The average good counter appears to have a smaller percentage of Type I than the average noncounter, but a small percentage of Type I does not necessarily result in a good counter. This is the result one might expect if the classification of diamonds as Type I or Type II indicated the relative density of some but not all of the electron and hole traps. For example, the classification Type II might indicate the absence of shallow electron traps at 0.1 to 0.2 ev below the conduction band (which are possibly related to the 8-micron absorption region) without giving information about deep electron traps at 0.5 ev or 0.7 ev below the conduction band.

Birefringence and luminescence appear to be of little value as criteria for selecting counting diamonds.

Trapping levels at 0.5 and 0.7 ev which are responsible for thermoluminescence are probably also responsible for part of the space charge field observed in diamond counters.

A donor level located about 3 ev below the conduction band is believed to be responsible for photoconductivity. When a sufficient number of electrons from this level are removed from the diamond, a semiconductor action may be set up which allows electrons to enter the crystal from the negative electrode when triggered by an incident quantum or high energy particle.

We wish to thank Mr. Louis Small, president of the Service Diamond Tool Company, for his cooperation in loaning us some of the diamonds used in this investigation.

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Magneto-Optic Rotation in Birefringent Media—Application of the Poincaré Sphere

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It is pointed out that the concept of the Poincaré sphere appreciably simplifies the mathematical treatment of phenomena accompanying the passage of polarized light through a medium which exhibits birefringence, optical activity or both simultaneously. This is exemplified by using the Poincaré sphere to evolve techniques which could be used for determining the true Faraday rotation in the presence of birefringence. When birefringence is present, measurements made with the half-shade at the polarizer and analyzer ends are not equivalent. In either arrangement, the errors introduced as a result of birefringence are largely reduced by taking the mean of two measurements for opposite directions of the field. Formulae are also derived by which the magnitudes of the error can be calculated for the particular experimental set up, knowing the value of the birefringence. In certain cases, even this need not be known, and the true rotation can be determined purely from measurements of the apparent rotations for two different azimuths of the incident plane of polarization.

INTRODUCTION

THE Poincaré sphere[1] is a convenient geometric representation in which the state of polarization of a light beam can be denoted by a point on the sphere. Its utility arises from the fact that the phenomena accompanying the propagation of polarized light through a medium exhibiting birefringence, optical activity, or both can be determined by means of simple geometric constructions on the sphere. In spite of this, comparatively few applications appear to have been made of this concept in original investigations. In, two papers published in this journal,[2,3] the Poincaré sphere was used for obtaining elegant methods for the determination of the state of polarization of light propagated in a birefringent medium and in the design of an accurate apparatus for the analysis of elliptically polarized light. The Poincaré sphere has been applied to the case when both birefringence and optical rotation coexist by Becquerel[4] in his studies on the paramagnetic rotation in tysonit and by Bruhat and Grivet[5] for the determination of the rotatory power of quartz at right angles to the optic axis. In an investigation of the latter, made by Szivessy and Schweers,[6] the Poincaré sphere was not made use of, and one has only to compare the large number of complicated formulas in their paper with the simplicity of Bruhat and Grivet's treatment to appreciate the advantage of the concept of the Poincaré sphere.

As far as the authors are aware, no other investigations have been reported making use of this concept until quite recently. Ramaseshan and Chandrasekharan[7] made a brief study of the influence of birefringence on measurements of Faraday effect, and Ramachandran

[1] H. Poincaré, Traite de la lumiere, Paris 2, 165 (1892).
[2] F. A. Wright, J. Opt. Soc. Am. 20, 529 (1930).
[3] C. A. Skinner, J. Opt. Soc. Am. 10, 490 (1925).
[4] J. Becquerel, Communs. Phys., Lab. Univ. Leiden No. 91C (1928); 211a (1930).
[5] G. Bruhat and P. Grivet, J. phys. et radium 6, 12 (1935).
[6] G. Szivessy and C. Schweers, Ann. Physik 1, 891 (1929).
[7] S. Ramaseshan and V. Chandrasekharan, Current (India) Sci. 20, 150 (1951). S. Ramaseshan, Proc. Indian Acad. Sci. 34A, 32 (1951).

and Chandrasekharan[8] measured the photoelastic constants of sodium chlorate, a cubic crystal exhibiting optical activity. Both these investigations were conducted in this laboratory and they revealed to the authors the elegance and simplicity of the geometric methods using the Poincaré sphere compared to the algebraic manipulations that have to be made from the electromagnetic theory.

The present paper contains a very brief account of the Poincaré sphere proper, particularly in relation to the propagation of light in a medium exhibiting, simultaneously, both optical activity and birefringence. Some properties of the sphere, which do not appear to have been noted before, are also discussed. Since a medium exhibiting magneto-optic rotation behaves like one possessing optical activity (so long as the direction of a ray is not reversed in the medium), the Poincaré

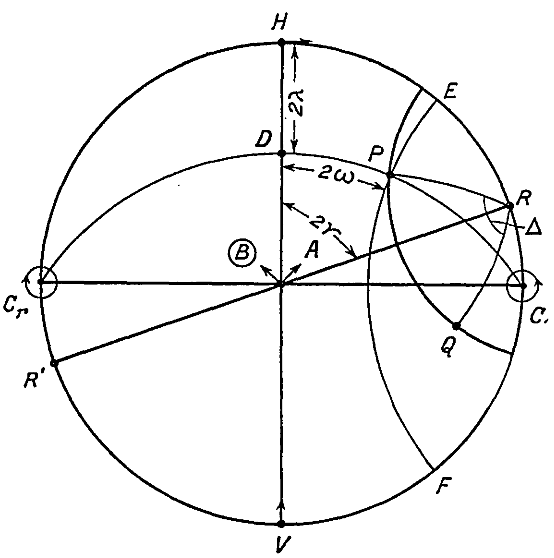

FIG. 1. Method of representing the state of polarization of light on the Poincaré sphere.

sphere can also be used for the study of the effect of birefringence on the Faraday rotation. Interesting results are thus obtained regarding the variation of the apparent rotation with the magnitude of the birefringence, with the azimuth of the incident plane polarized light and with the disposition of the measuring apparatus, such as half-shades, nicols, etc.

