Classroom

In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Magnetic Fields and Bohr's Quantization Rule

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This article deals with quantization, using Bohr's quantization rule, of orbits of charged particles moving in a magnetic field. It is pointed out that the momentum to be used in the Bohr–Sommerfeld quantization rule is not always the familiar mv but the so-called canonical or conjugate momentum which is obtained from the Lagrangian appropriate to the problem. Stating the well-known Lagrange equations of motion we arrive at the expression for the Lagrangian which will reproduce the familiar Newtonian equations for the problem at hand. This leads us to an expression for the canonical momentum. Magnetic flux quantization follows from the application of Bohr's rule using the canonical momentum instead of mv. Examples discussed include the Landau levels, and hydrogen atoms on the surface of neutron stars. The article concludes with comments on a justification of Bohr's rule.

Keywords
Magnetic flux quantization, Landau levels, Bohr quantization rule, semi-classical quantization.

1. Bohr Quantization Rule

Prior to the advent of quantum theory there existed a large amount of spectroscopic data which defied any

explanation. The spectrum of hydrogen, for example, contained several spectral lines (the experimental arrangement is such that light of a given frequency appears as a line which is actually the slit used) in the visible. The classical explanation was that the charge cloud in the atom oscillates at each of these frequencies and thus, based on Larmor's result emitted radiation at these frequencies. There was no clue as to why the atomic charge cloud chose to oscillate at these frequencies.

While dealing with the thermodynamics of black bodies, in 1900, Max Planck created a revolution by proposing his quantum hypothesis in which the energy of an oscillator, representing radiation of a given frequency, could have energy in multiples of a basic unit, the so-called quanta.

In 1913 Niels Bohr proposed his atomic model. The model was an extension of the Rutherford model of the atom with some additional prescriptions. An electron going around a proton, to take the simplest atom, has nonzero acceleration due to the (Coulomb) force of the nucleus. A result by Larmor then requires the accelerated electron to go on radiating energy, eventually sinking into the proton! Numerical estimate of the lifetime turns out to be tens of picoseconds (somewhat smaller than our own ages; not to think of the age of the universe!). Classically, the radius of an electron's orbit could take any continuous value. By a postulate Bohr ruled that the electron can move only in certain orbits and while it did so it did not radiate a la Larmor! The electron could, however, jump from one allowed orbit to another and give away (or absorb) the difference in the energies of the two orbits in the form of a photon. To appreciate the economy of thought, that such a prescription brings, consider a hypothetical atom which has 100 allowed energy levels. Let us number them according to increasing energies. When the electron jumps from any of the 99 higher energy orbits to the first energy orbit

Bohr's model, by shifting the emphasis from spectral lines to energy levels, with the lines related to the energy differences, grossly reduces the number of required 'explanations'.

we get 99 spectral lines. These 99 lines belong to 'series 1' as they all have the level 1 in common. We have 98 spectral lines in the 'series 2'; 97 in the 'series 3'; till the last 'series 99' which is the transition from the level 100 to the level 99. How many lines do we have in all? Sum of first 99 natural numbers is 4950. According to the classical theory the atom's electronic cloud would have been obliged to oscillate at 4950 frequencies! Surely a terminal case of malaria! Bohr's model, by shifting the emphasis from spectral lines to energy levels, with the lines related to the energy differences, grossly reduces the number of required 'explanations'. In the case of our hypothetical atom one needs to explain only 100 levels as opposed to the 4950 oscillations a classical theory would be required to explain. Bohr even had a prescription for the energy levels which is known as Bohr's quantization rule.

