

Mean-field dynamo action in renovating shearing flowsSanved Kolekar^{*} and Kandaswamy Subramanian[†]*IUCAA, Pune University Campus, Ganeshkhind, Pune 411007, India*S. Sridhar[‡]*Raman Research Institute, Sadashivanagar, Bangalore 560080, India*

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We study mean-field dynamo action in renovating flows with finite and nonzero correlation time (τ) in the presence of shear. Previous results obtained when shear was absent are generalized to the case with shear. The question of whether the mean magnetic field can grow in the presence of shear and nonhelical turbulence, as seen in numerical simulations, is examined. We show in a general manner that, if the motions are strictly nonhelical, then such mean-field dynamo action is not possible. This result is not limited to low (fluid or magnetic) Reynolds numbers nor does it use any closure approximation; it only assumes that the flow renovates itself after each time interval τ . Specifying to a particular form of the renovating flow with helicity, we recover the standard dispersion relation of the $\alpha^2\Omega$ dynamo, in the small τ or large wavelength limit. Thus mean fields grow even in the presence of rapidly growing fluctuations, surprisingly, in a manner predicted by the standard quasilinear closure, even though such a closure is not strictly justified. Our work also suggests the possibility of obtaining mean-field dynamo growth in the presence of helicity fluctuations, although having a coherent helicity will be more efficient.

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I. INTRODUCTION

Understanding the origin of coherent large-scale magnetic fields, observed in astrophysical systems from stars to galaxies, is of fundamental importance in astrophysics. The standard paradigm invokes the amplification of small seed magnetic fields due to dynamo action, typically involving large-scale shear flows combined with helical turbulence [1,3]. Recent numerical simulations have also raised the possibility that coherent fields can arise in nonhelical turbulent flows in the presence of shear [4], although the mechanism of how this happens is unclear [5,6].

The dynamics of large-scale magnetic fields is generally described by using the equations of mean-field electrodynamics. Here one defines the mean magnetic field $\overline{\mathbf{B}}$ and mean velocity field $\overline{\mathbf{U}}$ by suitable averaging over the small scales corresponding to the turbulent fluctuations. The mean-field evolution is governed by the averaged version of the induction equation, which results in an extra term, the turbulent electromotive force $\overline{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle$, where \mathbf{u} and \mathbf{b} are the fluctuating velocity and magnetic fields. Expressing $\overline{\mathcal{E}}$ in terms of the mean fields themselves is a closure problem, even for prescribed velocity fields. If the fluctuations can be assumed to be small, one can employ the quasilinear approximation or what is traditionally known as the first-order smoothing approximation (FOSA) to express the turbulent electromotive force $\overline{\mathcal{E}}$ in terms of a term proportional to $\overline{\mathbf{B}}$ (the α effect) and one proportional to the mean current density (turbulent diffusion). The α effect, which depends on the helicity of turbulence is crucial to amplify mean fields.

However, turbulent motions above some modest magnetic Reynolds number lead to a fluctuation dynamo and a rapid

growth of magnetic noise [1,7,8]. The growth rate of the fluctuation dynamo is typically larger than the growth rates associated with the mean-field dynamo. In the presence of this rapidly growing magnetic noise, the validity of the quasilinear approximation or FOSA becomes suspect. The question then arises whether one can indeed make sense of mean-field concepts like the alpha effect.

In this context, considering exactly solvable flow models becomes very useful. For example, if one assumes a random flow whose correlation time is exactly zero, dynamo action, on both small and large scales can be studied in great analytical detail [1,7,8]; however, such a flow is unphysical. Renovating flows discussed by several authors [9,10] (and references therein) provide, on the other hand, models involving flows with a finite nonzero correlation time but which are still analytically tractable. In renovating flows time is split into successive intervals of length τ and the stochastic component of the velocity in the different intervals is assumed to be statistically independent realizations of an underlying probability distribution (PDF). As the flow loses memory between different time intervals, the evolution of the moments of the magnetic field over any one time interval can be calculated by averaging over the underlying PDF. Considering random helical renovating flows, Gilbert and Bayly (GB) [10] showed that the magnetic field becomes increasingly intermittent with time. Nevertheless, the mean magnetic field can still grow with a growth rate which approaches that of a standard mean-field α^2 dynamo in the limit of small τ [9,10]. These works thus provide explicit demonstration that the growth of magnetic noise need not destroy the growth of the mean field even in the case of flows (which have this periodic loss of memory) with finite nonzero correlation times.

In the present work we generalize some of these results to renovating flows incorporating also a large-scale shear. Our primary motivation is to examine if the introduction of shear can lead to a mean-field growth even if the stochastic velocity field is nonhelical. It turns out that, once properly

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formulated in terms of renovating shearing waves, the details of our calculation share some crucial features with those of Gilbert and Bayly (GB) [10]. However, GB have not given these steps; so to aid the presentation of our results, we begin in Sec. II with a presentation of the main results of [10] for the mean-field evolution in renovating helical flows. In Sec. III we formulate the problem of renovating flows with a background linear shear, and prove a general result that there is no dynamo action when the flow is strictly nonhelical. We also consider a particular example of renovating shearing waves with helicity, caused by overdamped external forcing and recover the dispersion relation for the $\alpha^2\Omega$ dynamo. The final section summarizes and presents a discussion of our results.

II. RENOVATING HELICAL FLOWS WITHOUT SHEAR

We re-derive here the results of Gilbert and Bayly [10] on mean-field evolution in a model helical renovating flow, in the absence of background shear. Consider the induction equation for the evolution of the magnetic field,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B}]. \quad (1)$$

GB assumed the velocity field \mathbf{u} to have zero mean with only a turbulent component. In each renovating time interval τ , they took

$$\mathbf{u}(\mathbf{x}) = \mathbf{a} \sin(\mathbf{q} \cdot \mathbf{x} + \psi) + \mathbf{b}h \cos(\mathbf{q} \cdot \mathbf{x} + \psi). \quad (2)$$

with the conditions,

$$\mathbf{a} \cdot \mathbf{q} = 0, \quad \mathbf{b} = \mathbf{q} \times \mathbf{a}/q, \quad (3)$$

which implies incompressibility, that is, $\nabla \cdot \mathbf{u} = 0$. The parameter h satisfies $-1 \leq h \leq 1$ and determines the helicity of the flow. This helical flow is made random by choosing the parameters of the flow randomly and independently from an underlying PDF, for every time interval τ . The ensemble considered is the following: In each time interval $[(n-1)\tau, n\tau]$, (i) ψ is chosen uniformly random between 0 and 2π ; (ii) the propagation vector \mathbf{q} is uniformly distributed on a sphere of radius q ; (iii) for every fixed \mathbf{q} , the direction of \mathbf{a} is uniformly distributed in a circle of radius a in the plane perpendicular to \mathbf{q} . The parameters (q, a, h, τ) are nonrandom and completely describe the renovating flow. The randomness of ψ in condition (i) ensures statistical homogeneity, whereas conditions (ii) and (iii) ensure statistical isotropy of the flow.

The evolution of the magnetic field from time $(n-1)\tau$ to $n\tau$ is given by

$$B_i(\mathbf{x}, n\tau) = \int \mathcal{G}_{ij}(\mathbf{x}, \mathbf{y}) B_j(\mathbf{y}, (n-1)\tau) d^3y, \quad (4)$$

where $\mathcal{G}_{ij}(\mathbf{x}, \mathbf{y})$ is the Green's function of the induction equation Eq. (1). \mathcal{G}_{ij} is random due to the randomness of the turbulent velocity field. We take the ensemble average of this equation over the ensemble described above and note that the velocity in any time interval $[(n-1)\tau, n\tau]$ is uncorrelated with the initial magnetic field at time $(n-1)\tau$. Thus the average of the product of \mathcal{G}_{ij} and B_j can be written as the product of the averages, an important simplification arising from the loss of memory of renovating flows. The mean field $\overline{\mathbf{B}}$ then

evolves as

$$\overline{B}_i(\mathbf{x}, n\tau) = \int \overline{\mathcal{G}_{ij}(\mathbf{x}, \mathbf{y})} \overline{B}_j(\mathbf{y}, (n-1)\tau) d^3y. \quad (5)$$

Furthermore, from the statistical homogeneity of the renovating flow, one has $\overline{\mathcal{G}_{ij}(\mathbf{x}, \mathbf{y})} = \overline{\mathcal{G}_{ij}(\mathbf{x} - \mathbf{y})}$; then Eq. (5), which is a convolution in physical space, becomes a product in Fourier space. Defining the spatial Fourier transform,

$$\widehat{B}_i(\mathbf{k}, t) = \int d^3x \overline{B}_i(\mathbf{x}, t) e^{-i\mathbf{k} \cdot \mathbf{x}}, \quad (6)$$

we have in Fourier space,

$$\widehat{B}_i(\mathbf{k}, n\tau) = G_{ij}(\mathbf{k}) \times \widehat{B}_j(\mathbf{k}, (n-1)\tau), \quad (7)$$

where the response tensor, $G_{ij}(\mathbf{k})$, is defined by

$$G_{ij}(\mathbf{k}) = \int \overline{\mathcal{G}_{ij}(\mathbf{x} - \mathbf{y})} e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} d^3x. \quad (8)$$

The mean magnetic field will grow exponentially if its Fourier component is an eigenvector of the matrix $G_{ij}(\mathbf{k})$ with eigenvalue σ , whose magnitude is greater than one. For such an eigenvector we have

$$\widehat{B}_i(\mathbf{k}, n\tau) = \sigma^n \widehat{B}_i(\mathbf{k}, 0) \quad \text{or} \quad \widehat{B}_i(\mathbf{k}, t) = \sigma^{t/\tau} \widehat{B}_i(\mathbf{k}, 0), \quad (9)$$

and so the growth rate of this eigenmode is given by

$$\lambda = \frac{\ln(\sigma)}{\tau}. \quad (10)$$

Hence, the response tensor $G_{ij}(\mathbf{k})$ contains all the information about the growth or decay of the mean magnetic field. We now proceed to calculate it for the renovating velocity field of Eq. (2).