II. THE POINCARÉ SPHERE*

We shall always represent the sphere by its stereographic projection because suitable charts are available for making the calculations. The primitive circle is taken to be the "Greenwich meridian" or the great circle corresponding to longitudes 0° and 180°. All points on the upper hemisphere will be denoted by ordinary letters (e.g. A) while those in the lower hemisphere will be indicated by a circle around the letter.

A general point P (Fig. 1) represents a general elliptic vibration. The poles C_l and C_r represent left and right circularly polarized light, respectively. The equator $HAVB$ represents all plane vibrations, the point H of zero longitude standing for a horizontal vibration and a general point D, of longitude 2λ, representing a plane vibration whose direction is inclined at an angle λ to the horizontal. 2λ is taken to be positive for a counterclockwise rotation about C_l. Thus, the point A above has $\lambda=+45°$, while B, the correpsonding point below has $\lambda=-45°$. A general point P, with longitude 2λ and latitude 2ω, represents an elliptic vibration, whose major axis is at an angle λ to the horizontal and the ratio of whose axes b and a is given by

$$b/a=\tan\omega. \tag{1}$$

ω is taken to be positive measured from the equator towards C_l. Thus all left-rotating ellipses are on the hemisphere containing C_l and similarly all right-rotating ellipses are on the opposite hemisphere.

In a rotating birefringent medium, theory shows that two elliptic vibrations are propagated unchanged in type, along any direction of propagation. As in a non-optically active birefringent medium, these two vibrations are crossed with respect to each other. The two principal ellipses are propagated with different velocities so that plane polarized light incident on the medium emerges in general as elliptically polarized light. The state of polarization of light propagated in the medium can be obtained by the following simple construction. Let H and V be the two principal vibration directions of birefringence (i.e., in the absence of rotation) and let δ be the relative phase retardation introduced between the two per cm. Let ρ_0 be the true rotation† per cm (i.e., in the absence of birefringence). Draw the line RR' in the plane HC_lVC_r (Fig. 1) at an angle 2γ to HV where

$$\tan 2\gamma=2\rho_0/\delta_0. \tag{2}$$

Let

$$\Delta_0=(\delta_0^2+4\rho_0^2)^{\frac{1}{2}}. \tag{3}$$

Then, if P is the state of polarization of the incident light entering the medium of thickness t, then the state of polarization Q on emergence is obtained by rotating the sphere through an angle $\Delta=\Delta_0 t$ about the axis RR', which brings P to Q. The intermediate states of polarization lie on the small circular arc PQ.

The above construction follows from the electromagnetic theory of light propagation in an optically active birefringent medium.[9] Equation (3) is not strictly

[8] G. N. Ramachandran and V. Chandrasekharan, Proc. Indian Acad. Sci. 33A, 199 (1951).

* For proofs of the statements made in this section, see reference 1.

† The rotation is taken to be positive when it is counter-clockwise. This is contrary to the convention used by physical chemists, but agrees with the mathematical definition of a counter-clockwise angle as being positive. See also Szivessy.[9]

[9] G. Szivessy, *Handbuch der Physik* (Julius Springer, Berlin, Germany), Vol. 20, Ch. 11.

valid, for according to theory $\Delta_0^2=\delta_0^2+4\rho_0^2+$ a small correction term, but the deviations from Eq. (3) are only of the order of 10^{-6} and may be neglected in any practical case.

In the special case of a purely birefringent crystal, $\rho_0=0$ and $2\gamma=0$ and the sphere is to be rotated about HV, the rotation being anticlockwise looking from H to V, if H is the faster ray, and conversely. The vibrations which are propagated unchanged are the horizontal and vertical plane vibrations. Similarly, for a purely rotating crystal, $2\gamma=\pi/2$ and the circular vibrations C_l and C_r are propagated unchanged. The rotation of the sphere is anticlockwise looking from C_l to C_r if the medium is left-rotating and conversely. Thus, if elliptically polarized light (say P of Fig. 1) is incident, only the major axis of the ellipse is rotated, but its eccentricity is unchanged. The states of polarization of the light lie on the small circle EPF. When both rotation and birefringence coexist, the ellipses that are propagated unchanged are those represented by R and R'. It is readily seen that the axes of both are parallel to H and V and that the ratio of their axes are, respectively, $\tan\gamma$ and $-\cot\gamma$ (since $\omega(R)=\gamma$ and $\omega(R)=\gamma-\pi/2$), so that their major and minor axes are interchanged and they have opposite sense of rotation.

