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In Search Of A Covariant Quantum Measure

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Abstract. In the quantum measure approach, quantum theory is viewed as a generalisation of classical stochastic theory rather than of classical deterministic theory. An observable or "beable" is thus a *quantum measurable* set in an appropriate event algebra. Thus, the observables in quantum cosmology are covariant quantum measurable sets. In a quantum system with a strongly positive quantum measure or decoherence functional one can construct a vector valued measure whose domain is the algebra of events or physical questions. Using results in the literature, this allows us to specify when a vector valued measure on finite time events can be extended to a measure over infinite time, or equivalently, covariant events. For the class of complex percolation dynamics we show that there is a generic obstruction to this process and that covariant observables can be constructed only for a very restricted class of dynamics.

The Copenhagen interpretation of quantum theory imposes, at the most fundamental level, a dichotomy between the classical measuring device and the observed quantum system. This dichotomy is drawn into sharp and uncomfortable relief in quantum cosmology where there is no external measuring device that can collapse the wave function of the universe. Unlike most areas of physics where this problem can be shelved as a conceptual or philosophical issue, it is an unavoidable stumbling block in constructing a *physical* theory of quantum cosmology. Quantum cosmology, in addition, requires covariant observables, and is hence best suited for a sum-over-histories or path integral formulation, rather than "moment-of-time" formulations using operators and Hilbert spaces.

A typical example of the class of questions of interest to quantum cosmology is the "bounce-question": how likely is it for the universe to have undergone a cosmic bounce? In a sum-over-histories framework, the "bounce event" is the set of all possible spacetimes α_{bounce} which undergo a bounce. Is this class of spacetimes typical? For this we need to be able to put something like a measure on α_{bounce}. A classical or probability measure is required to satisfy the classical Kolmogorov sum rule

$$P(\gamma_1 \cup \gamma_2) = P(\gamma_1) + P(\gamma_2), \tag{1}$$

for distinct pairs of histories. However, we know that this rule does not work in quantum theory because of quantum interference. Instead one requires a generalised *quantum measure* on the set of covariantly defined events.

Of course, obtaining the probability of events analogous to α_{bounce} in a *classical* stochastic system is fairly straightforward. For example, the "return event" α_{return}: "the random walker who begins her walk from the origin eventually returns to the origin" is the set of paths that return to the origin, and can be assigned a probability. This return event bears more than a fleeting resemblance to the bounce event in quantum cosmology and suggests a deep parallel between quantum theory and classical stochastic theory. This is the viewpoint adopted in the quantum measure approach, where quantum theory is taken to be a generalisation of classical stochastic theory, rather than classical deterministic theory [1, 2, 3, 4, 5, 6]. Classical stochastic theory has a precise mathematical framework which we can generalise to quantum theory. In this framework the analogy between the return event and the bounce event manifests itself clearly.

Classical stochastic theory is most elegantly described in the language of of measure theory. It is defined by the triple $(\Omega, \mathcal{A}, \mu)$, where Ω is a *sample space* of histories or spacetime configurations, $\mathcal{A}$ is an *event algebra* or set of propositions about the system, and μ is a probability measure $\mu : \mathcal{A} \to [0, 1]$ satisfying Eqn (1). The event algebra $\mathcal{A}$ is a set of subsets of Ω closed under the finite set operations of union, intersection and complementation. $\mathcal{A}$ is said to be a *sigma algebra* if it is also closed under countable set operations.