A. The response tensor

In order to explicitly calculate the response tensor for the evolution of the mean field in the renovating helical flow, an important simplification was introduced by GB. The renovation time τ was split into two equal subintervals. In the first subinterval (step 1) resistivity was neglected and the field was just frozen-in and advected with the fluid, with twice the original velocity. In the second subinterval (step 2), induction by the velocity was neglected and the field was assumed to diffuse with twice the resistivity. Although such an assumption seems plausible in the limit of a short renovation time—as this would be one way of numerically integrating the induction equation—GB did not give any rigorous justification. For the present purpose, we adopt the same simplification as GB. Thus we consider the evolution of the magnetic field in these two steps and then Fourier transform the resulting averaged Green function.

Step 1. During the time interval 0 to $\tau/2$ we assume $\eta = 0$, and double the value of the velocity field. Then Eq. (1) becomes just the ideal induction equation,

$$\frac{d\mathbf{B}}{dt} \equiv \left(\frac{\partial}{\partial t} + 2\mathbf{u} \cdot \nabla \right) \mathbf{B} = (\mathbf{B} \cdot \nabla) 2\mathbf{u}, \quad (11)$$

with the standard Cauchy solution given by

$$B_i(\mathbf{r}, t) = \frac{\partial r_i}{\partial y_j} B_j(\mathbf{y}, t_0) = J_{ij}(\mathbf{r}) B_j(\mathbf{y}, t_0). \quad (12)$$

Here $\mathbf{B}(\mathbf{y}, t_0)$ is the initial magnetic field; $\mathbf{r}(t)$ is the position of a fluid element at time t , which was originally at a ‘‘Lagrangian’’ position \mathbf{y} at time t_0 . Note that the fluid elements follow the integral curves of the velocity field, with

$$d\mathbf{x}/dt = 2\mathbf{u} = 2\mathbf{a} \sin(\Phi) + 2\mathbf{b}h \cos(\Phi), \quad (13)$$

where we have substituted from Eq. (2), assumed twice the velocity for step 1 and defined the phase $\Phi = \mathbf{q} \cdot \mathbf{x} + \psi$. From the incompressibility condition, we have $d\Phi/dt = 2\mathbf{q} \cdot \mathbf{u} = 0$ and thus Eq. (13) can be integrated to give at time $t = \tau/2$,

$$\mathbf{r} = \mathbf{y} + \tau\mathbf{u} = \mathbf{y} + \mathbf{a}\tau \sin(\mathbf{q} \cdot \mathbf{y} + \psi) + \mathbf{b}\tau h \cos(\mathbf{q} \cdot \mathbf{y} + \psi). \quad (14)$$

Here we have used the constancy of the phase Φ and set it equal to its initial value $\Phi = \mathbf{q} \cdot \mathbf{y} + \psi$. Thus the Jacobian is

$$J_{ij}(\mathbf{r}) \equiv \frac{\partial r_i}{\partial y_j} = \delta_{ij} + a_i q_j \tau \cos(\mathbf{q} \cdot \mathbf{y} + \psi) - b_i q_j h \tau \sin(\mathbf{q} \cdot \mathbf{y} + \psi). \quad (15)$$

Step 2. During the time interval $\tau/2$ to τ the turbulent velocity field is zero and there is only diffusion present, with a resistivity 2η . The induction equation then reduces to a diffusion equation for the magnetic field as

$$\frac{\partial \mathbf{B}}{\partial t} = 2\eta \nabla^2 \mathbf{B}. \quad (16)$$

The solution of this equation is given in terms of the resistive Green’s function,

$$G^\eta(\mathbf{x} - \mathbf{r}) = \frac{1}{(4\pi\eta\tau)^{3/2}} \exp\left[-\frac{(\mathbf{x} - \mathbf{r})^2}{4\eta\tau}\right]. \quad (17)$$

The total Green’s function defined in Eq. (4) is simply the product of the two Green’s functions in the above two steps:

$$G_{ij}(\mathbf{x}, \mathbf{y}) = \mathcal{G}^\eta(\mathbf{x} - \mathbf{r}(\mathbf{y})) J_{ij}(\mathbf{r}(\mathbf{y})), \quad (18)$$

where we have written explicitly \mathbf{r} as a function of \mathbf{y} . The response tensor defined in Eq. (8) then becomes

$$G_{ij}(\mathbf{k}) = \overline{\int \frac{1}{(4\pi\eta\tau)^{3/2}} e^{-\frac{(\mathbf{x}-\mathbf{r})^2}{4\eta\tau}} J_{ij}(\mathbf{r}) e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} d^3x} = \overline{J_{ij}(\mathbf{r}(\mathbf{y})) e^{-i\mathbf{k}\cdot(\mathbf{r}(\mathbf{y})-\mathbf{y})} e^{-\eta\tau k^2}}, \quad (19)$$

where in the second step we have done the integral over \mathbf{x} . The overhead bars in Eq. (19) denote ensemble averages, and we will see below that due to the statistical homogeneity of the renovating flow, this averaged quantity does not depend on \mathbf{y} , but only on \mathbf{k} . Also note that in typical astrophysical systems, the resistive time scale will be much larger than the renovation time and the value of $\eta k^2 \tau$ is typically much smaller than unity, and so can safely be set to zero. A nonzero but small η will decrease the growth rate by a negligible amount.

We now evaluate the ensemble average in Eq. (19). Note that GB state the final result, omitting all intermediate steps. We give the detailed steps in Appendix A since they are of use in the case when shear is present. Here we list the important intermediate steps and the final expression. Let the angle between \mathbf{k} and \mathbf{q} be θ ; we will treat this as a colatitude and denote the azimuthal angle of \mathbf{q} by $\tilde{\phi}$. Let the component of \mathbf{k} perpendicular to \mathbf{q} make an angle ϕ with \mathbf{a} (see Fig. 1).

On averaging over phase ψ we get

$$G_{ij}(\mathbf{k}) = \delta_{ij} J_0(\tau a k \chi \sin \theta) - \frac{i h a q \tau}{\chi k \sin \theta} [\epsilon_{imn} k_m \hat{q}_n \hat{q}_j] J_1(\tau a k \chi \sin \theta), \quad (20)$$

where $\chi = (\cos^2 \phi + h^2 \sin^2 \phi)^{1/2}$ and the overhead bars denote ensemble averages over the remaining random variables

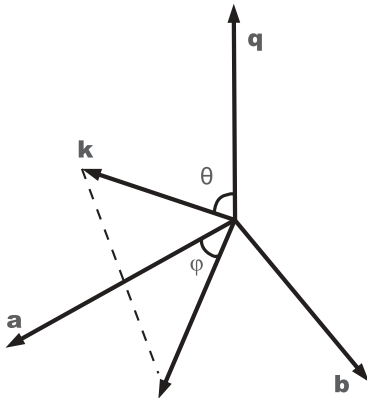


FIG. 1. The different angles defined in the text. The angle between \mathbf{k} and \mathbf{q} is θ . The vectors \mathbf{a} , \mathbf{b} , and \mathbf{q} are mutually perpendicular to each other. The component of \mathbf{k} perpendicular to \mathbf{q} makes an angle ϕ with \mathbf{a} .

θ , $\tilde{\phi}$, and ϕ . Averaging over direction of \mathbf{q} (i.e., averaging over the angles θ and $\tilde{\phi}$), we have

$$G_{ij}(\mathbf{k}) = \delta_{ij} g_0(\tau a k, h) + \frac{i h \tau a q \epsilon_{ijm} k_m}{2k} g_1(\tau a k, h), \quad (21)$$

where

$$g_0(s, h) = \frac{\overline{\sin(s\chi)}}{s\chi}; \quad g_1(s, h) = \frac{1}{\chi} \left(\frac{\overline{\sin(s\chi)}}{(s\chi)^2} - \frac{\overline{\cos(s\chi)}}{s\chi} \right), \quad (22)$$

and now the overhead bars denote ensemble averages over the random variable ϕ (for maximally helical flow with $h = \pm 1$, so $\chi = 1$, and the response tensor becomes independent of the random variable ϕ . In the rest of this paper h can take any value between -1 and $+1$). We recall that in step 1, we evolved the field for a time interval $\tau/2$, but with twice the velocity (i.e., $2a$ instead of a); nevertheless the combination $a\tau$ which appears in the above equation remains unaffected.

One of the eigenvectors of G_{ij} is \mathbf{k} with eigenvalue unity. But this eigenvector can be ignored since the magnetic field

mode must be orthogonal to \mathbf{k} (i.e., we must have $\nabla \cdot \mathbf{B} = 0$). The relevant eigenvectors of G_{ij} are found to be $(-i, 1, 0)^T$ and $(i, 1, 0)^T$, with the corresponding eigenvalues, σ_+ and σ_- , given by

$$\sigma_{\pm} = g_0(\tau ak, h) \mp \frac{\tau aqh}{2} g_1(\tau ak, h). \quad (23)$$

The growth rates λ_{\pm} are given by

$$\lambda_{\pm} = \frac{1}{\tau} \ln(\sigma_{\pm}) = \frac{1}{\tau} \ln \left[g_0(\tau ak, h) \mp \frac{\tau aqh}{2} g_1(\tau ak, h) \right]. \quad (24)$$

Here we have divided by the full time interval τ to get the growth rate. Following GB it is readily verified that both \pm mean-field modes can grow for sufficiently large renovation times. GB also demonstrated that the magnetic field becomes increasingly intermittent, in the sense that higher order single-point moments of the field grow faster. Therefore, as advertised, the mean-field dynamo operates efficiently in this case even in the presence of strongly growing magnetic noise. It is of interest to look at the growth rate in the limit of small renovation times such that $ak\tau \ll 1$ (this limit can also be looked upon as a long wavelength, or small k , limit at a fixed τ), when

$$\lambda_{\pm} \simeq \mp k \frac{ha^2\tau q}{6} - k^2 \frac{(1+h^2)a^2\tau}{12} \simeq \pm \alpha k - \eta_t k^2, \quad (25)$$

where $\alpha = -(1/3)\overline{\mathbf{u} \cdot (\nabla \times \mathbf{u})}(\tau/2)$ and $\eta_t = (1/3)\overline{\mathbf{u} \cdot \mathbf{u}}(\tau/2)$ are the turbulent transport coefficients.¹ Thus the growth rate for the case of small $ak\tau$ is identical to the growth rate of the standard α^2 mean-field dynamo usually obtained using the quasilinear approximation or FOSA. We now turn to consider the influence of shear.

