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The Koslowski–Sahlmann representation: quantum configuration space

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Abstract

The Koslowski–Sahlmann (KS) representation is a generalization of the representation underlying the discrete spatial geometry of loop quantum gravity (LQG), to accommodate states labelled by smooth spatial geometries. As shown recently, the KS representation supports, in addition to the action of the holonomy and flux operators, the action of operators which are the quantum counterparts of certain connection dependent functions known as 'background exponentials'. Here we show that the KS representation displays the following properties which are the exact counterparts of LQG ones: (i) the abelian * algebra of $SU(2)$ holonomies and '$U(1)$' background exponentials can be completed to a C^* algebra, (ii) the space of semianalytic $SU(2)$ connections is topologically dense in the spectrum of this algebra, (iii) there exists a measure on this spectrum for which the KS Hilbert space is realized as the space of square integrable functions on the spectrum, (iv) the spectrum admits a characterization as a projective limit of finite numbers of copies of $SU(2)$ and $U(1)$, (v) the algebra underlying the KS representation is constructed from cylindrical functions and their derivations in exactly the same way as the LQG (holonomy-flux) algebra except that the KS cylindrical functions depend on the holonomies *and* the background exponentials, this extra dependence being responsible for the differences between the KS and LQG algebras. While these results are obtained for compact spaces, they are expected to be of use for the construction of the KS representation in the asymptotically flat case.

Keywords: loop quantum gravity, representations, quantum configuration space
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1. Introduction

Loop quantum gravity (LQG) is an attempt at canonical quantization of a classical Hamiltonian description of gravity in terms of an $SU(2)$ connection and its conjugate electric field on a Cauchy slice. The electric field plays the role of a triad and thereby endows the slice with a spatial geometry. One of the key results of LQG is that the corresponding quantum geometry associated with LQG states has a fundamental discreteness. The smooth classical geometry of space is then expected to arise through a suitably coarse grained view of this discrete geometry [1].

One may enquire as to whether it is possible to describe the effective smoothness of classical spatial geometry directly at the quantum level without explicit recourse to any coarse graining. Koslowski answered this question affirmatively by slightly modifying the standard LQG representation [2]. He did this through an assignation of an additional smooth triad field label to every kinematic LQG state in conjunction with a modification of the action of flux operators so as to make them sensitive to the additional label. As demonstrated in detail by Sahlmann [3], the area and volume operators then acquire, in addition to the standard LQG type discrete contributions, a 'smooth' contribution determined by this additional label.

As explained in detail elsewhere [5], our interest in the Koslowski–Sahlmann (KS) representation arises from the possibility of using it to explore asymptotic flatness in canonical quantum gravity. As a precursor to such an exploration, it is of interest to obtain a detailed understanding of the KS representation for the simpler case of compact spatial topology. Accordingly, building on the work of Koslowski and Sahlmann, we initiated a study of the KS representation in [4, 5].

In [4], it was shown that in addition to the holonomy and flux operators of LQG, the KS representation also supports the action of the quantum correspondents of certain classical functions of the connection called 'background exponentials'. Each such exponential $\beta_{\bar{E}}(A)$ is labelled by a background electric field $\bar{E}^a_i$ and defined as $\beta_{\bar{E}}(A) := \exp\,(i \int_\Sigma \bar{E}^a_i A^i_a)$. Building on this work and that of Sahlmann, in [5], we studied the imposition of gauge and diffeomorphism invariance in the KS representation.

In this work we further study the KS representation with a view to providing structural characterizations similar to those developed for LQG. We refer here to the beautiful developments in the field, mainly in the nineties, which provided a characterization of the LQG Hilbert space as that of square integrable functions on a quantum configuration space of 'generalized' connections, the square integrability being defined with respect to a suitable 'Ashtekar–Lewandowski' measure on this space [6–11]. Moreover, this quantum configuration space can be viewed as a projective limit space [8–11, 13, 14]. In this work we prove that exact counterparts of these characterizations exist for the KS representation. The layout of our paper, including a detailed description of our results, is as follows.

The results in the paper depend on the use of structures which relate to the algebraic properties of holonomies and background exponentials. The structures associated with holonomies are semianalytic edges, piecewise semianalytic curves, the groupoid of paths and its subgroupoids. These holonomy related structures are used to show the classic LQG results mentioned above (see for example [10, 11]). The structures associated with background exponentials are the vector space of semianalytic $SU(2)$ electric fields, its abelian group structure under addition, and the subgroups of this group which are generated by sets of *rationally independent* semianalytic electric fields. These sets of rationally independent semianalytic electric fields are the background exponential related counterparts of the sets of *independent* edges, the latter serving as sets of independent 'probes' of the space of connections in the LQG context [7, 10, 11]. Section 2 serves to review the above holonomy related structures (leaning heavily on the exposition of [10–12]) as well as to define the background exponential related ones. Section 2 also establishes our notation for the rest of the paper.

Most of our results depend on the validity of a key 'master lemma'. This lemma states that, given a set of independent probes and a corresponding set of elements in $SU(2)$ (one for each independent edge) and $U(1)$ (one for each rationally independent electric field), there exists a semianalytic connection such that the evaluation of the relevant set of holonomies and background exponentials on this connection reproduces the given set of group elements to arbitrary accuracy. In section 3, we provide a precise statement of this lemma and describe the idea behind its proof. The proof itself is technically involved and relegated to an appendix.

Section 4 is devoted to the derivation of C^* algebraic results for the KS representation. First, we show that the abelian Poisson bracket algebra of holonomies and background exponentials, $\mathcal{HBA}$, can be completed to a C^* algebra, $\overline{\mathcal{HBA}}$. From general C^* algebraic arguments [15], one concludes that the classical configuration space of connections is densely embedded in the Gel'fand spectrum Δ of $\overline{\mathcal{HBA}}$, so that the spectrum may be thought of as a space of 'generalized' connections. In order to understand elements of this space better, we show that every element of the spectrum is in correspondence with a pair of homomorphisms, one homomorphism from the path groupoid to $SU(2)$ and the other from the abelian group of electric fields to $U(1)$. The first homomorphism corresponds to the algebraic structure provided by the holonomies and the second to that provided by the background exponentials.

Next we turn our attention to the definition of a measure on the spectrum which allows the identification of the spectrum as the quantum configuration space for the KS representation. We show that the KS 'vacuum expectation' value defines a positive linear function (PLF) on $\overline{\mathcal{HBA}}$. Standard theorems then imply that this function defines a measure $d\mu_{KS}$ on the spectrum Δ and that the KS Hilbert space is isomorphic to the space $L^2(\Delta, d\mu_{KS})$. Next, we consider the electric flux operators. These operators map the finite span of KS spinnets into itself. We define the action of these operators on $L^2(\Delta, d\mu_{KS})$ through the identification of KS spin network states with appropriate cylindrical functions on the spectrum. The compatibility of the measure $d\mu_{KS}$ with the adjointness properties of the flux operators so defined, follows immediately from the fact that these adjointness properties are implemented in the KS representation.

Section 5 is devoted to the projective limit characterization of the quantum configuration space. We show that the spectrum Δ is homeomorphic to an appropriate projective limit space $\bar{\mathcal{A}}$ whose fundamental building blocks are products of finite copies of $SU(2)$ and $U(1)$. Once again, the $U(1)$ copies capture the structure provided by the background exponentials whereas the $SU(2)$ copies correspond to, as in LQG, the structure provided by the holonomies. Following Velhinho [10], we show this through the identification of the C^*-algebraic and the projective limit notions of cylindrical functions together with an appropriate application of the Stone–Weierstrass theorem. In this context, as we shall show, the C^* algebraic notion of cylindrical functions corresponds to polynomials in the holonomies *and* background exponentials and their projective analogues to the same polynomials with holonomies replaced by $SU(2)$ elements and background exponentials by $U(1)$ elements. Next, we show that the Haar measures on these building blocks define a consistent family of cylindrical measures on $\bar{\mathcal{A}} \equiv \Delta$ and that this family derives from the KS measure $d\mu_{KS}$ on Δ. We use this characterization of $d\mu_{KS}$ to show that, similar to LQG [14], $\mathcal{A}$ lies in a zero measure set within $\bar{\mathcal{A}}$.

Section 6 focuses on the analysis of the algebraic structure underlying the KS representation. Our motivation for such an analysis stems from recent work by Stottmeister and Thiemann (ST) [19] in which they point out that the KS representation is not a consistent representation of the standard holonomy-flux algebra of LQG. ST point out that any representation of the holonomy-flux algebra must satisfy an infinite number of identities involving flux operators and their commutators. They provide an explicit and beautiful example relating the double commutator of a triplet of fluxes to a single flux [19] and they show that the KS representation does not satisfy this identity. Since the holonomies and fluxes are well defined operators in the KS representation, this 'Stottmeister–Thiemann' obstruction calls into question the existence of a consistent algebraic structure underlying the KS representation. In section 6 we explicitly construct exactly such a consistent algebraic structure.

We proceed as follows. Recall that the construction of the standard holonomy-flux algebra, including the precise non-commutativity of the fluxes, is based on the work of Ashtekar, Corichi and Zapata (ACZ) [18]. The fluxes are functions on the phase space of gravity. ACZ studied the action of the Hamiltonian vector fields associated with these fluxes on functions of connections constructed out of holonomies. This action is that of a derivation and ACZ captured the non-commutativity of the flux operators in LQG through the non-commutativity of these (classical) derivations. This in turn led to the construction of the holonomy-flux algebra in terms of (cylindrical) functions of holonomies and their derivations. Accordingly, we generalize the ACZ considerations to the KS case wherein the space of connection dependent functions is built out of not only the holonomies but also the background exponentials. The ACZ arguments so generalized indicate the identification of the algebraic structure of the Poisson bracket between fluxes with the commutator of derivations on this (enlarged) space of (cylindrical) functions. This implies that similar to the considerations of [16], the KS counterpart of the holonomy flux algebra is generated by these functions, their derivations (which are obtained through the action of the flux Hamiltonian vector fields) and multiple commutators of these derivations. We call this algebra as the holonomy-background exponential-flux algebra. We show, from the considerations of sections 3 and 4, that the KS representation is indeed a representation of the holonomy-background exponential-flux algebra.

This algebra is different from the usual holonomy-flux algebra by virtue of the extra structure provided by the background exponentials. In particular, the derivations corresponding to the flux Hamiltonian vector fields have, so to speak, an extra set of $U(1)$ components in addition to the usual $SU(2)$ ones. It is this extra structure, directly traceable to the background exponential functions, which is responsible for the evasion of the ST obstruction. In other words: (i) there *is* a consistent algebraic structure underlying the KS representation, namely the holonomy-background exponential-flux algebra, (ii) this structure is different from the standard holonomy-flux algebra and (iii) this structure does not necessarily support the flux commutator identities which are satisfied by representations of the holonomy-flux algebra; in particular, as we explicitly show, it does not satisfy the triple flux commutator identity of [19]. This concludes our description of section 6.

Finally, section 7 contains a brief discussion of our results as well as some remarks on the asymptotically flat case.

2. Preliminaries

All differential geometric structures of interest will be based on the semianalytic, C^k, $k \gg 1$ category. The classical configuration space $\mathcal{A}$ is given by $su(2)$-valued one-forms $A_a = A_a^i \tau_i$ on a compact (without boundary) three-manifold Σ. τ_i, $i = 1, 2, 3$ are $su(2)$ generators with $[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$ and $A_a \in \mathcal{A}$ represents $SU(2)$ connections of a trivial bundle. The elementary configuration space functions are

$$h_e[A]_C^D := \left(\mathcal{P}e^{\int_e A}\right)_C^D, \tag{2.1}$$

$$\beta_E[A] := e^{i\int_\Sigma E\cdot A}, \tag{2.2}$$

where $C, D = 1, 2$, $h_e[A]_C^D \in SU(2)$ is the $j = 1/2$ holonomy of the connection A along an edge e (see section 2.1); $E \cdot A \equiv E_i^a A_a^i$ with $E^a = E_i^a \tau^i$ a non-dynamical, unit density weight, smearing electric field. $\beta_E[A] \in U(1)$ is referred to as a 'background exponential'.

2.1. Holonomy related structures

The probes associated with holonomies are *paths*. A path p is an equivalence class of oriented piecewise semianalytic curves on the manifold, where two curves are equivalent if they differ by orientation preserving reparametrizations and retracings. $b(p)$ and $f(p)$ denote the beginning and end points of a path p. Two paths p and p' such that $f(p) = b(p')$ can be composed to form a new path denoted by $p'p$. Thus $b(p'p) = b(p)$ and $f(p'p) = f(p')$. Under this composition rule, the set of all paths, $\mathcal{P}$, becomes a groupoid. An edge e is a path p that has a representative curve which is semianalytic, with the image $\tilde{e}$ of this representative curve being a submanifold with boundary. Paths can always be written as finite compositions of edges. See [10, 11] for more precise definitions.

For a given connection $A \in \mathcal{A}$, the holonomy along a path p, $h_p[A] \in SU(2)$, satisfies $h_{p'p}[A]_C^D = h_{p'}[A]_C^{C'} h_p[A]_{C'}^D$, where $f(p) = b(p')$. Thus A defines a *homomorphism* from $\mathcal{P}$ to $SU(2)$. The set of *all* homomorphisms from $\mathcal{P}$ to $SU(2)$ is denoted by $\mathrm{Hom}(\mathcal{P}, SU(2))$, and corresponds to the space of generalized connections in LQG [10].

A set of edges $e_1, \ldots, e_n$ is said to be independent if their intersections can only occur at their endpoints, i.e. if $\tilde{e}_i \cap \tilde{e}_j \subset \{b(e_i), b(e_j), f(e_i), f(e_j)\}$. We denote by $\gamma := (e_1, \ldots, e_n)$ an ordered set of independent edges and by $\mathcal{L}_\mathrm{H}$ the set of all such ordered sets of independent edges. Given $\gamma, \gamma' \in \mathcal{L}_\mathrm{H}$, we say that $\gamma' \geqslant \gamma$ iff all edges of γ can we written as composition of edges (or their inverses) of γ'. Equivalently, if $\mathcal{P}_\gamma$ denotes the subgroupoid of $\mathcal{P}$ generated by edges of γ, then $\gamma' \geqslant \gamma$ iff $\mathcal{P}_\gamma$ is a subgrupoid of $\mathcal{P}_{\gamma'}$. It then follows that (i) $\gamma \geqslant \gamma$ and (ii) $\gamma'' \geqslant \gamma', \gamma' \geqslant \gamma \Rightarrow \gamma'' \geqslant \gamma$. Thus '$\geqslant$' defines a *preorder* [23] in $\mathcal{L}_\mathrm{H}$.

A preorder is weaker than a partial order in that it does not necessarily entail anti-symmetry, i.e. $a \geqslant b, b \geqslant a$ does not necessarily imply $a = b$. For example, $\gamma' \geqslant \gamma,\ \gamma \geqslant \gamma'$ does not imply that $\gamma = \gamma'$ since the two relevant sets of edges may differ in the ordering of their elements or by the substitution of an edge by its inverse.

Next, note that semianalyticity of the edges implies that given $\gamma, \gamma' \in \mathcal{L}_\mathrm{H}$ there always exists γ'' such that $\gamma'' \geqslant \gamma$ and $\gamma'' \geqslant \gamma'$ [10, 11]. Thus $(\mathcal{L}_\mathrm{H}, \geqslant)$ is a directed set[1].