The latitude 2ω and the longitude 2λ of a general elliptic vibration may be determined in various ways. One method is to use a quarter-wave plate for reducing the ellipse to a plane vibration and then measure the inclination of this to the axes of the quarter-wave plate. The azimuth of the axes gives λ, and the tangent of the angle between plane vibration and the major axis gives the ratio of the axes and hence ω. Obviously, because of the use of a quarter-wave plate, this method is applicable only to a particular wavelength. It can, however, be adapted such that any birefringent plate (not necessarily a quarter wave plate) is used as a compensator.[3] A simpler method would be the following.[8] One measures the azimuth of the major axis of the ellipse by using a biquartz half-shade and a linear analyzer.‡ A Babinet compensator is then introduced with its axes at 45° to the major axis and the shift s in the fringes is measured. If f is the fringe-width, then one has

$$2\omega=2\pi s/f. \tag{4}$$

III. DEFINITION OF TERMS USED LATER

It is necessary to evolve suitable terms for various quantities so as to simplify the discussion in the later sections. We shall use the term "principal directions" to denote the axes of the ellipses that are propagated unchanged in the medium. If the rotatory power of the medium becomes zero, these would correspond to the principal directions of the purely birefringent crystal thus produced. Similarly, "birefringence" means the difference in refractive indices of the two waves, when rotation is taken to be zero. The quantity Δ will be denoted by "composite phase difference," i.e., the phase difference between the two elliptic vibrations, while δ the corresponding phase difference caused by birefringence alone, $(=\delta_0 t)$ will be called "pure phase difference." When birefringence is present, the angle by which the major axis of the ellipse is rotated (ψ) would be different from the rotation (ρ) which it would have suffered if there were no birefringence. To distinguish between the two, ψ would be called the "apparent rotation" and ρ the "true rotation." The true rotation per unit distance traversed (ρ_0) is clearly the "rotatory power" of the medium.

Reverting to the Poincaré sphere, "light of polarization P" would mean that its polarization is represented by the point P. Similarly, a "polarizer P," is an apparatus which transmits light of polarization P when unpolarized light is incident on it, and an "analyzer P" is an apparatus which completely transmits light of polarization P, but only transmits a portion of light of any other state of polarization. For a general point P, these would be elliptic polarizers and elliptic analyzers.§ When P lies on the equator, or coincides with the poles of the Poincaré sphere, they become linear and circular polarizers and analyzers, respectively.

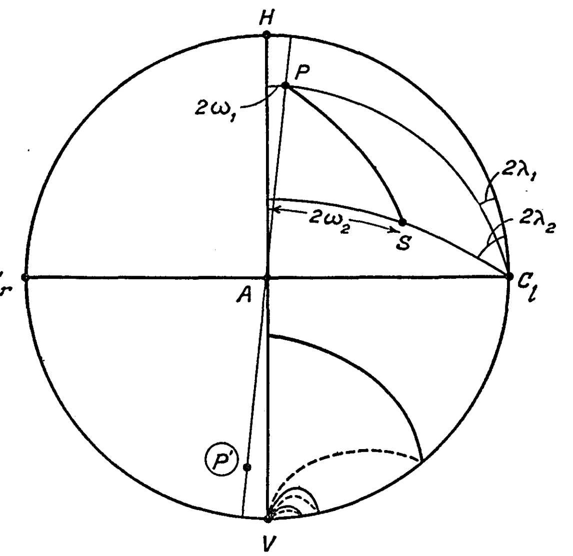

FIG. 2. Top half—Construction for determining the intensity of light transmitted by an elliptic analyzer. Bottom half—Locus of points representing the emergent light in Chauvin's experiment.

IV. A THEOREM ON THE POINCARÉ SPHERE

Suppose light of unit intensity having the polarization $S(\lambda_2, \omega_2)$ (Fig. 2) is incident on an elliptic analyzer $P(\lambda_1, \omega_1)$. It can be shown that the intensity trans-

‡ We shall use this term, instead of "nicol" or "polaroid" to denote an apparatus which transmits linearly polarized light. See next section on notation.

§ This is to be distinguished from the "elliptic analyzer" described by Ramachandran and Chandrasekharan.[8] Their set-up would be called crossed elliptic analyzers, according to our nomenclature.

mitted is $\cos^2(PS/2)$, where PS is the length of the great circular arc between P and S. As the proof of this is elementary, but tedious, only the main steps will be given here. The ratio of the axes of the ellipse S is $\tan\omega_2$ and the azimuth of the major axis is λ_2. The two axes thus have lengths $\cos\omega_2$ and $\sin\omega_2$, there being a phase difference of $\pi/2$ between the vibrations in the two directions. These can be resolved along the axes of the ellipse P. The effect of the analyzer would be to change the phase difference by $\pi/2$ and to transmit fractions $\cos\omega_1$ and $\sin\omega_1$ of the components along the major and minor axes, respectively. In this way, the intensity transmitted is found to be

$$\cos^2(\omega_1-\omega_2)\cos^2(\lambda_1-\lambda_2)+\sin^2(\omega_1+\omega_2)\sin^2(\lambda_1-\lambda_2). \quad (5)$$

This can be put in the form

$$\tfrac{1}{2}[1+\sin2\omega_1\sin2\omega_2+\cos2\omega_1\cos2\omega_2\cos2(\lambda_1-\lambda_2)] \quad (6)$$

$$=\tfrac{1}{2}(1+\cos PS)=\cos^2 PS/2, \quad (7)$$

from spherical trigonometry.