III. RENOVATING SHEAR FLOWS

We now investigate the evolution of the mean magnetic field in scenarios when there is a mean shear flow over and above the background turbulence. Shear flows and turbulence are ubiquitous in astrophysical systems. Recent work suggests that the presence of shear may open new pathways to the operation of large-scale dynamos [4–6]. For simplicity we consider the background mean velocity to be a linear shear flow. Let

¹In these expressions for α and η_t , the factor $\tau/2$ appears instead of τ , because the transport coefficients are really time integrals of correlation functions; see Eqs. (6.19) and (6.20) in [1]. For renovating flows in which the velocity field \mathbf{u} acts over the full interval τ [i.e., if we were dealing with Eq. (1), without the two-step prescription of GB], this implies averaging t over the time interval $(0, \tau)$, which is equal to $\tau/2$. However, we get the same result even for the two-step prescription of GB, because two effects contribute in precisely opposite ways: When the velocity field is doubled in value, the transport coefficients quadruple because they are quadratic in the velocities; however, the doubled velocity field is ON for only the first half of any time interval τ . Hence, averaging “ t ” over the time interval $(0, \tau)$ now implies integrating t over the time interval $(0, \tau/2)$ and then dividing it by the full interval τ , which gives $\tau/8$. Thus we obtain $4 \times \tau/8 = \tau/2$.

$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the unit vectors of a Cartesian coordinate system in the laboratory frame, $\mathbf{x} = (x_1, x_2, x_3)$ the position vector. Without loss of generality, we choose this mean velocity to be in the \mathbf{e}_2 direction and varying linearly with x_1 . Thus the velocity field is given by

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_{\text{sh}} + \mathbf{u}_{\text{turb}} = Sx_1\mathbf{e}_2 + \mathbf{u}_{\text{turb}}, \quad (26)$$

where S is the constant rate of shear coefficient. The turbulent velocity field \mathbf{u}_{turb} is composed of renovating shearing waves with quite general amplitudes at this point. In particular, we take

$$\mathbf{u}_{\text{turb}}(\mathbf{x}, t) = \mathbf{A}(t, \mathbf{q}) \sin[\mathbf{Q}(t) \cdot \mathbf{x} + \psi] + \mathbf{C}(t, \mathbf{q}) \cos[\mathbf{Q}(t) \cdot \mathbf{x} + \psi], \quad (27)$$

where the wave vector is a shearing wave vector of the form $\mathbf{Q} = (Q_1, Q_2, Q_3) \equiv (q_1 - Sq_2(t - t_i), q_2, q_3)$, and $\mathbf{q} = (q_1, q_2, q_3)$ its initial value at the beginning of each renovation period, that is, at $t = t_i$. Note that \mathbf{q} is chosen randomly from a specified PDF (see below) for each renovating period. We will see that such a form of \mathbf{u}_{turb} naturally arises when we consider Fourier modes of the velocity which satisfy the momentum equation in a background linear shear flow. We will also later adopt explicit forms of $\mathbf{A}(t, \mathbf{q})$ and $\mathbf{C}(t, \mathbf{q})$; but several of the conclusions that we arrive at are quite generally insensitive to the functional forms of \mathbf{A} and \mathbf{C} . We also assume the turbulence to be incompressible with $\nabla \cdot \mathbf{u}_{\text{turb}} = 0$, which implies

$$\mathbf{Q}(t) \cdot \mathbf{A}(t) = 0; \quad \mathbf{Q}(t) \cdot \mathbf{C}(t) = 0. \quad (28)$$

Thus the amplitudes have to shear in an opposite sense to the wave vector so as to maintain incompressibility. The shearing wave vector can be written in a compact form as $Q_j = q_j \gamma_{ij}(-(t - t_i))$, where $\gamma_{ij}(t)$ is the shearing matrix defined by

$$\gamma_{ij}(t) = \delta_{ij} + \delta_{i2}\delta_{j1}St. \quad (29)$$

The helicity of the turbulent velocity field $\mathbf{u}_{\text{turb}}(\mathbf{x}, t)$ is

$$H = \mathbf{u}_{\text{turb}} \cdot (\nabla \times \mathbf{u}_{\text{turb}}) = \mathbf{C} \cdot (\mathbf{Q} \times \mathbf{A}), \quad (30)$$

and this vanishes unless both \mathbf{C} or \mathbf{A} are nonzero.

As in the previous section we consider the turbulent flow to be a pulsed renovating flow. The turbulent velocity field is assumed to be ON for a time interval $\tau/2$, with twice its amplitude and with diffusion absent. For the next $\tau/2$ interval, the turbulent velocity field is OFF and only the diffusion is present with resistivity 2η . On an average, the turbulent velocity field is then correlated only for a time interval τ . The mean shear flow on the other hand is always present for the full time interval τ . The turbulent flow is randomized by considering an ensemble similar to that assumed for the renovating flow without shear. In each time interval $[(n-1)\tau, n\tau]$ (i) ψ is chosen uniformly random between 0 and 2π ; (ii) the propagation vector \mathbf{q} is uniformly distributed on a sphere of radius q . The randomness of \mathbf{A} is decided by the explicit form of \mathbf{A} itself, which we fix later, when solving for the explicit form of the response tensor. For the general analysis we will not require it. The response tensor will be essential here, too, to determine the growth or decay of the mean magnetic field modes.

A. The response tensor

We compute the response tensor for the evolution of the mean magnetic field again in two steps.

Step 1. During the time interval $t = 0$ to $t = \tau/2$, $\eta = 0$ and Eq. (1) reduces again to the ideal induction equation, whose solution is as before the Cauchy solution:

$$B_i(\tilde{\mathbf{r}}, t) = \frac{\partial \tilde{r}_i}{\partial y_j} B_j(\mathbf{y}, 0) = J_{ij}(\tilde{\mathbf{r}}) B_j(\mathbf{y}, 0). \quad (31)$$

Here $\tilde{\mathbf{r}}$ gives the location of the fluid element at time t , which was at time $t = 0$ at the location \mathbf{y} . These positions are now on the integral curve of the sheared and turbulent velocity field, and so the trajectory now obeys

$$d\mathbf{x}/dt = \mathbf{u}_{\text{sh}} + 2\mathbf{u}_{\text{turb}} = Sx_1 \mathbf{e}_2 + 2\mathbf{A}(t, \mathbf{q}) \sin \tilde{\Phi} + 2\mathbf{C}(t, \mathbf{q}) \cos \tilde{\Phi}. \quad (32)$$

We have substituted here from Eq. (26), assumed twice the turbulent velocity for step 1 and defined the phase $\tilde{\Phi} = [\mathbf{Q}(t) \cdot \mathbf{x} + \psi]$. Note that we have not doubled the shear velocity, as we keep the shear flow throughout the full period $(0, \tau)$. From the incompressibility condition, we have $\mathbf{Q} \cdot \mathbf{u}_{\text{turb}} = 0$. Therefore,

$$d\tilde{\Phi}/dt = \dot{\mathbf{Q}} \cdot \mathbf{x} + \mathbf{Q}(t) \cdot \dot{\mathbf{x}} = -Sq_2 x_1 + Sq_2 x_1 + 2\mathbf{Q} \cdot \mathbf{u}_{\text{turb}} = 0. \quad (33)$$

The constancy of $\tilde{\Phi}$ can be used to express it in terms of the initial position of the fluid element \mathbf{y} , and the initial wave vector $\mathbf{Q}(t=0) = \mathbf{q}$, that is, we can write $\tilde{\Phi} = \mathbf{q} \cdot \mathbf{y} + \psi$. Then Eq. (32) can be integrated to give

$$\begin{aligned} \tilde{r}_i &= \gamma_{ij}(t) r_j, \\ r_j &= \tilde{a}_j(t, q_k) \sin(\mathbf{q} \cdot \mathbf{y} + \psi) + \tilde{c}_j(t, q_k) \\ &\quad \times \cos(\mathbf{q} \cdot \mathbf{y} + \psi) + y_j, \end{aligned} \quad (34)$$

where r_j is a sheared position vector that will be of use later, and the coefficients \tilde{a}_j and \tilde{c}_j are defined by

$$\begin{aligned} \tilde{a}_j(t, q_i) &= \gamma_{jp}(-t) \int_0^t 2A_p(t', q_i) dt' \\ &\quad + S\delta_{j2} \int_0^t \int_0^{t'} 2A_1(t'', q_i) dt'' dt', \\ \tilde{c}_j(t, q_i) &= \gamma_{jp}(-t) \int_0^t 2C_p(t', q_i) dt' \\ &\quad + S\delta_{j2} \int_0^t \int_0^{t'} 2C_1(t'', q_i) dt'' dt'. \end{aligned} \quad (35)$$

Therefore the Jacobian matrix is

$$J_{ij}(\tilde{\mathbf{r}}(\mathbf{y}), t) = \frac{\partial \tilde{r}_i}{\partial y_j} = \gamma_{ip}(t) [\delta_{pj} + \tilde{a}_p(t) q_j \cos(\mathbf{q} \cdot \mathbf{y} + \psi) - \tilde{c}_p(t) q_j \sin(\mathbf{q} \cdot \mathbf{y} + \psi)]. \quad (36)$$

We need this Jacobian to be evaluated at the time $t = \tau/2$.

