a region where the gravitational field is uniform. Normally the gravitational field is different in different locations so the laboratory should be very narrow to be able to neglect the fluctuations. So the equivalence principle can also be stated as

Experiments in a sufficiently small freely falling laboratory, over a sufficiently short time, give results that are indistinguishable from those of the same experiment in an inertial frame in empty space.

b).Clocks in a Gravitational Field:

Let us consider a uniformly accelerating rocket with an acceleration 'g' along the z axis. Let observer A is at the head of the rocket and observer B is at the tail of the rocket. Let us assume observer A sends light signals to observer B frequently with the time difference between the successive signals as $\Delta\tau_A$ and the time difference between the reception of successive signals be $\Delta\tau_B$. Let the height of the rocket be 'h'. Then the position of the observers can be given as a function of time as

$z_A(t) = \frac{1}{2}gt^2$ and $z_B(t) = \frac{1}{2}gt^2 + h$

Let us consider the emission of two successive light pulses by A and their reception by B. Let $t = 0$ and $t = \Delta\tau_A$ be the times of emission of the signals and let t_1 be the time needed to reach B. The distance travelled by the first pulse is

$$z_A(0) - z_B(t_1) = ct_1$$

The distance travelled by the second pulse is shorter and given by

$$z_A(\Delta\tau_A) - z_B(t_1 + \Delta\tau_B) = c(t_1 + \Delta\tau_B - \Delta\tau_A)$$

Inserting the time dependent functions of positions in the above equations gives

$$h - \frac{1}{2} g t_1{}^2 = c t_1$$

$$h - \frac{1}{2} g t_1{}^2 - g t_1 \Delta\tau_B = c(t_1 + \Delta\tau_B - \Delta\tau_A)$$

Simplifying the above equations give

$$\Delta\tau_B = \Delta\tau_A \left(1 - \frac{gh}{c^2}\right)$$

gh is the gravitational potential difference between A and B.

Thus the time interval is depending on the gravitational potential at the location of the clock.

3.GEODESICS

A geodesic is a generalized notion of straight line in a curved space. The particles which are free from any influences other than the curvature of the spacetime are called free or freely falling in general relativity.

a).Geodesic Equation:

The Path followed by a particle in space-time is called its "*Geodesic*". The equations governing the motion of test particles and light rays in general curved spacetime are called "*Geodesic Equations*". These can be obtained from Variational calculus

Variational Principle for Free Test Particle Motion[1]
The world line of a free test particle between two timelike separated points extremizes the proper time between them.

The proper time along a timelike world line between two points A and B in spacetime is

$$\tau_{AB} = \int_A^B \left[-g_{\alpha\beta}(x) dx^\alpha dx^\beta\right]^{1/2}$$

The world line can be described parametrically by giving the four coordinates x^α as a function of a parameter σ that varies from 0 to 1 along the path from A to B. Then the proper time can be written as

$$\tau_{AB} = \int_0^1 d\sigma \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}\right]^{1/2}$$

Therefore the world lines that extremize the proper time should satisfy the Lagrange's equations $\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{dx^\alpha}{d\sigma}\right)}\right) - \frac{dL}{dx^\alpha} = 0$

For the Lagrangian

$$L\left(\frac{dx^\alpha}{d\sigma}, x^\alpha\right) = \left(-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right)^{1/2}$$

Where $g_{\alpha\beta}$ is the metric of the spacetime.

The general form of the geodesics in an arbitrary curved spacetime can be written as

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$$

There are four equations because there are four values for α.The coefficients $\Gamma^\alpha{}_{\beta\gamma}$ are called the *Christoffel symbols* and are constructed from the metric and its derivatives.They can be taken to be symmetric in the lower two indices.