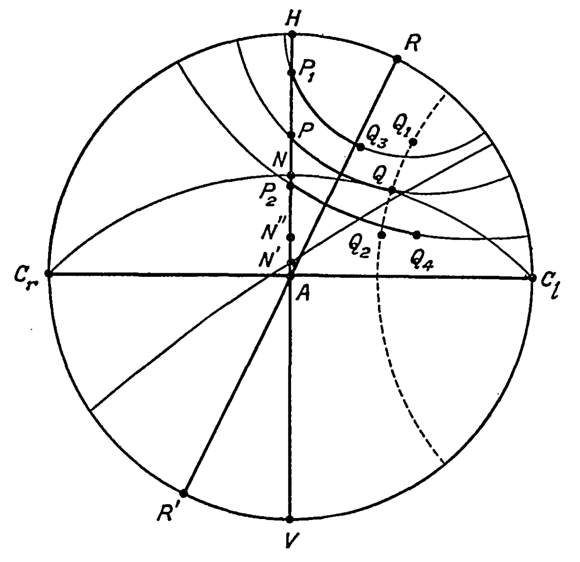

FIG. 3. The difference between putting the half-shade at the polarizer and the analyzer ends, illustrated on the Poincaré sphere.

An interesting consequence of this result is that the locus of points on the Poincaré sphere representing elliptic vibrations of which the elliptic analyzer P would transmit the same intensity, is a small circle with P as pole. In particular, it is seen that a linear analyzer of any azimuth would transmit half the intensity of circularly polarized light and a linear analyzer at 45° to H and V would transmit half the intensity of any elliptic vibration whose axes are vertical and horizontal. Further, an analyzer P would transmit zero intensity of light having a polarization represented by the antipodal point (P'). Thus, for every elliptic polarizer, there is a unique elliptic analyzer, which may be said to be "crossed" with respect to it, on the analogy of crossed nicols. Points R and R' in Fig. 1 thus represent ellipses which are "crossed" with respect to each other, as they should be because of their orthogonal property.

V. FARADAY EFFECT AND BIREFRINGENCE

A medium exhibiting magneto-optic rotation behaves similarly to one possessing natural optical activity. The only difference is that the sense of rotation is different for opposite directions of travel in the former case, while it is the same with optical activity. So long as one is interested in the phenomena accompanying light propagation only in a particular direction, the theory aforementioned can be applied *in toto* and used to evaluate the results when birefringence is also present. It has been known for a long time that the Faraday rotation of an isotropic solid, measured in the usual way with a half-shade at the analyzer end, appears to be less than its true value when the medium is birefringent.[10] Thus, Chauvin[11] measured the rotation in directions slightly inclined to the optic axis of calcite with the incident light polarized along a principal direction. With increasing magnitude of birefringence, he showed that the apparent rotation not only diminished in magnitude, but actually reversed in sign and also exhibited several reversals in sign. Chauvin explained these on the basis of the theory of light propagation in an optically active, birefringent crystal.

Obviously, therefore, measurements of the Faraday effect in a cubic crystal would be vitiated by the presence of accidental birefringence. Formerly, it has been the practice to choose for measurement as perfect a specimen as possible, judged by the least restoration of light under crossed polarizers, and to assume that the value obtained with this specimen is correct. As mentioned before, the errors introduced by birefringence in such measurements were considered by Ramaseshan and Chandrasekharan,[7] but not in great detail. The succeeding sections will deal with this problem, both in regard to the technique of measurement and the results obtained.

Before proceeding, we may indicate how the result of Chauvin can be simply explained from a geometric construction on the Poincaré sphere. Assuming the rotation ρ to be constant, and δ to increase continuously, the inclination 2γ of the axis of rotation continuously decreases while Δ increases, so that the state of polarization executes the spiral shown in the bottom half of Fig. 2. Obviously, the azimuth λ of the major axis first decreases from ρ, reverses sign, and then oscillates with successive reversal of sign, finally tending to zero, as it should for a purely birefringent crystal.

VI. EFFECT OF PUTTING THE HALF-SHADE AT THE POLARIZER AND ANALYZER ENDS

The apparatus used for the measurement of the Faraday effect is very similar to those used in polarimetry. In some polarimeters, the half-shade is put at

[10] H. Schutz, Handbuch Exp. Phys. **26**, 48 (1936).
[11] M. Chauvin, Compt. rend. **102**, 972 (1886).

the polarizer end and in others at the analyzer end, and different designs are used for the half-shade.|| All the methods lead to identical results, so long as the medium is purely rotating, but when birefringence is present, they are different in effect, as will be shown as follows.

Three types of half-shades are in common use; (a) biquartz, (b) Lippich, and (c) Laurent. The last of these employs a half-wave plate and hence is useful only for one wavelength. In the Lippich type, the two halves are not of the same intensity, so that it is difficult to work out a general theory for it. In the special design of Jellet, this defect is remedied and effects obtained with it are identical with those with a biquartz half-shade. We shall therefore consider only the latter. If β is the half-shade angle its effects would be to rotate the azimuth of the incident light through angles of $+\beta/2$ and $-\beta/2$ in the two halves.