Which of the classical orbits are allowed? Bohr postulated that only those orbits are allowed for which the angular momentum of the electron is a multiple of a fundamental unit. With Bohr's restriction to circular orbits the angular momentum is given by mvR and the Bohr's quantization condition becomes $mvR = n\hbar = n\frac{h}{2\pi}$. By shifting the 2π to the other side, it can be combined with the radius R of the orbit to suggest the circumference of the orbit. If we denote by q the coordinate along the orbit and by $p = mv$ the familiar (dynamical in technical nomenclature) momentum, Bohr's condition can be written as follows:

$$p \oint \mathrm{d}q = nh,$$

where the circle on the integral sign indicates an integral along the entire orbit. In this problem the magnitude of the momentum is constant along the orbit so it can well be taken inside:

$$\oint p \, \mathrm{d}q = nh.$$

Soon after Bohr, Wilson and Sommerfeld, independently, came up with a generalization of the Bohr quantization condition. Their rule amounts to the above equation but allows the possibility that the magnitude of p can vary. In this form it applies to elliptic orbits as well. Since Sommerfeld applied this rule to the hydrogen atom, the extension:

$$\oint p\,(q)\ \mathrm{d}q = nh$$

is usually called Bohr–Sommerfeld rather than Bohr–Sommerfeld–Wilson quantization rule (we have stressed now that the momentum can depend on the coordinate q along the orbit).

In this article we will use this generalization of the Bohr's rule and, having once mentioned its inventors, call it the Bohr rule for brevity.

Bohr's rule has a natural geometrical interpretation. A particle can be (fully) described classically by specifying its position and momentum. If we construct a two-dimensional space with position and momentum as the two coordinates then specifying these two coordinates, amounts to a single point in this, so-called, 'phase space' (for a particle moving in 3 dimensions the phase space has 6 dimensions; for N particles the phase space has $6N$ dimensions). The integral involved in Bohr's rule is just the area (for 2D phase space) inside the classical orbit as plotted in the phase space. The right-hand side (RHS) of the rule, then, tells us that this area is quantized in units of the Planck's constant. The phase space area between the consecutive allowed orbits is h (for N particles this will be h^N). Another way of stating the result is that there is one quantum state per area h (or the appropriate power of h) in the phase space. This rule for counting the number of quantum states in a given region of phase space is very crucial in statistical physics. Bohr's rule (or WKB, an approximation

Geometrical Interpretation of Bohr Rule:
The area, in phase space, occupied by allowed orbits is quantized. This interpretation justifies counting of quantum states used in quantum statistical mechanics.

In the presence of magnetic fileds the momentum, to be used in Bohr's rule, is not the familiar mv but what is called the canonical momentum.

to Schrödinger's equation) is one instance which justifies this counting for a wide class of (slowly varying 1D) potentials.

When Bohr's rule is applied to the hydrogen atom the momentum used is $p = mv$. This is fine as, for the hydrogen problem, it is the canonical momentum. However, when magnetic fields are present the canonical momentum turns out to be not the familiar momentum. To get the canonical momentum one needs to obtain the lagrangian for the problem. By comparing Newton's equations and Lagrange's (Euler–Lagranges') equations, we obtain the expression for the Lagrangian and hence the canonical momentum for a charged particle in the presence of a magnetic field. Our derivation brings out the well-understood ambiguity in the expressions for both these quantities and also points out the step at which the ambiguity enters into the picture.

2. Classical Motion of Electron in Magnetic Field

Throughout this article we denote the charge of electron as e ($e < 0$; it is $-\,|e|$ with $|e|$ as the fundamental unit of charge (1.6×10^{-19} C)). The magnetic field is in the positive z direction and the motion is in the x–y plane. We also assume that the reader has a picture of the circular orbits in mind wherein the direction of the magnetic field and motion are consistent. Newton's equation of motion (second law) for an electron moving in the magnetic field B (default positive z direction) is

$$m\vec{a} = e\vec{v} \times (B\,\hat{z}).$$

Let us resolve all the vectors into their Cartesian components as follows:

$$\begin{aligned} \vec{a} &= \ddot{x}\,\hat{x} + \ddot{y}\,\hat{y}\ , \\ \vec{v} &= \dot{x}\,\hat{x} + \dot{y}\,\hat{y}\ . \end{aligned}$$

