The Celestial Mechanics of Newton

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Newton's law of universal gravitation laid the physical foundation of celestial mechanics. This article reviews the steps towards the law of gravitation, and highlights some applications to celestial mechanics found in Newton's *Principia*.

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1. Introduction

Newton's *Principia* consists of three books; the third dealing with the *The System of the World* puts forth Newton's views on celestial mechanics. This third book is indeed the heart of Newton's "natural philosophy", which draws heavily on the mathematical results derived in the first two books. Here he systematises his mathematical findings and confronts them against a variety of observed phenomena culminating in a powerful and compelling development of the universal law of gravitation.

Newton lived in an era of exciting developments in Natural Philosophy. Some three decades before his birth Johannes Kepler had announced his first two laws of planetary motion (AD 1609), to be followed by the third law after a decade (AD 1619). These were empirical laws derived from accurate astronomical observations, and stirred the imagination of philosophers regarding their underlying cause.

Mechanics of terrestrial bodies was also being developed around this time. Galileo's experiments were conducted in the early 17th century leading to the discovery of the laws of free fall and projectile motion. Galileo's *Dialogue* about the system of the world was published in 1632.

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Celestial mechanics, astronomy, Newton, history.

René Descartes, in his *Principia philosophiae* (1644), attempted to provide a physical basis to many observed phenomena including planetary motion. He proposed ten physical laws, the first two of which were nearly identical to the first two laws of Newton, but the rest were inaccurate. Descartes echoed the prevalent view that forces may act on a body only by contact with another body, and proposed that the planets are carried along in their orbits by "corporal vortices" in an aethereal medium revolving around the Sun. This view must have gained a fair degree of popularity, because Newton spends a significant amount of effort to demonstrate that this could not be the case.

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Newton's *Principia* is not merely a collection of brilliant mathematical solutions of numerous problems in mechanics, it also represents a *tour de force* in objectivity and rational thought. The *Principia* establishes gravity as a force that "acts at a distance" and not by contact. Yet Newton was, personally, deeply uncomfortable with the notion of action at a distance. He did not let these personal views interfere with the deductive logic of the *Principia*, as he abided by the self-imposed "rules for the study of natural philosophy", with which Book 3 of the *Principia* begins. These are very much the same rules which govern our scientific method till today. For example, his rule 4 states that *"In experimental philosophy, propositions gathered from phenomena by induction should be considered either exactly or very nearly true notwithstanding any contrary hypotheses, until yet other phenomena make such propositions either more exact or liable to exceptions"*. To make the inductions from observed phenomena precise, Newton had a very powerful tool at hand – his mathematical genius. It is therefore not surprising that he chose to name his book after that of Descartes, but paraphrasing it *Philosophiae Naturalis Principia Mathematica*, or "Mathematical Principles of Natural Philosophy".

2. Route to the Law of Gravitation

It may be said that the most important steps in the discovery of the law of gravitation were the following.

2.1. *Expression of Centripetal Force for Circular Motion*

Newton was triggered to think in terms of centripetal rather than centrifugal forces by an exchange of letters with Robert Hooke in 1679–80.

Till the late 1670s, Newton did not formulate a clear idea of a centripetal force. For bodies in circular motion he, like many others of his time, believed in the existence of an 'outward' or centrifugal force, an idea contained in the work of Descartes. The discovery of the expression v^2/r for the centrifugal acceleration is credited to Christiaan Huygens, who published this result in his book *Horologium Oscillatorium* in 1673. Newton, however, had independently worked this out in the 1660s. Newton was triggered to think in terms of centripetal rather than centrifugal forces by an exchange of letters with Robert Hooke in 1679–80. By the time of the publication of the first edition of the *Principia* in 1687, Newton's derivation of this result took the following form.

In *Figure* 1, let bodies B and b, revolving in the circumferences of circles BD and bd, describe the (infinitesimal) arcs BD and bd in the same time. In absence of centripetal force the bodies would have described tangents BC and bc respectively. Motion due to centripetal forces are then represented by the nascent spaces CD and cd respectively, executed in the same time by the two bodies. These displacements are directed towards the centre of the circle and since both are described in the same interval of time, starting from zero initial velocity in that direction, they must be proportional to the centripetal force at the respective locations. Let figure tkb be similar to the figure DCB. Then CD$/kt$ = arc BD/arc bt and also kt/cd = (arc bt/arc bd)^2. The latter follows from a lemma proved by Newton, using a