(i) Analyzer End

Let the principal directions be vertical and horizontal and let linearly polarized light (P, Fig. 3) be incident at an azimuth 2λ. Let the effect of the medium be to bring it to Q by rotation through an angle Δ about RR'. The effect of the biquartz would be to split Q into two ellipses Q_1 and Q_2 lying in the same latitude, but differing in longitude by 2β. From the construction of section 4, the locus of points representing analyzers for which the intensities of the two halves are equal is the great circle C_lNC_r bisecting the arc Q_1Q_2. Therefore, the setting of the plane analyzer when this happens is given by N, the intersection of this great circle with the equator. (Actually, the setting would not be N, but the antipodal point to N, i.e., 90° away, when the intensity is least, but no confusion would arise by calling it N. The corresponding setting for the incident light would be P itself according to this convention and not its antipodal point.) Clearly, N has the same longitude as Q, the midpoint of Q_1Q_2.

(ii) Polarizer End

In this case, the light incident on the medium from the two halves will have the states of polarizations P_1 and P_2 and the emergent light will have the polarizations Q_3 and Q_4 and by the same construction as before, the setting of the linear analyzer for equality in the two halves will be N', which is different from N. Indeed, it can be very far away from N, and the longitude of N' need not even lie in between those of Q_3 and Q_4.

This is illustrated in Figs. 4 and 5. We consider the case when $2\gamma=30°$, i.e., $2\rho/\delta=1/\sqrt{3}$ and the half-shade angle $\beta=20$. The composite phase retardation is 63°, and we assume that the mean azimuth of the half-shade is H. Fig. 4 gives the variation in intensity of the light transmitted by the linear analyzer at various azimuths.

|| See Weissberger, *Physical Methods in Organic Chemistry* (Interscience Publishers, Inc., New York, 1945), Vol. 1, for details regarding various types of apparatus.

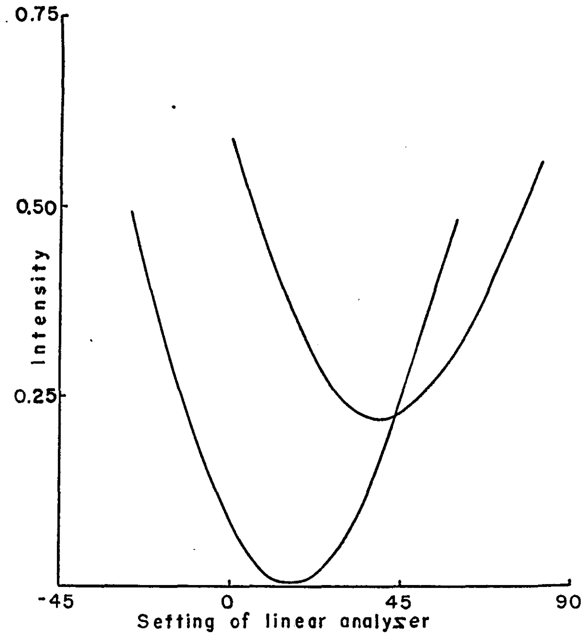

FIG. 4. Graph showing the variation of the intensity I_1 and I_2 of the two halves when the half-shade is put at the polarizer end and the linear analyzer is rotated.

It will be noticed that the minima for the two halves occur at about 20° and 40°, respectively, while the two are equal only at 44°. Figure 5 shows $\Delta I/I$ (i.e., $(I_1-I_2)/\frac{1}{2}(I_1+I_2)$) plotted against λ, and it will be seen that there is a sharp minimum at the azimuth of equality.

It is interesting to note that in both cases (i) and (ii) the setting N and N' of the analyzer is independent of the half-shade angle β. This follows from the construction that N (or N') should lie on the great circle bisecting Q_1Q_2 (or Q_3Q_4).

Considering now the effective rotation ψ in the two cases, the following formulas can readily be derived.

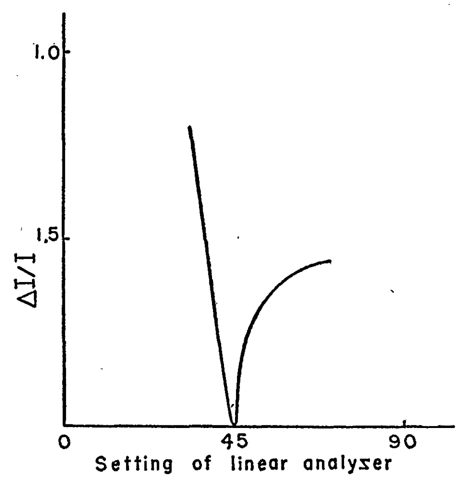

FIG. 5. Graph showing the variation of $\Delta I/I$ with the setting of the analyzer.

We shall not give the proof here. Let α be the azimuth of the polarizer in case (i) or the mean azimuth of the two beams in case (ii). Then we have

Case (i)

$$\alpha=0° \quad \tan 2\psi=\sin 2\gamma \sin\Delta/(\cos^2 2\gamma+\sin^2 2\gamma \cos\Delta), \tag{8}$$

$$\alpha=45° \quad \tan 2\psi=\sin 2\gamma \tan\Delta. \tag{9}$$

Case (ii)

$$\alpha=0° \quad \tan 2\psi=\sin 2\gamma \tan\Delta, \tag{10}$$

$$\alpha=45° \quad \tan 2\psi=\sin 2\gamma \sin\Delta/(\cos^2 2\gamma+\sin^2 2\gamma \cos\Delta.) \tag{11}$$

It will be noticed that the formulas for $\alpha=0$ and $45°$ have been interchanged in the two cases. We shall now prove that this is a particular case of the more general result, *viz*., that the value of 2ψ in case (i) for an azimuth α of the incident light is the same as that in case (ii) with an azimuth $45+\alpha$ and vice versa.

