J. Phys. A: Math. Gen., Vol. 12, No. 10, 1979. Printed in Great Britain

The Kerr black hole in thermal equilibrium and the ν vacuum

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Received 19 December 1978

Abstract. The ν vacuum discussed recently by Fulling (1977) in his general study of alternative vacuum states in space–times with horizons is investigated specifically for the scalar and Dirac fields in Kerr space–time. An explicit evaluation of energy flux shows that, except for the effects due to super-radiance of scalar waves, this vacuum corresponds to a black hole in thermal equilibrium with its environment.

1. Introduction

It is by now well known that field quantisation in curved space is not in general unique, but depends on the definition of positive frequency and the specification of the vacuum state (DeWitt 1975, Unruh 1976). In a recent paper Fulling (1977) has reviewed and investigated alternative vacuum states in static space–times with horizons. In the context of black hole evaporation (Hawking 1975), the principal vacua of interest are

(*a*) The η vacuum, specified by the absence of 'particles' on the past null infinity and the past horizon with positive frequency defined via the Killing vector $\eta \equiv \partial/\partial t$,

(*b*) The ξ vacuum, which differs from the η vacuum in its definition of positive frequency for modes originating from the past horizon; the positive frequency in this case is defined via the null (Killing) vector $\xi \equiv \partial/\partial U$,

(*c*) The ν vacuum defined by boundary conditions on the entire horizon, past and future. (See § 2.)

For the scalar field, it is known that the η vacuum is relevant to the case of a primordial black hole whereas the ξ vacuum is appropriate for the problem of a realistic black hole formed by gravitational collapse. In an earlier work (Iyer and Kumar 1978, 1979) the η and ξ quantisation schemes were extended to the Dirac field in Kerr space–time, following Chandrasekhar's (1976) separation of the massive spin-half equation in this metric. In particular it was shown that the η vacuum gives rise to the Unruh–Starobinsky effect for Dirac quanta whereas the ξ vacuum yields the thermal Hawking emission with appropriate fermion statistics.

The purpose of this paper is to exhibit the ν quantisation scheme for scalar and Dirac fields in the Kerr space–time. An explicit evaluation of asymptotic energy flux shows that except for super-radiant effects of scalar waves in the Kerr metric, the ν vacuum represents a black hole in thermal equilibrium with its surroundings. The spin-half case corresponds to perfect thermal equilibrium as expected, since the Dirac wave does not super-radiate.

2. The ν quantisation in Kerr metric

2.1. The scalar field

In the ν quantisation scheme we define positive frequency for modes originating (entering) from the past (into the future) horizon via the null vector $\partial/\partial U$ ($\partial/\partial V$) where U and V are the Kruskal like coordinates for the Kerr metric:

$$\begin{aligned} U &= -4M \exp(-\kappa_+ u) \\ V &= 4M \exp(\kappa_+ v) \end{aligned} \tag{2.1}$$

with

$$u = t - r', \qquad v = t + r'$$

where

$$\mathrm{d}r'/\mathrm{d}r = (r^2 + a^2)/\Delta, \qquad \kappa_+ = (r_+ - r_-)/2(r_+^2 + a^2),$$
$$r_\pm = M \pm (M^2 - a^2)^{1/2}.$$

Here M is the mass and Ma the angular momentum of the black hole. The above coordinates span the usual exterior region ($U<0$, $V>0$) denoted by (+). The extended manifold contains another exterior region ($U>0$, $V<0$) denoted by (−). The positive-frequency ν modes can be equivalently characterised by their analytic properties in the U and V planes. The positive frequency ν mode is one that on the past (future) horizon is analytic and bounded in the lower half $U(V)$ plane. On the past horizon this definition is the same as for the ξ scheme. For the ν scheme, however, the boundary conditions are specified on the entire horizon, past and future, unlike the ξ scheme where the surfaces of initial data are the past horizon and the past (null) infinity. The ν vacuum is a state that is vacuous in the neighbourhood of the entire horizon. Thus the field expansion in the ν scheme requires the construction of alternative modes satisfying the above boundary conditions.