2.2. Background exponential related structures

The probes associated with the background exponentials are electric fields, and we denote by $\mathcal{E}$ the set of all (semianalytic) electric fields. We will often see $\mathcal{E}$ as an Abelian group with composition law given by addition: $(E, E') \to E + E'$. For a given connection $A \in \mathcal{A}$, the background exponential function (2.2) satisfies $\beta_{E'}[A]\beta_E[A] = \beta_{E'+E}[A]$ and thus defines an element in $\mathrm{Hom}(\mathcal{E}, U(1))$ (the set of homomorphism from $\mathcal{E}$ to $U(1)$). A set of electric fields $E_1, \ldots, E_N$ will be said to be independent, if they are algebraically independent, i.e. if they are

[1] The label set $\mathcal{L}_\mathrm{H}$ differs slightly from the one used in [10, 11] where labels are given by subgroupoids $\mathcal{P}_\gamma$, regardless of the choice of 'generator' γ. One can nevertheless use $\mathcal{L}_\mathrm{H}$ in the projective limit characterization of the quantum configuration space. See appendix C for details.

independent under linear combinations with integer coefficients[2]:

$$\sum_{I=1}^{N} q_I E_I = 0,\ q_I \in \mathbb{Z} \Longleftrightarrow q_I = 0,\ I = 1, \ldots, N. \tag{2.3}$$

We denote by $\Upsilon = (E_1, \ldots, E_N)$ an ordered set of independent electric fields. The set of all ordered sets of independent electric fields is denoted by $\mathcal{L}_{\mathrm{B}}$. Let $\mathcal{E}_\Upsilon = \mathbb{Z}E_1 + \ldots + \mathbb{Z}E_N \subset \mathcal{E}$ denote the subgroup of $\mathcal{E}$ generated by Υ. We then define $\Upsilon' \geqslant \Upsilon$ iff $\mathcal{E}_\Upsilon$ is a subgroup of $\mathcal{E}_{\Upsilon'}$, or equivalently if the electric fields in Υ can be written as algebraic combinations of those in Υ'. As in the edge case, it follows that $\geqslant$ is a preorder relation.

In appendix B.1 it is shown that given any finite set of electric fields (not necessarily independent), there always exists a finite set of algebraically independent electric fields that generates the original set. Applying this result to the set $\Upsilon \cup \Upsilon'$ for given $\Upsilon, \Upsilon' \in \mathcal{L}_{\mathrm{B}}$, we find $\Upsilon'' \in \mathcal{L}_{\mathrm{B}}$ satisfying $\Upsilon'' \geqslant \Upsilon$ and $\Upsilon'' \geqslant \Upsilon'$. Thus $(\mathcal{L}_{\mathrm{B}}, \geqslant)$ is a directed set.

2.3. Combined holonomy and background exponential structures

The combined set of labels associated with holonomies and background exponentials is given by pairs $l = (\gamma, \Upsilon) \in \mathcal{L}_{\mathrm{H}} \times \mathcal{L}_{\mathrm{B}} =: \mathcal{L}$ with preorder relation given by $(\gamma', \Upsilon') \geqslant (\gamma, \Upsilon)$ iff $\gamma' \geqslant \gamma$ and $\Upsilon' \geqslant \Upsilon$. $\mathcal{L}$ is then a directed set, which will be used in section 5 to construct the projective limit description of the KS quantum configuration space.

Given $l = (e_1, \ldots, e_n, E_1, \ldots, E_N) \in \mathcal{L}$ we define the group

$$G_l := SU(2)^n \times U(1)^N, \tag{2.4}$$

and the map

$$\pi_l \colon \mathcal{A} \to G_l, \tag{2.5}$$

$$A \mapsto \pi_l[A] := \Big(h_{e_1}[A], \ldots, h_{e_n}[A], \beta_{E_1}[A], \ldots, \beta_{E_N}[A]\Big). \tag{2.6}$$

Most of the results in the present work rely on the result that $\pi_l[\mathcal{A}] \subset G_l$ is dense in G_l for any label $l \in \mathcal{L}$. This is shown in section 3 and appendix A.

2.4. KS representation

The KS Hilbert space, $\mathcal{H}_{\mathrm{KS}}$, is spanned by states of the form $|s, E\rangle$, where s is an LQG spin network and E a background electric field. The inner product is given by

$$\langle s', E' | s, E\rangle = \langle s|s'\rangle_{\mathrm{LQG}} \delta_{E',E}, \tag{2.7}$$

where $\langle s|s'\rangle_{\mathrm{LQG}}$ is the spin network LQG inner product and $\delta_{E',E}$ the Kronecker delta.

Holonomies (2.1) and background exponentials (2.2) act by

$$\hat{h}_{e\ D}^{\ C} |s, E\rangle = |\hat{h}_{e\ \ \ B}^{\mathrm{LQG}\,A} s, E\rangle, \tag{2.8}$$

$$\hat{\beta}_{E'}|s, E\rangle = |s, E' + E\rangle. \tag{2.9}$$

Above, we have used the notation of [4] wherein given an LQG operator $\hat{O}$ with action $\hat{O}|s\rangle = \sum_I O_I^{(s)} |s_I\rangle$ in standard LQG, we have defined the state $|\hat{O}s, E\rangle$ in the KS representation through

[2] It is easy to verify that (2.3) is equivalent to rational independence, i.e. the analogue of condition (2.3) with $q_I \in \mathbb{Q}$.

$$|\hat{O}s,\, E\rangle := \sum_I O_I^{(s)}\,|s_I,\, E\rangle. \tag{2.10}$$

The action of fluxes is given by

$$\hat{F}_{S,f}|s,\, E\rangle = |\hat{F}_{S,f}^{\mathrm{LQG}}s,\, E\rangle + F_{S,f}(E)|s,\, E\rangle, \tag{2.11}$$

where f^i is the $su(2)$-valued smearing scalar on the surface S and $F_{S,f}(E) = \int_S \mathrm{d}S_a f^i E_i^a$ the flux associated with the background electric field E_i^a.

The KS representation supports a unitary action of spatial diffeomorphisms and gauge transformations, which can be used to construct a diffeomorphism and $SU(2)$ gauge invariant space via group averaging techniques [3, 5].

3. The master lemma

In this section we state the master lemma and describe the idea behind its proof. We conclude with a summary of the steps in the proof. These steps are implemented in detail in appendix A.

Statement of the lemma: let $e_1, \ldots, e_n$ be n independent edges. Let $E_1, \ldots E_N$ be N rationally independent semianalytic $SU(2)$ electric fields. Let G be the product group $\mathrm{G} := SU(2)^n \times U(1)^N$. Define the map ϕ: $\mathcal{A} \to \mathrm{G}$ through

$$\phi(A) := \Big(h_{e_1}[A],\, \ldots,\, h_{e_n}[A],\, \beta_{E_1}[A],\, \ldots,\, \beta_{E_N}[A]\Big). \tag{3.1}$$

Then the image $\phi(\mathcal{A})$ of $\mathcal{A}$ is dense in G.

Idea behind the proof: let $\mathrm{g} \in \mathrm{G}$ so that $\mathrm{g} = (g_1, \ldots, g_n, u_1, \ldots, u_N)$ where $g_\alpha \in SU(2)$, $\alpha = 1, \ldots, n$ and $u_I \in U(1)$, $I = 1, \ldots, N$. Then it suffices to show that for any given $\mathrm{g} \in \mathrm{G}$ and any $\delta > 0$, there exists an element $A^{\mathrm{g},\delta} \in \mathcal{A}$ such that

$$\left| h_{e_\alpha}\left[A^{\mathrm{g},\delta}\right]_C^{\ D} - g_{\alpha C}^{\ \ D} \right| \leqslant C_1 \delta \quad \forall \quad \alpha = 1, \ldots, n \text{ and } C, D = 1, 2. \tag{3.2}$$

$$\left| \beta_{E_I}\left[A^{\mathrm{g},\delta}\right] - e^{i\theta_I} \right| \leqslant C_2 \delta \quad \forall \quad I = 1, \ldots, N, \tag{3.3}$$

where C_1, C_2 are δ-independent constants, C, D are $SU(2)$ matrix indices and $u_I =: e^{i\theta_I}$, $\theta_I \in \mathbb{R}$. We shall show that the above equations hold with $C_1 = 0$ and an appropriate choice of C_2. In what follows we shall drop the superscript g in $A^{\mathrm{g},\delta}$ to avoid notational clutter.

First we construct a connection $A^{B,\delta}$ which satisfies equation (3.3). This is done using the rational independence of the set of electric fields in conjunction with standard results on the Bohr compactification of $\mathbb{R}^m$ [21]). In general, of course, the evaluation of the edge holonomies on this connection will not satisfy equation (3.2).

On the other hand, from standard LQG results [7], given *any* set of n group elements we are guaranteed the existence of a connection whose holonomies along the n independent edges $\{e_\alpha\}$ reproduce these group elements *exactly*. Further, these LQG results imply that such a connection A^ϵ can be constructed for any positive ϵ such that it vanishes everywhere except around balls of radius ϵ, each such ball intersecting the interior of each edge in an ϵ size segment (where ϵ is a coordinate distance, measured in fixed coordinate charts). Moreover, since the connection samples only an ϵ size segment of each edge, it can be shown that the connection is of order $1/\epsilon$. Clearly the *three dimensional integral* of such a connection yields order ϵ^2 contributions.

Were we to add such a connection to $A^{B,\delta}$ above, then, for small enough ϵ, it would have a negligible effect on the conditions (3.3). However, the connection $A^{B,\delta}$ has, in general, support on the set of edges $\{e_\alpha\}$ and hence contributes to the edge holonomies. The idea then is to carefully choose the connection A^ϵ so that its contributions together with those from $A^{B,\delta}$ yield the set of elements $\{g_\alpha\}$. In order to do this we need to cleanly seperate the contributions of $A^{B,\delta}$ from those of A^ϵ. This would be easy to do if $A^{B,\delta}$ and A^ϵ had mutually exclusive supports; if this were so the integral over each edge e_α of $A^{B,\delta} + A^\epsilon$ would seperate into contributions over segments of this edge where each segment supports either $A^{B,\delta}$ or A^ϵ but not both. We could then write each edge holonomy of $A^{B,\delta} + A^\epsilon$ in terms of compositions of holonomies along the segments of each edge, each segment holonomy being evaluated solely with respect to $A^{B,\delta}$ or solely with respect to A^ϵ. We could then choose A^ϵ so as to 'undo' the contributions from $A^{B,\delta}$ and yield the required group elements g_α

Indeed, as shown in the appendix, we can choose the supports of $A^{B,\delta}$ and A^ϵ such that each edge e_α can be written as $s_\alpha^1 \circ s_\alpha \circ s_\alpha^2$ with $\tilde{s}_\alpha$ in the support of A^ϵ and $\tilde{s}_\alpha^1$, $\tilde{s}_\alpha^2$ in the support of $A^{B,\delta}$ so that $h_{e_\alpha}[A^{B,\delta} + A^\epsilon]$ takes the form $h_{s_\alpha^1}[A^{B,\delta}]h_{s_\alpha}[A^\epsilon]h_{s_\alpha^2}[A^{B,\delta}]$. We then choose A^ϵ such that $h_{s_\alpha}[A^\epsilon] = (h_{s_\alpha^1})^{-1}g_\alpha(h_{s_\alpha^2})^{-1}$ so that conditions (3.2) are satisfied with $C_1 = 0$.

To obtain $A^{B,\delta}$ with the desired support we first construct a connection $\bar{A}^{B,\delta}$ which satisfies the conditions (3.3) and then multiply it with a semianalytic function of appropriate support. To do so, recall that the support of A^ϵ is in balls of size ϵ. We construct a ball of size 2ϵ around each such ball. Then the desired function is constructed so as to equal unity outside these balls of size 2ϵ, and vanish inside the ϵ size balls which support A^ϵ. Since the modification of $\bar{A}^{B,\delta}$ is only in regions of order ϵ^3, for small enough ϵ these modifications contribute negligibly to the background exponentials and one can as well use $A^{B,\delta}$ instead of $\bar{A}^{B,\delta}$ to satisfy the conditions (3.3).

The technical implementation of the proof then proceeds along the following steps which are detailed in appendix A.

(i) Using standard results from Bohr compactification of $\mathbb{R}^m$, we construct a connection $\bar{A}^{B,\delta}$ which satisfies (3.3) for some C_2.

(ii) For sufficiently small ϵ and for appropriately chosen ϵ- independent charts, we show the existence of balls $B_\alpha(2\epsilon)$, $\alpha = 1,..,n$ of coordinate size 2ϵ such that

$$B_\alpha(2\epsilon) \bigcap B_\beta(2\epsilon) = \varnothing \;\text{ iff }\; \alpha \neq \beta, \tag{3.4}$$

$$B_\alpha(2\epsilon) \bigcap \tilde{e}_\beta = \varnothing \;\text{ iff }\; \alpha \neq \beta, \tag{3.5}$$

$$\bar{B}_\alpha(2\epsilon) \bigcap \tilde{e}_\alpha \;\text{ is a semianalytic edge.} \tag{3.6}$$

(iii) We construct a real semianalytic function f_ϵ such that $|f_\epsilon| \leqslant 1$ on Σ with

$$f_\epsilon = 1 \;\text{ on }\; \Sigma - \bigcup_\alpha B_\alpha(2\epsilon) \tag{3.7}$$

$$= 0 \;\text{ on }\; \bigcup_\alpha B_\alpha(\epsilon), \tag{3.8}$$

where $B_\alpha(\epsilon)$ denotes the ϵ size ball with the same centre as $B_\alpha(2\epsilon)$.

(iv) From (3.6) it follows that

$$e_\alpha = s_\alpha^1 \circ s_\alpha \circ s_\alpha^2 \tag{3.9}$$

with

$$\tilde{s}_\alpha := \tilde{e}_\alpha \bigcap \bar{B}_\alpha(\epsilon) \tag{3.10}$$

$$\tilde{s}_\alpha^1 \bigcup \tilde{s}_\alpha^2 = \tilde{e}_\alpha \bigcap \left(\Sigma - B_\alpha(\epsilon)\right). \tag{3.11}$$

Define:

$$h_{s_\alpha^i}\left[A^{B,f}\right] =: g_\alpha^i, \quad i = 1, 2 \tag{3.12}$$

where $A^{B,f} := f_\epsilon \bar{A}^{B,\delta}$. Then we construct a connection A^ϵ supported in $\cup_\alpha B_\alpha(\epsilon)$ such that

$$h_{s_\alpha}\left[A^\epsilon\right] = \left(g_\alpha^1\right)^{-1} g_\alpha \left(g_\alpha^2\right)^{-1} \tag{3.13}$$

(v) We define $A^\delta := A^{B,f} + A^\epsilon$ and show that conditions (3.2) and (3.3) are satisfied with $C_1 = 0$ and some C_2.