If given a metric 'g' then the christoffel symbols can be calculated from the equation

$$g_{\alpha\delta}\Gamma^\delta{}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\delta}}{\partial x^\beta} - \frac{\partial g_{\alpha\delta}}{\partial x^\alpha}\right)$$

<u>b).Solving the Geodesic Equation</u>:

Conservation laws reduce the number of equations that need to be solved. One first integral that is always available comes from the normalization of the four-velocity vector. It can be written as

$$\boldsymbol{u}.\boldsymbol{u} = g_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} = -1$$

It is the only integral available for the most general metric. Further conservation laws arise from symmetries. For example symmetry in displacement in time gives rise to energy conservation and symmetry in displacement in space leads to linear momentum conservation e.t.c.

Killing Vectors are used to identify the symmetries. For example in spherical polar coordinates azimuthal symmetry is specified by the killing vector ξ having components

$\xi^\alpha = (0,0,0,1)$

Symmetry implies a conserved quantity along a geodesic. Let us suppose that L is independent of coordinate x^1 then $\partial L/\partial x^1 = 0$. That implies

$$\frac{d}{d\sigma}\left[\frac{\partial L}{\partial(dx^1/d\sigma)}\right] = 0$$

$$\frac{\partial L}{\partial(dx^1/d\sigma)} = -g_{1\beta}\frac{1}{L}\frac{dx^\beta}{d\sigma} = -g_{1\beta}\frac{dx^\beta}{d\tau} = -g_{\alpha\beta}\xi^\alpha u^\beta = -\boldsymbol{\xi}.\mathbf{u}$$

In an arbitrary coordinate system, a conserved quantity along a geodesic can be represented as

$$\boxed{\boldsymbol{\xi}.\mathbf{u}=\text{contant}}$$

c).Null Geodesics:

Null geodesics are the geodesics followed by the light rays. The same procedure is followed to solve the geodesic equations as specified earlier only with few changes. The changes are:

1. The affine parameter λ is not the spatial distance.
2. The normalization of velocity four-vector changes as

$$\boldsymbol{u}.\boldsymbol{u} = g_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} = 0$$

4.GEOMETRY OUTSIDE A SPHERICAL STAR

The simplest curved spacetimes in general relativity are the ones with the most symmetry. The geometry outside a spherical star is one of those and it is called as the *Schwarzschild geometry*. It is named after Karl Schwarzschild who solved the vacuum Einstein equation to find it in 1916.

a).Schwarzschild Geometry:

The line element in a particular suitable set of coordinates summarizing the Schwarzschild geometry is given by (G=c=1 units)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

The G=c=1 system of units is called geometrized units. The coordinates are called the *Schwarzschild coordinates* and the corresponding metric $g_{\alpha\beta}$is called the *Schwarzschild metric*. These are the following properties of this metric

1. Time Independent:
   The metric is independent of time. The killing vector associated with this symmetry is $\xi^\alpha = (1,0,0,0)$.
2. Spherically Symmetric:
   The 2D surface with t=r=constant is given by the line element
   $$d\Sigma^2 = r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$
   This surface has the geometry of a sphere with radius r in flat 3D space. Thus Schwarzschild geometry is symmetric under rotations about the z-axis. The killing vector associated with this symmetry is

$$\eta^{\alpha} = (0,0,0,1)$$

Here the coordinate r is not the radius of a sphere. Rather it is related with the area of 2D spheres of fixed r and t by the standard formula

$$r = (A/4\pi)^{1/2}$$

3. Mass M:
   For small values of M the coefficient of dr^2 can be expanded and we end up getting static weak field metric. Thus M can be identified as the total mass of the source of curvature.
4. Schwarzschild Radius:
   The metric becomes singular at $r = 0$ and $r = 2M$. The latter is called the Schwarzschild radius. It is the characteristic length scale for the curvature in the Schwarzschild geometry.
   The metric can be explicitly written as
   $g_{\alpha\beta}$= diag(-(1-2M/r), $(1-2M/r)^{-1}$,r^2,$r^2\sin^2\theta$)

b).Gravitational Redshift:

