Classroom

In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Rotational Rectification of an Alternating Magnetic Field*

As an all-time student and a some-time teacher of physics, the author is of the view that problem solving should inform all classroom learning. Problems chosen, however, must be wholesome – demanding clarity of ideas involved in their formulation, physics of approximations made in their solution, followed by some lateral thinking. This is illustrated in this article through a specific classroom exercise.

* Dedicated to the memory of my inimitable teacher at IIT/Kgp, Prof. G S Sanyal (1922–2011) who always taught as something we just forgot.

1. Introduction: The Physical Problem

Consider a metallic ring (circular loop of a thin wire) made to spin about a diameter held normal to an externally applied magnetic field. The field is approximated as static and spatially uniform over the laboratory (experimental) scale, e.g., Earth's magnetic field, as shown in *Figure* 1.

The problem now is to understand physically, and solve mathematically for, the motion of the magnetic dipole (detector) in the laboratory frame under the combined torque exerted by the primary static magnetic field B and the secondary time-varying field $b(t)$, generated by

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Magnetic flux linkage, Faraday's law, Biot–Savart law, electrical impedance, adiabatic approximation.

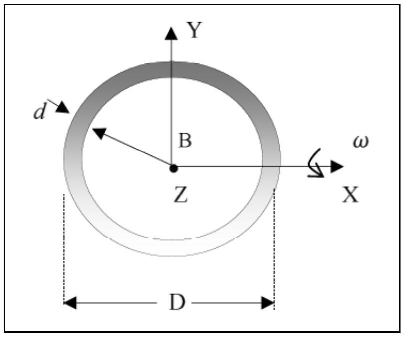


Figure 1. A schematic showing the experimental geometry. A circular ring of thin (*d*«*D*) wire spinning at angular velocity ω about a diameter parallel to X-axis in a uniform static magnetic field **B** directed along the Z-axis. A magnetic dipole (not shown) is well pivoted at the ring centre and serves as a field detector. Note that both the ring and the dipole can rotate only about the X-axis.

the current $I(t)$ induced in the metallic ring due to its rotation ω relative to B. It is essentially a dynamo-motor problem, but one with a strangeness of proportion as we shall see.

The causal chain of physical reasoning involved can be traced as follows: (a) The magnetic field lines (B) threading the ring subtend a time-varying magnetic flux linkage $\phi(t)$ equal to the field strength B times the instantaneous area of the spinning ring projected normal to $\mathbf{B}$ giving $\phi(t) = AB\cos(\omega t)$; (b) the harmonically time-varying flux linkage generates an electromagnetic force (emf) $V(t) = -\frac{\partial \phi}{\partial t}$ in the ring (Faraday's law, or the *universal flux rule*); (c) this motional emf in turn generates a harmonic current $I(t)$ in the ring determined by the electrical ring impedance coming from its resistance R and inductance L. *This is the dynamo action*; (d) the ring current $I(t)$ in turn generates a harmonic magnetic

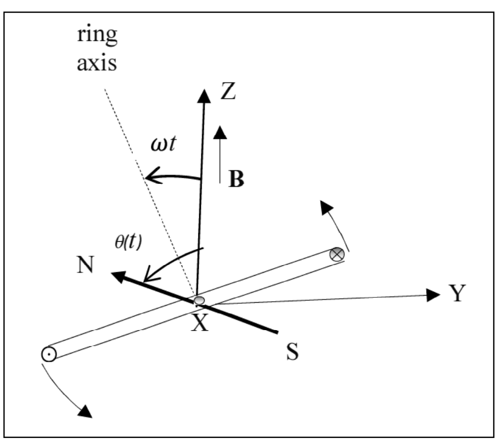


Figure 2. *Figure* 1 redrawn so as to give the angles and the sense of ring motion. Also shown is the magnetic dipole (S → N) pivoted freely at the ring centre.

field $b(t)$ (Biot–Savart law) co-rotating with the ring; the corresponding field in the laboratory frame exerts a mechanical torque on the magnetic dipole pivoted at the ring centre; (e) the magnetic dipole ($\boldsymbol{\mu}$) with a moment of inertia (m) is deflected by an angle $\theta(t)$ under the influence of the static primary field ($\boldsymbol{B}$) competing with the rotationally rectified time-varying secondary field $b(t)$. *This is the motor action.*

All we have to do now is to translate the above physical cause-effect sequence into the corresponding mathematical equations, and then solve these for certain quantities of interest within well-defined approximations suggested by the physics of the problem. This is to be followed by physical interpretation of the results so obtained, and some lateral thinking on possible generalization.