The expression for the force becomes $\vec{F} = e\,\vec{v}\times(B\,\hat{z}) = e\,B\,(\dot{y}\,\hat{x} - \dot{x}\,\hat{y})$. Substituting these resolutions into the

above vector form of Newton's equation of motion and using the fact that the basis vectors are linearly independent we get the component form of Newton's equations of motion.

$$m\,\ddot{x} = e\,B\,\dot{y}\,,$$
$$m\,\ddot{y} = -\,e\,B\,\dot{x}\,.$$

These coupled differential equations have the general solution (see *Box* 1):

$$x = X + R\,\cos\,(\omega\,t + \phi)\,,$$
$$y = Y + R\,\sin\,(\omega\,t + \phi)\,,$$

provided the so-called cyclotron or Larmor frequency ω is given by

$$\omega = \frac{|\,e\,|\,B}{m}.$$

Box 1. Solving Newton's Equations of Motion

One way to solve this set of coupled equations will be to take a derivative of the first equation with respect to time which will result in the right-hand side having a second time-derivative of y which is then replaced using the second equation. The resulting equation is $\dddot{x} = -\left(\frac{eB}{m}\right)^2 \dot{x}$. This is our good old friend, the simple harmonic oscillator (SHO), though for the x component of velocity. One gets a similar (SHO) equation for the y component of velocity. These equations can be readily integrated once to get expressions for the x and the y coordinates of the particle. Another interesting way is to combine x and y coordinates into a single complex number $z = x + i\,y$ and use the above two differential equations to get a (naturally) single differential equation for z: $m\,\ddot{z} = -i\,eB\,\dot{z}$. Note that this is a first order equation for $\dot{z}$. One could also think of this system as a first order system for a vector (as a computer would be told to do) and get a first order differential equation in a matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{eB}{m}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}\,.$$

This way of solving the problem is quite interesting as one ends up with an exponential, which for a matrix is defined by the series, of a matrix (incidentally it is the same 2×2 matrix that appears on the right-hand side of the above differential equation). The special feature of this matrix is that all its powers are either (+ or −) the same matrix or (+ or −) unit matrix. This helps to sum the entire series nicely in terms of sin and cos functions.

You may verify the above solution if you find it easier than solving those equations upfront. The result is the same as what we get (school physics) if we argue that the motion is going to be circular and use the toolbox of circular motion ($mv^2 = |e|vB$ and so on). The reason why we have not chosen that shortcut to arrive at the cyclotron frequency is that we are actually hunting for the Lagrangian.

3. Lagrangian and Canonical Momentum for a Charged Particle Moving in a Uniform Magnetic Field

Newton's form of mechanics is local in operation. At every instant we are told by the equation how to move to the next position. After reaching that position we know how to move next. A chain of displacements linked together by local directions. It is like a treasure hunt. Every time you crack a clue you get the next clue. You wouldn't know that the treasure was right under your chair till you went around following the directions given in the clues. All the clues, although present, are not accessible at the same time. There is another formulation of mechanics, the Lagrangian mechanics, which operates on global specifications. It is like telling someone to go along any path, but reach in the shortest time. On a flat ground the person will be forced to follow a straight path. Lagrangian mechanics is based on Hamilton's principle which states that among all paths possible the path actually followed has the lowest (extremum to be precise) value of the integral (the action):

$$S\left[q(x,t)\right] = \int_a^b L(q,\dot{q},t)\mathrm{d}t\,.$$

The way to go about this is to consider a candidate path $q = q(t)$, i.e., the coordinate as function of time. The paths considered are such that they all begin, at time a, at the same point, say A, and end, at time b,

in the same point, say B (which could be same as A). From this function $q(t)$ one can estimate the speed $\dot{q}t$); evaluate the Lagrangian L at every instant using these values for that instant. After doing the integration one gets the action S. It, therefore, depends on the entire path. A function gives a number for an input number. To emphasize the fact that the value of S depends on the entire path, it is called a functional (function input; number output). This action is least (or technically an extremum) for the true path which satisfies the following Euler–Lagrange (EL) equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \ .$$