The change in observed frequency at infinity with respect to the emitted frequency is called as gravitational redshift. The energy of a photon measured by an observer with four-velocity $\mathbf{u_{obs}}$ is

$$\boldsymbol{E} = -\boldsymbol{p}.\boldsymbol{u_{obs}}$$

But the energy of a photon can also be represented by the equation

$$\boldsymbol{E} = \hbar\omega$$

The spatial components of the four-velocity are zero for a stationary observer. From normalizing condition we get

$$u^{t}{}_{obs}(r) = \left(1 - \frac{2M}{r}\right)^{-1/2}$$

Thus

$u^{\alpha}{}_{obs}(r) = (1 - 2M/r)^{-1/2}\ \xi^{\alpha}$

The frequency of photon measured by the stationary observer at a radius R is

$$\hbar\omega_{*} = \left(1 - \frac{2M}{r}\right)^{-1/2} (-\boldsymbol{\xi}.\mathbf{p})_{R}$$

Similarly at infinity

$$\hbar\omega_{\infty} = (-\boldsymbol{\xi}.\mathbf{p})_{\infty}$$

But $(-\boldsymbol{\xi}.\mathbf{p})$ is a conserved quantity. Therefore we get $\omega_{\infty} = \omega_{*}\left(1 - \frac{2M}{r}\right)^{1/2}$.

c).Particle Orbits:

i).Conserved Quantities:

In Schwarzschild geometry the quantities that are conserved are **ξ.u** and **η.u** because the metric is independent of t and ϕ respectively.

Let

$$e = -\boldsymbol{\xi}.\boldsymbol{u} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

$$l = \boldsymbol{\eta}.\boldsymbol{u} = r^2 \sin^2\theta \frac{d\varphi}{d\tau}$$

'e' is called as energy per unit rest mass and 'l' is called as angular momentum per unit rest mass.

ii).Effective Potential and Radial Equation:

We have already told that normalization of velocity four-vector gives an integral. So, let's normalize the four-velocity.

$$\boldsymbol{u}.\boldsymbol{u} = -1$$

$$-\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1$$

Now we can eliminate $\frac{dt}{d\tau}$ and $\frac{d\varphi}{d\tau}$ using the conserved quantities 'e' and 'l'. Doing so we get

$$-\left(1 - \frac{2M}{r}\right)^{-1} e^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{r^2} = -1$$

$$\frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right]$$

By defining a constant $\epsilon \equiv \frac{e^2-1}{2}$ and the effective potential as

$$V_{eff}(r) = \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right] = \frac{-M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}$$

The equation can be compactly written as

$$\boxed{\epsilon = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)}$$

The extrema of the effective potential can be found by solving the equation $\frac{dV_{eff}(r)}{dr} = 0$. There is one local minima and one local maxima, whose radii are given as

$$r_{\substack{min \\ max}} = \frac{l^2}{2M}\left[1 \pm \sqrt{1 - 12\left(\frac{M}{l}\right)^2}\right]$$

If $l/M < \sqrt{12}$ there is no extrema and the effective potential is always negative. If $l/M > \sqrt{12}$ the effective potential has one maxima and one minima. The maxima lie above $V_{eff}(r) = 0$ if $l/M > 4$ and otherwise lies below it. The nature of the orbit depends on the relation between ϵ and V_{eff}.

<u>iii).Radial Plunge Orbits</u>:

The simplest example is the radial free fall of a particle from infinity – l=0. The particle can start from infinity with various values of its kinetic energy corresponding to different values of ϵ. If the particle starts at rest from infinity then dt/dτ=1 at infinity and e=1.

We have

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0$$

We endup getting $u^\alpha = \left((1 - 2M/r)^{-1}, -(2M/r)^{1/2}, 0,0\right)$. Solving for r gives

$$r(\tau) = (3/2)^{2/3}(2M)^{1/3}(\tau_* - \tau)^{2/3}$$

Where τ_* is an arbitrary constant which fixes the proper time when r=0.