2. Mathematical Derivation

The instantaneous magnetic flux $\phi(t)$ linking the spinning ring (the circular loop of thin wire) at time t is given by

$$\phi(t) = BA\cos\omega t, \qquad (1)$$

with $A(= \pi D^2/4)$ the area of the ring and D the ring diameter. The emf induced in the ring due to the time-varying magnetic flux linkage is

The Faraday law of electromagnetic induction (the universal flux rule) giving emf generated due to rate of change of the magnetic flux linking the loop spinning about a diameter held perpendicular to Earth's magnetic field.

$$V(t) = -\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = AB\omega\sin(\omega t). \qquad (2)$$

Here $\mathrm{d}/\mathrm{d}t$ denotes the total time derivative. In the present case, however, it is just the motional effect inasmuch as the field B is static.

The induced ring current $I(t)$ is then described by the well-known resistive-reactive circuit equation

$$L\frac{\mathrm{d}I(t)}{\mathrm{d}t} + RI(t) = V(t) \qquad (3)$$

with R = ring resistance, and L = ring self-inductance.

This system of three coupled linear differential equations (1–3) can be readily solved for the ring current $I(t)$ for any given initial $(t = 0)$ condition. We are, of course, not interested in the short-time transients, but rather in the long-time steady-state solution which is expected to be periodic in time with the circular frequency ω. For this one can straightforwardly adopt the well-known technique commonly used for harmonically driven linear LCR circuits. We write

Equivalent electrical circuit for the inductive (L) – resistive (R) system shown in *Figure* 2.

$$\phi(t) = \mathrm{Re}\phi(\omega)e^{i\omega t} \tag{4}$$

with Re denoting the real part and $\phi(\omega) = BA$, which is the maximum flux linkage due to the external magnetic field and is real. Here the cisoid $e^{i\omega t} = \cos\omega t + i\sin\omega t$ with $i = \sqrt{-1}$. (It is interesting to note in passing that while physicists use the symbol i after Euler, engineers prefer the symbol j following Steinmetz, who is said to have *created electricity from the square-root of* -1. Besides, the symbol j avoids confusing it with the symbol for the current, usually taken to be i, or I).

With this notation in hand, one can at once write

$$V(t) = \mathrm{Re}V(\omega)e^{i\omega t}, \quad \text{with} \quad V(\omega) = -i\omega\phi(\omega), \tag{5}$$

$$I(t) = \mathrm{Re}I(\omega)e^{i\omega t}, \quad \text{with} \quad I(\omega) = \frac{V(\omega)}{Z(\omega)}, \tag{6}$$

where $Z(\omega) \equiv R + i\omega L$ is the electrical impedance of the ring of resistance R(ohm) and inductance L(Henry). All through we will be using the SI system of units.

More explicitly,

$$I(t) = \frac{\omega AB}{\sqrt{R^2 + (\omega L)^2}} \sin\left(\omega t - \arctan\frac{\omega L}{R}\right), \tag{7}$$

Electromagnetically induced current in the spinning conducting loop.

where $\arctan\frac{\omega L}{R}$ is the reactive phase-shift with respect to the emf $V(t)$ generated in the spinning ring.

Secondary magnetic field produced along the rotating loop-axis by the loop current: Ampere's Law.

Next, we have to derive the secondary magnetic field generated by this ring current at the ring centre, where the magnetic dipole (detector) is pivoted. This is given by the Biot–Savart law as

$$b(t) = \frac{\mu_0 I(t)}{D} \equiv \frac{\mu_0 \omega A B}{D\sqrt{R^2 + (\omega L)^2}} \sin\left(\omega t - \arctan\frac{\omega L}{R}\right). \tag{8}$$

This time-varying field is directed along the ring axis, and co-rotates with it as shown in *Figure* 2. Here μ_0 is the magnetic permeability of space, $\mu_0 = 4\pi \times 10^{-7}$ Hm^{-1} (Henry metre^{-1}).

Finally, we turn to the rotational motion of the magnetic dipole, pivoted at the ring centre in the laboratory (inertial) frame, under the forcing torque exerted by the primary static field B and the time-periodic dynamically generated secondary field $b(t)$. This motor action is described by

$$M\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = -mB\sin\theta(t)$$
$$+\frac{\mu_0 m\omega AB}{DR\sqrt{1+\left(\frac{\omega L}{R}\right)^2}}\sin\left(\omega t - \arctan\frac{\omega L}{R}\right)\sin(\omega t - \theta(t)), \tag{9}$$

where M is the moment of inertia of the dipole. (Note that the angle θ as used here is the negative of the usual definition in spherical coordinate system). It can be readily seen that the first term on the RHS is the same as that for a nonlinear pendulum with a natural frequency of $(mB/N)^{1/2}$. The second term is an unusual driving term that depends on θ. In the absence of this driving term, $\theta = 0$ is a stable equilibrium with the frequency of small perturbations equal to $(mB/M)^{1/2}$. If $\sin(\omega t - \theta)$ in the second term (rotationally rectified term) on RHS is set to one, it reduces to the usual driven oscillator. It is the rotationally rectified term that causes the equilibrium θ_0 to deviate from zero.