If the motion is in a two-dimensional plane, with x and y the two coordinates, then we get two equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \quad ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y}.$$

These must be equivalent to the Newtonian equations of motion for the problem derived above. We will use this requirement to get a Lagrangian for charged particle in a magnetic field as follows:

We want the Euler equation for the x coordinate, namely $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$, to be equivalent to the corresponding Newton's equation, namely $m\,\ddot{x} = e\,B\,\dot{y}$. We guess that L must have a term $\frac{1}{2}m\dot{x}^2$ so that prior to the time derivative, on the left-hand side (LHS), we will get $\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = m\,\dot{x}$. The time derivative will then give us the LHS of Newton's equation. If we want to get the RHS of this Newton's equation from the RHS of the EL equation we anticipate L to have a term like $\dot{y}x$ so that the RHS of the EL will reduce to that of the Newtonian equation, just from this term. Similarly the y-equation prompts us to add a term of the form $-\dot{x}y$. Let us, therefore, try a Lagrangian of the form

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + e\ B\ (\alpha\ \dot{y}\ x - \beta\ \dot{x}\ y)\ .$$

Using this Lagrangian we can reproduce both the Newtonian equations if $\alpha + \beta = 1$. It is not possible to fix α and β individually. There is a deeper reason why this cannot be done. The additional piece (other than the familiar kinetic energy) is the well-known velocity dependent potential which is essentially a scalar product of the magnetic vector potential $\vec{A}$ and the velocity $\vec{v}$. In fact, this additional piece is $e\,\vec{A}\cdot\vec{v}$. As is well known one can have many vector potentials corresponding to the same magnetic field (see *Box* 2). If we set $\alpha = \frac{1}{2} + \gamma$ and $\beta = \frac{1}{2} - \gamma$, which covers the most general possible pair of α and β subject to the condition that their sum is unity, and use them in the above expression for the Lagrangian, the γ term in it is $eB\gamma\frac{\mathrm{d}}{\mathrm{d}t}(xy)$ which is a total derivative with respective to time. The action principle, stated above, involves an integral over time. When this term is integrated we get $eB\gamma xy|_{\mathrm{A}}^{\mathrm{B}}$. The value of this term depends only on the values of x and y at the end points; but as stated earlier a rule of the game is to vary

Box 2. Freedom in Choosing Magnetic Vector Potential

To see that this is so let us recall the relation between the magnetic vector potential and the magnetic field. The vector potential $\vec{A}$ has to be such that its curl should be $B\,\hat{z}$. Now, by definition the curl of an arbitrary vector can be written in the following determinantal form:

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} .$$

While expanding the determinant keep in mind that the middle row elements, the derivatives, operate on the third row elements. Since we are dealing with a magnetic field along the z direction we must have the components of $\vec{A}$ such that $\vec{B}$ has no x or y component. This can be easily arranged if $\vec{A}$ has no z component and the other two components do not depend on z. So we are left with the z component which is related to the vector potential $\vec{A}$ by $B = \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x$. We can get a contribution from the first term if A_y depends on x linearly and from the second term if A_x depends on y linearly (nonlinear dependences will result in a non-uniform magnetic field). So the form of the vector potential is $\vec{A} = B\,(\alpha\,x\,\hat{y} - \beta\,y\,\hat{x})$. It does not matter whether the first or the second term contributes. Only the overall contribution, which turns out to be $B\,(\alpha + \beta)$, is required to be B giving us $\alpha + \beta = 1$.

the paths as one may wish, keeping the end points fixed. So this term remains inert through the variation; obviously does not influence the equations of motion.