Step 2. During $\tau/2$ to τ the turbulent velocity field is zero and there is diffusion present along with shear. The induction equation then reduces to the following form:

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_{\text{sh}} \cdot \nabla \right) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u}_{\text{sh}} + 2\eta \nabla^2 \mathbf{B}. \quad (37)$$

The sheared Green's function for this equation is [3]

$$\mathcal{G}_{ij}^\eta \left(\mathbf{x}, \tilde{\mathbf{r}}, \frac{\tau}{2} \right) = \frac{\gamma_{ij}(\tau/2)}{(4\pi\eta\tau)^{3/2}} (\det|\mathbf{D}|)^{1/2} \exp\left(\frac{-\tilde{x}_i D_{ij} \tilde{x}_j}{4\eta\tau} \right), \quad (38)$$

where $\tilde{x}_i = \gamma_{ij}(-\tau/2)x_j - \tilde{r}_i$, and D_{ij} is a symmetric matrix whose inverse is given by

$$\mathbf{D}^{-1} = \begin{bmatrix} 1 & -S\tau/4 & 0 \\ -S\tau/4 & 1 + (S\tau)^2/12 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (39)$$

Note that in the above computation we have taken the initial time to be $t = 0$ and the final time as $t = \tau$; but the same steps and calculations go through for any renovating time interval $[(n-1)\tau, n\tau]$, with the initial time being $t = (n-1)\tau$, the final time being $t = n\tau$, and the initial wave vector $\mathbf{q} = \mathbf{Q}((n-1)\tau)$. Hence during any such time interval τ , the magnetic field at time $t = n\tau$ is related to the magnetic field at time $t = (n-1)\tau$ by

$$\begin{aligned} B_i(\mathbf{x}, n\tau) &= \int \mathcal{G}_{ik}^\eta \left(\mathbf{x}, \tilde{\mathbf{r}}, \frac{\tau}{2} \right) J_{kj} \left(\tilde{\mathbf{r}}(\mathbf{y}), \frac{\tau}{2} \right) \\ &\quad \times B_j(\mathbf{y}, (n-1)\tau) d^3 \tilde{\mathbf{r}}. \end{aligned} \quad (40)$$

We would like to calculate the response tensor starting from the above evolution equation. For this we first define the Fourier transform of $\mathbf{B}_i(\mathbf{x}, t)$ by expressing it in terms of the shearing waves,

$$\hat{B}_i(\mathbf{k}, t) = \int B_i(\mathbf{x}, t) e^{-i\mathbf{K}(t) \cdot \mathbf{x}} d^3 x, \quad (41)$$

where we have defined the sheared wave vector $K_j(t) = k_i \gamma_{ij}(-(t-t_i)) \equiv (k_1 - Sk_2(t-t_i), k_2, k_3)$ and $\mathbf{k} = \mathbf{K}(t_i)$ is the initial wave vector at time t_i , which for each step we take to be the time $(n-1)\tau$. We will see that the evolution of the mean magnetic field is especially transparent when the field is expanded in terms of shearing waves in Fourier space.

We take the Fourier transform of Eq. (40), and change the integration variable from $\tilde{\mathbf{r}}$ to \mathbf{y} . The Jacobian for this transformation is unity as the flow velocity which maps \mathbf{y} to $\tilde{\mathbf{r}}$ is divergence free i.e. $d^3 \tilde{\mathbf{r}} = d^3 \mathbf{y}$. We get

$$\begin{aligned} \hat{B}_i(\mathbf{k}, n\tau) &= \int B_i(\mathbf{x}, n\tau) e^{-i\mathbf{K}(n\tau) \cdot \mathbf{x}} d^3 x = \int \int \mathcal{G}_{ik}^\eta \left(\mathbf{x}, \tilde{\mathbf{r}}, \frac{\tau}{2} \right) J_{kj} \left(\tilde{\mathbf{r}}(\mathbf{y}), \frac{\tau}{2} \right) B_j(\mathbf{y}, (n-1)\tau) e^{-i\mathbf{K}(n\tau) \cdot \mathbf{x}} d^3 x d^3 y \\ &= \int \int \int \mathcal{G}_{ik}^\eta \left(\mathbf{x}, \tilde{\mathbf{r}}, \frac{\tau}{2} \right) J_{kj} \left(\tilde{\mathbf{r}}(\mathbf{y}), \frac{\tau}{2} \right) \hat{B}_j(\mathbf{l}, (n-1)\tau) e^{i\mathbf{l} \cdot \mathbf{y}} e^{-i\mathbf{K}(n\tau) \cdot \mathbf{x}} \frac{d^3 l}{(2\pi)^3} d^3 x d^3 y, \end{aligned} \quad (42)$$

where we have also expressed the initial field at time $(n-1)\tau$ in terms of its Fourier transform. To do the integral over \mathbf{x} , we use the identity $\gamma_{jm}(t)\gamma_{mp}(-t) = \delta_{jp}$ to write

$$\exp(-i\mathbf{K} \cdot \mathbf{x}) = \exp(-iK_j\gamma_{jm}(\tau/2)[\gamma_{mp}(-\tau/2)x_p - \tilde{r}_m] - iK_j\gamma_{jm}(\tau/2)\tilde{r}_m) = \exp[-iK_j\gamma_{jm}(\tau/2)(\tilde{x}_m + \tilde{r}_m)], \quad (43)$$

and change the integration variable from x_m to $\tilde{x}_m = \gamma_{mp}(-\tau/2)x_p - \tilde{r}_m$. Since $\det|\boldsymbol{\gamma}| = 1$, we can write $d^3x = d^3\tilde{x}$. The integration over $\tilde{\mathbf{x}}$ now becomes a Fourier transform of the sheared resistive Greens function [3] and we get

$$\begin{aligned} \int \mathcal{G}_{ik}^\eta\left(\mathbf{x}, \tilde{\mathbf{r}}, \frac{\tau}{2}\right) e^{-i\mathbf{K} \cdot \mathbf{x}} d^3x &= \int \frac{\gamma_{ik}(\tau/2)}{(4\pi\eta\tau)^{3/2}} (\det|\mathbf{D}|)^{1/2} \exp\left(\frac{-\tilde{x}_n D_{nm} \tilde{x}_m}{4\eta\tau}\right) e^{-i\mathbf{K}\boldsymbol{\gamma}_{\tau/2}\tilde{\mathbf{x}}} e^{-i\mathbf{K}\cdot[\boldsymbol{\gamma}_{\tau/2}\tilde{\mathbf{r}}]} d^3\tilde{x} \\ &= \gamma_{ik}(\tau/2) \exp\{-\eta\tau K_n \gamma_{nm}(\tau/2) D_{mp}^{-1} \gamma_{jp}(\tau/2) K_j\} e^{-i\mathbf{K}\cdot[\boldsymbol{\gamma}_{\tau/2}\tilde{\mathbf{r}}]} = \mathcal{G}_{ik}(\tilde{\mathbf{k}}) e^{-i\tilde{\mathbf{k}} \cdot \tilde{\mathbf{r}}}, \end{aligned} \quad (44)$$

where $\tilde{k}_m = K_j \gamma_{jm}(\tau/2) = k_j \gamma_{jm}(-\tau/2)$. Then we have

$$\hat{B}_i(\mathbf{k}, n\tau) = \gamma_{ik}(\tau/2) \exp(-\eta\tau \tilde{k}_p D_{pr}^{-1} \tilde{k}_r) \int \int J_{kj}\left(\tilde{\mathbf{r}}(\mathbf{y}), \frac{\tau}{2}\right) \hat{B}_j(\mathbf{l}, (n-1)\tau) e^{i\mathbf{l} \cdot \mathbf{y}} e^{-i\tilde{\mathbf{k}} \cdot \tilde{\mathbf{r}}} \frac{d^3l}{(2\pi)^3} d^3y. \quad (45)$$

Taking the ensemble average of Eq. (45), we get for the mean-field evolution,

$$\overline{\hat{B}_i(\mathbf{k}, n\tau)} = \gamma_{ik}(\tau/2) \exp(-\eta\tau \tilde{k}_p D_{pr}^{-1} \tilde{k}_r) \int \left[\int e^{-i\tilde{\mathbf{k}} \cdot \tilde{\mathbf{r}}} J_{kj}\left(\tilde{\mathbf{r}}(\mathbf{y}), \frac{\tau}{2}\right) e^{i\mathbf{l} \cdot \mathbf{y}} d^3y \right] \overline{\hat{B}_j(\mathbf{l}, (n-1)\tau)} \frac{d^3l}{(2\pi)^3}. \quad (46)$$

Here we have assumed as before that the velocity field during the time interval $(n\tau, (n-1)\tau)$ and the initial magnetic field at time $(n-1)\tau$ are statistically independent. Note the dependence on the stochastic parameters comes only through $\tilde{\mathbf{r}}(\mathbf{y})$. As $\tilde{k}_m = k_j \gamma_{jm}(-\tau/2)$ and $\tilde{r}_m = \gamma_{mj}(\tau/2)r_j$, we have $\tilde{\mathbf{k}} \cdot \tilde{\mathbf{r}} = \mathbf{k} \cdot \mathbf{r}$ and the quantity in the square bracket can be written as

$$\int e^{-i\tilde{\mathbf{k}} \cdot \tilde{\mathbf{r}}} J_{kj}\left(\tilde{\mathbf{r}}(\mathbf{y}), \frac{\tau}{2}\right) e^{i\mathbf{l} \cdot \mathbf{y}} d^3y = \int e^{-i\mathbf{k} \cdot (\mathbf{r}-\mathbf{y})} J_{kj}\left(\tilde{\mathbf{r}}(\mathbf{y}), \frac{\tau}{2}\right) e^{-i\mathbf{k} \cdot \mathbf{y}} e^{i\mathbf{l} \cdot \mathbf{y}} d^3y = \overline{e^{-i\mathbf{k} \cdot (\mathbf{r}-\mathbf{y})} J_{kj}\left(\tilde{\mathbf{r}}(\mathbf{y}), \frac{\tau}{2}\right)} (2\pi)^3 \delta(\mathbf{l} - \mathbf{k}). \quad (47)$$

In the last step we have used the fact that the averaged quantity is independent of y_i , as can be easily seen by doing the averaging first over the random phase ψ of the turbulent field; this also follows from the statistical homogeneity of the turbulence. The y_i independence is explicitly shown by calculation below. We then have for the evolution of the mean field,

$$\overline{\hat{B}_i(\mathbf{k}, n\tau)} = G_{ij}(\mathbf{k}) \overline{\hat{B}_j(\mathbf{k}, (n-1)\tau)}, \quad (48)$$

where the response tensor $G_{ij}(\mathbf{k})$ is now

$$G_{ij}(\mathbf{k}) = \overline{\gamma_{il}(\tau) [\delta_{lj} + \tilde{a}_l q_j \cos(\mathbf{q} \cdot \mathbf{y} + \psi) - \tilde{c}_l q_j \sin(\mathbf{q} \cdot \mathbf{y} + \psi)] e^{-i\mathbf{k} \cdot (\mathbf{r}-\mathbf{y})} \exp(-\eta\tau k_p M_{pr} k_r)}, \quad (49)$$

where $M_{ij} = \gamma_{im}(-\tau/2) D_{mp}^{-1} \gamma_{jp}(-\tau/2)$ and the coefficients \tilde{a}_l and \tilde{c}_l are shorthand for $\tilde{a}_l(\tau/2, \mathbf{q})$ and $\tilde{c}_l(\tau/2, \mathbf{q})$, given in Eq. (35). One can see that the form of the response tensor in Eq. (49) with shear is similar to the form of the response tensor in Eq. (19) without shear and reduces to the latter when $S = 0$. Similar to the case without shear, we see that the effects of resistive dissipation appear only as a separate exponential term. Since it is small in astrophysical systems of interest, henceforth we set $\eta = 0$.