To construct the ν modes consider the function in the extended manifold given by

$$\begin{aligned} \hat{\phi}_{\omega m\lambda\epsilon} &\equiv p(\tilde{\omega})\phi_{\omega m\lambda\epsilon+} && \text{in } (+) \\ &\equiv p(-\tilde{\omega})\phi_{\omega m\lambda\epsilon-} && \text{in } (-) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} p(\tilde{\omega}) &\equiv \frac{\exp(\pi\tilde{\omega}/2\kappa_+)}{(2\sinh(\pi|\tilde{\omega}|/\kappa_+))^{1/2}}, \qquad \tilde{\omega} = \omega - m\Omega, \\ \Omega &= a/2Mr_+. \end{aligned} \tag{2.3}$$

λ is a separation constant appearing in the angular equation, (+) (−) labels the exterior region where the solution is confined. $\epsilon = \mathrm{I}, \mathrm{II}$ denotes the two linearly independent solutions characterised by their behaviour at infinity and at the horizon:

$$\begin{aligned} \phi_{\omega m\lambda \mathrm{I}} &\xrightarrow[r\to\infty]{} \frac{\exp(-\mathrm{i}\omega t)\exp(\mathrm{i}m\phi)s_{\omega m\lambda}(\theta)}{(2\pi|k|)^{1/2}r}(\exp(-\mathrm{i}kr') + A_{\mathrm{I}}\exp(\mathrm{i}kr')) \\ &\xrightarrow[r\to r_+]{} \frac{\exp(-\mathrm{i}\omega t)\exp(\mathrm{i}m\phi)s_{\omega m\lambda}(\theta)}{(2\pi|k|)^{1/2}(r_+^2 + a^2)^{1/2}}(B_{\mathrm{I}}\exp(-\mathrm{i}\tilde{\omega}r')) \end{aligned} \tag{2.4a}$$

$$\begin{aligned}\phi_{\omega m\lambda \mathrm{II}} &\xrightarrow[r\to\infty]{} \frac{\exp(-\mathrm{i}\omega t)\exp(\mathrm{i}m\phi)s_{\omega m\lambda}(\theta)}{(2\pi|\tilde{\omega}|)^{1/2}r}(B_{\mathrm{I}}\exp(\mathrm{i}kr'))\\ &\xrightarrow[r\to r_+]{} \frac{\exp(-\mathrm{i}\omega t)\exp(\mathrm{i}m\phi)s_{\omega m\lambda}(\theta)}{(2\pi|\tilde{\omega}|)^{1/2}(r_+^2+a^2)^{1/2}}(\exp(\mathrm{i}\tilde{\omega}r')+A_{\mathrm{II}}\exp(-\mathrm{i}\tilde{\omega}r'))\end{aligned}\tag{2.4b}$$

where

$$k=\omega(1-\mu^2/\omega^2)^{1/2}.$$

By construction of wave packets it is seen that the type I mode is localised at past infinity and evolves to an ingoing piece at the future horizon and another outgoing piece at future infinity. The type II mode emanates from the past horizon and evolves to an ingoing piece at the future horizon and another outgoing piece at future infinity.

For $\epsilon=\mathrm{II}$ the ν mode is the same as the ξ mode. Employing equation (2.4*b*) for the behaviour of the solution near the horizon it is readily established (Iyer and Kumar 1978) that on the past (future) horizon $\hat{\phi}_{\omega m\lambda_{\mathrm{II}}}$ of equation (2.2) is the restriction near real $U(V)$ in the lower half $U(V)$ plane of an analytic and bounded function with a cut along the negative real $U(V)$ axis *for all values of* $\tilde{\omega}$. For $\epsilon=\mathrm{I}$ the mode vanishes on the past horizon; however its piece on the future horizon again satisfies the required analytic property in the V plane. Thus equation (2.2) provides us with positive frequency modes satisfying the boundary conditions of the ν scheme. (Note that one could consider a ν mode entirely localised on the future horizon. Such a mode, however, will correspond to two disjoint pieces in the past, one at the past horizon and another at past null infinity, and will obviously not be orthogonal to the type II mode.) It is seen similarly that $\hat{\phi}^*_{\omega m\lambda\epsilon}$ are the negative frequency ν modes *for all* $\tilde{\omega}$.

We define the generalised inner product for scalar solutions in the extended manifold by

$$(\phi_1,\phi_2)\equiv\langle\phi_{1+},\phi_{2+}\rangle-\langle\phi_{1-},\phi_{2-}\rangle\tag{2.5}$$

where $\phi_{1\pm}(\phi_{2\pm})$ are the restrictions of $\phi_1(\phi_2)$ to the $(\pm)$ regions and $\langle\ ,\ \rangle$ denote the usual inner product:

$$\langle\phi_2,\phi_1\rangle=\frac{1}{2\mathrm{i}}\int(-g)^{1/2}g^{t\alpha}[\phi^*_{2,\alpha}\phi_1-\phi^*_2\phi_{1,\alpha}]\,\mathrm{d}^3x.\tag{2.6}$$