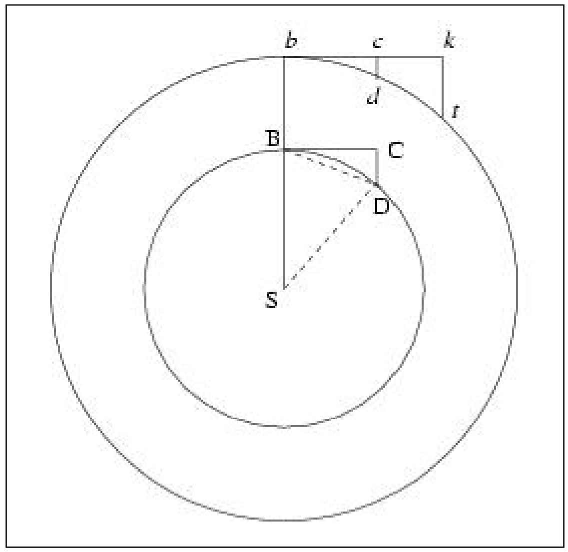


Figure 1. Centripetal force in uniform circular motion.

known property of the circle: (chord BD)^2 = BS$\times$CD (*Figure* 1), and taking limits to infinitesimal displacements. Hence CD/cd=(arc BD$\times$arc bt)/(arc bd)^2. Now arc bt=(Sb/SB)$\times$(arc BD), and therefore the ratio of forces becomes that of (arc BD)^2/SB to (arc bd)^2/Sb. Since arcs BD and bd are described in the same time interval, their lengths are proportional to the respective velocities, giving centripetal force $\propto$ v^2/r.

2.2. *The* $1/r^2$ *Force Law*

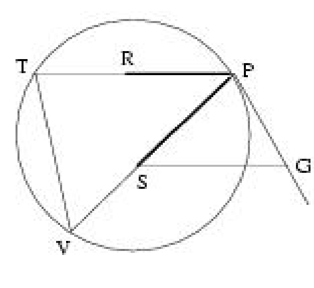
Once the expression for centrifugal acceleration was known, it was a simple matter to combine this with Kepler's third law to derive that for the planets, the centrifugal force varies inversely as the square of their distance from the Sun, since orbital speed $v = 2\pi r/T$, and $T^2 \propto r^3$ (here T is the orbital period of the planet). Eliminating T one finds $v \propto 1/r^{1/2}$, and hence $v^2/r \propto 1/r^2$. Indeed this was done by several people in the late 1670s/early 1680s, including Robert Hooke, Christopher Wren and Edmund Halley. Newton, on the other hand, had discovered this result much earlier, in the 1660s, since he had already derived the v^2/r law before its publication by Huygens. At this time he even estimated that the centrifugal force at the Earth's surface is much smaller than the gravitational force and that at the Moon's orbit, the centrifugal force measures about 1/4000 that of the Earth's surface gravity. The true significance of these results, however, was clear to Newton only after he realised that the force needed to keep bodies in orbit is centripetal, not centrifugal.

2.3. *Kepler's Laws*

While several contemporaries of Newton had arrived at the $1/r^2$ dependence of the centripetal forces acting on planets, none except Newton had the mathematical mastery to demonstrate that all the laws of Kepler follow from a centrally directed force that varies in magnitude inversely as the square of the distance. Newton showed that Kepler's first law, stating the constancy of areal velocity, follows from the fact that the force is central in nature. This proof is reproduced in the article by H S Mani in this issue. Kepler's second law, that the orbits of planets are ellipses with the Sun at the focus, was derived by Newton in his characteristically ingenious fashion. First, he inverted the problem to show that if a body moves in an elliptical orbit around a centre of force located at its focus, the centripetal force required is proportional to $1/r^2$. He provided two different sets of proofs, one of which is outlined below.

To start with, the expression for centripetal force (C.F.) directed towards any arbitrary point within a circular orbit is computed. In the diagram of *Figure* 2, if P be the particle executing the orbit and S be the centre of force, then the magnitude of centripetal force is inversely proportional to $\mathrm{SP}^2 \times \mathrm{PV}^3$ (the proof of this has been reproduced in the article by H S Mani in this issue). Newton then draws the following corollary. If the particle executes the same circular motion due to the action of a centripetal force directed towards a different centre R, then the ratio of these two forces would be

Figure 2. Ratio of centripetal forces directed towards different centres within a circular orbit.



$$\frac{\mathrm{C.F._S}}{\mathrm{C.F._R}} = \frac{\mathrm{RP}^2 \times \mathrm{PT}^3}{\mathrm{SP}^2 \times \mathrm{PV}^3} = \frac{\mathrm{SP} \times \mathrm{RP}^2}{\mathrm{SG}^3}. \tag{1}$$

Here SG is a line parallel to RP which intersects at G the tangent drawn through P. The above result follows by noting that triangles PSG and TPV are similar, giving PT/PV = SP/SG.