<u>iv).Stable Circular Orbits</u>:

Stable circular orbits occur at the radii $r = r_{min}$. The radius of this orbit decreases with decreasing l/M , but stable circular orbits are not possible at arbitrarily small radii. The innermost stable circular orbit occurs when $l/M = \sqrt{12}$ at the radius

$$\boxed{r_{ISCO} = 6M}$$

For a particle orbiting a gravitational source at r_{ISCO} we can prove that

$$\Omega^2 = \frac{M}{r^3}$$

The above equation is nothing but the kepler's law.

<u>d).Light Ray Orbits</u>:

The orbit of a light ray can also be calculated just as done for the particle but we have to consider that the affine parameter is different from the one used for particle orbits and also the normalization of velocity four vector gives **u.u**=0 rather than **u.u**= -1 for particle orbits. We endup getting $\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$ where $W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$.

5.GRAVITATIONAL COLLAPSE AND BLACK HOLES

a).Stars:

The total life of a star is an interplay two opposing forces.

1. The contracting gravitational force
2. The expanding forces of gases heated by *thermonuclear burning.*

A star is born when a cloud of interstellar gas consisting mostly of hydrogen and helium that is momentarily cooler, denser and lower in kinetic energy than its surroundings collapses due to gravitation. Compressional heating raises the temperature of the core high enough to ignite the nuclear fission reactions to release energy. The star then reaches a steady state where the energy lost to radiation is balance by that produced by the nuclear fission of hydrogen. Our sun is at present in this state.

As time proceeds eventually, a significant fraction of hydrogen in the star's core is exhausted and there is no longer enough thermonuclear force to provide the energy lost to radiation. Gravitational contraction resumes. Again the compressional heating raises the temperature until the reaction which burns helium to other elements ignite. The star becomes brighter and its surface temperature changes. This cycle repeats itself until the majority of the element left in the star is iron or its neighbouring atoms in the periodic table. These *iron peak nuclei* are therefore called the ashes of thermonuclear burning.

A star which runs out of thermonuclear fuel has two possibilities.

1. Equilibrium star supported against the force of gravity by a non-thermal source of pressure.
2. Star never reaches equilibrium and the end state is ongoing gravitational collapse.

b).The Schwarzschild Black Hole:

i).Eddington-Finkelstein Coordinates:

Let's consider an idealized case where the collapsing body and the spacetime around it are spherically symmetric. According to Newton's the Newtonian gravitational potential outside a spherically symmetrical body is given by $-GM/r$, whether or not the body is changing with time. A similar theorem in general relativity shows that, even though the mass distribution is time dependent, the geometry outside a spherically symmetric gravitational collapse is time-independent Schwarzschild geometry.

As the collapse proceeds, more and more of the Schwarzschild geometry is uncovered. Now we have to face the singularities in the Schwarzschild metric at $r=2M$ and $r=0$ and the significance in the change in sign of g_{tt} and g_{rr} at $r=2M$. The singularity at $r=2M$ turns out not to be a singularity in geometry of spacetime, but a singularity in Schwarzschild coordinates. To show this let use another coordinate system to represent the Schwarzschild geometry.

Let us make a transformation so as to change the coordinate 't' in Schwarzschild geometry as

$$t = v - r - 2M \log\left|\frac{r}{2M} - 1\right|$$

Starting either from $r < 2M$ or $r > 2M$ and transforming t to v in the line element given by Schwarzschild metric gives the result

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

This is called as Eddington-Finkelstein Coordinate system. The fact that the above metric is obtained by starting from the Schwarzschild metric starting from either $r < 2M$ or $r > 2M$ shows that these two regions, although separated by a singularity in Schwarzschild metric, are in fact smoothly connected.