4. Bohr Quantization Rule in the Presence of Magnetic fields: Magnetic Flux Quantization

Since the choices of α and β are arbitrary subject to their sum being unity, we choose both equal to half. It remains to be seen that the result is independent of this special choice (the so-called symmetric gauge). The Lagrangian is then:

$$L = \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2) + \frac{e\,B}{2}(x\,\dot{y} - \dot{x}\,y)\,.$$

Transforming to the cylindrical coordinates, (the system has a circular symmetry), using

Lagrangian for a charged particle in magnetic field.

$$\begin{aligned} &x = r\ \cos\theta\,,\\ &y = r\ \sin\theta\,,\\ &\text{i.e., } \dot{x} = \dot{r}\ \cos\theta - r\ \dot{\theta}\ \sin\theta\ ,\\ &\text{and}\quad \dot{y} = \dot{r}\ \sin\theta + r\ \dot{\theta}\ \cos\theta\,, \end{aligned}$$

we get the following Lagrangian in the cylindrical coordinates

$$L = \frac{1}{2}m\,(\dot{r}^2 + r^2\,\dot{\theta}^2) + \frac{eB}{2}r^2\,\dot{\theta}\ .$$

It can be seen that the Lagrangian does not depend on the angle θ (it is what is called an ignorable coordinate) and by the Euler–Lagrange equation the conjugate momentum is a constant of motion:

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = m\,r^2\,\dot{\theta} + \frac{eB}{2}\,r^2\,.$$

We also know that r and $\dot{\theta}$ are constants, with our choice of origin at the center of the circular orbit. We are now in a position to do the integral required for the application of the Bohr's rule:

$$\begin{aligned} \oint p\mathrm{d}q \equiv \oint p_\theta\,\mathrm{d}\theta = \int_0^{2\pi} p_\theta\,\mathrm{d}\theta = p_\theta \int_0^{2\pi}\mathrm{d}\theta = p_\theta\ 2\pi \\ = 2\pi m r^2\dot{\theta} + \pi r^2 eB\,. \end{aligned}$$

Although the magnetic vector potential, (hence) the Lagranigan and the canonical momentum, are not unique, all choices yield the same value of the $\oint p\,\mathrm{d}q$ integral; non-unique part, so to say, gives zero on integrating over the entire range of θ!

Using the formula for the angular frequency $\dot{\theta}$ (which is the same as ω), and (most importantly) remembering that the charge of electron is negative, i.e., $-|e|$ we see that the second term is half the first one with a minus sign. That this result remains the same for other choices of the vector potentials, $\gamma \neq 0$ in our notation, is left to the reader as an exercise.

Bohr's quantization rule, therefore, becomes:

$$\oint p\ \mathrm{d}q\ = \pi r^2\ |e|\, B = |e|\ \Phi_{\mathbf{B}} = nh\,,$$

where we have combined the orbit area πr^2 and the magnetic field strength B into the magnetic flux, Φ_{B}, threading the orbit. Bohr's quantization rule then implies that the allowed orbits are such that the magnetic flux threading them is quantized in units of $h/|e|$:

$$\Phi_{\mathbf{B}} = n\ \frac{h}{|e|}\,.$$

5. Landau Levels and Hydrogen Atoms Near the Surface of a Neutron Star

The energy levels that a charged particle is allowed to have in the presence of a constant magnetic field are known as the Landau levels. We have seen above that Bohr's rule yields a discrete set of values for the radius of the orbit. Now the speed and the radius are related by the equation of motion (the circular motion form):

$$\frac{mv^2}{r} = |e|\, Bv$$

$$\text{or} \quad mv = |e|\, Br\,.$$

The energy of the electron is given by

$$E = \frac{(mv)^2}{2m} = \frac{e^2B^2r^2}{2m}\,.$$

By using Bohr's quantization rule, derived above, and the expression for the Larmor frequency, the expression for the energy becomes:

$$E = \frac{e^2 B^2\, r^2}{2m} = \frac{|e|\, B}{m}\frac{1}{2}\frac{|e|\, B\pi r^2}{\pi} = \omega\frac{n\, h}{2\pi}\,;$$