4. C^*-algebraic considerations

4.1. C^* algebra $\overline{\mathcal{HBA}}$

We denote by $\mathcal{HBA}$ the *-algebra of functions of $\mathcal{A}$ generated by the elementary functions (2.1) and (2.2), with * relation given by complex conjugation. A generic element of $\mathcal{HBA}$ takes the form:

$$a[A] = \sum_{i=1}^{M} c_i \beta_{E_i'}[A] h_{e_1'^i}[A]_{C_1^i}^{D_1^i} \dots h_{e_{m_i}'^i}[A]_{C_{m_i}^i}^{D_{m_i}^i} \in \mathcal{HBA}, \tag{4.1}$$

for given M complex numbers c_i, M electric fields E_i', $\sum_{i=1}^{M} m_i$ edges $e_k'^i$, and choices of matrix elements for the $SU(2)$ holonomies, $C_k^i, D_k^i \in \{1, 2\}$. Since holonomies and background exponentials are bounded functions of $\mathcal{A}$, elements of $\mathcal{HBA}$ are bounded. Thus, the sup norm is well defined on $\mathcal{HBA}$:

$$\| a \| := \sup_{A \in \mathcal{A}} |a[A]|, \quad a \in \mathcal{HBA}. \tag{4.2}$$

Being a sup norm, it is compatible with the product and complex-conjugation star relations, so that upon completion we obtain a unital C^* algebra denoted by $\overline{\mathcal{HBA}}$. By Gel'fand theory $\overline{\mathcal{HBA}}$ can be identified with the C^* algebra of continuous functions on a compact, Hausdorff space Δ, $\overline{\mathcal{HBA}} \simeq C(\Delta)$. It will be useful for later purposes to denote by $\mathrm{Cyl}(\Delta) \subset C(\Delta)$ the subalgebra of continuous functions corresponding to $\mathcal{HBA}$ in the Gel'fand identification. $\mathrm{Cyl}(\Delta)$ will be referred to as the space of cylindrical functions of Δ.

Finally, the fact that $\mathcal{HBA}$ separates points in $\mathcal{A}$ implies that $\mathcal{A}$ is topologically dense in Δ [15]. Hence Δ represents a space of generalized connections.

4.2. Characterization of elements of $\mathcal{HBA}$

It will be useful to characterize elements of $\mathcal{HBA}$ by identifying independent edges and electric fields involved in any given algebra element as follows.

Let $a \in \mathcal{HBA}$ so that it is of the form (4.1).

Let $(E_1, \dots, E_N)$, $N \leqslant M$, be a set of independent electric fields as defined in section 2 in terms of which all E_J''s in (4.1) can be written as integral linear combinations:

$$E'_J = \sum_{I=1}^{N} k_J^I E_I, \quad I = 1, \dots, N, \quad k_J^I \in \mathbb{Z} \tag{4.3}$$

(such an algebraically independent generating set always exists, see appendix B.1). From (4.3) it follows that the background exponentials in (4.1) can be replaced by appropriate products of background exponentials (and their complex conjugates) associated with these independent electric fields.

Let $(e_1, \dots, e_n)$, $n \leqslant \sum_{i=1}^{M} m_i$ be a set of independent edges as defined in section 2 such that all edges in (4.1) can be obtained as compositions of them or their inverses. It follows that the edge holonomies in (4.1) can be replaced by products of holonomies (and their complex conjugates) along these independent edges.

With these replacements the element a acquires the form of a polynomial in the holonomies along the independent edges, background exponentials associated with the independent electric fields, and their complex conjugates. Thus we have shown that with $l := (e_1, \dots, e_n, E_1, \dots, E_N)$, $l \in \mathcal{L}$, the algebra element (4.1) takes the form:

$$a[A] = a_l\big(\pi_l[A]\big), \tag{4.4}$$

where $\pi_l[A] \equiv (h_{e_1}[A], \dots, h_{e_n}[A], \beta_{E_1}[A], \dots, \beta_{E_N}[A]) \in G_l \equiv SU(2)^n \times U(1)^N$ as in equation (2.6) and a_l: $G_l \to \mathbb{C}$ is a function that depends polynomially on the $SU(2)$ and $U(1)$ entries and their complex conjugates. Clearly, there exist many choices of l for which equation (4.4) holds.

It is then useful to define the notion of *compatibility* of a and l. Given $a \in \mathcal{HBA}$ so that a is necessarily of the form (4.1), let l be such that all edges and all electric fields in (4.1) can be obtained in terms of compositions of edges and integral linear combinations of electric fields in l. Then we shall say that l is *compatible* with a, that a is *compatible* with l and that a, l are *mutually compatible*. In this language what we have shown above is that given $a \in \mathcal{HBA}$ and l which is compatible with a, there exists a function a_l: $G_l \to \mathbb{C}$ with polynomial dependence on its $SU(2)$ and $U(1)$ entries and their complex conjugates such that (4.4) holds.

A key result which we shall need, and which follows directly from the lemma, may then be stated as follows:

Let $l \in \mathcal{L}$ and $a \in \mathcal{HBA}$ such that l is compatible with a. Then the function a_l as constructed above is the *unique* continuous function on G_l whose restriction to $\pi_l[\mathcal{A}] \subset G_l$ agrees with $a[A]$[3]. Accordingly, we shall say that (given mutually compatible a, l) a_l is *uniquely determined* by a, l.

Next, note that if l is compatible with a and l'' is such that $l'' \geqslant l$, then l'' is compatible with a so that we have

$$a[A] = a_{l''}\big(\pi_{l''}[A]\big) = a_l\big(\pi_l[A]\big). \tag{4.5}$$

It is of interest to elucidate the relationship between a_l and $a_{l''}$. We proceed as follows.

Since $l'' \geqslant l$, edges $e_i \in l$ can be written as compositions of edges in l''. Let us denote this relation by: $e_i = \tilde{p}_i(e''_1, \dots)$, where $\tilde{p}_i$ denotes a particular composition of edges (and their inverses) in l''. This corresponds to a relation on the holonomies of the form:

$$h_{e_i}[A] = h_{\tilde{p}_i(e''_1, \dots)}[A] = p_i(h_{e''_1}[A], \dots), \tag{4.6}$$

[3] The result follows immediately from the fact that a_l is manifestly continuous on G_l, that equation (4.4) gives the values of a_l on the set $\pi_l[\mathcal{A}] \subset G_l$ and that, by the lemma of section 3, this is a dense subset of G_l.

where p_i: $SU(2)^{n''} \to SU(2)$ is the map determined by interpreting the composition rules of $\tilde{p}_i$ as matrix multiplications. For example, if $e_1 = e_2'' \circ (e_1'')^{-1}$ then $p_1(g_1'', \ldots, g_{n''}'') = g_2''(g_1'')^{-1}$. Similarly, electric fields $E_I \in l$ can be written as integer linear combinations of electric fields in l'':

$$E_I = \tilde{P}_I\left(E_1'', \ldots\right) := \sum_{J=1}^{N''} q_I^J E_J'', \quad q_I^J \in \mathbb{Z}, \quad I = 1, \ldots, N. \tag{4.7}$$

Associated with (4.7) there is the map P_I: $U(1)^{n''} \to U(1)$ given by $P_I(u_1'', \ldots, u_{n''}'') = \Pi_{J=1}^{n''}(u_J'')^{q_I^J}$ so that

$$\beta_{E_I}[A] = \beta_{\tilde{P}_I\left(E_1'', \ldots\right)}[A] = P_I\left(\beta_{E_1''}[A], \ldots\right). \tag{4.8}$$

The above maps combine in a map[4]

$$p_{l,l''} := \left(p_1, \ldots, p_n, P_1, \ldots, P_N\right) : G_{l''} \to G_l \tag{4.9}$$

(that is 'block diagonal' in the $SU(2)$ and $U(1)$ entries). Equations (4.6) and (4.8) can then be summarized as

$$\pi_l[A] = p_{l,l''}\left(\pi_{l''}[A]\right). \tag{4.10}$$

Substituting (4.10) in the last term of (4.5) we find:

$$a_{l''}\left(\pi_{l''}[A]\right) = a_l\left(p_{l,l''}\left(\pi_{l''}[A]\right)\right). \tag{4.11}$$

Thus $a_{l''}$ and $a_l \circ p_{l,l''}$ coincide on the dense subset $\pi_{l''}[\mathcal{A}] \subset G_{l''}$. Since $a_{l''}$ and $a_l \circ p_{l,l''}$ are continuous functions on $G_{l''}$ we conclude that

$$a_{l''} = a_l \circ p_{l,l''}. \tag{4.12}$$

We conclude this section by noting one more consequence of the lemma:
The algebra norm (4.2) of $a \in \mathcal{HBA}$ as in (4.4) coincides with the sup norm on G_l of a_l:

$$\| a \| = \sup_{A \in \mathcal{A}} \left| a_l\left(\pi_l[A]\right)\right| = \sup_{g \in G_l} \left| a_l(g)\right|. \tag{4.13}$$

4.3. Characterization of Δ

One of the characterizations of the quantum configuration space in standard LQG is given by the set $\mathrm{Hom}(\mathcal{P}, SU(2))$ of homomorphisms from the path groupoid $\mathcal{P}$ to $SU(2)$ [10, 11]. The analogue space associated with the background exponentials is given by $\mathrm{Hom}(\mathcal{E}, U(1))$, the set of homomorphisms from the abelian group $\mathcal{E}$ of semianalytic electric fields (with abelian product given by addition) to $U(1)$.

We will now establish a one-to-one correspondence between Δ and $\mathrm{Hom}(\mathcal{P}, SU(2)) \times \mathrm{Hom}(\mathcal{E}, U(1))$.

First we show that any element $\phi \in \Delta$ defines an element of $\mathrm{Hom}(\mathcal{P}, SU(2)) \times \mathrm{Hom}(\mathcal{E}, U(1))$. Recall that from Gel'fand theory, ϕ is a C^* algebraic homomorphism from the C^* algebra $\overline{\mathcal{HBA}}$ to the C^* algebra of complex numbers $\mathbb{C}$. Let $\mathcal{HA}$ and $\mathcal{BA}$ be the *-algebras generated, respectively, by only the holonomies and by only the background exponentials. The sup norm on $\mathcal{HBA}$ defines a norm on each of $\mathcal{HA}$ and $\mathcal{BA}$ and the two algebras can then be completed in their norms so defined to yield the C^* algebras $\overline{\mathcal{HA}}$

[4] This map will be of later use in the projective limit construction of section 5.

and $\overline{\mathcal{BA}}$. Clearly $\overline{\mathcal{HA}}$ and $\overline{\mathcal{BA}}$ are subalgebras of $\overline{\mathcal{HBA}}$ with $\overline{\mathcal{HA}}$ being exactly the holonomy C^* algebra of LQG.

Let $\phi_{\rm H} := \phi|_{\overline{\mathcal{HA}}}$ be the restriction of ϕ to $\overline{\mathcal{HA}}$. $\phi_{\rm H}$ is a homomorphism from $\overline{\mathcal{HA}}$ to $\mathbb{C}$. By the standard LQG description [10, 11], $\phi_{\rm H}$ defines an element $s_\phi \in \mathrm{Hom}(\mathcal{P}, SU(2))$ given by

$$s_\phi(p)_C^{\ D} := \phi\left(h_{p\ C}^{\ \ D}\right). \tag{4.14}$$

Similarly, it is easy to verify that $\phi_{\rm B} := \phi|_{\overline{\mathcal{BA}}}$ defines an element $u_\phi \in \mathrm{Hom}(\mathcal{E}, U(1))$ given by

$$u_\phi(E) := \phi\left(\beta_E\right), \tag{4.15}$$

since $u_\phi(0) = 1$, $u_\phi(E_1 + E_2) = u_\phi(E_1)u_\phi(E_2)$ and $\overline{u_\phi(E)} = u_\phi(-E)$, implying $u_\phi(E) \in U(1)$ (see [11, 12] for the analogue statement in the context of $\mathbb{R}$-Bohr).

Conversely, given $s \in \mathrm{Hom}(\mathcal{P}, SU(2))$ and $u \in \mathrm{Hom}(\mathcal{E}, U(1))$ we want to find $\phi \in \Delta$ such that $u_\phi = u$ and $s_\phi = s$. Following the same strategy as in LQG [10–12], we first find a homomorphism ϕ: $\mathcal{HBA} \to \mathbb{C}$, and then show it is bounded and hence extends to $\overline{\mathcal{HBA}}$.

Given a general element $a \in \mathcal{HBA}$, it can always be written in the form (4.4) for any l compatible with a. Accordingly, we choose some compatible $l = (e_1, \ldots, e_n, E_1, \ldots, E_N)$ and define ϕ on $\mathcal{HBA}$ by:

$$\phi(a) := a_l\left(s(e_1), \ldots, s(e_n), u\left(E_1\right), \ldots, u\left(E_N\right)\right). \tag{4.16}$$

Since a_l is uniquely defined (see section 4.2), there is no ambiguity in this definition if l is specified. However, there are infinitely many l which are compatible with a. We now show that $\phi(a) \in \mathbb{C}$ given by (4.16) is *independent* of the choice of such l. Accordingly, let l', l be compatible with a. Then we need to show that

$$a_l\left(s(e_1), \ldots, u\left(E_1\right), \ldots\right) = a_{l'}\left(s\left(e_1'\right), \ldots, u\left(E_1'\right), \ldots\right). \tag{4.17}$$

This can be shown by writing a_l and $a_{l'}$ in terms of a finer label l'' such that $l'' \geqslant l'$ and $l'' \geqslant l$ according to (4.12), and using the homomorphism properties of s and u. Let p_i: $SU(2)^{n''} \to SU(2)$ and $P_I := U(1)^{n''} \to U(1)$ be the maps described in section 4.2 determined by the way probes of l are written in terms of those of l''. The homomorphism property of s and u implies that

$$s(e_i) = p_i\left(e_1'', \ldots\right), \tag{4.18}$$

$$u\left(E_I\right) = P_I\left(E_1'', \ldots\right), \tag{4.19}$$

substituting these relations in the rhs of (4.16) and using the result (4.12) we find

$$a_l\left(s(e_1), \ldots, u\left(E_1\right), \ldots\right) = a_{l''}\left(s\left(e_1''\right), \ldots, u\left(E_1''\right), \ldots\right). \tag{4.20}$$

Repeating the argument for the set l', one concludes

$$a_{l'}\left(s\left(e_1'\right), \ldots, u\left(E_1'\right), \ldots\right) = a_{l''}\left(s\left(e_1''\right), \ldots, u\left(E_1''\right), \ldots\right). \tag{4.21}$$

Hence (4.17) follows and (4.16) is independent of the choice of compatible l.

Next we show that ϕ so defined is a homomorphism to $\mathbb{C}$. By choosing any fixed l it trivially follows that:

(a) ϕ maps the zero element of $\mathcal{HBA}$ to $0 \in \mathbb{C}$.

(b) ϕ maps the unital element of $\mathcal{HBA}$ to $1 \in \mathbb{C}$.

(c) given any complex number C and algebra element $a \in \mathcal{HBA}$, $\phi(Ca) = C\phi(a)$.

Further note that there exists a 'fine enough' l which is simultaneously compatible with a given set of elements $a, b, ab, a+b, a^* \in \mathcal{HBA}$. From the continuity of $a_l, b_l, (ab)_l, (a+b)_l, (a^*)_l$ on G_l, the continuity preserving property of the operations of addition, multiplication and complex conjugation on the space of continuous functions on G_l and the uniqueness of the specification of any $c_l: G_l \to \mathbb{C}$ given mutually compatible $c \in \mathcal{HBA}$, $l \in \mathcal{L}$ (see section 4B), it follows that:

(d) $\phi(ab) = \phi(a)\phi(b)$, $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(a^*) = \phi(a)^*$.

Properties (a)–(d) show that equation (4.16) defines a homomorphism from $\mathcal{HBA}$ to $\mathbb{C}$. Next, we note that due to the lemma and equation (4.13) ϕ is bounded since:

$$|\phi(a)| = \left| a_l\left(s(e_1), \ldots, u\left(E_1\right), \ldots\right)\right| \leqslant \sup_{g \in G_l}\left|a_l(g)\right| = \|a\|. \tag{4.22}$$

It then follows that ϕ uniquely extends to a homomorphism from $\overline{\mathcal{HBA}}$ to $\mathbb{C}$ (see [11] around equation 6.2.71 for discussion of extension of bounded homomorphisms to completed algebra).