At large values of r the metric approaches the flat metric with t replaced by v-r. But r=0 is a true singularity. It is a place of infinite spcetime curvature and infinite gravitational forces.

ii).Light Cones of the Schwarzschild Geometry:

The key to understanding the Schwarzschild geometry as a black hole is the behaviour of radial light rays. Since these are moving radially $d\theta = d\varphi = 0$ and $ds^2 = 0$

$$-\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr = 0$$

$$\left[-\left(1 - \frac{2M}{r}\right) dv + 2dr\right] dv = 0$$

i.e $v = constant$ or $-\left(1 - \frac{2M}{r}\right) dv + 2dr = 0$

By solving we get $v - 2\left(r + 2M\, log\left|\frac{r}{2M} - 1\right|\right) = constant$

$v = constant$ lines are incoming light rays. For the second solution for large values of r it represents outgoing rays. If $r < 2M$ the same equation represents incoming rays. The curve $r = 2M$ represents stationary light rays. The $r = 2M$ divides the spacetime into two regions:

1. The region outside $r = 2M$ from which light can escape to infinity.
2. The region inside $r = 2M$, where gravity is so strong that even light cannot escape.

This is the defining feature of black hole geometry. The surface $r = 2M$ is called the *event horizon* of a black hole.

iii).Geometry of the Horizon and Singularity:

The horizon $r = 2M$ is a three-dimensional null surface in the spacetime. Its normal vector points in the r-direction and is a null vector. The v=constant slice of the horizon is a two surface with the geometry of a sphere with area $A=16\pi M^2$, which is called the area of the horizon. The area does not change with v in the time independent Schwarzschild geometry. However it would change if matter fell into the black hole in a spherically symmetric way.

c).Collapse to a Black Hole:

i).Two Observers – The Inside Story:

To understand the consequences of a spherical collapse, consider two observers. One observer rides on the surface of a star down to r=0 whereas the other observer remains outside the horizon at a large fixed radius $r = r_R$. The geometry outside the star is the well-known Schwarzschild geometry. Suppose the falling observer carries a clock and communicates with the distant observer by sending out light signals at equally spaced times according to his clock.

The pulses emitted by the falling observer seize to reach the distant observer once the star has crossed $r = 2M$. So there is no way that the observer from inside the horizon can communicate with the distant observer and the distant observer can never receive any information from inside the horizon.

ii).The Observers – The Outside Story:

The distant observer never sees the star cross the radius $r = 2M$. The last signal to reach the distant observer is the one emitted just before the star crosses this radius. Furthermore the pulses emitted by the falling observer are received at increasingly longer intervals at the clock of the distant observer. The photons get red-shifted and the red-shift reaches infinity as the star reaches $r = 2M$.

d).Nonspherical Gravitational Collapse:

Realistic collapse situations are not exactly spherical. A pre-collapse star may be distorted by rotation.

1. *Formation of a singularity:*
   Once the collapsing star crosses the $r = 2M$ there is no escape from singularity even if the star is non-spherical.
2. *Formation of an Event Horizon:*
   Even if the star is non-spherical once the singularity is formed it is hidden from the distant observer so there is an event horizon formed around the non-spherical black hole.
3. *Area Increase:*
   As described earlier the area of the black hole increases when mass falls into it in a spherically symmetric way. However, even if the mass falls in non-spherically symmetric fashion, the area of the horizon increases.
   This is a consequence of the area increase theorem for black holes. It reminds us of the entropy in thermodynamics. Quantum mechanics can be used to show that the entropy of a black hole is proportional to its area.

$$\boxed{Entropy\ Of\ a\ black\ hole\ \propto Area\ of\ the\ blackhole}$$

6.COSMOLOGICAL MODELS

a).Homogeneous, Isotropic Spacetimes:

A homogeneous, isotropic spacetime is one for which the geometry is spherically symmetric about any point in space and the same at one point in space as at any other. Homogeneity and isotropy are the properties of space and not of spacetime. Homogeneous, isotropic spacetimes have a family of preferred three-dimensional special slices on which the three-dimensional geometry is homogeneous and isotropic.