$$E_n = n\, \hbar\, \omega\,.$$

An interesting situation arises when we ask what happens if a hydrogen atom is placed in a magnetic field. This problem is discussed in textbooks under the title 'Zeeman effect'. The magnetic fields considered there have, however, much smaller effect on the electron's motion when compared to the Coulomb field of the nucleus. But there are real situations, such as the atmospheres of white dwarfs and neutron stars where the magnetic field is very strong. The magnetic field on the surface of a neutron star could be as high as 10^8 Tesla (compare this to few tens of Tesla that can be produced in labs). If we calculate the radius of the first orbit using the above flux quantization we get 3 picometers! Much smaller than the half angstrom radius of hydrogen atom! So the magnetic field is providing the major component of the centrifugal force needed and the Coulomb force is a perturbation. The situation is even more interesting as the magnetic field does not do anything along its direction. Across the magnetic field we expect the shape of the orbital to have a diameter of six picometer, but along the field it is the Coulomb field which is responsible for binding and we may expect the orbital to have one angstrom width. The hydrogen atom will be like a needle with its length few thousand times its width. Properties of matter in such extreme environments need to be worked out, *abinitio*, from quantum mechanics (Dirac's equation which takes care of relativistic effects and the spin of electron). The collective behavior of such atoms could also be very interesting. We have no possible help from laboratories here; the fields are beyond their reach.

Properties of matter in very high magnetic fields could be very different.

6. Justification of Bohr's Rule

Bohr's quantization rule brought understanding, in the context of spectral lines. It was the dawn of a new theory. The fact that his rule gave the spectral lines was a clear proof that the development was in the right direction.

But surely there must be an explanation of his ad hoc rule. Why should the angular momentum be quantized? A prince (in marked contrast to usual fairy tales!) came up, in his doctoral thesis, with an explanation! The prince suggested that all matter particles had waves associated with them, the wavelength of which is related to the momentum. The classical circular motion equation relates the particle's momentum to the radius of its orbit. So, only one of them can be considered independent. Now imagine a wave going around the circumference of such an orbit. Like the waves in a flute or a violin string, only certain wavelengths will satisfy a resonance condition: those for which the circumference is an integer multiple of the wavelength given by the prince's formula. When the details are worked out the prince's formula for the wavelength and the resonance condition do amount to the quantization of angular momentum. The prince had explained Bohr's rule in terms of something deeper: wave nature of matter.

The prince's story didn't end there. When his thesis was reviewed in a meeting in Zurich in 1925, Peter Debye, of the lattice fame, remarked to Schrödinger, that it makes no sense to talk about waves without a wave equation. To this task Schrödinger set himself in the romantic setting of an alpine villa on the next Christmas vacation. There he first derived a relativistic equation which is now known as the Klein–Gordon equation. This equation has certain difficulties associated with it and, therefore, Schrödinger derived a non-relativistic wave equation, now known after him, and solved the hydrogen

atom problem. After coming back he communicated his papers and then told his wife (during all this exciting time she wasn't with him on the vacation; historians are not sure who was; they certainly know four or five candidates who could have been but were definitely not with him during this vacation) that he will get a Nobel Prize, which he did in 1933.

Now that the Schrödinger equation is the truth can one derive Bohr's quantization rule, as some approximation, from it? Usually, an approximation referred to as WKB (Wenzel, Brilloin and Kramers) is cited as a derivation of Bohr's rule. The end result of the WKB can be written in terms of the classical orbit

$$\oint p\,(q)\,\mathrm{d}q = \left(n + \frac{1}{2}\right)\,h$$

and, for large quantum numbers, is essentially the same as the Bohr's quantization rule. This analysis provides a rigorous justification of Bohr's rule.