Equation of angular motion of the magnetic dipole (detector) pivoted at the current loop centre as caused by the magnetic torque.

Note the multiplicative factor $\sin(\omega t - \theta(t))$ on the right-hand side of (9) that projects the instantaneous component of the induced field $b(t)$ (which is along the ring axis and co-rotates with it) normal to the instantaneous magnetic-dipole moment $\boldsymbol{\mu}$ with $\theta(t) \neq \omega t$ in general. This phase difference is crucial to the effect we are about to discuss.

With this, our mathematical derivation of the equation of motion of the dipole is completed. Next, we turn to its solution. One may, of course, always resort to numerical computation. Indeed, the interested reader is strongly encouraged to solve (9) on her/his laptop for a detailed solution in the parameter space of interest, and thus gain a comprehensive appreciation of the range of possible behaviour. For now, however, we will press on with our analytical approach basing on certain physically motivated approximations.

3. Adiabatic Approximation and the Dipole Deflection

In (9) we can identify certain terms that are *slow-varying* in time relative to some other terms that are *fast varying*. Thus, e.g., the driving terms such as $\sin\omega t$ and $\cos\omega t$ are fast variables while the angular variable $\theta(t)$ for the driven dipole μ (with its mechanical moment of inertia M) is expected to be a relatively slow variable. It is a good approximation then to replace the fast variables by their time-averaged values, i.e., $\langle\cos\omega t\rangle = 0 = \langle\sin\omega t\rangle$ and $\langle\cos^2\omega t\rangle = \frac{1}{2} = \langle\sin^2\omega t\rangle$ in (9), eliminating thus the fast variables in favour of slow variables. This is referred to as the *adiabatic approximation* – well known in physics.

In this adiabatic approximation, (9) reduces to

$$\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = -(\alpha - \beta\sin\gamma)\sin\theta(t) + \beta\cos\gamma\cos\theta(t) \quad (10)$$

Adiabatic approximation for slow dipole rotation and a relatively fast spinning of the loop.

The steady-state time-averaged deflection of the magnetic dipole (detector) in the adiabatic approximation.

with

$$\alpha = mB/M,\ \beta = \frac{\mu_0 m\omega AB}{2DRM\sqrt{1+\left(\frac{\omega L}{R}\right)^2}}, \text{and}$$
$$\gamma = \arctan\frac{\omega L}{R}.$$

The $\sin\theta(t)$ term on the RHS is the effect of the static primary field B. The $\cos\theta(t)$ term is clearly due to the *rotationally rectified* field $b(t)$ (in the adiabatic approximation), and is directed normal to the primary field.

Equation (10) can be reduced to quadrature by multiplying both sides by $\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}$ and integrating with respect to time. The result is, however, not expressible in terms of elementary function. We are, however, interested in the steady (static) deflection θ_0 of the dipole detector which can be experimentally measured. Equation (10) gives this null (fixed) point θ_0 at which the right-hand side vanishes, i.e.,

$$\frac{\sin\theta_0}{\cos\left(\theta_0 - \arctan\frac{\omega L}{R}\right)} = \frac{\mu_0\omega(\pi D^2/4)}{2DR\sqrt{1+\left(\frac{\omega L}{R}\right)^2}}. \qquad (11)$$

The null point θ_0 is the dipole deflection away from $\mathbf{B}$ at which the dipole (detector) will come to rest. In fact, it will oscillate about θ_0 with a small amplitude (for some typical choice of parameters), but any mechanical friction not considered here explicitly, will dampen it to rest at θ_0 in the long-time limit.

4. Estimated Fixed-Point Deflection θ_0

We now turn to a quantitative estimate of the deflection θ_0 expected for some typical values of the parameters involved, namely,

(a) Geometrical: Ring diameter $D = 0.25$ m.

(b) Electrical: Resistance of the nichrome ring

$$R = \frac{\pi D \rho_{\text{nichrome}}}{(\pi d^2/4)} \simeq 1\Omega\,,$$

with $\rho_{\text{nichrome}} \simeq 10^{-6}\,\Omega\text{m}$.

(c) Ring Inductance $L = \mu_0\left[\frac{D}{2}\left(\ln\left(\frac{8D}{d}\right) - 1.75\right)\right]H$, giving inductive reactance ωL (at $\omega \equiv 2\pi f$ with $f = 50\text{Hz}$) $\simeq 0.3\text{m}\,\Omega$. Clearly, the ring resistance $\gg$ the ring reactance.