Finally it is easy to verify explicitly that (4.16) satisfies $u_\phi = u$ and $s_\phi = s$, thus establishing the correspondence between Δ and $\mathrm{Hom}(\mathcal{P}, SU(2)) \times \mathrm{Hom}(\mathcal{E}, U(1))$.

4.4. Realization of the KS Hilbert space as the space $L^2(\Delta, \mu_{\mathrm{KS}})$

We now use the KS representation of $\mathcal{HBA}$ (see section 2.4 and [5]) to construct a PLF on $\overline{\mathcal{HBA}}$. Given $a \in \mathcal{HBA}$, let us denote by $\hat{a}$ the corresponding operator in the KS Hilbert space $\mathcal{H}_{\mathrm{KS}}$. We define the PLF by

$$\omega(a) := \langle 0, 0 | \hat{a} | 0, 0 \rangle, \tag{4.23}$$

where $|0, 0\rangle \in \mathcal{H}_{\mathrm{KS}}$ is the KS 'vacuum' state corresponding to the trivial spin network and vanishing background electric field. As in LQG [10–12], the PLF can be written as an integral over the group elements: For $a \in \mathcal{HBA}$ given by (4.4), we have (see appendix B.2 for a proof):

$$\omega(a) = \int_{G_l} a_l(g)\mathrm{d}\mu_l, \tag{4.24}$$

for any $l \in \mathcal{L}$ compatible with a. Here $\mathrm{d}\mu_l$ is the Haar measure on the group G_l normalized so that $\int_{G_l} \mathrm{d}\mu_l = 1$. Boundedness of ω follows from the lemma via equation (4.4):

$$|\omega(a)| = \left| \int_{G_l} a_l(g)\mathrm{d}\mu_l \right| \leqslant \sup_{g \in G_l}\left|a_l(g)\right| = \|a\|. \tag{4.25}$$

Thus ω uniquely extends to $\overline{\mathcal{HBA}} \simeq C(\Delta)$ [11]. The Riesz–Markov theorem then implies the existence of a regular measure μ_{KS} on Δ such that

$$\omega(a) = \int_\Delta a \mathrm{d}\mu_{\mathrm{KS}}, \tag{4.26}$$

where in the rhs of (4.26) a is seen as an element of $C(\Delta)$ via the Gel'fand identification. By construction it follows that $\mathcal{H}_{\mathrm{KS}} \simeq L^2(\Delta, \mu_{\mathrm{KS}})$, since $\mathcal{H}_{\mathrm{KS}}$ can be identified with the GNS Hilbert space associated with ω, and the two constructions lead to the same representation [10].

Elements $a \in \overline{\mathcal{HBA}} \simeq C(\Delta)$ have now a dual interpretation: when seen as elements of $C(\Delta)$ we will interpret them as 'wavefunctions' in the L^2 representation, i.e. *vectors* on the Hilbert space. When seen as elements of $\overline{\mathcal{HBA}}$, we will usually associate them with *operators*

$\hat{a}$ on the Hilbert space $\mathcal{H}_{\mathrm{KS}}$. In the 'wavefunction' picture, $\mathrm{Cyl}(\Delta) \simeq \mathcal{HBA}$ plays a special role: It is a dense subspace of $L^2(\Delta, \mathrm{d}\mu_{\mathrm{KS}})$, which serves as a dense domain for the definition of the unbounded flux operators $\hat{F}_{S,f}$. In the next section we discuss the action of fluxes (2.11) in this 'wavefunction' picture.

4.5. Action of fluxes on $L^2(\Delta, \mu_{\mathrm{KS}})$

In the L^2 description of $\mathcal{H}_{\mathrm{KS}}$, the KS spinnet $|s, E\rangle$ corresponds to the 'wavefunction' $T_s\beta_E \in \mathrm{Cyl}(\Delta)$, where $T_s[A] \in \mathcal{HBA}$ is the spin network function associated with s [11], $\beta_E[A] \in \mathcal{HBA}$ the background exponential function (2.2), and T_s, β_E the respective elements in $\mathrm{Cyl}(\Delta)$ under the Gel'fand identification $\mathcal{HBA} \simeq \mathrm{Cyl}(\Delta)$.

Since the action of the flux operator $\hat{F}_{S,f}$ on $|s, E\rangle$ yields the finite linear combination of KS spinnets (2.11), we can translate this action as a map $\hat{F}_{S,f}$: $\mathrm{Cyl}(\Delta) \to \mathrm{Cyl}(\Delta)$. In this description, equation (2.11) takes the form

$$\hat{F}_{S,f}\left(T_s\beta_E\right) = \left(\hat{F}^{\mathrm{LQG}}_{S,f} T_s\right)\beta_E + F_{S,f}(E)T_s\beta_E. \tag{4.27}$$

Here $(\hat{F}^{\mathrm{LQG}}_{S,f} T_s) \in \mathrm{Cyl}(\Delta)$ denotes the finite linear combination of spin networks, $\hat{F}^{\mathrm{LQG}}_{S,f}|s\rangle$, obtained by the action of the flux operator labelled by (S, f) in the standard LQG representation. Recall that since $T_s[A] \in \mathcal{HBA}$, it is a polynomial in a set of independent edge holonomies. Using the algebraic identification of $\mathcal{HBA}$ with $\mathrm{Cyl}(\Delta)$, it then follows from standard LQG that the correspondent in $\mathcal{HBA}$ of $(\hat{F}^{\mathrm{LQG}}_{S,f} T_s) \in \mathrm{Cyl}(\Delta)$ is $-iX^{\mathrm{H}}_{S,f}(T_s[A]) \in \mathcal{HA} \subset \mathcal{HBA}$ where $X^{\mathrm{H}}_{S,f}$ is a 'derivative operator' whose action on $T_s[A]$ is built out of that of left and right invariant vector fields of $SU(2)$, on the $SU(2)$ valued edge holonomies underlying $T_s[A]$ [20].

It is useful for the purposes of section 6 to note that we may also re-express $X^{\mathrm{H}}_{S,f}(T_s[A])$ as the classical Poisson bracket $\{T_s[A], F_{S,f}\}$ (see for example [11, 18] as well as equation (6.3) below). This sort of re-expression extends to both the terms in the right-hand side of equation (4.27) so that the correspondent in $\mathcal{HBA}$ of the right-hand side can be written as the Poisson bracket $-i\{T_s[A]\beta_E[A], F_{f,S}\}$.

Next note that since *any* element of $\mathcal{HBA} \simeq \mathrm{Cyl}(\Delta)$ is a polynomial in the holonomies and background exponentials, the Peter–Weyl theorem (see for example [11]) implies that any such element can be re-expressed as a finite linear combination of KS spin net functions, i.e. $a[A]$ can be written as an expansion $\sum_i c_i T_{s_i}[A]\beta_{E_i}[A]$ for suitably defined spinnets s_i and electric fields E_i. Denoting the wave function in $\mathrm{Cyl}(\Delta)$ corresponding to $a[A] \in \mathcal{HBA}$ by a, it then follows from the previous paragraph and equation (4.27) that the $\mathcal{HBA}$ correspondent of $\hat{F}_{S,f}a$ is $-i\{a[A], F_{S,f}\}$[5].

5. The quantum configuration space as a projective limit

As in LQG, the quantum configuration space Δ admits a characterization as a projective limit space. In section 5.1 we describe the projective limit space, denoted by $\bar{\mathcal{A}}$, and show that it is homeomorphic to Δ. In section 5.2 we discuss measure theoretic aspects of $\bar{\mathcal{A}}$.

[5] This result can also be interpreted as follows. Since the KS representation is a representation of the classical Poisson algebra [5], it follows that given $a[A] \in \mathcal{HBA}$ and $\{a[A], F_{S,f}\} \in \mathcal{HBA}$, we have that $[\hat{a}, \hat{F}_{S,f}] = i\widehat{\{a, F_{S,f}\}}$. The 'wavefunction' associated with $a \in \mathcal{HBA}$ corresponds to the vector $\hat{a}|0\rangle \in \mathcal{H}_{\mathrm{KS}}$ where $|0\rangle$ is the KS vacuum. Using the fact that $\hat{F}_{S,f}|0\rangle = 0$ and the above commutation relation, one concludes that the $\mathcal{HBA}$ element associated with the wave function $(\hat{F}_{S,f}a)$ is given by $-i\{a[A], F_{S,f}\}$.

5.1. Topological identification of $\bar{\mathcal{A}}$ with Δ

The ingredients in the construction of $\bar{\mathcal{A}}$ are: (i) the directed set $\mathcal{L}$ and the family of compact spaces $\{G_l, l \in \mathcal{L}\}$ defined in section 2.3; (ii) the continuous projections $p_{l,l''}: G_{l''} \to G_l,\ l'' \geqslant l$ described in section 4.2, equation (4.9).

Recall that $p_{l,l''}$ is determined by the way probes in l are written in terms of probes in l''. These maps are surjective[6] and it is easy to verify that if $l'' \geqslant l' \geqslant l$ then $p_{l,l''} = p_{l,l'} \circ p_{l',l''}$. Thus $(\mathcal{L}, \{G_l\}, \{p_{ll'}\})$ satisfy the required conditions for the construction of a projective limit space [13, 23].

Let us describe the main features of $\bar{\mathcal{A}}$ (see appendix C for additional details as well as for a comparison with the usual construction in LQG). A point in $\bar{\mathcal{A}}$ is given by an assignment of points $x_l \in G_l$ for each $l \in \mathcal{L}$ satisfying the consistency condition $x_l = p_{l,l'}(x_{l'})$ whenever $l' \geqslant l$. $\bar{\mathcal{A}}$ is a compact Hausdorff space (see [11] and appendix C). We denote by $\{x_{l'}\}$ an element of the projective limit space. The canonical projections

$$p_l: \bar{\mathcal{A}} \to G_l, \tag{5.1}$$

$$\{x_{l'}\} \mapsto x_l, \tag{5.2}$$

satisfy $p_l = p_{l,l'} \circ p_{l'}$ for $l' \geqslant l$, are continuous[7], and as shown in [13] surjective[8].

Note that given a connection $A \in \mathcal{A}$, the points $\pi_l[A] \in G_l$ satisfy the projective consistency conditions with respect to the projections $p_{l,l'}$ by virtue of equation (4.10). This implies that every $A \in \mathcal{A}$ defines an element of $\bar{\mathcal{A}}$. Since holonomies and background exponentials seperate points in $\mathcal{A}$, it follows that this definition is unique so that there is a natural injection of $\mathcal{A}$ in $\bar{\mathcal{A}}$. It follows that with this injection, we have that

$$p_l(A) = \pi_l(A) \in G_l. \tag{5.3}$$

We now show, following [10, 13], that $\bar{\mathcal{A}}$ and Δ are homeomorphic by identifying their corresponding algebras of continuous functions, $C(\bar{\mathcal{A}})$ and $C(\Delta)$.

Let $\mathrm{Pol}(G_l)$ denote the set of functions on G_l that depend polynomially in their entries and their complex conjugates. This is the space of the functions a_l of section 4.2. Define

$$\mathrm{Cyl}\left(\bar{\mathcal{A}}\right) := \bigcup_{l\in L}\ p_l^*\ \mathrm{Pol}\left(G_l\right) \subset C\left(\bar{\mathcal{A}}\right), \tag{5.4}$$

where p_l^* is the pullback of the projections (5.1). Since the p_l's are continuous, elements of $\mathrm{Cyl}(\bar{\mathcal{A}})$ are continuous functions on $\bar{\mathcal{A}}$. An element $f \in \mathrm{Cyl}(\bar{\mathcal{A}})$ is always of the form $f = f_l \circ p_l$ for some $l \in \mathcal{L}$ and some $f_l \in \mathrm{Pol}(G_l)$. As in section 4 we will say that such l and f_l are *compatible* with f. If l is compatible with f with corresponding $f_l \in \mathrm{Pol}(G_l)$ then $l' \geqslant l$ is also compatible, with corresponding function $f_l \circ p_{l,l'} \in \mathrm{Pol}(G_{l'})$ (this follows from the property $p_l = p_{l,l'} \circ p_{l'}$). Note also that if l is compatible with f, the surjectivity of p_l implies the uniqueness of f_l, i.e. f_l is the *only* function on G_l such that $f = f_l \circ p_l$.

From the following four properties: (i) $\mathrm{Cyl}(\bar{\mathcal{A}})$ is a * subalgebra of $C(\bar{\mathcal{A}})$[9]; (ii) the constant function belongs to $\mathrm{Cyl}(\bar{\mathcal{A}})$; (iii) $\mathrm{Cyl}(\bar{\mathcal{A}})$ separates points in $\bar{\mathcal{A}}$ (since the 'coordinates' x_l belong to $\mathrm{Cyl}(\bar{\mathcal{A}})$); and (iv) $\bar{\mathcal{A}}$ is compact and Hausdorff, it follows from the

[6] The master lemma implies that $p_{l,l'}(G_{l'})$ is dense in G_l. Continuity of $p_{ll'}$ implies compactness of $p_{l,l'}(G_{l'})$. Since G_l is Hausdorff $p_{l,l'}(G_{l'})$ is closed and hence $p_{l,l'}(G_{l'}) = G_l$.

[7] The topology of $\bar{\mathcal{A}}$ corresponds to the weakest topology such that the maps (5.1) are continuous [11].

[8] We thank José Velhinho for pointing us to [13].

[9] This follows from the fact that for given functions $f, g \in \mathrm{Cyl}(\bar{\mathcal{A}})$, one can always find a common compatible label l so that the operations of linear combinations, products and complex conjugation in $\mathrm{Cyl}(\bar{\mathcal{A}})$ can be recast as the corresponding operations in $\mathrm{Pol}(G_l)$. For instance: $fg = (p_l^* f_l)(p_l^* g_l) = p_l^*(f_l g_l)$.

Stone–Weierstrass theorem that the completion of $\mathrm{Cyl}(\bar{\mathcal{A}})$ in the sup norm coincides with $C(\bar{\mathcal{A}})$ [10].

We now show that $\mathrm{Cyl}(\bar{\mathcal{A}})$ is isomorphic to $\mathcal{HBA}$. Given $f \in \mathrm{Cyl}(\bar{\mathcal{A}})$ and $l \in \mathcal{L}$, $f_l \in \mathrm{Pol}(G_l)$ compatible with f, we define the map

$$T: \ \mathrm{Cyl}\big(\bar{\mathcal{A}}\big) \to \mathcal{HBA} \tag{5.5}$$

$$f = f_l \circ p_l \mapsto T(f) = f_l \circ \pi_l. \tag{5.6}$$

From the properties of compatible labels and functions described in this section for elements of $\mathrm{Cyl}(\bar{\mathcal{A}})$ and in section 4 for elements of $\mathcal{HBA}$, it follows that $T(f)$ in (5.6) is independent of the choice of l and that T is an algebra homomorphism. Furthermore, by virtue of equation (4.13) and the surjectivity of $p_l \; \forall \; l$, it follows that $\| T(f) \|_{\mathcal{HBA}} = \| f \|_{\mathrm{Cyl}(\bar{\mathcal{A}})}$.