Newton remarks that this result is applicable to any curved orbit, not just a circle. Since we are describing just the instantaneous motion at P, the result for a circle would hold for any curve with the same radius of curvature.

Newton then estimates the centripetal force directed to the centre of an elliptical orbit. In *Figure* 3, let P be the particle moving in the elliptical orbit sketched, under the action of a force directed towards its centre C. Let RPZ be the tangent at P and PG, DK be conjugate diameters (DK parallel to RPZ). Let Q be the point on the orbit at a later time. Complete parallelogram QvPR and drop perpendicular QT on PC. Also drop perpendicular PF on DK. Now for any curved trajectory the centripetal force is proportional to $\mathrm{QR}/(\mathrm{PC}^2 \times \mathrm{QT}^2)$ (lim PQ$\rightarrow$0),

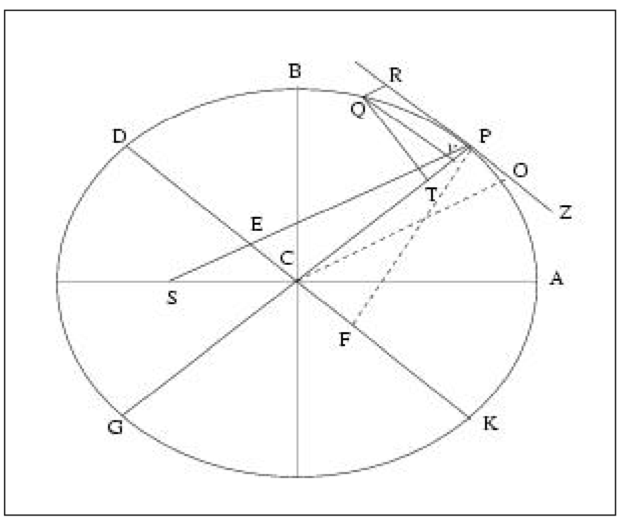


Figure 3. Diagram to compute centripetal force directed to the centre or the focus of an elliptical orbit.

as Newton had already shown (see equation (6) in the article by H S Mani in this issue). Newton now uses two properties of the ellipse: $\mathrm{P}v \times v\mathrm{G}/\mathrm{Q}v^2 = \mathrm{PC}^2/\mathrm{CD}^2$ and $\mathrm{CD} \times \mathrm{PF} = \mathrm{BC} \times \mathrm{CA}$. Using similar triangles QvT and PCF one finds $\mathrm{Q}v^2/\mathrm{QT}^2 = \mathrm{PC}^2/\mathrm{PF}^2$. From these relations and noting that Pv=QR, it is evident that

$$\frac{\mathrm{QR}}{\mathrm{QT}^2 \times \mathrm{PC}^2} = \frac{\mathrm{PC}^2}{\mathrm{CD}^2 \times \mathrm{PF}^2 \times v\mathrm{G}} = \frac{\mathrm{PC}^2}{\mathrm{BC}^2 \times \mathrm{CA}^2 \times v\mathrm{G}}.$$

As PQ$\to$0, $v\mathrm{G}\to\mathrm{PG} = 2\mathrm{PC}$. Hence the centripetal force is proportional to $\mathrm{PC}^2/(\mathrm{BC}^2 \times \mathrm{CA}^2 \times 2\mathrm{PC})$. But since BC and CA for the ellipse are given, the force is $\propto$ PC, or the distance from the centre.

An important result derived by Newton is that the major axis of an elliptical orbit is stationary only if the force is strictly inverse square.