In terms of the generalised inner product, we have the following orthonormality relations:

$$\begin{aligned}
&(\hat{\phi}_{\omega m\lambda\mathrm{II}},\hat{\phi}_{\omega'm'\lambda'\mathrm{II}})=\delta(\omega-\omega')\delta_{mm'}\delta_{\lambda\lambda'}; &&\kappa=\pm1\\
&(\hat{\phi}^*_{\omega m\lambda\mathrm{II}},\hat{\phi}^*_{\omega'\mu'\lambda'\mathrm{II}})=-\delta(\omega-\omega')\delta_{mm'}\delta_{\lambda\lambda}'; &&\kappa=\pm1\\
&(\hat{\phi}_{\omega m\lambda\epsilon},\hat{\phi}^*_{\omega'm'\lambda'\epsilon'})=0; &&\kappa=\pm1\\
&(\hat{\phi}_{\omega m\lambda\epsilon},\hat{\phi}_{\omega'm'\lambda'\epsilon'})\\
&\quad=0; \qquad \epsilon\neq\epsilon',\ \kappa=\pm1\\
&(\hat{\phi}_{\omega m\lambda\mathrm{I}},\hat{\phi}_{\omega'm'\lambda'\mathrm{I}})\\
&\quad=\delta(\omega-\omega')\delta_{mm'}\delta_{\lambda\lambda'}; &&\kappa=\pm1,\ \omega\tilde{\omega}>0\\
&\quad=-\delta(\omega-\omega')\delta_{mm'}\delta_{\lambda\lambda'}; &&\kappa=\pm1,\ \omega\tilde{\omega}<0\\
&(\hat{\phi}^*_{\omega m\lambda\mathrm{I}},\hat{\phi}^*_{\omega'\mu'\lambda'\mathrm{I}})\\
&\quad=-\delta(\omega-\omega')\delta_{mm'}\delta_{\lambda\lambda'}; &&\kappa=\pm1,\ \omega\tilde{\omega}>0\\
&\quad=\delta(\omega-\omega')\delta_{mm'}\delta_{\lambda\lambda'}; &&\kappa=\pm1,\ \omega\tilde{\omega}<0
\end{aligned}\tag{2.7}$$

where

$$\begin{aligned} \kappa &= +1 \qquad \text{if } \epsilon = I,\ \omega > \mu \\ &\qquad\qquad \epsilon = \mathrm{II},\ \tilde{\omega} > 0,\ |\omega| > \mu \\ &= -1 \qquad \text{if } \epsilon = \mathrm{I},\ \omega < -\mu \\ &\qquad\qquad \epsilon = \mathrm{II},\ \tilde{\omega} < 0,\ |\omega| > \mu. \end{aligned}$$

The field expansion in terms of the ν modes is

$$\begin{aligned} \Phi = &\sum_{m,\lambda} \int_{\substack{\kappa=\pm 1 \\ \omega\tilde{\omega}>0}} \mathrm{d}\omega\, [\hat{a}_{\omega m\lambda \mathrm{I}} \hat{\phi}_{\omega m\lambda \mathrm{I}} + \hat{b}^{\dagger}_{\omega m\lambda \mathrm{I}} \hat{\phi}^{*}_{\omega m\lambda \mathrm{I}}] \\ &+ \sum_{m,\lambda} \int_{\substack{\kappa=\pm 1 \\ \omega\tilde{\omega}<0}} \mathrm{d}\omega\, [\hat{a}_{\omega m\lambda \mathrm{I}} \hat{\phi}^{*}_{\omega m\lambda \mathrm{I}} + \hat{b}^{\dagger}_{\omega m\lambda \mathrm{I}} \hat{\phi}_{\omega m\lambda \mathrm{I}}] \\ &+ \sum_{m,\lambda} \int_{\kappa=\pm 1} \mathrm{d}\omega\, [\hat{a}_{\omega m\lambda \mathrm{II}} \hat{\phi}_{\omega m\lambda \mathrm{II}} + \hat{b}^{\dagger}_{\omega m\lambda \mathrm{II}} \hat{\phi}^{*}_{\omega m\lambda \mathrm{II}}] \end{aligned} \tag{2.8}$$

Note the change in the identification of annihilation and creation operators for type I modes with $\omega\tilde{\omega} < 0$ in equation (2.8). This follows from the change in sign of the norm for this case (see equation 2.7). Inverting equation (2.8) and using the definition of creation operators in the conventional η scheme we obtain the following transformations:

$$\begin{aligned} \hat{a}_{\omega m\lambda \mathrm{I}} &= p(\tilde{\omega}) a_{\omega m\lambda \mathrm{I}+} - p(-\tilde{\omega}) b^{\dagger}_{\omega m\lambda \mathrm{I}-}; && \omega > 0,\ \tilde{\omega} > 0 \\ &= -p(\tilde{\omega}) b^{\dagger}_{-\omega-m\lambda \mathrm{I}+} + p(-\tilde{\omega}) a_{-\omega-m\lambda \mathrm{I}-}; && \omega < 0,\ \tilde{\omega} < 0 \\ &= -p(\tilde{\omega}) b^{\dagger}_{\omega m\lambda \mathrm{I}+} + p(-\tilde{\omega}) a_{\omega m\lambda \mathrm{I}-}; && \omega > 0,\ \tilde{\omega} < 0 \\ &= p(\tilde{\omega}) a_{-\omega-m\lambda \mathrm{I}+} - p(-\tilde{\omega}) b^{\dagger}_{-\omega-m\lambda \mathrm{I}-}; && \omega < 0,\ \tilde{\omega} > 0 \end{aligned} \tag{2.9}$$

$$\begin{aligned} \hat{b}_{\omega m\lambda \mathrm{I}} &= p(\tilde{\omega}) b_{\omega m\lambda \mathrm{I}+} - p(-\tilde{\omega}) a^{\dagger}_{\omega m\lambda \mathrm{I}-}; && \omega > 0,\ \tilde{\omega} > 0 \\ &= -p(\tilde{\omega}) a^{\dagger}_{-\omega-m\lambda \mathrm{I}+} + p(-\tilde{\omega}) b_{-\omega-m\lambda \mathrm{I}-}; && \omega < 0,\ \tilde{\omega} < 0 \\ &= -p(\tilde{\omega}) a^{\dagger}_{\omega m\lambda \mathrm{I}+} + p(-\tilde{\omega}) b_{\omega m\lambda \mathrm{I}-}; && \omega > 0,\ \tilde{\omega} < 0 \\ &= p(\tilde{\omega}) b_{-\omega-m\lambda \mathrm{I}+} - p(-\tilde{\omega}) a^{\dagger}_{-\omega-m\lambda \mathrm{I}-}; && \omega < 0,\ \tilde{\omega} > 0. \end{aligned} \tag{2.10}$$

Here $a_{\omega m\lambda \mathrm{I}\pm}$, $b_{\omega m\lambda \mathrm{I}\pm}$ are the corresponding operators in the usual η scheme. For type II, the transformation between the ν and η operators is the same as between the ξ and η operators (Iyer and Kumar 1979) and need not be written down here. The consistency of our construction can be checked by verifying that the ν operators also satisfy the standard commutation relations. From equations (2.9) and (2.10) we obtain

$$\begin{aligned} [\hat{a}_{\omega m\lambda\epsilon}, \hat{a}^{\dagger}_{\omega' m'\lambda'\epsilon'}] &= \delta(\omega - \omega')\delta_{mm'}\delta_{\lambda\lambda'}\delta_{\epsilon\epsilon'}; && \kappa = \pm 1, \\ [\hat{b}_{\omega m\lambda\epsilon}, \hat{b}^{\dagger}_{\omega' m'\lambda'\epsilon'}] &= \delta(\omega - \omega')\delta_{mm'}\delta_{\lambda\lambda'}\delta_{\epsilon\epsilon'}; && \kappa = \pm 1. \end{aligned} \tag{2.11}$$

All other commutators vanish.

The scalar ν vacuum is defined by

$$\hat{a}_{\omega m\lambda\epsilon}|0\rangle_{\nu} = \hat{b}_{\omega m\lambda\epsilon}|0\rangle_{\nu} = 0, \qquad \kappa = \pm 1. \tag{2.12}$$

2.2. *The Dirac field*

This case differs crucially from the scalar case in that the norm of the Dirac wave is positive definite for both the positive and negative frequencies. We retain this feature in the extended manifold by defining the generalised inner product as

$$(\psi_1, \psi_2) \equiv \langle \psi_{1+}, \psi_{2+} \rangle + \langle \psi_{1-}, \psi_{2-} \rangle \tag{2.13}$$

where the usual time-independent inner product is

$$\langle \psi_1, \psi_2 \rangle = \int (-g)^{1/2} \bar{\psi}_1 \gamma^t \psi_2 \, \mathrm{d}^3 x. \tag{2.14}$$