A quantum measure space is, in analogy, defined by the triple $(\Omega, \mathcal{A}, \mu)$, where Ω and $\mathcal{A}$ are defined as above, and the *quantum measure* $\mu : A \to \mathbb{R}^+$ is required to satisfy, instead of Eqn (1), the weaker Quantum Sum Rule (QSR) [1]

$$\mu(\alpha \cup \beta \cup \gamma) = \mu(\alpha \cup \beta) + \mu(\alpha \cup \gamma) + \mu(\beta \cup \gamma) - \mu(\alpha) - \mu(\beta) - \mu(\gamma) \tag{2}$$

for mutually disjoint sets $\alpha, \beta, \gamma \in \mathcal{A}$. How is one to interpret μ? To begin with if $\mu(\alpha) = 0$ for some $\alpha \in \mathcal{A}$ then it is possible to say with certainty that the event α *does not happen*, independent of whether the system undergoes a measurement or not. In order to make more definite post or pre-dictions, however, a more full fledged interpretation is required, and this is provided by the Anhomomorphic logic or Coevent interpretation [7, 8][1]. We will not discuss this interpretation here, since we are concerned with the more primitive task of constructing a covariant quantum measure space.

Let us return to the example of the classical random walk. Since we wish to ask questions that are not limited to a certain final time, the sample space Ω is the set of all future-infinite paths, starting out at the origin, say, at time $t = 0$. The dynamics is constructed by assigning transition probabilities from some initial state (x_i, t_i) to a final state (x_f, t_f). A transition probability is obtained by simply summing up the probability $P(\gamma_i)$ of each path γ_i with the same initial and final states. This gives a "classical" sum over histories for the transition probability. For the return event, however, one needs to consider paths of indefinite time. This event is obtained by taking the union of all paths γ_i^t that return to the origin for the first time at some time t, $\Gamma^t = \cup_{i \in I(t)} \gamma_i^t$ (where $I(t)$ indexes the set of returning paths at time t), and then taking the union $\cup_{t=1}^{\infty} \Gamma^t$, i.e., over *all* possible times t. Is such a set measurable?

In order to be able to frame this question precisely, one needs to first define the event algebra more carefully. The measure we have described is defined on finite time paths which are not subsets of Ω. However for each finite time or "truncated" path γ^t we can assign a *cylinder set* $\mathrm{cyl}(\gamma^t) \equiv \{\gamma \in \Omega | \gamma(t') = \gamma^t(t') \, \forall \, 0 \leq t' \leq t\}$ which is a subset of Ω. The set of cylinder sets generates an event algebra $\mathcal{A}$ over which the probability measure μ is defined, i.e., $\mu(\mathrm{cyl}(\gamma^t)) = P(\gamma^t)$. However, since $\mathcal{A}$ is constructed purely out of finite set operations, the return event does not belong to $\mathcal{A}$ but rather to $S_{\mathcal{A}}$, which is the (unique) "sigma-algebra completion" of $\mathcal{A}$ [10, 11]. Indeed, all events which are independent of truncation time belong to $S_{\mathcal{A}}$ but not to $\mathcal{A}$. This is an important point, since independence from truncation time is one of the main ingredients required for constructing covariant observables in quantum cosmology. In order to construct observables from $S_{\mathcal{A}}$, however, μ must extend from A to a measure on $S_{\mathcal{A}}$, and in order for this extension to make sense, it should be unique. Thus, we can ask the precise question: does the rule Eqn (1) still hold for countable unions? For classical measures, the

[1] It is useful to point out that in a certain approach to classical proability theory based on the so-called "Cournot principle" [9], events with vanishing probabilities are used to determine all events of interest.

Caratheodary-Hahn theorem ensures that a finitely additive probability measure $\mu : \mathcal{A} \to [0,1]$ extends to a *unique* countably additive probability measure $\overline{\mu} : S_{\mathcal{A}} \to [0,1]$, thus making it possible to give an unambiguous answer to the return question.

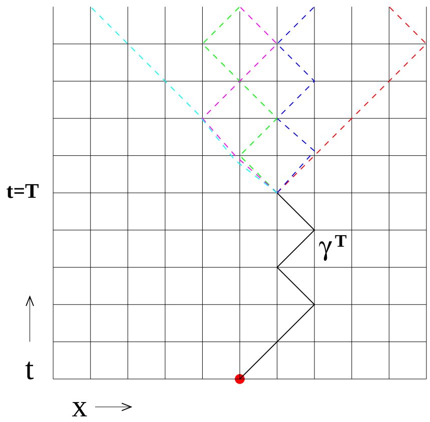

Figure 1. The cylinder set cyl(γ^T) constructed from a truncated history γ^T is a subset in the space of all future infinite paths. We show an example of this set for the classical random walk.