Going from $\mathcal{HBA}$ to $\mathrm{Cyl}(\bar{\mathcal{A}})$, recall from section 4.2 that given $a \in \mathcal{HBA}$, we can find a compatible $l \in \mathcal{L}$ such that $a[A] = a_l(\pi_l[A])$ with $a_l \in \mathrm{Pol}(G_l)$. For any such a_l we define $f := a_l \circ p_l$. We now show that this definition is independent of the choice of compatible l. Accordingly let l, l' be compatible with a. Consider any l'' compatible with a with $l'' \geqslant l, l'$. From equation (4.12) we have that $a_l \circ p_{ll''} = a_{l'} \circ p_{l'l''} = a_{l''}$ from which it follows that $a_l \circ p_l = a_l \circ p_{ll''} \circ p_{l''} = a_{l'} \circ p_{l'}$. Thus $f := a_l \circ p_l$ defines the same element of $\mathrm{Cyl}(\bar{\mathcal{A}})$ regardless of the choice of l. The uniqueness of a_l given a, l (see section 4.2) then implies that this map from $\mathcal{HBA}$ to $\mathrm{Cyl}(\bar{\mathcal{A}})$ is injective. Finally, it can easily be verified that this map is the inverse map of (5.6).

It follows that $\mathrm{Cyl}(\bar{\mathcal{A}})$ and $\mathcal{HBA}$ are equivalent as normed, * algebras. Hence their completions are isomorphic. By Gel'fand theory it follows that $\bar{\mathcal{A}}$ and Δ are homeomorphic, which completes our characterization of the quantum configuration space as a projective limit space.

As an application of this characterization, we demonstrate a curious 'cartesian' structure of $\bar{\mathcal{A}}$. Note that the master lemma of section 3 is a statement of a certain 'algebraic independence' of the '$U(1)$ probes' (namely the background exponentials) and the '$SU(2)$ probes' (namely the edge holonomies). This suggests that the quantum configuration space may admit a split into a 'holonomy' related part and a 'background exponential part'. Indeed, a product structure of this sort is implied by the characterization of section 4.3 wherein we showed that as a point set Δ could be identified with $\mathrm{Hom}(\mathcal{P}, SU(2)) \times \mathrm{Hom}(\mathcal{E}, U(1))$. However, no topological information is available in this characterization. We would like to see if the product structure persists when $\mathrm{Hom}(\mathcal{P}, SU(2))$, $\mathrm{Hom}(\mathcal{P}, SU(2))$ are equipped with suitably defined topologies so that the Gel'fand topology of Δ can be realized as a product topology. We found it difficult to show this using C^* algebraic methods because the norm on $\overline{\mathcal{HBA}}$ intertwines the properties of the holonomy and the background exponential structures. We now show that the projective limit characterization of Δ allows an immediate demonstration of the desired result.

Recall from section 2.3 that $\mathcal{L} = \mathcal{L}_{\mathrm{H}} \times \mathcal{L}_{\mathrm{B}}$ with $\mathcal{L}_{\mathrm{H}}$ and $\mathcal{L}_{\mathrm{B}}$ described in sections 2.1 and 2.2 respectively. Each label set can be separately used to construct the projective limit spaces $\bar{\mathcal{A}}_{\mathrm{H}}$ and $\bar{\mathcal{A}}_{\mathrm{B}}$. The relevant ingredients for $\bar{\mathcal{A}}_{\mathrm{H}}$ are the compact spaces $G_\gamma = SU(2)^n$ with n the number of edges in γ, and the maps $p_{\gamma\gamma'}$ as described after equation (4.6) (see also appendix C). Similarly the relevant ingredients for $\bar{\mathcal{A}}_{\mathrm{B}}$ are the spaces $G_\Upsilon := U(1)^N$ with N the number of electric fields in Υ and corresponding projections $p_{\Upsilon,\Upsilon'}$ as described after equation (4.7). Let $p_\gamma: \bar{\mathcal{A}}_{\mathrm{H}} \to G_\gamma$ and $p_\Upsilon: \bar{\mathcal{A}}_{\mathrm{B}} \to G_\Upsilon$ be the canonical projections analogous to (5.1).We now demonstrate that $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_{\mathrm{H}} \times \bar{\mathcal{A}}_{\mathrm{B}}$ are homeomorphic.

We first construct a bijection between the two spaces. Given $\{x_l\} \in \bar{\mathcal{A}}$, each $x_l \in G_l$ is given by a pair $x_\gamma \in G_\gamma$ and $x_\Upsilon \in G_\Upsilon$ where $l = (\gamma, \Upsilon)$ so that $G_l = G_\gamma \times G_\Upsilon$. The consistency condition on the x_l's implies consistency of the x_γ's and x_Υ's so that $\{x_\gamma\}$ defines an element in $\bar{\mathcal{A}}_{\rm H}$ and $\{x_\Upsilon\}$ an element in $\bar{\mathcal{A}}_{\rm B}$. Conversely, given $\{x_\gamma\} \in \bar{\mathcal{A}}_{\rm H}$ and $\{x_\Upsilon\} \in \bar{\mathcal{A}}_{\rm B}$ the corresponding point in $\bar{\mathcal{A}}$ is given by $x_l \equiv x_{(\gamma,\Upsilon)} := (p_\gamma(\{x_{\gamma'}\}), p_\Upsilon(\{x_{\Upsilon'}\}) \in G_l$. $\{x_l\}$ so defined satisfies the consistency conditions and hence defines an element of $\bar{\mathcal{A}}$ which corresponds to the inverse of the previous mapping.

We now show that this bijection between $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_{\rm H} \times \bar{\mathcal{A}}_{\rm B}$ is a homeomorphism. Recall that the topologies of $\bar{\mathcal{A}}$, $\bar{\mathcal{A}}_{\rm H}$, $\bar{\mathcal{A}}_{\rm B}$ are generated by inverse projections p_l^{-1}, p_γ^{-1}, p_Υ^{-1} of open sets in G_l, G_γ, G_Υ. Given $l = (\gamma, \Upsilon)$ and open sets $U_\gamma \subset G_\gamma$, $U_\Upsilon \subset G_\Upsilon$, the bijection between $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_{\rm H} \times \bar{\mathcal{A}}_{\rm B}$ identifies the open set $p_l^{-1}(U_\gamma \times U_\Upsilon) \in \bar{\mathcal{A}}$ with the open set $p_\gamma^{-1}(U_\gamma) \times p_\Upsilon^{-1}(U_\Upsilon) \in \bar{\mathcal{A}}_{\rm H} \times \bar{\mathcal{A}}_{\rm B}$. Since the product topology on G_l is generated by rectangle sets and since

$$
\begin{aligned}
p_l^{-1}\Big(\cup_\alpha \, \big(U_\gamma^\alpha \times U_\Upsilon^\alpha\big)\Big) &= \cup_\alpha \; p_l^{-1}\big(U_\gamma^\alpha \times U_\Upsilon^\alpha\big),\\
p_l^{-1}\Big(\cap_i \, \big(U_\gamma^i \times U_\Upsilon^i\big)\Big) &= \cap_i \; p_l^{-1}\big(U_\gamma^i \times U_\Upsilon^i\big),
\end{aligned} \tag{5.7}
$$

for some (possibly non-denumerable) label set α and finite label set i, it follows that the bijection is indeed a homeomorphism thus completing the proof.

To relate this result with that of section 4, let us denote by $\Delta_{\rm H}$ the spectrum of $\overline{\mathcal{HA}}$ and $\Delta_{\rm B}$ that of $\overline{\mathcal{BA}}$. The arguments of the present and previous sections may be reproduced for each of the algebras $\mathcal{BA}$ and $\mathcal{HA}$ to conclude that $\bar{\mathcal{A}}_{\rm B} \simeq \Delta_{\rm B} \simeq \mathrm{Hom}(\mathcal{E}, U(1))$[10] and $\bar{\mathcal{A}}_{\rm H} \simeq \Delta_{\rm H} \simeq \mathrm{Hom}(\mathcal{P}, SU(2))$ (the latter being the standard characterizations in LQG). Thus, the product structure presented here coincides with that of section 4.

5.2. Measure theoretic aspects of the projective limit space

Recall from appendix C that we have two equivalent projective limit constructions of $\bar{\mathcal{A}}$. The first, which we have used hitherto in this section, is based on the preordered directed set of labels $\mathcal{L} = \{l\} = \{(\gamma, \Upsilon)\}$. The second is based on the partially orderered directed label set $\hat{\mathcal{L}} = \{\hat{l}\} = \{(\hat{\gamma}, \hat{\Upsilon})\}$ where, as detailed in appendix C, $\hat{\gamma}$ corresponds to the path subgroupoid of $\mathcal{P}$ generated by the edges in γ and $\hat{\Upsilon}$ to the abelian subgroup of $\mathcal{E}$ generated by the electric fields in Υ. In what follows we shall follow the argumentation of [10]. Since this reference uses a partially ordered label set in its analysis, we shall use the second, partially ordered directed label set, characterization of $\bar{\mathcal{A}}$.

Let the Haar measure on $G_{\hat{l}}$ be $\mu_{\hat{l}}$. Note that the Haar measure $\mu_{\hat{l}}$ is a regular Borel measure. As shown in appendix C.3, the set of measures $\{\mu_{\hat{l}}\}$ satisfies the consistency condition $(p_{\hat{l},\hat{l}'})_* \mu_{\hat{l}'} = \mu_{\hat{l}}$ whenever $\hat{l}' \geqslant \hat{l}$, where $(p_{\hat{l},\hat{l}'})_* \mu_{\hat{l}'}$ is the push-forward measure. Let $C(\bar{\mathcal{A}}) \equiv C(\Delta) \equiv \overline{\mathcal{HBA}}$ be the C^* algebra of continuous functions on $\bar{\mathcal{A}}$. Our demonstration above that $\mathrm{Cyl}(\bar{\mathcal{A}}) = \mathcal{HBA}$ implies that $\mathrm{Cyl}(\bar{\mathcal{A}})$ is dense in $C(\bar{\mathcal{A}})$. It then follows from proposition 4, section 3.2 of [10] that the consistent family of (regular Borel) Haar measures $\{\mu_{\hat{l}}\}$ defines a unique regular Borel measure on $\bar{\mathcal{A}}$. A natural question is if this projective limit measure is the same as the measure $\mu_{\rm KS}$[11]. We now show that the answer to this question is in the affirmative. The proof of proposition 4 in [10] uses the fact that the consistent set of measures define a PLF on $C(\bar{\mathcal{A}})$. From appendices B.2 and C.3, it follows that this PLF is exactly the KS PLF of equation (4.23). It immediately follows from this fact, together with the

[10] In order to show this we use the fact that, just as for $\mathcal{HA}$, elements of $\mathcal{BA}$ seperate points in $\mathcal{A}$.

[11] Note that the identification of $\bar{\mathcal{A}}$ with Δ as a topological space implies the identification of corresponding Borel algebras so that this question is well posed.

unique association of the measure μ_{KS} with the KS PLF via the Riesz–Markov theorem, that the projective limit measure is μ_{KS}.

As an application of the projective limit characterization of μ_{KS}, it is straightforward to check that a simple adaptation of the proof of Marolf and Mourão [14] for the LQG case shows that, as in LQG, the classical configuration space $\mathcal{A}$ lies in a set of measure zero in the quantum configuration space $\bar{\mathcal{A}}$. The main feautures of this adaptation are as follows (we assume familiarity with the notation and contents of [14]):

(a) Choose the subsets $\triangle^{\{\epsilon_i\}_{i=1}^n}$ of $SU(2)^n$ as in [14]. For each $q \in (0, q_0]$ (for some fixed q_0, $0 < q_0 < 1$), choose $\{\epsilon_i\}_{i=1}^{\infty} = \{\epsilon_i^{(q)}\}_{i=1}^{\infty}$ as in [14].
(b) For each fixed q, replace the set of shrinking (independent) hoops $\{\beta_i\}$ of [14] by a set of shrinking edges $\{e_i\}$. To specify this edge set, fix a coordinate chart $\{x, y, z\}$ around some point $p_0 \in \Sigma$ so that p_0 is at the origin and let e_i be the straight line along the z-axis from the origin to $(0, 0, \epsilon_i^{(q)}\delta_i)$ where δ_i is as defined in [14].
(c) Use the small edge expansion for the edge holonomy of a connection $A \in \mathcal{A}$ to show that its edge holonomy is confined to a neighbourhood of the identity of size $\epsilon_i^{(q)}$ for sufficiently large i.

It is then easy to see that the desired result follows by a repetition of the proof of Marolf and Mourão (without the complications of quotienting by the action of gauge transformations, since we are interested in $\mathcal{A}$ rather than $\mathcal{A}/\mathcal{G}$).

We leave other applications of projective techniques (such as the definition of a host of projectively consistent 'differential geometric' structures [11, 13]) to future work.

6. The holonomy-background exponential-flux algebra

Our construction of the holonomy-background exponential-flux algebra parallels that of the holonomy-flux algebra in [16, 18] and we assume familiarity with those works. The only minor difference between our treatment and theirs is that we restrict attention to polynomial cylindrical functions of the form (4.1) whereas the cylindrical functions of [16, 18] comprise all continuous functions rather than only polynomials. In section 6.1 we construct the holonomy-background exponential-flux algebra. In section 6.2 we use an identity of ST [19] to illustrate the difference between the holonomy-background exponential-flux algebra and the LQG holonomy flux algebra. In section 6.3 we show that the KS representation is a representation of the holonomy-background exponential-flux algebra.