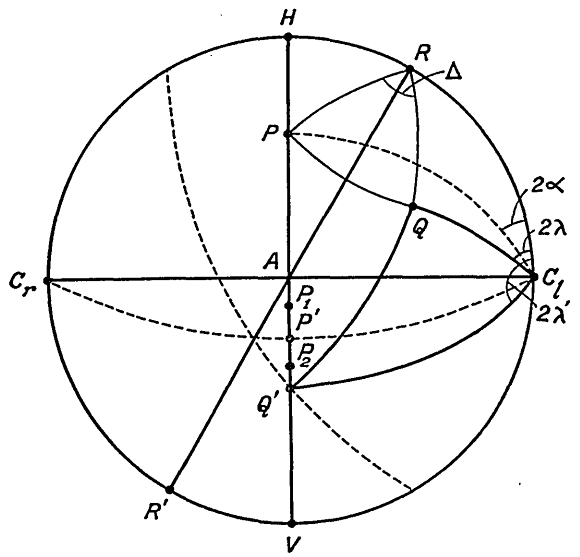

FIG. 6. Construction for proving the result in Sec. 6.

In Fig. 6, let P (longitude 2α) be the state of polarization of the incident light in case (i). If Δ is the composite phase retardation, then the emergent light will have the polarization Q (longitude 2λ) and the apparent rotation is $\psi=(\lambda-\alpha)$. Now consider case (ii) with the mean longitude of the two halves of the half-shade (P_1 and P_2) at P' (longitude $\alpha+\pi/2$). Obviously, the locus of points representing elliptic analyzers which would transmit equal intensities of the two halves is the great circle $C_lP'C_r$ *whose pole is* P. The effect of the passage of light through the medium is to rotate both P_1 and P_2 through an angle Δ about the axis RR'. The great circle $C_lP'C_r$, of equal intensity, would also therefore be rotated through the same angle Δ about the same axis. This great circle is best drawn by considering the fact that its pole (P) would also be rotated through the angle about RR' and would thus be brought to Q. The required great circle thus has for its pole the point Q, and its intersection with the equator gives Q', and the azimuth λ' of the plane analyzer for equality of intensity.

The following are clear from the diagram:

$$\text{arc}QQ'=\pi/2, \quad \text{arc}Q'C_l=\pi/2.$$

Hence

$$\angle Q'CQ=\pi/2 \quad \text{or} \quad 2\lambda'=2\lambda+\pi/2,$$

i.e.,

$$\lambda'=\lambda+\pi/4.$$

Since

$$\alpha'=\alpha+\pi/4,$$

we have

$$\psi'=\lambda'-\alpha'=\lambda-\alpha=\psi.$$

This interesting result, which was entirely unexpected, has in fact been verified by experiment. A rectangular block of optical glass ($3\times3\times2$ cm) was placed in a magnetic field and was subjected to different stresses. The light traveled along the thickness (2 cm) and the pressure was applied on the pair of faces 3×2 cm. In the absence of any strain, the glass exhibited a Faraday rotation of 17.40° for the field employed. The field was kept the same, and the glass was subjected to the successive loads shown in Table I. In each case, the birefringence introduced (δ) was directly measured with a Babinet compensator in the absence of the field (column 2). The succeeding columns contain the measured values of ψ, each of which is the average of two measurements made with the field reversed in between. This reduces the errors caused by nonuniformity of the strain (see Sec. 7). The results of the theory are verified to be true from the table. When ψ is of the order of 30° or more, the agreement is not satisfactory. This is a result of two causes: (a) the measurements themselves are not accurate because the ratio of the axes (b/a) of the ellipse approaches unity, and (b) a Lippich half-shade had to be used at the polarizer end owing to the nature of the experimental set up. In this, the two halves are not of equal intensity and the geometrical constructions outlined before are not exactly valid. It is, however, possible to work out the results from theory for this case. Thus, in Fig. 3, the setting with the Lippich would be N'', where $N''Q_3/N''Q_4=$ the ratio of the intensities of the two halves. Obviously, no general result can be derived in this case as the results would depend on the value of the ratio, the half-shade angle, etc.

Table I also shows another result, mentioned earlier, that the apparent rotation *increases* with birefringence when $\alpha=45°$.

VII. EFFECT OF REVERSING THE FIELD ON THE MEASUREMENT OF APPARENT ROTATION

Equations (8) to (11) of the previous section show how the apparent rotation varies with birefringence when the incident light is polarized at 0° or 45° to the principal directions. In what follows, we shall consider only case (i) with the half-shade at analyzer end, since the formulas for case (ii) are readily deducible from these, utilizing the results of the previous section.

Suppose, linearly polarized light is incident on the medium at an arbitrary azimuth α, and let the major axis of the emergent ellipse have an azimuth λ_1. Then, it can be shown that

$$\cot 2\lambda_1 = \frac{\cos^2 2\gamma + \sin^2 2\gamma \cos\Delta - \sin 2\gamma \sin\Delta \tan 2\alpha}{\tan 2\alpha \cos\Delta + \sin 2\gamma \sin\Delta}. \quad (12)$$

ψ, the apparent rotation, is equal to $(\lambda_1 - \alpha)$ and can be calculated. Suppose now, the field is reversed, the birefringence remaining the same. Then the azimuth λ_2 of the emergent ellipse is given by

$$\cot 2\lambda_2 = \frac{\cos^2 2\gamma + \sin^2 2\gamma \cos\Delta + \sin 2\gamma \sin\Delta \tan 2\alpha}{\tan 2\alpha \cos\Delta - \sin 2\gamma \sin\Delta}. \quad (13)$$