The Flat Robertson – Walker Metric:

The simplest example of a homogeneous, isotropic geometry can be described by the line element

$$\boxed{ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)}$$

Where *a(t)* is a function of the time coordinate *t* called the scale factor. The above equation is called the *flat Robertson – Walker metric*, not because the spacetime is flat but because the geometry of the special slices is flat. It's a *Friedman – Robert – Walker* (FRW) model when the scale factor obeys the Einstein equation.

b).The Cosmological Redshift:

The flat Robertson – Walker geometry is time dependent. The energy of a particle will change as it moves in this geometry similarly to the way it would if it moved in a time – dependent potential. Let's now derive the simple relation that gives its form.

Let's rewrite the line element in polar coordinates:

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]$$

Let's fix our position as the origin. Consider an observer in a galaxy coordinate distance $r = R$ away. Suppose the observer at the distant galaxy emits a photon with frequency ω_e at time t_e which we receive at the present time t_0. Let ω_0 be the received frequency.

The pulse emitted travels on a radial null curve for which

$$ds^2 = 0 = --dt^2 + a^2(t)dr^2$$

Between the time interval t_e and t_0 the pulse travelled a spatial coordinate distance R. Therefore

$$R = \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

This relation connects the times of emission and reception with the coordinate distance travelled.

Suppose the observer in the distant galaxy emits a series of pulses separated by some short interval of time δt_e,i.e with the circular frequency $\omega_e=2\pi/\delta t_e$. The time interval δt_0 between the pulses at reception can be calculated as

$$\int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a(t)} = R = \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

Assuming that δt_e and δt_0 are small, we get

$$\frac{\delta t_0}{a(t_0)} - \frac{\delta t_e}{a(t_e)} = 0$$

This in terms of frequency can be represented as:

$$\frac{\omega_0}{\omega_e} = \frac{a(t_e)}{a(t_0)}$$

Although derived for a spatially flat FRW model, this relation holds in any of the homogeneous, isotropic models. In an expanding universe where $a(t)$ grows with t, the ratio $\frac{a(t_e)}{a(t_0)}$ will be less than one and the received frequency will be less than the emitted one. This is called the "*Cosmological Redshift*".

$$1 + z \equiv \frac{\lambda_0}{\lambda_e} = \frac{\omega_e}{\omega_0} = \frac{a(t_0)}{a(t_e)}$$

'z' is called the *red shift.*

For example, consider a galaxy a small distance away at the time of reception so that its coordinate separation R is small. Its distance at reception is

$$d = a(t_0)R$$

Any light ray from the galaxy travels along the null path. The coordinate time that it travels is

$(\Delta t)^2 = a^2(t_0)R^2 + (terms\ of\ order\ R^2)$. The time of travel is also d and $t_e=t_0-d$, both neglecting R^2 corrections.

$$z \equiv \frac{\Delta\lambda}{\lambda} = \left[\frac{\dot{a}(t_0)}{a(t_0)}\right] d$$

This is Hubble's law and it gives the connection of Hubble constant to the geometry of spacetime

$$H_0 \equiv H(t_0) \equiv \frac{\dot{a}(t_0)}{a(t_0)}$$

The inverse of H_0 is called the Hubble time, t_H. Hubble time is a convenient unit of time in cosmology.

7.CONCLUSION

As you can see *General Relativity* is the only theory at present which can explain the various complex phenomena observed in the universe. Its applications range from microscopic to macroscopic. All the astronomical phenomena can be predicted and explained by this theory.

It explains the evolution of a star. It is only a classical theory and a Quantum Mechanical formalism of General Relativity is not yet on firm footing. In future there can be another theory of which general relativity may become a special case like classical mechanics became a special case of the present general relativity.

Thus general relativity is a very important theory and plays a vital role in understanding the formation and evolution of our universe.

8.REFERENCES

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