To obtain a positive frequency ν mode for the Dirac field, consider, in the extended manifold, the spinor defined by

$$\begin{aligned} \hat{\psi}_{(1)\omega m\lambda\epsilon} &\equiv q(\tilde{\omega}) \psi_{\omega m\lambda\epsilon+} && \text{in } (+) \\ &\equiv q(-\tilde{\omega}) \psi_{\omega m\lambda\epsilon-} && \text{in } (-) \end{aligned} \tag{2.15}$$

where

$$q(\tilde{\omega}) \equiv \frac{\exp(\pi\tilde{\omega}/2\kappa_+)}{(2\cosh(\pi\tilde{\omega}/\kappa_+))^{1/2}} \tag{2.16}$$

and $\psi_{\omega m\lambda\epsilon\pm}$ are the Dirac modes in the two exterior regions. In Chandrasekhar's representation the separated form of Dirac wave in the Kerr metric is given by

$$\begin{aligned} \psi_{\omega m\lambda\epsilon} = \exp(-\mathrm{i}\omega t) \exp(\mathrm{i}m\phi)(2^{-1/2}\rho^{*-1} S^-(\theta) R^-(r), \Delta^{-1/2} S^+(\theta) R^+(r), \\ -\Delta^{-1/2} S^-(\theta) R^+(r), -2^{-1/2} \rho^{-1} S^+(\theta) R^+(r))^T \end{aligned} \tag{2.17}$$

where

$$\rho = r + \mathrm{i}a\cos\theta$$

Here $S^{\pm}_{\omega m\lambda}(\theta)$ are the angular functions and $R^{\pm}_{\omega m\lambda\epsilon}(r)$ the radial functions satisfying coupled equations. The asymptotic behaviour of the radial function $R^+(r)$ is:

$$R^+_{\mathrm{I}} \xrightarrow[r\to\infty]{} N_{\mathrm{I}}[\exp(-\mathrm{i}\alpha) - (\omega/\mu)[1 - (1 - \mu^2/\omega^2)^{1/2}] B_{\mathrm{I}} \exp(\mathrm{i}\alpha)]$$

$$R^+_{\mathrm{II}} \xrightarrow[r\to\infty]{} N_{\mathrm{II}}[-(\omega/\mu)[1 - (1 - \mu^2/\omega^2)^{1/2}] B_{\mathrm{II}} \exp(\mathrm{i}\alpha)]$$

where

$$\begin{aligned} \alpha &= \omega(1 - \mu^2/\omega^2)^{1/2}[r' - \mu^2 M(\omega^2 - \mu^2)^{-1} \ln r] \\ N_{\mathrm{I}} &= \{2\pi(\omega^2/\mu^2)(1 - \mu^2/\omega^2)^{1/2}[1 - (1 - \mu^2/\omega^2)^{1/2}]\}^{-1/2} \\ N_{\mathrm{II}} &= \pi^{-1/2} \end{aligned} \tag{2.18}$$

$R^-(r)$ can be obtained from the coupled radial equations. The behaviour of $R^{\pm}_{\mathrm{I,II}}(r)$ near the horizon can be similarly written down. (For complete details see Iyer and Kumar 1978). On the future horizon both type I and II modes behave as

$$\psi_{\omega m\lambda\epsilon\pm} \sim \exp(\mathrm{i}m\phi^+) \exp[-(\mathrm{i}\tilde{\omega}/\kappa_+) \ln|V|] \tag{2.19}$$

where $\phi^+ = \phi - \Omega t$ and $\sim$ denotes the spinor part. Using equation (2.19) in equation (2.15) it is readily established that on the future horizon $\hat{\psi}_{(1)\omega m\lambda\epsilon}$ is the restriction near real V in the lower half V plane of an analytic and bounded function with a cut along the negative real V axis *for all values of* $\tilde{\omega}$. Similarly on the past horizon the type II mode goes as

$$\psi_{\omega m\lambda \mathrm{II}\pm} \sim \exp(\mathrm{i}m\phi^{+}) \exp[(\mathrm{i}\tilde{\omega}/\kappa_{+}) \ln|U|] \tag{2.20}$$

(Type I vanishes here) from which a similar analytic property in the U plane can be established. Thus $\hat{\psi}_{(1)\omega m\lambda\epsilon}$ for both $\epsilon = \mathrm{I}, \mathrm{II}$ are positive frequency ν modes *for all* $\tilde{\omega}$. In the same manner $\hat{\psi}_{(2)-\omega-m\lambda\epsilon}$ defined by

$$\begin{aligned} \hat{\psi}_{(2)-\omega-m\lambda\epsilon} &\equiv q(\tilde{\omega})\psi_{-\omega-m\lambda\epsilon+} && \text{in } (+) \\ &\equiv -q(-\tilde{\omega})\psi_{-\omega-m\lambda\epsilon-} && \text{in } (-) \end{aligned} \tag{2.21}$$

are seen to be negative frequency ν modes *for all* $\tilde{\omega}$.