We now average over the phase ψ of the turbulent velocity field, which leads us to the following expression:

$$G_{ij}(\mathbf{k}) = \gamma_{il}(\tau) \sigma_{lj}(\mathbf{k}), \quad \sigma_{lj}(\mathbf{k}) = \left[\overline{\delta_{lj} J_0(\sqrt{(\mathbf{k} \cdot \tilde{\mathbf{a}})^2 + (\mathbf{k} \cdot \tilde{\mathbf{c}})^2}) - i \frac{(\tilde{a}_l q_j \mathbf{k} \cdot \tilde{\mathbf{c}} - \tilde{c}_l q_j \mathbf{k} \cdot \tilde{\mathbf{a}})}{\sqrt{(\mathbf{k} \cdot \tilde{\mathbf{a}})^2 + (\mathbf{k} \cdot \tilde{\mathbf{c}})^2}} J_1(\sqrt{(\mathbf{k} \cdot \tilde{\mathbf{a}})^2 + (\mathbf{k} \cdot \tilde{\mathbf{c}})^2})} \right], \quad (50)$$

where the overhead bar now refers to the averaging over the directions of \mathbf{q} and over the randomness of \mathbf{A} . One can reach some conclusions about the decay of the mean magnetic field at this stage of the averaging itself. The mean magnetic field evolves as

$$\overline{B_i(\mathbf{k}, n\tau)} = \gamma_{il}(\tau) \sigma_{lj}(\mathbf{k}) \overline{B_j(\mathbf{k}, 0)}. \quad (51)$$

The growth or decay of the mean-field mode is governed by the product of the two matrices, the shearing matrix $\boldsymbol{\gamma}(\tau)$ and the shear-turbulence tensor $\boldsymbol{\sigma}(\mathbf{k})$. It is known that the field grows linearly due to the continuous shearing of the background fluid which causes the $B_1(\mathbf{k})$ component of the field to be continuously advected along the \mathbf{e}_2 direction. This is reflected by the presence of the shearing matrix $\boldsymbol{\gamma}(\tau)$ in the expression of the response tensor $G(\mathbf{k})$ and is completely

natural as well as expected. The turbulent stretching of the field lines due to the transfer of energy from the turbulent pulses of the fluid can, too, lead to the growth of the mean field and the shear-turbulence tensor $\boldsymbol{\sigma}(\mathbf{k})$ contains precisely this information through its dependence on the random parameters. Hence, it is of interest to look at the structure of the $\boldsymbol{\sigma}(\mathbf{k})$ tensor and calculate its eigenvalues depending on which the mean field grows or decays exponentially.

When helicity is zero, then either $\tilde{\mathbf{a}}$ or $\tilde{\mathbf{c}}$ is zero from Eqs. (30) and (35). Then the second term in $\sigma_{lj}(\mathbf{k})$ vanishes and we get

$$\sigma_{lj}(\mathbf{k}) = \delta_{lj} \overline{J_0(\mathbf{k} \cdot \tilde{\mathbf{a}})}, \quad (52)$$

where we have chosen $\tilde{\mathbf{c}} = 0$ without any loss of generality. In this case, the eigenvalue of $\boldsymbol{\sigma}(\mathbf{k})$ is just $\sigma = \overline{J_0(\mathbf{k} \cdot \tilde{\mathbf{a}})}$. Note

that the maximum value of the Bessel function $J_0(x)$ is unity. Hence after averaging over all the possible values of $\mathbf{k} \cdot \tilde{\mathbf{a}}$ as per the ensemble chosen, we must necessarily obtain $\overline{J_0(\mathbf{k} \cdot \tilde{\mathbf{a}})} < 1$. This shows that in the absence of helicity the mean-field modes eventually decay with a decay rate $\lambda = (\tau)^{-1} \log \sigma$ [see Eq. (10)]. Therefore, quite generally, there is no mean-field dynamo if the turbulent velocity is strictly nonhelical, even in the presence of shear.

B. Forced overdamped shearing wave

We now solve for the form of the $\sigma_{ij}(\mathbf{k})$ by taking a particular form of the \mathbf{u}_{turb} in Eq. (27) obeying the following

forced, damped Euler equation:

$$\left(\frac{\partial}{\partial t} + Sx_1 \frac{\partial}{\partial x_2} \right) \mathbf{u}_{\text{turb}} + Su_{\text{turb}}^1 \mathbf{e}_2 + (\mathbf{u}_{\text{turb}} \cdot \nabla) \mathbf{u}_{\text{turb}} = -\nabla p - \frac{\mathbf{u}_{\text{turb}}}{\tau_d} + \mathbf{f}, \quad (53)$$

where τ_d is a given damping time and $\mathbf{f}(\mathbf{x}, t)$ is the external forcing which is assumed to satisfy $\nabla \cdot \mathbf{f} = 0$. In the approximation $\partial \mathbf{u}_{\text{turb}} / \partial t \ll \mathbf{u}_{\text{turb}} / \tau_d$, the wave is assumed to be overdamped, saturating quickly in time τ_d to its terminal velocity. In Eq. (B13) of Appendix B the following solution is derived:

$$\mathbf{u}_{\text{turb}}(\mathbf{x}, t) = \mathbf{A}(t, \mathbf{q}) \sin[\mathbf{Q}(t) \cdot \mathbf{x} + \psi] + \mathbf{C}(t, \mathbf{q}) \cos[\mathbf{Q}(t) \cdot \mathbf{x} + \psi], \quad \text{where}$$

$$A_{1,3} = a_{1,3} + S\tau_d a_1 \left[\frac{Q_{1,3} Q_2}{Q^2 - S\tau_d Q_1 Q_2} \right], \quad A_2 = a_2 + S t a_1 - S\tau_d a_1 \left[\frac{Q_1^2 + Q_3^2}{Q^2 - S\tau_d Q_1 Q_2} \right];$$

$$C_{1,3} = h \left\{ c_{1,3} + S\tau_d c_1 \left[\frac{Q_{1,3} Q_2}{Q^2 - S\tau_d Q_1 Q_2} \right] \right\}, \quad C_2 = h \left\{ c_2 + S t c_1 - S\tau_d c_1 \left[\frac{Q_1^2 + Q_3^2}{Q^2 - S\tau_d Q_1 Q_2} \right] \right\}, \quad (54)$$

where $Q_j = q_i \gamma_{ij}(-t)$, $\mathbf{q} \cdot \mathbf{a} = 0$, $\mathbf{c} = \hat{\mathbf{q}} \times \mathbf{a}$, and h such that $-1 \leq h \leq 1$ determines the helicity of the flow. Here the forcing function is related to the constants $\{a_1, a_2, a_3\}$ and $\{c_1, c_2, c_3\}$. In addition, note that the vectors $\mathbf{A}(t, q)$, $\mathbf{C}(t, q)$, and $\mathbf{Q}(q, t)$ contain the effects of shear, but have been expressed in terms of a , c , and q which are statistically isotropic. In the limit of very strong damping, that is, $\tau_d \rightarrow 0$, the above expressions simplify and can be written compactly as

$$A_i = \gamma_{ij}(t) a_j, \quad C_i = h \gamma_{ij}(t) c_j. \quad (55)$$

The turbulent velocity field now becomes

$$[\mathbf{u}_{\text{turb}}(\mathbf{x}, t)]_i = \gamma_{ij}(t) a_j \sin[\mathbf{Q}(t) \cdot \mathbf{x} + \psi] + h \gamma_{ij}(t) c_j \cos[\mathbf{Q}(t) \cdot \mathbf{x} + \psi]. \quad (56)$$

Substituting Eq. (55) in Eq. (35), we find that $\tilde{\mathbf{a}} = \mathbf{a}\tau$ and $\tilde{\mathbf{c}} = h\mathbf{c}\tau$; hence the response tensor in Eq. (50) becomes

$$G_{ij}(\mathbf{k}) = \gamma_{il}(\tau) \left[\overline{\delta_{lj} J_0(\sqrt{(\mathbf{k} \cdot \mathbf{a}\tau)^2 + (\mathbf{k} \cdot h\mathbf{c}\tau)^2}) - i \frac{(a_l q_j h \tau \mathbf{k} \cdot \mathbf{c} - c_l q_j h \tau \mathbf{k} \cdot \mathbf{a})}{\sqrt{(\mathbf{k} \cdot \mathbf{a})^2 + (\mathbf{k} \cdot h\mathbf{c})^2}} J_1(\sqrt{(\mathbf{k} \cdot \mathbf{a}\tau)^2 + (\mathbf{k} \cdot h\mathbf{c}\tau)^2})} \right]. \quad (57)$$