In a theory of quantum cosmology, any infinite time event like the bounce event has no time-label attached to it and is hence a candidate for a covariant observable, so long as the measure itself is invariant. It is therefore of importance to know whether analogs of the Caratheodary-Hahn extension theorem exist for quantum measures. In order to be able to address this in a precise manner, we find it useful to consider only those quantum measures which are derived from a *decoherence functional* $D : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$. Here, D is (i) Hermitian (ii) finitely bi-additive and (iii) strongly positive [4]. This last condition is the statement that the eigenvalues of D constructed as a matrix over a set of histories $\{\alpha_i\}$ are non-negative. The quantum measure for any $\alpha \in \mathcal{A}$ is then given by $\mu(\alpha) \equiv D(\alpha, \alpha) \geq 0$. Strong positivity of D means that a histories Hilbert space $\mathcal{H}$ can be constructed from $\mathcal{A}$, via the GNS procedure, with the inner product given by the decoherence functional [4, 12]. As we now show, the quantum measure can be expressed as a finitely additive *vector* measure over $\mathcal{H}$. This is an important technical step since it brings the quantum measure back into the folds of standard measure theory.

We briefly review the construction of $\mathcal{H}$ from [12]. Let V be the space of complex valued functions on A which are non-zero only on a finite number of elements of A. V is the free vector space over $\mathcal{A}$, with inner product

$$\langle u, v \rangle_V \equiv \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{A}} u^*(\alpha) v(\beta) D(\alpha, \beta). \tag{3}$$

The histories Hilbert space $\mathcal{H}$ is then constructed by taking the set of Cauchy sequences $\{u_i\}$ in V and quotienting by the equivalence relation

$$\{u_i\} \sim \{v_i\} \quad \text{if} \quad \lim_{i \to \infty} \parallel u_i - v_i \parallel_V = 0, \tag{4}$$

where the norm is given by the inner product. Thus,

$$\begin{aligned} [\{u_i\}] + [\{v_i\}] &\equiv [\{u_i + v_i\}] \\ \lambda[\{u_i\}] &\equiv [\{\lambda u_i\}], \\ \langle [\{u_i\}], [\{v_i\}] \rangle &\equiv \lim_{i \to \infty} \langle u_i, v_i \rangle_V, \end{aligned} \tag{5}$$

for all $[\{u_i\}], [\{v_i\}] \in \mathcal{H}$, $\lambda \in \mathbb{C}$. In [12] it was shown that for a large class of examples, this Hilbert space is canonically isomorphic to the standard Hilbert space.

The *quantum vector measure* $\mu_{\mathsf{v}} : A \to \mathcal{H}$ is given by

$$\mu_{\mathsf{v}}(\alpha) \equiv [\chi_\alpha] \in H, \tag{6}$$

where $[\cdot]$ denotes the equivalence class under (4) and χ_α denotes the constant Cauchy sequence $\{\chi_\alpha\}$ for the indicator function $\chi_\alpha : A \to \{0, 1\}$

$$\chi_\alpha(\beta) = \begin{cases} 1 & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta \neq \alpha. \end{cases} \tag{7}$$

Thus,

$$\langle \mu_{\mathsf{v}}(\alpha), \mu_{\mathsf{v}}(\beta) \rangle = D(\alpha, \beta), \tag{8}$$

with the inner product taken in $\mathcal{H}$. The finite bi-additivity of D implies that μ_{v} is finitely additive

$$\mu_{\mathsf{v}}\left(\bigcup_{i=1}^{n} \alpha_i\right) = \sum_{i=1}^{n} \mu_{\mathsf{v}}(\alpha_i), \tag{9}$$

for n mutually disjoint sets $\alpha_i \in A$. Hence μ_{v} is a vector measure on A.