With respect to the generalised inner product, equation (2.13) the Dirac ν modes constructed above satisfy the orthonormality relations

$$(\hat{\psi}_{(a)\omega m\lambda\epsilon}, \hat{\psi}_{(b)\omega' m'\lambda'\epsilon'}) = \delta(\omega - \omega')\delta_{mm'}\delta_{\lambda\lambda'}\delta_{\epsilon\epsilon'}\delta_{ab}; \qquad \kappa = \pm 1, \qquad a, b = 1, 2. \tag{2.22}$$

An arbitrary Dirac field may now be expanded in terms of the ν modes as

$$\Psi = \sum_{m,\lambda,\epsilon} \int_{\kappa=\pm 1} \mathrm{d}\omega\, [\hat{a}_{\omega m\lambda\epsilon}\hat{\psi}_{(1)\omega m\lambda\epsilon} + \hat{b}^{\dagger}_{\omega m\lambda\epsilon}\hat{\psi}_{(2)-\omega-m\lambda\epsilon}] \tag{2.23}$$

from which, using equation (2.22), we obtain

$$\hat{a}_{\omega m\lambda\epsilon} = (\hat{\psi}_{(1)\omega m\lambda\epsilon}, \Psi); \qquad \kappa = \pm 1 \tag{2.24a}$$

$$\hat{b}^{\dagger}_{\omega m\lambda\epsilon} = (\hat{\psi}_{(2)-\omega-m\lambda\epsilon}, \Psi); \qquad \kappa = \pm 1. \tag{2.24b}$$

Using equations (2.13), (2.15) and (2.21), the ν operators $\hat{a}$ and $\hat{b}$ are related to the creation and annihilation operators of the conventional η scheme by the following Bogoliubov transformations:

$$\begin{aligned} \hat{a}_{\omega m\lambda\epsilon} &= q(\tilde{\omega})a_{\omega m\lambda\epsilon+} + q(-\tilde{\omega})b^{\dagger}_{\omega m\lambda\epsilon-}; && \kappa = +1 \\ \hat{a}_{\omega m\lambda\epsilon} &= q(\tilde{\omega})b^{\dagger}_{-\omega-m\lambda\epsilon+} + q(-\tilde{\omega})a_{-\omega-m\lambda\epsilon-}; && \kappa = -1 \\ \hat{b}_{\omega m\lambda\epsilon} &= q(\tilde{\omega})b_{\omega m\lambda\epsilon+} - q(-\tilde{\omega})a^{\dagger}_{\omega m\lambda\epsilon-}; && \kappa = +1 \\ \hat{b}_{\omega m\lambda\epsilon} &= q(\tilde{\omega})a^{\dagger}_{-\omega-m\lambda\epsilon+} - q(-\tilde{\omega})b_{-\omega-m\lambda\epsilon-}; && \kappa = -1 \end{aligned} \tag{2.25}$$

The consistency of our construction is established by checking that equation (2.25) yields the canonical anticommutation relations for the ν operators.

$$\begin{aligned} \{\hat{a}_{\omega m\lambda\epsilon}, \hat{a}^{\dagger}_{\omega' m'\lambda'\epsilon'}\} &= \delta(\omega-\omega')\delta_{mm'}\delta_{\lambda\lambda'}\delta_{\epsilon\epsilon'}; && \kappa = \pm 1, \\ \{\hat{b}_{\omega m\lambda\epsilon}, \hat{b}^{\dagger}_{\omega' m'\lambda'\epsilon'}\} &= \delta(\omega-\omega')\delta_{mm'}\delta_{\lambda\lambda'}\delta_{\epsilon\epsilon'}; && \kappa = \pm 1. \end{aligned} \tag{2.26}$$

All other anti-commutators vanish.

The Dirac ν vacuum is defined by

$$\hat{a}_{\omega m\lambda\epsilon}|0\rangle_{\nu} = \hat{b}_{\omega m\lambda\epsilon}|0\rangle_{\nu} = 0, \qquad \kappa = \pm 1 \tag{2.27}$$

3. Asymptotic energy flux in the ν vacuum

3.1. The scalar field

The stress–energy tensor density of a scalar field in an arbitrary curved space–time is:

$$T^{\mu\nu}=\tfrac{1}{2}(-g)^{1/2}[(g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}-g^{\alpha\beta}g^{\mu\nu})\,\partial_\alpha\Phi^\dagger\,\partial_\beta\Phi+\mu^2g^{\mu\nu}\Phi^\dagger\Phi] \tag{3.1}$$