6.1. Construction of the classical and quantum algebras.

Let $a \in \mathcal{HBA}$ and let $l \in \mathcal{L}$ be compatible with a (see section 4). Then it is straightforward to verify that:

$$\left\{a[A],\, F_{S,f}\right\} =: X_{S,f}\, a[A], \tag{6.1}$$

$$X_{S,f} := X^{\mathrm{H}}_{S,f} + X^{\mathrm{B}}_{S,f}, \tag{6.2}$$

$$X^{\mathrm{H}}_{S,f} := \sum_{e_i \in l} \left(X^{\mathrm{H}}_{S,f} h_{e^i C_i}{}^{D^i} \right) \frac{\partial}{\partial h_{e^i C_i}{}^{D^i}}, \tag{6.3}$$

$$X^{\mathrm{B}}_{S,f} := \sum_{E_I \in l} \left(X^{\mathrm{B}}_{S,f} \beta_{E_I} \right) \frac{\partial}{\partial \beta_{E_I}}. \tag{6.4}$$

Here $(X^{\mathrm{H}}_{S,f} h_{e_i})$ is defined exactly as in LQG so that its evaluation involves the appropriate action of $SU(2)$ invariant vector fields on $h_{e_i} \in SU(2)$. The evaluation of $(X^{\mathrm{B}}_{S,f} \beta_{E_I})$ involves the analogous action of the $U(1)$ invariant vector field on $\beta_{E_I} \in U(1)$:

$$X^{\mathrm{B}}_{S,f} \beta_{E_I} = \int_S \mathrm{d}S_a f^i E^a_{iI} \frac{\partial}{\partial \theta} e^{i\theta} \Big|_{e^{i\theta} = \beta_{E_I}} = iF_{S,f}\left(E_I\right) \beta_{E_I}. \tag{6.5}$$

From equations (6.2)–(6.5) it follows that the operators $X_{S,f}$, $X^{\mathrm{H}}_{S,f}$, $X^{\mathrm{B}}_{S,f}$ all act as *derivations* on $\mathcal{HBA}$, i.e. they map $\mathcal{HBA}$ into itself and their actions obey the Leibniz rule (see equation (6.8) below). It is also easy to check that

$$\left[X^{\mathrm{B}}_{S,f}, X^{\mathrm{B}}_{S',f'} \right] = \left[X^{\mathrm{B}}_{S,f}, X^{\mathrm{H}}_{S',f'} \right] = 0 \quad \text{on} \quad \mathcal{HBA}. \tag{6.6}$$

Next, note that if we choose $a \in \mathcal{HBA}$ such that it depends only on the holonomies, $X_{S,f}$ acts exactly as in LQG. In other words, the action of $X_{S,f}$ restricted to $\mathcal{HA} \subset \mathcal{HBA}$ ($\mathcal{HA}$ is defined in 4.3) is exactly the LQG action. It then follows, similar to the LQG case, that while the commutator of a pair of derivations $[X_{S,f}, X_{S',f'}]$ on $\mathcal{HBA}$ is itself, in general, not of the form $X_{S'',f''}$, this commutator still acts as a derivation on $\mathcal{HBA}$. As in the LQG case, consider, the finite span of objects V_{deriv} of the form:

$$aX_{S,f},\ a\left[X_{S_1,f_1}, X_{S_2,f_2} \right],\ a\left[X_{S_1,f_1}, \left[... \left[X_{S_{n-1},f_{n-1}}, X_{S_n,f_n} \right] ... \right] \right]. \tag{6.7}$$

(Note that the classical correspondents of these objects are $aF_{S,f}$, $a\{F_{S_1,f_1}, F_{S_2,f_2}\}$, $a\{F_{S_1,f_1}, \{...\{F_{S_{n-1},f_{n-1}}, F_{S_n,f_n}\}...\}\}$, where $a \in \mathcal{HBA}$). It follows that every $Y \in V_{\mathrm{deriv}}$ acts as a derivation on $\mathcal{HBA}$ i.e.

$$Y: \mathcal{HBA} \to \mathcal{HBA}, \quad Y(ab) = Y(a)b + aY(b), \quad \forall\ a, b \in \mathcal{HBA}. \tag{6.8}$$

Next, define the vector space $\mathfrak{A} = \mathcal{HBA} \times V_{\mathrm{deriv}}$. We define the $*$ operation on $\mathfrak{A}$ by:

$$(a, Y)^* = \left(\bar{a}, \bar{Y} \right), \quad \text{where} \quad \bar{Y}(b) := \overline{Y\left(\bar{b}\right)}, \tag{6.9}$$

where $\bar{a}$ denotes the complex conjugate of a. From the definitions (6.2)–(6.5), it follows that $\bar{X}_{S,f} = X_{S,f}$. This can then be used to show that multiple commutators of the $X_{S,f}$'s are also invariant under the '‐' relation. It then follows straightforwardly that the $*$ operation maps $\mathfrak{A}$ to itself and defines a $*$ relation on $\mathfrak{A}$. Finally, we define the Lie bracket [,] on $\mathfrak{A}$ by

$$[(a, Y), (a', Y')] = (Y'(a) - Y(a'), [Y, Y']), \tag{6.10}$$

where $[Y, Y'] \in V_{\mathrm{deriv}}$ is the commutator of the derivations Y, Y' on $\mathcal{HBA}$. We refer to $(\mathfrak{A}, [\ ,\ ])$ as the *classical holonomy-background exponential-flux algebra*. It is the exact counterpart of the (classical) ACZ holonomy-flux algebra (referred to as $(\mathfrak{A}_{\mathrm{class}}, \{\ ,\ \})$ in [16]) underlying the standard LQG representation. The classical Lie algebra $\mathfrak{A}$ can be converted to its quantum counterpart $\hat{\mathfrak{A}}$ through the steps of section 2.5 of [16].

In the next section we comment on the differences between the algebra $\mathfrak{A}$ and its LQG counterpart $\mathfrak{A}_{\mathrm{class}}$ [16]. For notational convenience, we shall denote $\mathfrak{A}_{\mathrm{class}}$ by $\mathfrak{A}^{\mathrm{LQG}}$.

6.2. On the difference between $\mathfrak{A}$ and $\mathfrak{A}^{\mathrm{LQG}}$

As is apparent from the previous section, the construction of $\mathfrak{A}$ differs from that of $\mathfrak{A}^{\mathrm{LQG}}$ due to the added structure provided by the background exponentials. In more detail, the generators of $\mathfrak{A}$ (see equation (6.7)) differ from those of $\mathfrak{A}^{\mathrm{LQG}}$ (equation (19) in [16]) in two ways. First, the cylindrical function a in (6.7) depends on the background exponentials as well as the holonomies so that $a \in \mathcal{HBA}$ whereas the cylindrical function Ψ in (19) of [16] depends only on the holonomies so that $\Psi \in \mathcal{HA}$. The second difference is that the derivation $X_{S,f}$ and its commutators in (6.7) inherit their algebraic properties from their realization as derivations on $\mathcal{HBA}$ whereas the corresponding LQG objects in (19) of [16] inherit theirs from their realization as derivations on $\mathcal{HA}$. In contrast to the first, the second difference is a bit subtle. To see it explicitly, we turn our attention to the beautiful example considered by ST in [19].

Accordingly, consider $(0, [X_{S,f_1}[X_{S,f_2}, X_{S,f_3}]]) \in \mathfrak{A}^{\mathrm{LQG}}$. From the ST identity [19], we have that

$$\left[X_{S,f_1}\left[X_{S,f_2}, X_{S,f_3}\right]\right] = \frac{1}{4} X_{S,[f_1,[f_2,f_3]]} \tag{6.11}$$

where both the left-hand side and the right-hand side are derivations on $\mathcal{HA}$. This implies the identification of the elements $(0, [X_{S,f_1}[X_{S,f_2}, X_{S,f_3}]])$ and $(0, \frac{1}{4} X_{S,[f_1,[f_2,f_3]]})$ in the algebra $\mathfrak{A}^{\mathrm{LQG}}$.

Let us now consider $(0, [X_{S,f_1}[X_{S,f_2}, X_{S,f_3}]])$ as an element of $\mathfrak{A}$. From equations (6.2) and (6.6) it follows that

$$\left(0, \left[X_{S,f_1}\left[X_{S,f_2}, X_{S,f_3}\right]\right]\right) = \left(0, \left[X^{\mathrm{H}}_{S,f_1}\left[X^{\mathrm{H}}_{S,f_2}, X^{\mathrm{H}}_{S,f_3}\right]\right]\right). \tag{6.12}$$

From (6.11), (6.3) it immediately follows that

$$\left[X^{\mathrm{H}}_{S,f_1}\left[X^{\mathrm{H}}_{S,f_2}, X^{\mathrm{H}}_{S,f_3}\right]\right] = \frac{1}{4} X^{\mathrm{H}}_{S,[f_1,[f_2,f_3]]}. \tag{6.13}$$

Now, from (6.2) it follows that $X^{\mathrm{H}}_{S,[f_1,[f_2,f_3]]} \neq X_{S,[f_1,[f_2,f_3]]}$ because of the missing '$U(1)$' contribution, $X^{\mathrm{B}}_{S,[f_1,[f_2,f_3]]}$, which in turn means that in contrast to the LQG case, the two elements $(0, [X_{S,f_1}[X_{S,f_2}, X_{S,f_3}]]_{\mathrm{LQG}})$ and $(0, \frac{1}{4} X_{S,[f_1,[f_2,f_3]]})$ are *not* identified in the algebra $\mathfrak{A}$.

6.3. The KS representation and $\hat{\mathfrak{A}}$

As shown in [16] elements in $\hat{\mathfrak{A}}$ are of the form $\hat{a}\hat{X}_{S_1,f_1}\hat{X}_{S_2,f_2}..\hat{X}_{S_n,f_n}$ where $\hat{a}$ and $\hat{X}_{S_i,f_i}$ are the quantum correspondents of $a \in \mathcal{HBA}$ and F_{S_i,f_i}. The algebraic properties of $\hat{a}$ derive from those of $a \in \mathcal{HBA}$ and the algebraic properties of $\hat{X}_{S_i,f_i}$ derive from the algebraic properties of X_{S_i,f_i} as derivations on $\mathcal{HBA}$. Thus the algebraic properties of elements in $\hat{\mathfrak{A}}$ derive from those of $\mathcal{HBA}$ and derivations thereon.

Next we note the following:

(1) There is an algebraic isomorphism between cylindrical functions on Δ, $\mathrm{Cyl}(\Delta)$ and $\mathcal{HBA}$ (see section 4).

(2) The KS representation is an $L^2(\Delta, \mathrm{d}\mu_{\mathrm{KS}})$ representation. The space of cylindrical functions $\mathrm{Cyl}(\Delta)$ is dense in $L^2(\Delta, \mathrm{d}\mu_{\mathrm{KS}})$. The operators $\hat{a}$, $\hat{F}_{S,f}$ map $\mathrm{Cyl}(\Delta)$ into itself and their actions can be inferred from the algebraic isomorphism between $\mathrm{Cyl}(\Delta)$ and $\mathcal{HBA}$ (see section 4.5). In particular, given $\Psi \in \mathrm{Cyl}(\Delta)$ with correspondent $\Psi[A] \in \mathcal{HBA}$:

  (i) $\hat{a}$ acts by multiplication so that $\hat{a}\Psi := a\Psi$, where $a \in \mathrm{Cyl}(\Delta)$ is the correspondent of the element $a[A] \in \mathcal{HBA}$.

(ii) From section 4.5 and equation (6.1), the correspondent of $\hat{F}_{S,f}\Psi$ in $\mathcal{HBA}$ is $-iX_{S,f}\Psi[A]$ so that the algebraic properties of $\hat{F}_{S,f}$ are determined by those of the derivation $X_{S,f}$ on $\mathcal{HBA}$.

(3) The KS inner product implements the * relations on $\hat{a}$, $\hat{F}_{S,f} \in \hat{\mathfrak{A}}$ as adjointness relations.

The discussion in the first paragraph of this section together with (1)–(3) above show that the KS representation is indeed a representation of $\hat{\mathfrak{A}}$.

7. Discussion

In this work we have shown that the KS representation admits structural characterizations which are the counterparts of LQG ones. An immediate question is whether these characterizations can also be used to show a LOST-Fleischhack type [16, 17] uniqueness theorem based on the holonomy-background exponential-flux algebra of section 6. If so, the KS representation would then be the unique representation of this algebra with a cyclic, diffeomorphism invariant and $SU(2)$ gauge invariant state and at the kinematic level, there would be little to choose between the KS and the LQG representations. A key question is then if any progress is possible in the KS representation with regard to the quantum dynamics. While [3, 5] suggest that there is no fundamental obstruction to the imposition of $SU(2)$ gauge invariance and spatial diffeomorphism invariance, we do not know if the Hamiltonian constraint can be defined in this representation. Of course, if one takes the view that the KS representation is some sort of effective description for smooth spatial geometry, and that the underlying fundamental description is that of LQG, this question is moot.

The results presented in this work and summarized in detail in section 1 have been derived in the context of compact (without boundary) spatial topology. However, our main interest in the KS representation is in its possible application to asymptotically flat quantum gravity[12]. In contrast to the compact case, in the asymptotically flat case the classical connection and its conjugate triad field are required to satisfy detailed boundary conditions at spatial infinity.

The triad field is required to approach a fixed flat triad at spatial infinity. It is difficult to tackle the triad boundary conditions in an LQG like representation because of the contrast of the discrete spatial geometry underlying LQG with the smooth, asymptotically flat spatial geometry in the vicinity of spatial infinity. In particular, the spatial geometry needs to be excited in a non-compact region, which means that the LQG spin networks of the compact case need to be generalized so as to have infinitely many edges and vertices [24]. Moreover, a suitably coarse grained view of the quantum spatial geometry in the vicinity of spatial infinity must coincide with an asymptotically flat one. On the other hand, the KS representation offers the possibility of already accounting for smooth spatial geometry without coarse graining, and, asymptotically flat spatial geometry without the consideration of KS spinnet graphs with infinitely many edges [5]; in brief this may be achieved by restricting attention to KS spin net states whose graphs have a finite number of compactly supported edges but whose triad label satisfies the asymptotic conditions. The boundary conditions on the connection are, however, much harder to tackle. This is so because the natural spinnet basis has much more controllable behaviour with respect to the electric flux operators both in the LQG and the KS representations. Since we already have a possibility of encoding the triad boundary conditions in the KS representation, let us focus on the issue of connection boundary conditions in the KS (as opposed to the LQG) representation.

[12] For an application to the case of parametrized field theory, see [25, 26].

As we have shown, in the compact case, one way in which the quantum configuration space is tied to the classical configuration space is that the latter embeds into the former as a dense set. One may hope that something similar happens in the asymptotically flat case, namely, the quantum configuration space is the topological completion of the space of semianalytic connections *satisfying the asymptotic conditions*. While this would be one way in which the classical boundary conditions on the connections leave their imprint in the quantum theory, one may worry that since the quantum configuration space is larger than the classical one, perhaps a (for example, measure theoretically,) large number of quantum connections could be thought of as violating these boundary conditions. We now argue that this fear could be misplaced. In the holonomy-background exponential algebra, $\mathcal{HBA}$, classical connections are integrated against one dimensional edges to give holonomies and against three dimensional background fields to give background exponentials. If, as mentioned in the previous paragraph, in the asymptotically flat case we restrict the edges of interest to be confined to compact regions, it is only the background fields which sense the asymptotic behaviour of the connection. Let us then focus only on the structures associated with the background exponentials. A preliminary analysis indicates that the fields which label the background exponential functions satisfy the 'maximally' permitted asymptotic behaviour which allows their integrals with respect to (classical) connections to be well defined. If it transpires that the quantum connections, as in the compact case, define homomorphisms from the abelian group of fields, subject to this 'maximal' asymptotic behaviour, the very existence of these homomorphisms could be reasonably interpreted as the imposition of classical asymptotic behaviour on the quantum configuration space.

Thus, the KS representation offers hope that the complications arising due to asymptotic flatness are not insurmountable, at least at the level of quantum kinematics. We are at present engaged in working out the ideas sketched above. To conclude, we remark that if these efforts meet with success, they may also shed light on how to generalize LQG to asymptotically flat spacetimes so as to retain the most remarkable feature of the theory, namely the fundamental discreteness of space.