As before, let the angle between \mathbf{k} and \mathbf{q} be θ ; we will treat this as a colatitude and denote the azimuthal angle of \mathbf{q} by $\tilde{\phi}$. Let the component of \mathbf{k} perpendicular to \mathbf{q} make an angle ϕ with \mathbf{a} . Then, on averaging over the phase ψ we can write

$$G_{ij}(\mathbf{k}) = \gamma_{il}(\tau) \left\{ \overline{\delta_{lj} J_0(\tau a k \chi \sin \theta) - \frac{i h a q \tau}{\chi k \sin \theta} [\epsilon_{lmn} k_m \hat{q}_n \hat{q}_j] J_1(\tau a k \chi \sin \theta)} \right\}, \quad (58)$$

where $\chi = (\cos^2 \phi + h^2 \sin^2 \phi)^{1/2}$ and the overhead bar denotes ensemble averages over the remaining random variables θ , $\tilde{\phi}$, and ϕ . Comparing with Eq. (20), we see that the form of the response tensor is identical to its form when shear is absent with the only difference being an overall $\gamma_{il}(\tau)$ factor. Since the ensemble average is done at one arbitrary instant, the final form of $G_{ij}(\mathbf{k})$ is identical to Eq. (21) with the extra γ_{ij} factor. Hence, we have

$$G_{ij}(\mathbf{k}) = \gamma_{il}(\tau) \sigma_{ij}(k), \quad \sigma_{ij}(k) = \delta_{ij} g_0(\tau a k, h) + \frac{i h \tau a q \epsilon_{l j m} k_m}{2k} g_1(\tau a k, h), \quad g_0(s, h) = \frac{\overline{\sin(s\chi)}}{s\chi};$$

$$g_1(s, h) = \frac{1}{\chi} \left(\frac{\overline{\sin(s\chi)}}{(s\chi)^2} - \frac{\overline{\cos(s\chi)}}{s\chi} \right), \quad (59)$$

and now the overhead bars denote ensembles averaging over the random variable ϕ (for maximally helical flow with $h = \pm 1$, so $\chi = 1$, and the response tensor becomes independent of the random variable ϕ). The shear turbulence tensor σ has the two nontrivial eigenvectors $(-i, 1, 0)^T$ and $(i, 1, 0)^T$, with the corresponding eigenvalues σ_+ and σ_- given by

$$\sigma_{\pm} = g_0(\tau a k, h) \mp \frac{\tau a q h}{2} g_1(\tau a k, h). \quad (60)$$

For zero helicity, the second term vanishes and dynamo action is absent, as was shown in the general case in the previous section. Moreover, even if there were mirror-symmetric fluctuations in h , this would not lead to a dynamo. This is because $g_0(s, h)$ and $g_1(s, h)$ are even in h , while the coefficient of the second term of the response tensor in Eq. (59) is linear (and hence odd) in h . Thus on averaging the response tensor over any symmetric PDF of h with zero mean only the first term of G_{ij} survives and there is no dynamo. This conclusion is similar to that obtained by GB for the case without shear.

IV. $\alpha^2\Omega$ DYNAMO

Let us look at the expression in Eq. (59) of the response tensor in the case of small correlation times, when $ak\tau \ll 1$. Then to quadratic order in τ , we get

$$\begin{aligned} G_{ij}(\mathbf{k}) &= \delta_{ij} \left[1 - \frac{(1+h^2)}{12} (\tau ak)^2 \right] \\ &\quad + \delta_{i2}\delta_{j1} S\tau + i \frac{ha^2\tau^2 q}{6} \epsilon_{ijm} k_m \\ &= \delta_{ij} (1 - \eta_t \tau k^2) + \delta_{i2}\delta_{j1} S\tau - i\tau\alpha\epsilon_{ijm} k_m, \end{aligned} \quad (61)$$

where $\alpha = -(1/3) [\mathbf{u}_{\text{turb}} \cdot (\nabla \times \mathbf{u}_{\text{turb}})] (\tau/2)$ and $\eta_t = (1/3) [\mathbf{u} \cdot \mathbf{u}] (\tau/2)$ are the turbulent transport coefficients. The mean-field evolution equation [Eq. (48)] then becomes

$$\begin{bmatrix} B_{\tau 1} \\ B_{\tau 2} \\ B_{\tau 3} \end{bmatrix} = \begin{bmatrix} 1 - \eta_t \tau k^2 & -i\alpha\tau k_3 & i\alpha\tau k_2 \\ i\alpha\tau k_3 + S\tau & 1 - \eta_t \tau k^2 & -i\alpha\tau k_1 \\ -i\alpha\tau k_2 & i\alpha\tau k_1 & 1 - \eta_t \tau k^2 \end{bmatrix} \begin{bmatrix} B_{01} \\ B_{02} \\ B_{03} \end{bmatrix}, \quad (62)$$

which is the evolution equation for the $\alpha^2\Omega$ dynamo [1]. We seek solutions of the eigenvalue problem when $k_2 = 0$. Of the three eigenvalues, $\sigma_1 = (1 - \eta_t \tau k^2)$ is irrelevant, because the corresponding eigenvector does not satisfy the solenoidality condition $(\nabla \cdot \mathbf{B}) = 0$. The remaining two eigenvalues are

$$\sigma_{\pm} = 1 - \tau\eta_t k^2 \pm \tau(\alpha^2 k^2 - i\alpha k_3 S)^{1/2}, \quad (63)$$

corresponding to the eigenvectors,

$$(\alpha k_3, \pm i\sqrt{\alpha^2 k^2 - i\alpha k_3 S}, -\alpha k_1)^T. \quad (64)$$

The growth rates λ_{\pm} of these eigenmodes are given by

$$\lambda_{\pm} = \frac{1}{\tau} \ln(\sigma_{\pm}) = -\eta_t k^2 \pm (\alpha^2 k^2 - i\alpha k_3 S)^{1/2}, \quad (65)$$

which are the same as one would get in the case of the $\alpha^2\Omega$ dynamo [1].

V. DISCUSSION AND CONCLUSIONS

This paper presents studies of dynamo action in turbulent shear flows when the turbulence has a nonzero correlation time. Our goal is to study the dynamics of a system which is complex enough to be a useful model, yet tractable analytically; the renovating flows discussed earlier by several authors [9,10] (and references therein) provide just such a platform. Our contribution is to consider random, helical renovating flows in the context of a background linear shear flow.

We began with a review of the work of Gilbert and Bayly (GB) [10] on random helical renovating flows in the absence of a background shear flow. GB considered random flows, each of whose realizations was a plane, sinusoidal helical wave. The merit of choosing such simple random ensembles is that the trajectories of fluid elements, in the flow caused by each wave, can be integrated analytically. Thus the Green's function mapping the magnetic field from one time step to another can be obtained, and averaged over the underlying PDF of the random ensemble of flows. GB give the final result, while skipping almost all the intermediate steps. We found it useful not only to record these missing steps, but lay them out for the reader so that it becomes easier for us to present our analysis of the more complicated problem of renovating flows with shear.

We then formulated the problem of renovating flows in the presence of a background linear shear flow. Following GB, we considered an ensemble of random flows, each of whose members is a plane, sinusoidal helical wave. However, unlike in the case considered by GB, the wave cannot be time independent. In fact, each of these members must be a *shearing wave*, one whose amplitude and wave vector are both time dependent. Then the trajectories of fluid elements (in the flow caused by each shearing wave) were determined analytically, the Green's function (mapping the magnetic field from one time step to another) derived, and averaged over the underlying PDF of the random ensemble of flows, to obtain a general expression for the (averaged) response tensor. We showed that even without fully averaging the response tensor, for which one requires the explicit form of the time-dependent, shearing wave amplitude, it is still possible to prove a general result: that there is no dynamo action when the shearing waves are *strictly* nonhelical. This is an important and nontrivial result in view of the considerable current interest in the existence or otherwise of such nonhelical shear dynamos (see [4–6]).

We then considered a particular model in which the shearing waves were generated through external forcing of the linear shear flow. Working in the overdamped limit, we derived an explicit form for the response tensor. It is interesting to note that this form is closely related to the response tensor of GB; specifically, our response tensor is the product of the response tensor of GB with the shearing matrix. This was then applied to the case of $\alpha^2\Omega$ dynamos in the limit of small correlation times, and we recovered the standard dispersion relation for the $\alpha^2\Omega$ dynamo. Thus the growth of the mean field in sheared helical turbulence is as expected from quasilinear closures usually employed to derive the mean-field equations. This is obtained in spite of the fact that magnetic fluctuations in such renovating flows are expected to grow more rapidly (as shown explicitly by GB) than the mean field, wherein, one may imagine, that quasilinear closures break down. Our work therefore provides another illustration that rapidly growing fluctuations need not destroy the growth of the mean field.

Our result that there is no mean-field dynamo in strictly nonhelical turbulence, even in the presence of shear, raises the question as to what causes such growth in numerical simulations [4]. One possibility is the incoherent α -shear dynamo [11]. There seems to be evidence from such simulations for fluctuations in α (the first reference in [4]). Here the mean fields are defined as averages over two spatial directions, and

fluctuations in α over time are considered as meaningful. In the present context, we can define the required fluctuations as due to fluctuations of the parameter h from one renovation time to another, which is easier to justify as being physically meaningful. GB themselves considered fluctuations in h , but argued that the PDF of h needs to be skewed for there to be net growth.

Even in the presence of shear, if we averaged the response tensor say in Eq. (59) over a mirror-symmetric PDF of h with zero mean, the ‘‘helical’’ term would vanish and one would not have a dynamo. However if we think of the mean field as being defined before averaging over h , then one could study its dynamics under such fluctuations. In the presence of shear, we can see from Eq. (65), that both signs of h would cause

growth, but the eigenvector in Eq. (64) would get an extra phase shift. It would be interesting to work out exactly how such random changes to the eigenvector alters the efficiency of the dynamo. One would expect that a coherent h would lead to a more efficient dynamo rather than a fluctuating h .

The work here has focused on the mean-field evolution. The same model can also be used to generalize the Kazantsev model [8] for the fluctuation dynamo, to the case with finite correlation time and including shear.