It is common in Faraday effect measurements to determine the rotations for the two directions of the field and find the average which is $(\lambda_1 - \lambda_2)/2$. Now, from the aforenamed two equations,

$$\begin{aligned} \cot 2(\lambda_1 - \lambda_2) &= [(\cos^2 2\gamma + \sin^2 2\gamma \cos\Delta)^2 - \sin^2 2\gamma \sin^2\Delta \\ &\qquad + \tan^2 2\alpha(\cos^2\Delta - \sin^2 2\gamma \sin^2\Delta)] \quad (14) \\ &\quad \div 2 \sin 2\gamma \sin\Delta[\cos^2 2\gamma + \sin^2 2\gamma \cos\Delta \\ &\qquad + \tan^2 2\alpha \cos\Delta]. \end{aligned}$$

Although these equations are valid for any value of α, they find application particularly when α is near about 0° or 45°. This would cover the case when the polarizer is set parallel or at 45° to the principal directions, but when there is a slight mis-setting of either the polarizer or the specimen, or if the principal directions are not exactly the same at all points in the medium. To determine the order of magnitude of the errors introduced thereby in the apparent rotation, a few particular cases have been numerically evaluated and the results are shown in Tables II and III. The small deviation of α from 0° or 45° is made equal to 2°52′, so that $\tan 2\alpha = \frac{1}{10}$. ψ is the apparent rotation when α is exactly 0° or 45°; when it deviates from these, ψ_1 and ψ_2 are the apparent rotations for positive and negative fields, and $\bar{\psi}$ is the mean of the two ($\bar{\psi} = (\psi_1 - \psi_2)/2$).

Tables II and III show that the usual practice of taking the mean of the measurements for the two directions of the field largely eliminates the errors caused by slight mis-setting of the polarizer or specimen or as a result of variations in the strain axes in the specimen. The results have an important bearing on the method suggested by Ramaseshan[7] for determining photoelastic constants of isotropic solids by measuring the reduction in the apparent rotation due to strain. Since the strain directions may not be the same at all points in the medium, it would be expected that measurements made with one direction of the field alone would be highly erratic. In fact this was found to be so, but taking the mean, eliminated all the irregularities. The measurements of photoelastic constants are being reported elsewhere.

TABLE I. Effect of the position of the half-shade on measurements of apparent rotation.

Load in kg	δ in degrees	ψ in degrees: Analyzer end $\alpha=0°$	Analyzer end $\alpha=45°$	Polarizer end $\alpha=0°$	Polarizer end $\alpha=45°$
0	0.0	17.40	17.40	17.40	17.40
38.9	32.4	16.50	19.05	18.55	16.55
54.4	46.8	15.55	22.15	21.40	15.50
70.0	63.0	14.10	28.75	27.80	14.25
85.6	77.8	12.90	39.00	33.00	12.40
101.1	92.9	11.20	69.00	60.10	11.15

VIII. EVALUATION OF THE TRUE ROTATION IN THE PRESENCE OF BIREFRINGENCE

A general method of determining the true rotation (either natural or magneto-optic rotation) in the presence of birefringence is the following: Linearly polarized light is allowed to fall on the medium at an azimuth α to the principal directions and one measures the apparent rotation ψ by means of a half-shade at the analyzer end and the ratio of the axes ($\tan\omega$) of the emergent ellipse by a suitable method. Both δ and 2ρ can then be calculated. The formulas are:[8]

TABLE II. Effect of an error in setting on the apparent rotation, when the azimuth is near 0° ($\alpha = 2°52'$).

	$2\gamma=30°$, $2\rho/\delta=1/\sqrt{3}$				$2\gamma=60°$, $2\rho/\delta=\sqrt{3}$			
Δ	ψ_1	ψ_2	$\bar{\psi}$	ψ	ψ_1	ψ_2	$\bar{\psi}$	ψ
0	0°0′	0°0′	0°0′	0°0′	0°0′	0°0′	0°0′	0°0′
30	6°59′	−7°32′	7°15′	7°15′	12°46′	−12°57′	12°52′	12°51′
60	12°12′	−14°11′	13°12′	13°10′	24°53′	−25°22′	25°7′	25°6′
90	14°54′	−18°52′	16°53′	16°51′	36°48′	−37°14′	37°1′	36°57′
120	14°20′	−20°47′	17°33′	17°22′	50°7′	−49°37′	49°52′	49°43′
150	6°3′	−18°24′	12°13′	12°34′	68°7′	−65°4′	66°36′	66°21′
180	−8°30′	−8°30′	0°0′	0°0′	92°48′	−87°12′	90°0′	90°0′

$$\left.\begin{aligned} \tan 2\gamma &= [\cos 2\alpha - \cos 2\omega \cos 2(\alpha+\psi)]/\sin 2\omega \\ \cos\Delta &= 1 - (1-\cos 2\omega \cos 2\psi)/(1-\cos^2 2\gamma \cos^2 2\alpha) \\ \delta &= \Delta \cos 2\gamma; \quad 2\rho = \Delta \sin 2\gamma. \end{aligned}\right\}. \quad (15)$$

This method may be used when neither δ nor 2ρ can be measured independently.

If, however, pure birefringence can be measured independently as in Faraday effect studies, then allowance may be made for it, and the true rotation can be deduced from a measurement of ψ alone.