Few remarks are, however, in order. At the outset Bohr's rule, in the historical context, deals with two-dimensional motion whereas the WKB applies strictly to one dimension. There is a subtler difference between the two settings. Bohr's rule has a classical picture of orbit in mind. Now such classical orbits translate into what are known as coherent states in quantum mechanics. To make the point clear, let us consider a simple harmonic oscillator. The stationary states, fixed energy and number, have the property that their associated probability distribution is time independent. There is no way such probability distributions may look like a classical pendulum going from one end to the other. This is true no matter how large the quantum number n is. It is not true, therefore, that one recovers classical mechanics by simply going to large quantum numbers. To get something like an oscillating probability distribution, like the classical pendulum, it is necessary to superimpose

different states. Such a superposition of states has a time-dependent probability distribution and can look like a pendulum. Another way of looking at the problem is to note that classical trajectories, orbits, have well-defined position and momentum which is not permissible in quantum mechanics. If strictly classical is not possible, can we get something which is as classical as possible? Yes. These are special superpositions which have the lowest possible uncertainty (zero uncertainty would be the classical target). These, so-called, coherent states behave very much like the oscillating pendulum or the classical electron that Bohr imagined going around the nucleus. To get classical mechanics from quantum mechanics one has to construct such coherent states (with minimum uncertainty) in addition to going for large quantum numbers. Coming back to Bohr's rule we note that it must be a coherent state underlying his classical orbits. The WKB approximation, on the other hand deals with a stationary state and not a coherent state. A coherent state has fluctuations associated with it. For example, if the average energy of a oscillator in a coherent state (read classical) is n (in units of Planck's quanta) then there is fluctuation of order $\sqrt{n}$. Now $\sqrt{n}$ may be much smaller than n but it is much greater than 1. Why is it that a rule based on classical trajectories, which has fluctuations of order $\sqrt{n}$ in its properties, can be exact to order unity, the *difference* between adjacent levels. The quantization should not have survived an averaging over $\sqrt{n}$ states. It is just a coincidence that Bohr's rule, with its implicit coherent state picture, turns out to be the same as that of the WKB rule which deals explicitly with a fixed number state.

A real justification of Bohr's rule comes from a formulation of quantum mechanics, due to Dirac and Feynman. In this path integral formulation a particle propagates along all possible paths. However, the contributions, like the theory of diffraction in optics, along different paths

add up with different phases. The phase of a propagation by a certain path is just the value of the action for that path divided by Planck's constant (both have the same dimensions). Contributions from most paths cancel out as the action varies a lot (remember we are dividing by a small constant). However, the classical paths, as mentioned earlier, are paths of least action. Just as an ordinary function is flat (does not change its value) near its maxima or minima the action does not change much for paths near the classical path. The contributions from such paths, therefore, add up constructively. The relevant value of action , therefore, turns out to be that of the classical path and we get Bohr's rule as a byproduct. This path integral approach also answers the question, very valid in classical mechanics: how does the particle know the path along which the action is least? (Does it solve the EL equations before starting?). In path integral approach the answer is simple to state but difficult to grasp: it goes along all possible paths!

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Suggested Reading

[1] **The principle of least action has been discussed by R P Feynman in Chapter 19 of the second volume of his *Lectures on Physics*, Narosa Book Distributors Pvt. Ltd., 2008. This is one of the most lively discussions of the principle; a must read for a physics student.**

[2] **For all matters related to classical mechanics, such as Lagrangian mechanics, in particular the Lagrangian for a charged particle in magnetic field (a more general derivation than presented here) the classic text by H Goldstein, J Safko and C P Poole, *Classical Mechanics and Path Integrals*, 3rd Edition, Pearson, 2011, can be consulted.**

[3] **The path integral approach con be read from the book by R P Feynman and A R Hibbs, *Quantum Mechanics and Path Integrals*, Dover, 2010. Alternative good read on path integrals is the Dover book by I S Schulman, *Techniques and Applications of Path Integration*, Dover Books on Physics, 2005.**