The pre-fix "quantum" for the above vector measure refers only to the range H, which is the histories Hilbert space. This vector measure does indeed satisfy additivity, i.e., Eqn (1) and is thus a "bonafide" measure.

We can thus express the quantum measure space as the triple $(\Omega, \mathcal{A}, \mu_{\mathsf{v}})$, and our quest for infinite time or covariant events reduces to the question – is there a unique extension of the measure to $(\Omega, S_{\mathcal{A}}, \overline{\mu_{\mathsf{v}}})$? The answer is, in general, no. However, if μ_{v} satisfies sufficiently stringent convergence properties, then uniqueness is guaranteed by the Caratheodary-Hahn-Kluvanek theorem (see [13] for a complete statement). For a finite dimensional $\mathcal{H}$, one of these conditions refers to the total variation:

$$|\mu_{\mathsf{v}}(\alpha)| = \sup_{\pi(\alpha)} \sum_{\rho} \parallel \mu_{\mathsf{v}}(\alpha_\rho) \parallel, \tag{10}$$

where the supremum is over all finite partitions $\pi(\alpha) = \{\alpha_\rho\}$ of $\alpha \in \mathcal{A}$. $|\mu_{\mathsf{v}}|$ is itself a non-negative finitely additive pre-measure on $\mathcal{A}$ and is countably additive iff μ_{v} is (Prop. 9, Chapter 1.1, [13]). For finite $\mathcal{H}$, a necessary condition for an extension is that μ_{v} be of *bounded variation*, i.e., $|\mu_{\mathsf{v}}(\alpha)| < \infty$ for all $\alpha \in \mathcal{A}$.

We will now focus on a concrete example of quantum cosmology, namely a complex percolation dynamics for causal sets. In the causal set approach to quantum gravity, the spacetime continuum is replaced by a locally finite partially ordered set, the causal set [14]. In the continuum approximation of the theory, the order relation corresponds to the causal ordering and on an average, the number of discrete events in any causally convex region of the causal set corresponds to the continuum volume. We refer the reader to the literature for a more detailed introduction to causal sets [14, 15]. For the purpose of our analysis all that we will require is a measure theoretic characterisation of the dynamics.

We give a brief description of the classical sequential growth models of [16, 17]. Sequential growth means that a labelled causal set is grown element by element starting from a single element. Every new element is added with some probability either to the future of an existing element or left unrelated to it. This process generates labelled causal sets, but the probabilities can be chosen in a label independent way. In classical sequential growth, the probabilities are required to be Markovian, label independent and also satisfy a local causality condition

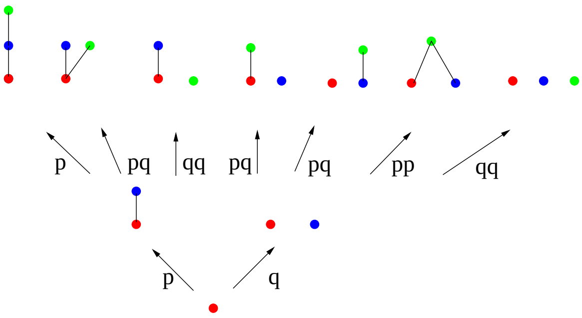

Figure 2. Transitive percolation dynamics for causal sets.

called Bell causality. Figure 2 shows an important example of sequential growth, the transitive percolation model, which will be our starting point for a quantum percolation model. Here, $p \in [0,1]$ is the probability that the new element is to the immediate future of an existing element and $q = 1 - p$ is then the probability that it is unrelated. If the growth continues for infinite time, it generates causal sets that are past finite.