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Appendix A. Proof of the master lemma

Step (i):

The N background electric fields E_i are rationally independent but not necessarily linearly independent. Let $m \leqslant N$ be the dimension of the linear span of the background electric fields and assume $\{E_1, \ldots, E_N\}$ is ordered so that the first m electric fields are linearly independent. The last $p := N - m$ electric fields can then be written as linear combinations of the first m

ones:

$$E_{m+j} = \sum_{\mu=1}^{m} k_j^{\ \mu} E_\mu, \quad j = 1, \ldots, p, \tag{A.1}$$

for some real constants $k_j^{\ \mu}$. Next, let A^ν, $\nu = 1, \ldots, m$, be m $su(2)$-valued one-forms satisfying[13]:

$$\int \mathrm{Tr}\left[E_\mu^a A_a^\nu\right] = \delta_\mu^\nu, \quad \mu, \nu = 1, \ldots, m, \tag{A.2}$$

and consider the m parameter family of one-forms:

$$A_{\vec{t}} := \sum_{\mu=1}^{m} t_\mu A^\mu. \tag{A.3}$$

The $U(1)^N$ part of the map (3.1) restricted to the m parameter family of connections (A.3) induces the following map from $\mathbb{R}^m$ to $U(1)^N$:

$$(t_1, \ldots, t_m) \mapsto \left(e^{it_1}, \ldots, e^{it_m}, e^{it_\mu k_1^\mu}, \ldots, e^{it_\mu k_p^\mu}\right). \tag{A.4}$$

Our aim is to use the map (A.4) to reproduce with arbitrary precision the given N phases $(e^{i\theta_1}, \ldots, e^{i\theta_N}) \in U(1)^N$. The first m phases can be exactly reproduced by taking:

$$t_\mu = \theta_\mu + 2\pi n_\mu, \quad n_\mu \in \mathbb{Z}, \ \ \mu = 1, \ldots, m. \tag{A.5}$$

We are then left with the m integers $\{n_\mu\}$ to approximate p phases, the relevant map being:

$$(n_1, \ldots, n_m) \mapsto \left(e^{i2\pi n_\mu k_1^\mu}, \ldots, e^{i2\pi n_\mu k_p^\mu}\right). \tag{A.6}$$

Now, the condition of rationally independence of the N electric fields translates into the following condition of rational independence of N vectors in $\mathbb{R}^m$: *The canonical basis $\vec{e}_i \in \mathbb{R}^m$, $i = 1, \ldots, m$ (with components $(\vec{e}_i)^\mu = \delta_i^{\ \mu}$), together with the vectors $\vec{k}_j \in \mathbb{R}^m$, $j = 1, \ldots, p$ (with components $(\vec{k}_i)^\mu = k_j^{\ \mu}$), are rationally independent.* The example (26.19 (e)) of [21] shows that, under this condition, the range of the map (A.6) is dense in $U(1)^p$[14]. This implies that given $\delta > 0$, we can find $\vec{t}^{(\delta)} \in \mathbb{R}^m$ such that $e^{it_\mu^{(\delta)}} = e^{i\theta_\mu}$, $\mu = 1, \ldots, m$ and $|e^{it_\mu^{(\delta)} k_j^\mu} - e^{i\theta_{m+j}}| < \delta$, $j = 1, \ldots, p = N - m$. Setting $\bar{A}^{B,\delta} := A_{\vec{t}^{(\delta)}}$ we obtain the desired connection satisfying (3.3).

Step (ii):

Let p_α be a point on the open edge $\tilde{e}_\alpha - \{b(e_\alpha), f(e_\alpha)\}$. Since Σ is Hausdorff, there exists an open neighbourhood U_α of p_α such that U_α seperates p_α from the points $b(e_\alpha), f(e_\alpha)$. Further, U_α can be chosen such that $U_\alpha \cap \tilde{e}_\beta = \emptyset$ for $\alpha \neq \beta$; else p_α is an accumulation point of a sequence in $\tilde{e}_\beta$, which, by virtue of the compactness of $\tilde{e}_\beta$, implies that $p_\alpha \in \tilde{e}_\beta \cap \tilde{e}_\alpha$, contradicting the condition that $\tilde{e}_\alpha$ and $\tilde{e}_\beta$ can only intersect at their endpoints. A similar argument implies that U_α, $\alpha = 1, \ldots, n$ can be chosen such that $U_\alpha \cap U_\beta = \emptyset$ if $\alpha \neq \beta$.

Finally, since $\tilde{e}_\alpha$ is a semianalytic manifold, it follows (see for example definition A.12 of [16]) that U_α can be chosen to be small enough that it is in the domain of a single semianalytic chart χ_α in which it takes the form of a ball of size τ within which $\tilde{e}_\alpha \cap U_\alpha$ is connected and

[13] To explicitly obtain m $su(2)$-valued one-forms A^ν satisfying (A.2), we introduce a semianalytic metric h_{ab} on Σ which defines an inner product on the space of background electric fields by $\langle E, E' \rangle := \int h^{-1/2} h_{ab}\, \mathrm{Tr}\, [\mathrm{E}^a \mathrm{E}'^b]$. Since $E_1, \ldots, E_m$ are linearly independent, and $\langle , \rangle$ positive definite, the $m \times m$ matrix $\langle E_\mu, E_\nu \rangle$, $\mu, \nu = 1, \ldots, m$ is invertible. Let $c_{\mu\nu}$ be its inverse, so that $\sum_{\rho=1}^{m} \langle E_\mu, E_\rho \rangle c_{\rho\nu} = \delta_{\mu\nu}$. It is then easy to verify that the one-forms $A_a^\nu := h^{-1/2} \sum_{\rho=1}^{m} c_{\rho\nu} h_{ab} E_\rho^b$ satisfy (A.2).

[14] This results also follows from theorem IV in section III.5 of [22].

runs along a coordinate axis. Thus, we have that $\chi_\alpha(U_\alpha) \subset \mathbb{R}^3$ with $\chi_\alpha(U_\alpha) = \{\vec{x} \in \mathbb{R}^3 | \parallel \vec{x} \parallel < \tau\}$, and that $\chi_\alpha(U_\alpha \cap \tilde{e}_\alpha) = ((-\tau, \tau), 0, 0)$.

In the χ_α coordinate chart, we denote balls of coordinate size δ centred at the origin by $B_\alpha(\delta)$. Accordingly we denote the above choice of U_α by $B_\alpha(\tau)$. Clearly by taking $\epsilon \ll \tau$ we have $B_\alpha(2\epsilon) \subset U_\alpha$.

Step (iii):

We need to specify f_ϵ in $B_\alpha(2\epsilon) - B_\alpha(\epsilon)$ such that f_ϵ is semianalytic. Consider the polynomial in $\mathbb{R}$ given by $g(y) := c\int_0^y (y'(1-y'))^K \mathrm{d}y'$, with c chosen so that $g(1) = 1$ and $K > k$. Then g interpolates between the constant 0 function for $y < 0$ and the constant 1 function for $y > 1$ in a C^K manner. Setting

$$\left(f_\epsilon \circ \chi_\alpha\right)(\vec{x}) = \begin{cases} 0 & \text{for } \parallel \vec{x} \parallel < \epsilon \\ 1 & \text{for } \parallel \vec{x} \parallel \geqslant 2\epsilon \\ g\left(\frac{1}{3}\left(\parallel x \parallel^2/\epsilon^2 - 1\right)\right) & \text{for } \epsilon \leqslant \parallel \vec{x} \parallel < 2\epsilon \end{cases} \tag{A.7}$$

does the job.

Step (iv):

We take A^ϵ with support on U_α given by

$$A^\epsilon|_{U_\alpha} = w_\alpha \chi_\alpha^*\left(a_\epsilon \mathrm{d}x\right). \tag{A.8}$$

Here $w_\alpha \in su(2)$ are constant (but ϵ and δ dependent) $su(2)$ elements satisfying

$$e^{w_\alpha} = \left(g_\alpha^1\right)^{-1} g_\alpha \left(g_\alpha^2\right)^{-1}, \quad \alpha = 1, \ldots, n, \tag{A.9}$$

and taken to be in the ball of radius 4π of $su(2)$ that maps onto $SU(2)$ under the exponential map.

a_ϵ: $\chi_\alpha(U_\alpha) \to \mathbb{R}$ is taken to be:

$$a_\epsilon(\vec{x}) = \begin{cases} c'\left(\epsilon^2 - \parallel \vec{x} \parallel^2\right)^K & \text{for } \parallel \vec{x} \parallel \leqslant \epsilon \\ 0 & \text{for } \parallel \vec{x} \parallel > \epsilon \end{cases} \tag{A.10}$$

with $K > k$ and c' chosen so that $\int_{-\epsilon}^{\epsilon} a_\epsilon(x, 0, 0)\mathrm{d}x = 1$. This last condition, together with the choice of w_α (A.9) guarantees the required condition (3.13).

Step (v):

A^ϵ was constructed so that the holonomies of $A^\delta := f_\epsilon \bar{A}^{B,\delta} + A^\epsilon$ along the edges e_α exactly reproduce the group elements g_α and hence (3.2) is satisfied with $C_1 = 0$. For the $U(1)$ elements we have

$$\left| e^{i\int E_I \cdot A^\delta} - e^{i\theta_I} \right| = \left| e^{i\int E_I \cdot \bar{A}^{B,\delta}} e^{-i\theta_I} - e^{i\int\left[\left(1-f_\epsilon\right)E_I \cdot \bar{A}^{B,\delta} - E_I \cdot A^\epsilon\right]} \right| \tag{A.11}$$

$$\leqslant \left| e^{i\int E_I \cdot \bar{A}^{B,\delta}} e^{-i\theta_I} - 1 \right| + \left| e^{i\int\left[\left(1-f_\epsilon\right)E_I \cdot \bar{A}^{B,\delta} - E_I \cdot A^\epsilon\right]} - 1 \right|. \tag{A.12}$$

From step (i) above, the first term in A.12 is bounded by δ. The phases in the second term can be bounded by:

$$\left| \int \left(1 - f_\epsilon\right) E_I \cdot \bar{A}^{B,\delta} \right| \leqslant \sum_{\alpha=1}^{n} \left| \int_{B_\alpha(2\epsilon)} \left(1 - f_\epsilon\right) E_I \cdot A_\delta^{(\beta)} \right| \tag{A.13}$$

$$\leqslant c_1(\delta)\epsilon^3 \tag{A.14}$$

for some constant $c_1(\delta)$, and

$$\left| \int E_I \cdot A^\epsilon \right| \leqslant \sum_{i=1}^{n} \left| \int_{\|\vec{x}\|<\epsilon} \mathrm{Tr}\left[\left((\chi_i)_* E_I \right)^x w_i \right] a_\epsilon \mathrm{d}x\mathrm{d}y\mathrm{d}z \right| \tag{A.15}$$

$$\leqslant c_2(\delta)\epsilon^2 \tag{A.16}$$

for some constant $c_2(\delta)$. Here we used the fact that Tr $[((\chi_i)_* E_I)^x w_i]$ has some ϵ independent bound and that $\int |a_\epsilon| \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ has an order ϵ^2 bound as follows from the condition on c' described after equation (A.10). By Taylor expanding, we conclude that, for given δ and sufficiently small ϵ, the second term in (A.12) has an ϵ^2 bound:

$$|e^{i\int[(1-f_\epsilon)E_I \cdot \bar{A}^{B,\delta} - E_I \cdot A^\epsilon]} - 1| < c(\delta)\epsilon^2, \tag{A.17}$$

for some constant $c(\delta)$. Thus, if for given δ we chose ϵ such that

$$\epsilon \ll (\delta/c(\delta))^{1/2}, \tag{A.18}$$

we achieve the desired bound (3.3) with $C_2 = 2$.

Appendix B. Assorted proofs

B.1. Generating set (4.3)

If the M electric fields $E'_1, \ldots, E'_M$, are algebraically independent, then $M = N$. If not, then there exists M integers q_i, $i = 1, \ldots, M$, not all of them zero, such that $\sum_i q_i E'_i = 0$. At least one q_i is different from zero, so for concreteness let $q_M \neq 0$. We can then solve for E'_M to get

$$E'_M = q_M^{-1} \sum_{i=1}^{M-1} E'_i. \tag{B.1}$$

Define

$$E_i^{(1)} := q_M^{-1} E'_i, \quad i = 1, \ldots, M-1. \tag{B.2}$$

Then the electric fields E'_i, $i = 1, \ldots, M$, can be expressed as integer linear combinations of the $M-1$ electric fields $E_i^{(1)}$ (B.2). If $E_1^{(1)}, \ldots, E_{M-1}^{(1)}$ are algebraically independent, we are done. Otherwise we apply the above procedure to the $M-1$ electric fields $E_i^{(1)}$ to obtain a new set of $M-2$ electric fields in terms of which the rest are expressed as integer linear combinations. The procedure is iterated until one obtains an algebraically independent set.

B.2. Equation (4.24)

By linearity of the PLF, it is enough to consider the special case where a_l takes the form

$$a_l(g_1, \ldots, g_n, u_1, \ldots, u_N) = \tilde{a}(g_1, \ldots, g_n)(u_1)^{m_1} \ldots (u_N)^{m_N}, \tag{B.3}$$

where $m_I \in \mathbb{Z}$, $I = 1, \ldots, N$. In such case, we have

$$\omega(a) = \left\langle 0, 0 | \tilde{a}\left(\hat{h}_{e_1}, \ldots, \hat{h}_{e_n}\right) \cdot 0, \sum_{I=1}^{N} m_I E_I \right\rangle \tag{B.4}$$

$$= \langle 0|\tilde{a}\Big(\hat{h}_{e_1}, \ldots, \hat{h}_{e_n}\Big)|0\rangle_{\text{LQG}}\delta_{0, \sum_{I=1}^{N} m_I E_I} \tag{B.5}$$

$$= \int_{SU(2)^n} \tilde{a}(g)\mathrm{d}\mu \prod_{I=1}^{N} \delta_{0,m_I}. \tag{B.6}$$

Here we used that standard rewriting of the LQG PLF in terms of $SU(2)$ integrals [11], the algebraic independence of the electric fields (2.3), and the basic inner product $\langle s_1, E_1|s_2, E_2\rangle = \langle s_1|s_2\rangle_{\text{LQG}}\delta_{E_1E_2}$.

As in the treatment of $\mathbb{R}$-Bohr [11, 12], we notice that the last factor in (B.6) corresponds to an integral over $U(1)^N$ with normalized Haar measure:

$$\int_{U(1)^N} \mathrm{d}\mu(u_1)^{m_1}\ldots(u_N)^{m_N} = \prod_{I=1}^{N} \int \frac{\mathrm{d}\theta_I}{2\pi} e^{\,i m_I \theta_I} \tag{B.7}$$

$$= \prod_{I=1}^{N} \delta_{0,m_I}. \tag{B.8}$$

Substituting the Kronecker deltas in (B.6) by (B.7) we recover (4.24) for the special case of f given by (B.3). By linearity, it follows that (4.24) holds for general algebra elements.

Appendix C. Projective limit

In this appendix we give further details on the projective limit space and clarify the relation between the use of preordered and partially ordered label sets. To simplify the discussion, we first describe in detail the case of standard LQG in section C.1. In section C.2 we give the partially ordered label set description of $\bar{\mathcal{A}}$. In section C.3 we show cylindrical consistency of the Haar measures on G_l and $G_{\hat{l}}$.

C.1. Relation between preorder and partially ordered label sets for holonomy probes.

In the usual construction, the label set is given by subgroupoids of $\mathcal{P}$ generated by finitely many edges. Let $\mathcal{L}_{\text{LQG}}$ be such label set so that $L \in \mathcal{L}_{\text{LQG}}$ denotes a subgroupoid of $\mathcal{P}$ generated by a finite number of edges. The relation $L' \geqslant L$ iff L is a subgroupoid of L', makes $\mathcal{L}_{\text{LQG}}$ a partially ordered directed set. The compact space associated with $L \in \mathcal{L}_{\text{H}}$ is:

$$\mathcal{A}_L := \text{Hom}(L, SU(2)), \tag{C.1}$$

and the projections $p_{LL'}$ are defined by restriction: $y_{L'} \in \mathcal{A}_{L'}$ induces a homomorphism on any subgroupoid $L \leqslant L'$ by simply restricting the action of $y_{L'}$ to L. Let us denote by $\bar{\mathcal{A}}_{\text{LQG}}$ the resulting projective limit space, as described in [10, 11].