If both 2ρ and δ are small, then it can be shown from Eqs. (8) and (9) that

TABLE III. Effect of an error in setting on the apparent rotation, when the azimuth is near 45° ($\alpha = 42°8'$).

	$2\gamma=30°$, $2\rho/\delta=1/\sqrt{3}$				$2\gamma=60°$, $2\rho/\delta=\sqrt{3}$			
Δ	ψ_1	ψ_2	$\bar{\psi}$	ψ	ψ_1	ψ_2	$\bar{\psi}$	ψ
0	0°0′	0°0′	0°0′	0°0′	0°0′	0°0′	0°0′	0°0′
30	7°44′	−8°20′	8°2′	8°3′	13°11′	−13°21′	13°16′	13°17′
60	19°4′	−21°31′	20°17′	20°27′	28°6′	−28°20′	28°14′	28°9′
90	45°29′	−44°38′	45°4′	45°0′	45°7′	−44°53′	45°0′	45°0′
120	73°20′	−65°57′	69°39′	69°33′	62°25′	−61°7′	61°46′	61°50′
150	86°14′	−77°46′	82°0′	81°57′	77°55′	−75°25′	76°40′	76°43′
180	94°17′	−85°42′	90°0′	90°0′	91°20′	−88°39′	90°0′	90°0′

(a) for the incident azimuth $\alpha=0°$ or $90°$

$$2\psi_0=2\rho(1-\delta^2/6)^7, \quad (16a)$$

and (b) for $\alpha=45°$,

$$2\psi_{45}=2\rho(1+\delta^2/3). \quad (16b)$$

The error in using these approximate formulas is less than 1 percent so long as δ and 2ρ do not exceed 30°.

It will be noticed from the forementioned formulas that the apparent rotation is *less* than the true rotation when $\alpha=0°$ or $90°$, but is *more* when $\alpha=45°$. It is interesting to see what would be the average value of 2ψ for all azimuths α of the incident light. Thus, from Eq. (12), $\tan 2\psi$ has the value given below for a particular value of 2α:

$$\tan 2\psi=\frac{\sin^2 2\gamma \sin\Delta-(1-\cos\Delta)\cos^2 2\gamma \tan 2\alpha+\sin 2\gamma \sin\Delta \tan^2 2\alpha}{\cos^2 2\gamma+\sin^2 2\gamma \cos\Delta+\cos\Delta \tan^2 2\alpha}.$$

The mean value of $\tan 2\psi$ over the range $-\pi/2$ to $+\pi/2$ of 2α can be shown to be the remarkably simple expression

$$(\tan 2\psi)_m=[\cot\Delta(\cot\Delta+\operatorname{cosec}\Delta \cot^2 2\gamma)]^{-\frac{1}{2}}. \quad (17)$$

With the same approximations as were used in deriving Eqs. (16a) and (16b), one obtains

$$2\psi_m=2\rho(1+\delta^2/12). \quad (18)$$

Although the deviation in the mean value of the apparent rotation is less than the deviation for either $\alpha=0$ or $\alpha=45°$, it does not vanish.

However, Eqs. (16a) and (16b) suggest a simple method of eliminating the effect of birefringence, *without measuring its value*. We have, from these,

$$(2\psi_0+\psi_{45})/3=\rho, \quad (19)$$

and this equation is correct to the third order in δ, the only terms that occur being δ^4 and higher powers.

The above formulas can be verified to be true from the data in Table I. The entries in the second row correspond to $\delta=32.4°$, which is near the limit of validity of the formulas. It is seen that the decrease in ψ for $\alpha=0$ is nearly half the increase for $\alpha=45°$ and their magnitudes are what are given by Eqs. (16a) and (16b). Further $(2\psi_0+\psi_{45})/3$ is 17.35°, which differs from the correct value 17.4° by less than 0.5 percent.

Another approximate formula, valid when 2ρ is small ($<30°$) and $\delta\gg 2\rho$ is

$$2\psi=2\rho \sin\delta/\delta, \quad (20)$$

if $\alpha=0$. For $\alpha=45°$, 2ψ increases indefinitely with increase of δ. The relation between the two is

$$\tan 2\psi=2\rho \tan\delta/\delta, \quad (21)$$

but this is not a convenient equation to use.

Equations (16) and (20) cover practically the whole range of values which occurs in Faraday effect studies, and can be used to deduce the true rotation from the measured apparent rotation.

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Energy Levels and Wavelengths of the Isotopes of Mercury-198 and -202

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In the region 6709–2302A the wavelengths of sixty lines in the first spectra of the even isotopes of mercury, $_{80}Hg^{198}$ and $_{80}Hg^{202}$, have been measured relative to 5460.7532A. Electrodeless tubes (see reference 3) containing 0.5 mg of mercury were excited by radio waves of 90- to 300-mc frequency. Comparison with the data of Meggers and Kessler (see reference 5) shows the median difference AOW−MK to be 0.0001A. A comparison of the wavelength and level systems of the two isotopes confirms our opinion that the spectrum of natural mercury may be useful as a source of secondary standards, since the isotope shift of the odd level $6p\ ^1P_1°$ is nearly the same as that of several even levels.

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IN continuation of the search for secondary standards of wavelength the spectra of mercury 198 and 202 have been observed throughout the region 6709–2262A. The method of observation was the same as that used to observe the spectra of neon and mercury[1] with the

[1] Burns, Adams, and Longwell, J. Opt. Soc. Am. 40, 339 (1950).