Since the growth process generates causal sets that are labelled by the sequence in which each element is born, the sample space Ω is the set of past finite labelled causal sets. As in the classical random walk, associated with every finite labelled causal set C^n is a cylinder set $\mathrm{cyl}(C^n)$ which is a subset of Ω containing labelled past finite causal sets whose first n elements form the sub-causal set C^n. These cylinder sets form an event algebra $\mathcal{A}$, with measure given by taking the product of probabilities in constructing C^n via the above sequential growth process. The covariant observables of the classical dynamics are, however measurable sets in the space Ω' of *unlabelled* past finite causal sets – in other words, the time foliation or labelling that is given by a specific growth process should not matter. One way to obtain these covariant observables is then to extend the measure from $\mathcal{A}$ to $S_{\mathcal{A}}$ on Ω and then consider the quotient sigma algebra $S'_{\mathcal{A}}$ over relabellings [19]. Thus, in order to get covariant observables from a a labelled growth dynamics, the extension from $(\Omega, \mathcal{A}, \mu)$ to $(\Omega, S_{\mathcal{A}}, \overline{\mu})$ is crucial before one can take the quotient $(\Omega', S'_{\mathcal{A}}, \overline{\mu}')$ over relabellings. While the Caratheodary-Hahn theorem guarantees this for the classical stochastic growth, as we will see, it is not always guaranteed for a quantum growth process.

In direct analogy with classical transitive percolation, we define a complex percolation model as follows. We allow $p \in \mathbb{C}$ which then gives an *amplitude* for transition, rather than a probability. In addition, we assume that

$$D(\mathrm{cyl}(C^n), \mathrm{cyl}(C'^n)) = \psi^*(C^n)\psi(C'^n) \qquad (11)$$

where $\psi(C^n)$ is the amplitude for the transition from the empty set to the n-element causal set C^n. ψ is thus a complex measure on $\mathcal{A}$, and hence also a vector measure. The associated histories Hilbert space $\mathcal{H}$ can be shown to also be one-dimensional, so that the quantum vector measure μ_{v} on $\mathcal{A}$ is simply a complex measure. In this case, standard results in measure theory [18] imply that in order for μ_{v} to extend uniquely to a μ'_{v} on $S_{\mathcal{A}}$, it must have a bounded *total variation*. This corresponds to an unconditional convergence of the vector measure over all partitions.

Using simple properties of the sequential growth dynamics it is possible to show that $p+q = 1$ and that for p, q not real, $p+q = 1+\zeta$ where $\zeta > 0$ [20]. The Markov sum rule is then used to show that the total variation is bounded only for p, q real. These are the, somewhat oddly labelled, “real” complex-percolation models. In these models, even though the quantum measure is not additive, the observables of the theory are identical to those of classical transitive percolation. In particular, the observables can be entirely characterised by "stem sets". A *stem* is the causal set analog of a "past set" and a *stem set* is a subset of the set of unlabelled causal sets containing all causal sets which have the same stem [19]. Thus, for example, it is possible to assign a measure to the event, "the causal set is originary", i.e., the set of all causal sets in Ω with an "initial" element to the past of all other elements.

The lack of an extension for generic complex percolation models however suggests that these models cannot take on covariant robes in a straightforward way. In classical sequential growth models, the extension to the full sigma algebra is guaranteed, and one can then use this to define a quotient measure space that is fully label invariant. Without an extension, it is not clear that there is a well defined or canonical procedure using which covariant questions can be extracted. One lesson that can be learned from this example is that breaking covariance by introducing a time foliation may not always allow a fully quantum covariant theory to be constructed. Of course, if one could begin with a quantum measure on an event algebra over unlabelled causal sets, then even if some classical events are not quantum measurable, the set of covariant observables would be non-trivial. However, to find a simple set of manipulable rules with which to define such a measure seems intrinsically harder without first using the set of labelled causal sets as an intermediary. Admittedly, this is a poorly investigated area, and it may be that an appropriate choice of label invariant event algebras (for example the stem sets of [19] described above) will reveal potentially new and interesting dynamical rules with which to construct covariant observables.

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