The corresponding ingredients in our construction are: the label set $\mathcal{L}_{\text{H}}$, the compact spaces $G_\gamma = SU(2)^n$ and the projections $p_{\gamma\gamma'}$ determined by the way edges in γ are decomposed in terms of edges of γ'', see section 4.2. It is easy to verify the compatibility of the projections with the relation '$\geqslant$' in the sense described in section 5. The corresponding projective limit space can be constructed completely analogous to $\bar{\mathcal{A}}_{\text{LQG}}$: the 'ambient' space $G_\infty := \prod_{\gamma \in \mathcal{L}_{\text{H}}} G_\gamma$ with the Tychonov topology (the weakest making the canonical projections to G_γ continuous) is compact and Hausdorff [11]. The projective limit space is the subset $\bar{\mathcal{A}}_{\text{H}} \subset G_\infty$ of points in G_∞ satisfying consistency conditions with the projections:

$$\bar{\mathcal{A}}_{\mathrm{H}} := \left\{ \{x_\gamma\} \in G_\infty \,\middle|\, p_{\gamma,\gamma'}(x_{\gamma'}) = x_\gamma, \;\; \forall\, \gamma' \geqslant \gamma \right\}. \tag{C.2}$$

$\bar{\mathcal{A}}_{\mathrm{H}}$ is given the topology induced by G_∞, and the same proof [11] that $\bar{\mathcal{A}}_{\mathrm{LQG}}$ is closed goes through here as well[15]. By the same arguments as for $\bar{\mathcal{A}}_{\mathrm{LQG}}$, it follows that $\bar{\mathcal{A}}_{\mathrm{H}}$ is a compact, Hausdorff space.

Let us see that $\bar{\mathcal{A}}_{\mathrm{LQG}}$ and $\bar{\mathcal{A}}_{\mathrm{H}}$ are homeomorphic. A bijection between the two spaces can be given as follows. Denote by $\mathcal{P}_\gamma \in \mathcal{L}_{\mathrm{LQG}}$ the subgroupoid generated by γ and let

$$\mathrm{Gen}(L) = \left\{ \gamma \in \mathcal{L}_{\mathrm{H}} \,\middle|\, \mathcal{P}_\gamma = L \right\}, \tag{C.3}$$

be the set of all possible 'generators' of a given $L \in \mathcal{L}_{\mathrm{LQG}}$. For $\gamma, \gamma' \in \mathrm{Gen}(L)$ we have that $\gamma' \geqslant \gamma$ and $\gamma \geqslant \gamma'$. $p_{\gamma,\gamma'}$ then defines a homeomorphism between $G_{\gamma'}$ and G_γ with inverse given by $p_{\gamma',\gamma}$. An element $y_L \in \mathcal{A}_L$ defines a point in G_γ, $\gamma \in \mathrm{Gen}(L)$ by [10]:

$$\rho_\gamma \colon \mathcal{A}_L \to G_\gamma, \quad \gamma = (e_1, \ldots, e_n) \in \mathrm{Gen}(L), \tag{C.4}$$

$$y_L \mapsto \big(y_L(e_1), \ldots, y_L(e_n) \big) =: y_L(\gamma). \tag{C.5}$$

The points $x_\gamma := y_L(\gamma) \in G_\gamma$ for each $\gamma \in \mathrm{Gen}(L)$ satisfy the consistency conditions

$$p_{\gamma\gamma'}(x_{\gamma'}) = x_\gamma, \quad p_{\gamma'\gamma}(x_\gamma) = x_{\gamma'}, \quad \gamma, \gamma' \in \mathrm{Gen}(L). \tag{C.6}$$

Conversely, any $x_\gamma \in G_\gamma$ determines a homomorphism in $\mathcal{A}_{\mathcal{P}_\gamma}$ by

$$\sigma_\gamma : \; G_\gamma \to \mathcal{A}_{\mathcal{P}_\gamma} \tag{C.7}$$

$$x_\gamma \mapsto y_{\mathcal{P}_\gamma} \,\Big|\, y_{\mathcal{P}_\gamma}(\gamma) = x_\gamma, \tag{C.8}$$

and given x_γ and $x_{\gamma'}$ satisfying (C.6), they define the same homomorphism. The continuous maps ρ_γ and σ_γ above are inverses of each other. Let:

$$\tilde{p}_\gamma := \rho_\gamma \circ p_L \colon \bar{\mathcal{A}}_{\mathrm{LQG}} \to G_\gamma \tag{C.9}$$

$$\tilde{p}_L := \sigma_\gamma \circ p_\gamma \colon \bar{\mathcal{A}}_{\mathrm{H}} \to \mathcal{A}_L, \quad \gamma \in \mathrm{Gen}(L). \tag{C.10}$$

Then it is easy to verify that the maps

$$\rho : \bar{\mathcal{A}}_{\mathrm{LQG}} \to \bar{\mathcal{A}}_{\mathrm{H}}, \quad \rho\big(\{x_{L'}\} \big) := \left\{ \tilde{p}_\gamma\big(\{x_{L'}\} \big) \right\} \tag{C.11}$$

$$\sigma : \bar{\mathcal{A}}_{\mathrm{H}} \to \bar{\mathcal{A}}_{\mathrm{LQG}}, \quad \sigma\big(\{x_{\gamma'}\} \big) := \left\{ \tilde{p}_L\big(\{x_{\gamma'}\} \big) \right\}, \tag{C.12}$$

are inverse of each other and that they provide a bijection between $\bar{\mathcal{A}}_{\mathrm{LQG}}$ and $\bar{\mathcal{A}}_{\mathrm{H}}$. Finally this bijection is clearly a homeomorphism: the topology on $\bar{\mathcal{A}}_{\mathrm{H}}$ generated by the projections p_γ coincides with that generated by the projections $\tilde{p}_L$ by virtue of continuity and invertibility of the maps ρ_γ and σ_γ.

C.2. Partially ordered directed label set for $\bar{\mathcal{A}}$

We describe here the partially ordered directed set relevant for $\bar{\mathcal{A}}$. Define an equivalence relation in $\mathcal{L}$ by $l \sim l'$ iff $l \geqslant l'$ and $l' \geqslant l$. Let $\mathcal{L}_{\mathrm{H}} = \mathcal{L}/\!\sim$ be the corresponding quotient space. We denote by $\hat{l}$ elements of $\mathcal{L}_{\mathrm{H}}$. On $\mathcal{L}_{\mathrm{H}}$ we can define the relation $\hat{l} \geqslant \hat{l}'$ iff $l \geqslant l'$ for some $l \in \hat{l}$ and $l' \in \hat{l}'$. It is easy to verify that this relation is well defined (independent of the

[15] Lemma 6.2.10 in [11] can be repeated to show that every convergent net $\{x_l^\alpha\}$ in G_∞ such that $\{x_l^\alpha\}$ is in $\bar{\mathcal{A}}_{\mathrm{H}}$ for any α, converges to a point in $\bar{\mathcal{A}}_{\mathrm{H}}$

choice of representatives l and l'), and that it defines a partial order on $\mathcal{L}_{\mathrm{H}}$, since by construction $\hat{l} \geqslant \hat{l}'$ and $\hat{l}' \geqslant \hat{l}$ imply $\hat{l} = \hat{l}'$. It is easy to verify that the directed set property of $\mathcal{L}$ implies that $\mathcal{L}_{\mathrm{H}}$ is a partially ordered directed set.

An intrinsic characterization of $\hat{l}$ can be given as in the previous section: $\hat{l} \approx (\mathcal{P}_\gamma, \mathcal{E}_\Upsilon) =: (\hat{\gamma}, \hat{\Upsilon})$, where $(\gamma, \Upsilon) \in \hat{l}$. The pair $(\mathcal{P}_\gamma, \mathcal{E}_\Upsilon)$ is independent of the choice of representative, and the $\geqslant$ relation defined above for $\mathcal{L}_{\mathrm{H}}$ corresponds to: $\hat{l} \geqslant \hat{l}'$ iff the pair of subgroupoids associated with $\hat{l}'$ are subgroupoids of the pair associated with $\hat{l}$.

We now describe the spaces and projections associated with the label set $\mathcal{L}_{\mathrm{H}}$. Given $l, l' \in \hat{l}$, we have maps $p_{ll'}: G_{l'} \to G_l$ and $p_{l'l}: G_l \to G_{l'}$. The consistency condition of the maps implies that $p_{ll'} \circ p_{l'l} = \mathrm{Id}_{G_l}$ and $p_{l'l} \circ p_{ll'} = \mathrm{Id}_{G_{l'}}$, so that $p_{ll'}$ is a diffeomorphism between $G_{l'}$ and G_l. The space $G_{\hat{l}}$ could then be defined as G_l for some fixed representative $l \in \hat{l}$. In the present case, however, the label sets have additional structure that allows for a more intrinsic definition of $G_{\hat{l}}$. By the same argument as in the previous section, it is easy to verify that an element $g \in G_l$ defines a pair of homomorphisms $\hat{g} = (g^{\mathrm{H}}, g^{\mathrm{B}}) \in \mathrm{Hom}(\mathcal{P}_\gamma, SU(2)) \times \mathrm{Hom}(\mathcal{E}_\Upsilon, U(1))$, with $l = (\gamma, \Upsilon)$. Further, $p_{l'l}(g) \in G_{l'}$ defines the same pair of homomorphisms $\hat{g}$ for any $l' \in \hat{l}$. Thus we set $G_{\hat{l}} := \mathrm{Hom}(\mathcal{P}_\gamma, SU(2)) \times \mathrm{Hom}(\mathcal{E}_\Upsilon, U(1))$ with $(\gamma, \Upsilon) \in \hat{l}$. The definition is independent of the choice of representative $(\gamma, \Upsilon) \in \hat{l}$. Finally, the projections $p_{\hat{l}\hat{l}'}$ can be defined, as in the previous section, by restriction of the homomorphisms to the corresponding subgroupoid. Such definition is then compatible with the projections $p_{ll'}$.

The discussion of the previous section can be easily adapted to the present case to conclude that the projective limit space associated with $(\mathcal{L}_{\mathrm{H}}, \{G_{\hat{l}}\}, \{p_{\hat{l}\hat{l}'}\})$ is homeomorphic to $\bar{\mathcal{A}}$.

C.3. Projective consistency of the Haar measures on G_l

Let $G_l = SU(2)^n \times U(1)^N$ where n and N are the number of independent edges and electric fields in l. Let μ_l be the normalized Haar measure on G_l so that μ_l is a product of Haar measures on the $SU(2)$ and $U(1)$ factors. We want to show that $(p_{l,l'})_*\mu_{l'} = \mu_l$ whenever $l' \geqslant l$ so that $\{\mu_l, l \in \mathcal{L}\}$ define a consistent family of measures. Recall that the maps $p_{ll'}$, described in section 4.3 are 'block diagonal' i.e. do not mix $SU(2)$ factors with $U(1)$ ones. Given $(\gamma', \Upsilon') \geqslant (\gamma, \Upsilon)$, $p_{ll'}$ is determined by maps

$$g_i = p_i\left(g'_1, \ldots, g'_{n'}\right), \quad i = 1, \ldots, n \tag{C.13}$$

$$u_I = P_I\left(u'_1, \ldots, u'_{N'}\right), \quad i = 1, \ldots, N. \tag{C.14}$$

The consistency condition then translates into two separate conditions, $p_*\mu_{n'} = \mu_n$ and $P_*\mu_{N'} = \mu_N$, where μ_n is the Haar measure on $SU(2)^n$, μ_N that of $U(1)^N$ and similarly for the primed quantities. The proof that $p * \mu_{n'} = \mu_n$ is the same as the one used in standard LQG to show the cylindrical consistency of the $SU(2)$ Haar measures, see [7, 10]. Such a proof is mainly group theoretical, and so it should not be difficult to adapt it to the $U(1)$ factors. Below we present an alternative proof for the cylindrical consistency of the $U(1)$ measures.

Let $C(U(1)^N)$ be the space of continuous functions on $U(1)^N$. Recall that $C(U(1)^N)$ is a C^* algebra with norm $\|f\| := \sup_{u \in U(1)^N} |f(u)|$, so that in particular it is a normed vector space. Define the linear functionals Γ and Γ' on this space by:

$$\Gamma(f) := \int_{U(1)^N} f(u_1, \ldots, u_N) \mathrm{d}\mu_N \tag{C.15}$$

$$\Gamma'(f) := \int_{U(1)^{N'}} f\left(P_1(u'), \ldots, P_N(u')\right) \mathrm{d}\mu_{N'} \tag{C.16}$$

where

$$P_I(u') = \Pi_{J=1}^{N'} (u'_J)^{q_I^J}, \tag{C.17}$$

with q_I^J the integers that determine how electric fields of Υ are written in terms of those in Υ', see equation (4.7). Showing $P * \mu_{N'} = \mu_N$ is then equivalent to showing $\Gamma(f) = \Gamma'(f) \ \ \forall f \in C(U(1)^N)$. First, we note that both Γ, Γ' are bounded and hence continuous with respect to the topology of $C(U(1)^N)$: clearly $|\Gamma(f)| \leqslant \| f \|$. For Γ' we have:

$$|\Gamma'(f)| \leqslant \sup_{u' \in U(1)^{N'}} |f(P(u'))| = \sup_{u \in U(1)^N} |f(u)| = \| f \|, \tag{C.18}$$

where the first equality is due to the fact that the map P: $U(1)^{N'} \to U(1)^N$ is surjective. Next, let $\mathrm{Pol}(U(1)^N) \subset C(U(1)^N)$ be the set of functions given by finite linear combinations of elements of the form $\Pi_{I=1}^N (u_I)^{m_I}$ with $m_I \in \mathbb{Z}$. Since $\mathrm{Pol}(U(1)^N)$ is a * sub algebra of $C(U(1)^N)$ and separates points, it follows by the Stone–Weierstrass theorem that $\mathrm{Pol}(U(1)^N)$ is dense in $C(U(1)^N)$. By the bounded linear transformation theorem [11] it is then enough to show that Γ and Γ' agree on this dense subset of $C(U(1)^N)$. Finally, by linearity we can focus attention on an element $\Pi_{I=1}^N (u_I)^{m_I}$. One finds (see equation (B.7)):

$$\Gamma\left(\Pi_{I=1}^N (u_I)^{m_I}\right) = \Pi_{I=1}^N \delta_{0,m_I} \tag{C.19}$$

and

$$\Gamma'\left(\Pi_{I=1}^N (u_I)^{m_I}\right) = \int_{U(1)^{N'}} \Pi_{I=1}^N \left(\Pi_{J=1}^{N'} \left(u'_J\right)^{q_I^J}\right)^{m_I} \tag{C.20}$$

$$= \Pi_{J=1}^{N'} \delta_{0, \sum_{I=1}^N m_I q_I^J} \tag{C.21}$$

$$= \Pi_{I=1}^N \delta_{0,m_I}, \tag{C.22}$$

where we used $\int_{U(1)} u^m = \delta_{0,m}$ and the rational independence of the electric fields:

$$m_I q_I^J = 0 \ \ \forall J \iff m_I q_I^J E'_J = 0 \iff m_I E_I = 0 \iff m_I = 0 \ \ \forall I. \tag{C.23}$$

We thus have shown that the measures $\{\mu_l\}$ represent a family of consistent measures on $(\mathcal{L}, \{G_l\}, \{p_{ll'}\})$. The measures $\mu_{\hat{l}}$ on $G_{\hat{l}}$ are defined by the push forward of maps σ_l: $G_l \to G_{\hat{l}}$ defined analogously as σ_γ in equation (C.7). The consistency of the measures $\{\mu_l\}$ immediately implies the consistency of the measures $\{\mu_{\hat{l}}\}$.

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