We can, therefore, approximate (11) to

$$\tan\theta_0 \simeq -\left(\frac{\pi\mu_0\omega D}{8R}\right). \qquad (12)$$

Thus, the magnitude of θ_0 (for small θ_0) increases linearly with the rotational frequency ω of the ring and the ring diameter, as also inversely with the ring resistance R. But there is a conflict in that for higher ω the inductive reactance comes into play; and for $\omega L \gg R$, the deflection θ_0 decreases again. Similarly, the gain expected from increasing the ring diameter D is offset by the concomitant increase in R (and L as well; see point (b) in this section). Thus, the deflection seems to remain small as a rule. For the above choice of parameters, we have $\theta_0 \simeq 1$ millidegree, which is rather small. One could consider reducing the wire diameter d and using a lower resistivity material like copper. Recalling that the inductance increases rather weakly (logarithmically) with decreasing wire diameter, while the resistance increases rapidly (as $1/d^2$), the above combination should favour greater deflection θ_0, and more so by allowing the use of higher rotational velocity ω. In fact, for a ring of copper ($\rho_{\text{Cu}} = 1.7 \times 10^{-8}\ \Omega\,\text{m}$) with other parameters remaining the same, we can obtain a reasonable deflection $\sim$1 degree. Of course, one could carry out further optimization – by use of a laptop.

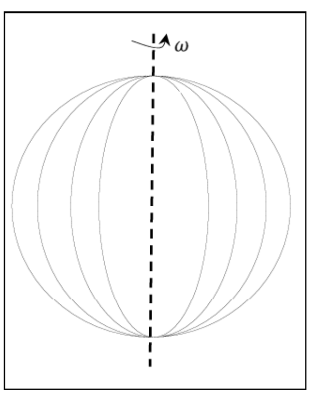
Steady-state time-averaged deflection of the dipole giving an absolute standard for resistance (R) in terms of length (D), time ($1/\omega$), and the vacuum permeability (μ_0).

It is to be noted that in (12) the external field B simply cancels out, as also does M, the moment of inertia, and the dipole moment m. (These will, of course, determine the slow oscillations about the fixed-point deflection θ_0). Most importantly, the electrical resistance R gets determined in terms of the geometrical quantities (lengths D, d), time $2\pi/\omega$, and the inductance L (which can again be expressed in terms of D, d and the permeability μ_0 of space). It was along these lines that *James Clerk Maxwell arrived at the first ever fundamental unit of resistance that served as an electrical standard for quite some time.*

5. Some Lateral Thinking

The conflicting effects of the parameters (resistance R inductance L, and the circular frequency ω) frustrates all simple attempts to maximize the measured quantity of interest, namely the deflection θ_0 in the laboratory! It is tempting to use a tight coil of many turns ($N \gg 1$) instead of the single ring; but the mutual inductance effectively gives an inductive reactance that scales as N^2, and any gain whatsoever may get offset. One can, however, think laterally – replacing the single ring by a number $N \gg 1$ of identical rings, electrically insulated and magnetically well-isolated, and arranged longitudinally as shown in *Figure* 3. This is an arrangement that, in the adiabatic approximation, enhances the steady torque due to the rotationally rectified $b(t)$ field N-fold, and thus gives an N-fold increase of the deflection θ_0, for small θ_0. But, again, very large N will bring the mutual inductance back into the adverse action.

Figure 3. The single conducting ring of *Figure* 1 replaced by a number ($N \gg 1$) of identical rings arranged longitudinally about a common diameter. Rings are electrically insulated and inductively well separated. This enhances the steady deflection θ_0 of the magnetic dipole (detector) N-fold.



Clearly, the limit $N \to \infty$ (without insulation) would correspond to replacing the rings by a thin conducting shell. In turn, replacing the shell by a conducting sphere may further change the physics drastically. One may be led to considering some variant of a self-excited dynamo theory for the magnetic fields associated with rotating planetary and other astrophysical rotating bodies. Many other questions come to mind, e.g., can the induced rotationally rectified field $b(t)$ *become* comparable with the static source field (B) in magnitude, and if so, for what choice of parameters. The inward-bound physics of this problem is indeed fascinating. Interested reader is encouraged to pursue some other lines of thought as well in this connection.

Suggested Reading

[1] The first ever electrical Resistance Standard in terms of measurements of length, time and mass only was proposed in 1863 by James Clerk Maxwell. It was based directly on the physics underlying the mechanisms discussed in this classroom exercise. See, Basil Mahon, *The Man Who Changed Everything*, pp.117–119, Wiley, UK, 2003.

[2] D J Griffiths, *Introduction to Electrodynamics*, Prentice Hall, India, 2006.

[3] D Halliday and R Resnick, *Physics*, Wiley-Eastern, New Delhi, 1966.