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APPENDIX A: CALCULATING THE AVERAGED RESPONSE TENSOR IN THE ABSENCE OF SHEAR

Below we give the details involved in going from Eq. (19) to Eq. (21).

$$G_{ij}(\mathbf{k}) = \overline{[\delta_{ij} + a_i q_j \tau \cos(\mathbf{q} \cdot \mathbf{y} + \psi) - b_i q_j h \tau \sin(\mathbf{q} \cdot \mathbf{y} + \psi)] e^{-i\mathbf{k} \cdot (\mathbf{a}\tau \sin(\mathbf{q} \cdot \mathbf{y} + \psi) + \mathbf{b}\tau h \cos(\mathbf{q} \cdot \mathbf{y} + \psi))}}. \quad (\text{A1})$$

Let the angle between \mathbf{k} and \mathbf{q} be θ . Let the component of \mathbf{k} perpendicular to \mathbf{q} make an angle ϕ with \mathbf{a} . Now \mathbf{k} can be written as

$$\mathbf{k} = \underbrace{\left[\mathbf{k} - \frac{\mathbf{q}(\mathbf{k} \cdot \mathbf{q})}{q^2} \right]}_{\text{perpendicular to } \mathbf{q}} + \underbrace{\frac{\mathbf{q}(\mathbf{k} \cdot \mathbf{q})}{q^2}}_{\text{along } \mathbf{q}}. \quad (\text{A2})$$

Since \mathbf{a} and \mathbf{b} both lie in the plane perpendicular to \mathbf{q} , and also since \mathbf{a} and \mathbf{b} are perpendicular to each other,

$$\mathbf{k} \cdot \mathbf{a} = \mathbf{k}_\perp \cdot \mathbf{a} = |\mathbf{k}_\perp| |\mathbf{a}| \cos \phi = (k \sin \theta) a \cos \phi, \quad \mathbf{k} \cdot \mathbf{b} = k a \sin \theta \sin \phi. \quad (\text{A3})$$

Now we average $G_{ij}(\mathbf{k})$ term by term. For the first term, we have

$$I_1 = \overline{\delta_{ij} e^{-i\mathbf{k} \cdot (\mathbf{a}\tau \sin(\mathbf{q} \cdot \mathbf{y} + \psi) + \mathbf{b}\tau h \cos(\mathbf{q} \cdot \mathbf{y} + \psi))}}. \quad (\text{A4})$$

The argument of the exponential is written as

$$-i\tau k a \sin \theta [\cos \phi \sin(\mathbf{q} \cdot \mathbf{y} + \psi) + h \sin \phi \cos(\mathbf{q} \cdot \mathbf{y} + \psi)] = -i\tau k a \chi \sin \theta \sin(\mathbf{q} \cdot \mathbf{y} + \psi + \alpha), \quad (\text{A5})$$

where $\chi = (\cos^2 \phi + h^2 \sin^2 \phi)^{1/2}$, $\chi \cos \alpha = \cos \phi$, and $\chi \sin \alpha = h \sin \phi$. Hence, we have

$$I_1 = \overline{\delta_{ij} e^{-i\tau k a \chi \sin \theta \sin(\mathbf{q} \cdot \mathbf{y} + \psi + \alpha)}}. \quad (\text{A6})$$

First we average over ψ . Since ψ goes over all the possible phases, on averaging we get

$$I_1 = \delta_{ij} \int_0^{2\pi} e^{-i(\tau k a \chi \sin \theta) \sin \zeta} \frac{d\zeta}{2\pi} = \delta_{ij} J_0(\tau a k \chi \sin \theta), \quad (\text{A7})$$

where we have used the following integral representation of the Bessel function of the first kind.

$$\int_0^\pi e^{i\beta \cos x} \cos(nx) dx = i^n \pi J_n(\beta). \quad (\text{A8})$$

We next average over the direction of \mathbf{q} . We keep \mathbf{k} fixed (say, along the z direction) and vary \mathbf{q} about \mathbf{k} over all the solid angles.

$$\begin{aligned} I_1 &= \delta_{ij} \int_0^\pi \int_0^{2\pi} J_0(\tau a k \chi \sin \theta) \frac{\sin \theta d\theta d\phi}{4\pi} = \frac{\delta_{ij}}{2} \int_0^\pi J_0(\tau a k \chi \sin \theta) \sin \theta d\theta \\ &= \delta_{ij} \int_0^{\frac{\pi}{2}} J_0(\tau a k \chi \cos \theta) \cos \theta d\theta = \delta_{ij} \frac{\sin(\tau a k \chi)}{\tau a k \chi}. \end{aligned} \quad (\text{A9})$$

Lastly we average over the direction of \mathbf{a} . Since \mathbf{a} can point in any direction in the plane perpendicular to \mathbf{q} , we average over the angle ϕ . We get

$$I_1 = \delta_{ij} g_0(\tau a k, h), \quad \text{where} \quad g_0(s, h) = \frac{\overline{\sin(s\chi)}}{s\chi}, \quad (\text{A10})$$

where the overhead bar now denotes the ensemble average over the random variable ϕ . We proceed to the remaining term of $G_{ij}(\mathbf{k})$ in Eq. (A1):

$$I_2 = \overline{[a_i q_j \tau \cos(\mathbf{q} \cdot \mathbf{y} + \psi) - b_i q_j h \tau \sin(\mathbf{q} \cdot \mathbf{y} + \psi)] e^{-i\mathbf{k} \cdot (\mathbf{a} \tau \sin(\mathbf{q} \cdot \mathbf{y} + \psi) + \mathbf{b} \tau h \cos(\mathbf{q} \cdot \mathbf{y} + \psi))}}. \quad (\text{A11})$$

We proceed in a similar way as above. We define slightly different parameters. The argument of the exponential is now written as

$$-i\tau k a \sin \theta [\cos \phi \sin(\mathbf{q} \cdot \mathbf{y} + \psi) + h \sin \phi \cos(\mathbf{q} \cdot \mathbf{y} + \psi)] = i\tau k a \chi \sin \theta \cos(\mathbf{q} \cdot \mathbf{y} + \psi - \alpha), \quad (\text{A12})$$

where $\chi = (\cos^2 \phi + h^2 \sin^2 \phi)^{1/2}$, $\chi \sin \alpha = -\cos \phi$, and $\chi \cos \alpha = -h \sin \phi$. Therefore we have

$$I_2 = \overline{[a_i q_j \tau \cos(\mathbf{q} \cdot \mathbf{y} + \psi) - b_i q_j h \tau \sin(\mathbf{q} \cdot \mathbf{y} + \psi)] e^{i\tau k a \chi \sin \theta \cos(\mathbf{q} \cdot \mathbf{y} + \psi - \alpha)}}. \quad (\text{A13})$$

First we average over ψ . Since ψ goes over all the possible phases, we write

$$I_2 = \int_0^{2\pi} e^{i(\tau k a \chi \sin \theta) \cos \zeta} [a_i q_j \tau \cos(\zeta + \alpha) - b_i q_j h \tau \sin(\zeta + \alpha)] \frac{d\zeta}{2\pi}. \quad (\text{A14})$$

Expanding $\cos(\zeta + \alpha)$ and $\sin(\zeta + \alpha)$, and keeping only the even terms under integration, we get

$$\begin{aligned} I_2 &= \int_0^{2\pi} e^{i(\tau k a \chi \sin \theta) \cos \zeta} [a_i q_j \tau \cos \alpha \cos \zeta - b_i q_j h \tau \sin \alpha \cos \zeta] \frac{d\zeta}{2\pi} = i\tau [a_i q_j \cos \alpha - b_i q_j h \sin \alpha] J_1(\tau a k \chi \sin \theta) \\ &= -\frac{i h \tau q_j}{\chi a k \sin \theta} [a_i (\mathbf{k} \cdot \mathbf{b}) - b_i (\mathbf{k} \cdot \mathbf{a})] J_1(\tau a k \chi \sin \theta), \end{aligned} \quad (\text{A15})$$

where we have used Eq. (A8) to arrive at the second expression and Eq. (A3) to obtain the last expression. This can be further simplified as

$$q_j [a_i (\mathbf{k} \cdot \mathbf{b}) - b_i (\mathbf{k} \cdot \mathbf{a})] = q_j [\mathbf{k} \times (\mathbf{a} \times \mathbf{b})]_i = q_j [\mathbf{k} \times a^2 \hat{\mathbf{q}}]_i = a^2 q \epsilon_{imn} k_m \hat{q}_n \hat{q}_j. \quad (\text{A16})$$

We next average over the direction of \mathbf{q} . We keep \mathbf{k} fixed along the z direction and vary \mathbf{q} about \mathbf{k} over all the solid angle, then we can write $\hat{\mathbf{q}} = \sin \theta \cos \tilde{\phi} \hat{\mathbf{i}} + \sin \theta \sin \tilde{\phi} \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}}$,

$$I_2 = -\int_0^\pi \int_0^{2\pi} \frac{i h \tau}{\chi a k \sin \theta} [a^2 q \epsilon_{imn} k_m \hat{q}_n \hat{q}_j] J_1(\tau a k \chi \sin \theta) \frac{\sin \theta d\theta d\tilde{\phi}}{4\pi}. \quad (\text{A17})$$

The $\tilde{\phi}$ dependence comes only through $\hat{q}_n \hat{q}_j$, hence we integrate it first over $\tilde{\phi}$. Note from Eq. (A16) that the z component of \hat{q}_n does not contribute to the integral in Eq. (A17). Then we get for the x, y components of \hat{q}_n, \hat{q}_j , $\int_0^{2\pi} \hat{q}_n \hat{q}_j \frac{d\tilde{\phi}}{2\pi} = \frac{\delta_{nj}}{2} \sin^2 \theta$. Thus,

$$\begin{aligned} -I_2 &= \int_0^\pi \frac{i h \tau}{\chi a k \sin \theta} [a^2 q \epsilon_{i3j} k_3] \sin^2 \theta J_1(\tau a k \chi \sin \theta) \frac{\sin \theta d\theta}{4} = \frac{i h \tau a q \epsilon_{i3j} k_3}{\chi 2k} \int_0^\pi (\sin \theta J_1(\tau a k \chi \sin \theta)) \frac{\sin \theta d\theta}{2} \\ &= \frac{i h \tau a q \epsilon_{i3j} k_3}{\chi 2k} \int_0^\pi \left[-\frac{d}{d(\tau a k \chi)} J_0(\tau a k \chi \sin \theta) \right] \frac{\sin \theta d\theta}{2} = \frac{i h \tau a q \epsilon_{i3j} k_3}{\chi 2k} \frac{d}{d(\tau a k \chi)} \int_0^\pi J_0(\tau a k \chi \sin \theta) \frac{\sin \theta d\theta}{2} \\ &= \frac{i h \tau a q \epsilon_{i3j} k_3}{\chi 2k} \frac{d}{d(\tau a k \chi)} \left[\frac{\sin(\tau a k \chi)}{\tau a k \chi} \right]. \end{aligned} \quad (\text{A18})$$

Lastly we average over the direction of \mathbf{a} . Since \mathbf{a} can point in any direction in the plane perpendicular to \mathbf{q} , we average over the angle ϕ . We then get

$$I_2 = \frac{i h \tau a q \epsilon_{ij3} k_3}{2k} g_1(\tau a k, h), \quad \text{with} \quad g_1(s, h) = \frac{1}{\chi} \overline{\left(\frac{\sin(s\chi)}{(s\chi)^2} - \frac{\cos(s\chi)}{s\chi} \right)}, \quad (\text{A19})$$

where the overhead bar now denotes the ensemble average over the random variable ϕ .