The expectation value of $T^{\mu\nu}$ in the ν vacuum, in the usual exterior region, using the expansion of equation (2.8) and (2.12) is

$$\begin{aligned}{}_\nu\langle 0|T^{\mu\nu}|0\rangle_\nu=\tfrac{1}{2}(-g)^{1/2}\Big[&(g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}-g^{\alpha\beta}g^{\mu\nu})\\
&\times\Big(\sum_{m,\lambda}\int_{\substack{\kappa=\pm1\\ \omega\tilde\omega>0}}\mathrm{d}\omega\,p^2(\tilde\omega)\phi_{\omega m\lambda\,\mathrm{I}+,\alpha}\phi^*_{\omega m\lambda\,\mathrm{I}+,\beta}\\
&+\sum_{m,\lambda}\int_{\substack{\kappa=\pm1\\ \omega\tilde\omega<0}}\mathrm{d}\omega\,p^2(\tilde\omega)\phi^*_{\omega m\lambda\,\mathrm{I}+,\alpha}\phi_{\omega m\lambda\,\mathrm{I}+,\beta}\\
&+\sum_{m,\lambda}\int_{\kappa=\pm1}\mathrm{d}\omega\,p^2(\tilde\omega)\phi_{\omega m\lambda\,\mathrm{II}+,\alpha}\phi^*_{\omega m\lambda\,\mathrm{II}+,\beta}\Big)\\
&+\mu^2g^{\mu\nu}\Big(\sum_{m,\lambda}\int_{\substack{\kappa=\pm1\\ \omega\tilde\omega>0}}\mathrm{d}\omega\,p^2(\tilde\omega)\phi_{\omega m\lambda\,\mathrm{I}+}\phi^*_{\omega m\lambda\,\mathrm{I}+}\\
&+\sum_{m,\lambda}\int_{\substack{\kappa=\pm1\\ \omega\tilde\omega<0}}\mathrm{d}\omega\,p^2(\tilde\omega)\phi^*_{\omega m\lambda\,\mathrm{I}+}\phi_{\omega m\lambda\,\mathrm{I}+}\\
&+\sum_{m,\lambda}\int_{\kappa=\pm1}\mathrm{d}\omega\,p^2(\tilde\omega)\phi_{\omega m\lambda\,\mathrm{II}+}\phi^*_{\omega m\lambda\,\mathrm{II}+}\Big)\Big]\end{aligned} \tag{3.2}$$

The outgoing energy flux across a surface at infinity is

$$\mathrm{d}E/\mathrm{d}t=\int_{r\to\infty}\mathrm{d}\theta\,\mathrm{d}\phi\,\langle 0|T^{rt}|0\rangle. \tag{3.3}$$

From equation (3.2), using the asymptotic behaviour of the modes, equation (2.4), we obtain

$$\mathrm{d}E/\mathrm{d}t=-\frac{1}{2\pi}\Big[\sum_{m,\lambda}\int_{\kappa=\pm1}\mathrm{d}\omega\,p^2(\tilde\omega)[(k\omega/|k|)(1-|A_\mathrm{I}|^2)-(k\omega/|\tilde\omega|)|B_\mathrm{II}|^2]\Big] \tag{3.4}$$

The Wronskian relations between the A's and B's following from the equation for the radial part of the scalar wave are

$$1-|A_\mathrm{I}(\omega,m,\lambda)|^2=(\tilde\omega/k)|B_\mathrm{I}(\omega,m,\lambda)|^2, \tag{3.5a}$$

$$kB_\mathrm{II}(\omega,m,\lambda)=\tilde\omega B_\mathrm{I}(\omega,m,\lambda). \tag{3.5b}$$

Substitution of equation (3.5) in (3.4) yields

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\frac{1}{\pi}\Big[\sum_{m\Omega>\mu}\int_\mu^{m\Omega}\mathrm{d}\omega\,p^2(\tilde\omega)\omega\sum_\lambda(1-|A_\mathrm{I}|^2)-\sum_{m\Omega<-\mu}\int_{m\Omega}^{-\mu}\mathrm{d}\omega\,p^2(\tilde\omega)\omega\sum_\lambda(1-|A_\mathrm{I}|^2)\Big] \tag{3.6}$$

which, using equation (3.5*a*), shows that

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{\text{scalar},\nu} > 0. \tag{3.7}$$

We thus find that the scalar ν vacuum has energy flux in the super-radiant modes. For $a = 0$ (Schwarzschild case), $\mathrm{d}E/\mathrm{d}t = 0$ and the ν vacuum represents a black hole in thermal equilibrium with the surroundings. (For a general discussion on black holes in thermal equilibrium see for example Bekenstein and Meisels (1977), Gibbons and Perry (1976, 1978).) In the large mass limit of the Kerr black hole (temperature $\to 0$) equation (3.6) reduces to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{\pi}\sum_{m\Omega>\mu}\int_{\mu}^{m\Omega}\mathrm{d}\omega\,\omega\sum_{\lambda}(1-|A_{\mathrm{I}}(-\omega,-m,\lambda)|^2) \tag{3.8}$$

The absence of thermal equilibrium for the scalar case in the classical super-radiant modes can be understood straightforwardly. For a Kerr black hole immersed in a thermal bath of scalar waves at the same temperature, the Hawking emission and absorption will be equal. However this will not include the (stimulated) *amplification* of scalar waves in the super-radiant modes and a net energy flux will result. If so, for a Dirac wave which does not exhibit super-radiance, the ν vacuum should then give rise to a vanishing energy flux, leading to perfect thermal equilibrium in all the modes. This is precisely what we demonstrate in the next section.