Combining the results for I_1 and I_2 , the response tensor is obtained to be

$$G_{ij}(\mathbf{k}) = \delta_{ij} g_0(\tau a k, h) + \frac{i h \tau a q \epsilon_{ijm} k_m}{2k} g_1(\tau a k, h). \quad (\text{A20})$$

APPENDIX B: FORCED OVERDAMPED SHEARING WAVE

The forced, damped Euler equation with a background linear shear is (in this appendix we use \mathbf{v} instead of \mathbf{u}_{turb} for brevity) as follows:

$$\left(\frac{\partial}{\partial t} + S x_1 \frac{\partial}{\partial x_2} \right) \mathbf{v} + S v_1 \mathbf{e}_2 + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p - \frac{\mathbf{v}}{\tau_d} + \mathbf{f}, \quad (\text{B1})$$

where τ_d is a given damping time, and $\mathbf{f}(\mathbf{x}, t)$ is the external forcing which is assumed to satisfy $\nabla \cdot \mathbf{f} = 0$. The pressure, $p(\mathbf{x}, t)$, is determined by requiring that Eq. (B1) preserve the incompressibility of the flow. We consider external forcing of

the form,

$$\mathbf{f} = \text{Re} \{ \mathbf{F}(t) \exp [i\mathbf{Q}(t) \cdot \mathbf{x}] \}; \quad \mathbf{Q}(t) \cdot \mathbf{F}(t) = 0, \quad (\text{B2})$$

which excites a single plane shearing wave:

$$\mathbf{v} = \text{Re} \{ \mathbf{W}(t) \exp [i\mathbf{Q}(t) \cdot \mathbf{x}] \}, \quad p = \text{Re} \{ P(t) \exp [i\mathbf{Q}(t) \cdot \mathbf{x}] \}. \quad (\text{B3})$$

Incompressibility (i.e., $\nabla \cdot \mathbf{v} = 0$) requires that

$$\mathbf{Q} \cdot \mathbf{W} = 0, \quad (\text{B4})$$

which makes the nonlinear term, $(\mathbf{v} \cdot \nabla) \mathbf{v}$, vanish because

$$(\mathbf{W} \cdot \nabla) \exp [\pm i\mathbf{Q}(t) \cdot \mathbf{x}] = \pm (i\mathbf{Q} \cdot \mathbf{W}) \exp [i\mathbf{Q}(t) \cdot \mathbf{x}] = 0. \quad (\text{B5})$$

When Eq. (B3) are substituted in Eq. (B1), we obtain

$$\frac{d\mathbf{W}}{dt} + i\mathbf{W} \left(\mathbf{x} \cdot \frac{d\mathbf{Q}}{dt} + Sx_1 Q_2 \right) + SW_1 \mathbf{e}_2 = -i\mathbf{Q}P - \frac{\mathbf{W}}{\tau_d} + \mathbf{F}. \quad (\text{B6})$$

Requiring that the \mathbf{x} dependent terms inside the parentheses vanish implies that $\mathbf{Q}(t)$ must be of the form,

$$Q_1 = q_1 - Stq_2; \quad Q_2 = q_2; \quad Q_3 = q_3, \quad (\text{B7})$$

where $\mathbf{q} = (q_1, q_2, q_3)$ is a constant wave vector. Then $\mathbf{W}(t)$ satisfies

$$\frac{d\mathbf{W}}{dt} + SW_1 \mathbf{e}_2 = -i\mathbf{Q}P - \frac{\mathbf{W}}{\tau_d} + \mathbf{F}, \quad (\text{B8})$$

where $Q^2 = \mathbf{Q} \cdot \mathbf{Q} = (q_1 - Stq_2)^2 + q_2^2 + q_3^2$. We consider the overdamped case when $|d\mathbf{W}/dt| \ll |\mathbf{W}/\tau_d|$, so we drop the time derivative term on the left side of Eq. (B8). P can now be eliminated by taking the dot product of Eq. (B8) with \mathbf{Q} and using $\mathbf{Q} \cdot \mathbf{W} = 0$. Then $\mathbf{W}(t)$ satisfies

$$SW_1 \mathbf{e}_2 = SW_1 \left(\frac{Q_2 \mathbf{Q}}{Q^2} \right) - \frac{\mathbf{W}}{\tau_d} + \mathbf{F}. \quad (\text{B9})$$

The solution is

$$\begin{aligned} W_1 &= \tau_d F_1 + S\tau_d \left[\frac{Q_1 Q_2}{Q^2 - S\tau_d Q_1 Q_2} \right] \tau_d F_1, \quad W_2 = \tau_d F_2 - S\tau_d \left[\frac{Q_1^2 + Q_3^2}{Q^2 - S\tau_d Q_1 Q_2} \right] \tau_d F_1, \\ W_2 &= \tau_d F_3 + S\tau_d \left[\frac{Q_2 Q_3}{Q^2 - S\tau_d Q_1 Q_2} \right] \tau_d F_1. \end{aligned} \quad (\text{B10})$$

Using $\mathbf{Q} \cdot \mathbf{F} = 0$, it can be verified that $\mathbf{Q} \cdot \mathbf{W} = 0$. The above solution for \mathbf{W} is valid for quite arbitrary forms of the forcing. Now we make a specific choice for $\mathbf{F}(t)$:

$$\tau_d F_1 = G_1; \quad \tau_d F_2 = G_2 + StG_1; \quad \tau_d F_3 = G_3, \quad (\text{B11})$$

where $\mathbf{G} = (G_1, G_2, G_3)$ is a constant complex vector that is orthogonal to \mathbf{q} (i.e., $\mathbf{q} \cdot \mathbf{G} = 0$). Then the dependence of \mathbf{W} on time t is given in explicit form as

$$\begin{aligned} W_1 &= G_1 + S\tau_d \left[\frac{Q_1 Q_2}{Q^2 - S\tau_d Q_1 Q_2} \right] G_1, \quad W_2 = G_2 + StG_1 - S\tau_d \left[\frac{Q_1^2 + Q_3^2}{Q^2 - S\tau_d Q_1 Q_2} \right] G_1, \\ W_2 &= G_3 + S\tau_d \left[\frac{Q_2 Q_3}{Q^2 - S\tau_d Q_1 Q_2} \right] G_1. \end{aligned} \quad (\text{B12})$$

We now write the velocity field in explicit real form, using arguments familiar from the discussion of the polarization of monochromatic plane electromagnetic waves. \mathbf{G} is a complex vector. If its square, $G^2 = \mathbf{G} \cdot \mathbf{G}$, has argument equal to 2ψ , then we may write $\mathbf{G} = \mathbf{E} \exp [i\psi]$ with $\mathbf{q} \cdot \mathbf{E} = 0$, where \mathbf{E} is a complex vector whose square, $E^2 = \mathbf{E} \cdot \mathbf{E}$, is a real quantity. We now express \mathbf{E} in explicit form as $\mathbf{E} = (hc - i\mathbf{a})$ with $\mathbf{q} \cdot \mathbf{a} = 0$ and $\mathbf{q} \cdot \mathbf{c} = 0$, where \mathbf{c} and \mathbf{a} are real vectors orthogonal to \mathbf{q} , and h is a real number; we can choose $|\mathbf{c}| = |\mathbf{a}|$ and $-1 \leq h \leq 1$. Since $E^2 = (h^2 c^2 - a^2 - 2ih \mathbf{c} \cdot \mathbf{a})$ has been chosen to be a real quantity, we must have $\mathbf{c} \cdot \mathbf{a} = 0$. In other words, \mathbf{a} and \mathbf{c} are mutually orthogonal vectors lying in the plane perpendicular to \mathbf{q} . Then the velocity field of the sheared plane wave of Eq. (B3) is given by

$$\begin{aligned} \mathbf{v}(\mathbf{x}, t) &= \mathbf{A}(t, \mathbf{q}) \sin [\mathbf{Q}(t) \cdot \mathbf{x} + \psi] + \mathbf{C}(t, \mathbf{q}) \cos [\mathbf{Q}(t) \cdot \mathbf{x} + \psi], \quad \text{where} \\ A_{1,3} &= a_{1,3} + S\tau_d a_1 \left[\frac{Q_{1,3} Q_2}{Q^2 - S\tau_d Q_1 Q_2} \right], \quad A_2 = a_2 + Sta_1 - S\tau_d a_1 \left[\frac{Q_1^2 + Q_3^2}{Q^2 - S\tau_d Q_1 Q_2} \right]; \\ C_{1,3} &= h \left\{ c_{1,3} + S\tau_d c_1 \left[\frac{Q_{1,3} Q_2}{Q^2 - S\tau_d Q_1 Q_2} \right] \right\}, \quad C_2 = h \left\{ c_2 + Stc_1 - S\tau_d c_1 \left[\frac{Q_1^2 + Q_3^2}{Q^2 - S\tau_d Q_1 Q_2} \right] \right\}, \\ \text{where } Q_1 &= q_1 - Stq_2, \quad Q_2 = q_2, \quad Q_3 = q_3; \quad \mathbf{q} \cdot \mathbf{a} = 0, \quad \mathbf{c} = \hat{\mathbf{q}} \times \mathbf{a}, \quad -1 \leq h \leq 1. \end{aligned} \quad (\text{B13})$$

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