3.2. *The Dirac field*

The energy–momentum tensor density for the spin-half field is given by

$$T_{\mu\nu} = \tfrac{1}{4}\mathrm{i}(-g)^{1/2}[\bar{\Psi}\gamma_\mu\nabla_\nu\Psi + \bar{\Psi}\gamma_\nu\nabla_\mu\Psi - (\nabla_\mu\bar{\Psi})\gamma_\nu\Psi - (\nabla_\nu\bar{\Psi})\gamma_\mu\Psi]. \tag{3.9}$$

Using the ν expansion of the Dirac field equation (2.23) and equation 2.27, we have, in the usual exterior region:

$$\begin{aligned}{}_{\nu}\langle 0|T_{\mu\nu}|0\rangle_{\nu} = \tfrac{1}{4}\mathrm{i}(-g)^{1/2}&\sum_{m,\lambda,\epsilon}\int_{\kappa=\pm1}\mathrm{d}\omega\, q^2(\tilde{\omega})\\ &\times[\bar{\psi}_{-\omega,-m,\lambda,\epsilon,+}\gamma_\mu\nabla_\nu\psi_{-\omega,-m,\lambda,\epsilon,+} + \bar{\psi}_{-\omega,-m,\lambda,\epsilon,+}\gamma_\nu\nabla_\mu\psi_{-\omega,-m,\lambda,\epsilon,+}\\ &-(\nabla_\mu\bar{\psi}_{-\omega,-m,\lambda,\epsilon,+})\gamma_\nu\psi_{-\omega,-m,\lambda,\epsilon,+} - (\nabla_\nu\bar{\psi}_{-\omega,-m,\lambda,\epsilon,+})\gamma_\mu\psi_{-\omega,-m,\lambda,\epsilon,+}]\end{aligned} \tag{3.10}$$

Going over to Chandrasekhar's representation the vacuum expectation value can be obtained in a straightforward manner. The final result is

$$\begin{aligned}{}_{\nu}\langle 0|T_{rt}|0\rangle_{\nu} \xrightarrow[r\to\infty]{} \frac{1}{2^{1/2}\pi}\Bigg[&\sum_m\int_{\kappa=\pm1}\mathrm{d}\omega\,\omega q^2(\tilde{\omega})\sum_{\lambda}(|S^+|^2+|S^-|^2)(|B_{\mathrm{I}}|^2-1)\\ &+\sum_m\int_{\kappa=\pm1}\mathrm{d}\omega\,\omega q^2(\tilde{\omega})\\ &\times\{1-(\omega^2/\mu^2)[1-(1-\mu^2/\omega^2)^{1/2}]^2\}\sum_{\lambda}(|S^+|^2+|S^-|^2)|B_{\mathrm{II}}|^2\Bigg]\end{aligned} \tag{3.11}$$

From equation (3.3) we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t}=\frac{1}{2^{1/2}\pi}\left[\sum_m\int_{\kappa=\pm1}\mathrm{d}\omega\,\omega q^2(\tilde{\omega})\sum_\lambda(1-|B_{\mathrm{I}}|^2)\right.$$
$$\left.-\sum_m\int_{\kappa=\pm1}\mathrm{d}\omega\,\omega q^2(\tilde{\omega})\sum_\lambda\pi^{-1}N_{\mathrm{I}}^{-2}|B_{\mathrm{II}}|^2\right]\tag{3.12}$$

The 'Wronskian' relations following from the radial equations for $R^{\pm}(r)$ are

$$1-|B_{\mathrm{I}}|^2=\pi N_{\mathrm{I}}^2|A_{\mathrm{I}}|^2\tag{3.13a}$$

$$\pi N_{\mathrm{I}}^2A_{\mathrm{I}}=B_{\mathrm{II}}\tag{3.13b}$$

which substituted in equation (3.11), proves that

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{\mathrm{Dirac},\nu}=0\tag{3.14}$$

thus establishing that the ν vacuum for Kerr spacetime is appropriate for the problem of a rotating black hole in thermal equilibrium with its surroundings.

Acknowledgments

One of us (BRI) acknowledges the award of a UGC (India) research fellowship for carrying out this work.

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