Finally, Newton derives the force directed towards the focus of the ellipse by using the corollary quoted above. In *Figure* 3, let S be the focus to which the force is directed. E is the point of intersection of SP with the diameter DK drawn parallel to the tangent RPZ. A line parallel to SP from C intersects the tangent RPZ at point O. By (1),

$$\frac{\mathrm{C.F.}_{\mathrm{S}}}{\mathrm{C.F.}_{\mathrm{C}}} = \frac{\mathrm{CO}^3}{\mathrm{CP} \times \mathrm{SP}^2} = \frac{\mathrm{PE}^3}{\mathrm{CP} \times \mathrm{SP}^2}. \qquad (2)$$

But since PE = AC = constant (property of an ellipse), and $\mathrm{C.F.}_{\mathrm{C}} \propto \mathrm{CP}$, one concludes that $\mathrm{C.F.}_{\mathrm{S}} \propto 1/\mathrm{SP}^2$, or the force is inversely proportional to the square of the distance from the focus.

Newton then extends this proof to hyperbolic and parabolic figures and also solves the inverse problem showing that the shape of the orbit under inverse square force is a conic.

Another important result he derives is that the major axis of the elliptical orbit is stationary only if the force is strictly inverse square. Any departure from this would lead to apsidal motion (revolving orbit). He was able to quantitatively estimate the amount of apsidal motion expected, given a degree of departure from the inverse square law.

2.4. *The Moon Test*

The centripetal force that keeps the Moon in its orbit is the same gravity we experience at the Earth's surface.

As mentioned above, Newton already knew that the centripetal acceleration at the orbit of the Moon is about 1/4000 the acceleration due to gravity at the Earth's surface. The distance to the Moon from the centre of the Earth was known to him to be about 60.5 Earth radii which would imply a decrease of an inverse square force from the Earth's surface to the Moon's orbit by a factor 3660, pretty close to 4000 (in fact the discrepancy was due to the use of an inaccurate value for the radius of the Earth). This made Newton realise that the centripetal force that keeps the Moon in its orbit is the same gravity we experience at the Earth's surface. It was then a matter of a series of carefully crafted arguments to extend this notion of gravity to satellites of Jupiter and Saturn, then to planets moving around the Sun, the universality of the inverse square law thus established. In fact, he used the lack of noticeable apsidal motion in the planetary orbits and the observed fact that the apogee of the Moon moves very slowly, to argue that the inverse square law must hold to a great degree of accuracy. He correctly attributed the small motion of Moon's apogee to the orbital perturbation caused by the Sun's gravity.

Galileo's law of free fall and Newton's third law of motion laid the foundation for the universal theory of gravitation.

The above results, added to Galileo's law of free fall and Newton's third law of motion (which, combined, led to Newton's equivalence principle – that the same mass is responsible for both inertia and gravity) laid the foundation for the universal theory of gravitation.

3. Applications to Celestial Mechanics

Having established the law of universal gravitation, Newton applied it to a variety of problems.

In the *Principia*, Newton computed many details of lunar motion, explained every known variation of the Moon's motion and predicted others. He also computed the mutual perturbation of the orbits of Jupiter and Saturn, although it is unclear if there was any convincing evidence for this at that time.

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Assuming the Earth–Sun distance to be known, Newton applied his theory of gravitation to determine the ratio of masses of the Sun, Earth, Jupiter and Saturn, making use of the observed motion of the satellites around the planets and of course the motion of the planets around the Sun.

Newton then formulated a theory of tides based on the gravitational attraction of the Moon and the Sun. This was a tremendous intellectual leap forward, since this is the first time any physical cause was attributed to tidal phenomena. It was also a triumph since this theory succeeded in explaining many of the general features of the tides. Comparing the height of the tides attributed to the Moon and the Sun, Newton estimated the ratio of the mass of the Moon to that of the Sun.

The *Principia* ends with the theory of gravitation applied to cometary orbits. Newton devises a method to compute (fit) the orbit of a comet based on observations carried out at a few epochs, and then to predict the location of the comet's appearance at later times. He applies this to several comets, including the Halley's comet which appeared in 1682. His predictions, based on this physical theory (as opposed to the prevalent method of empirical interpolations and extrapolations used by his contemporary astronomers), were accurate within a couple of arcminutes, as demonstrated in the results tabulated in the *Principia*.

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4. Conclusion

There is no doubt that Newton's *Principia* was a monumental intellectual achievement of a magnitude so far unsurpassed. Newton tied all known laws of terrestrial mechanics and celestial mechanics into a firm, single physical basis. The basis of his conclusions were primarily the observations of bodies in the solar system accumulated over centuries. There is more to the cosmos than the solar system, however. Newton was well aware of this and did, at times, speculate on the fate of the universe. In the *Principia* he states that fixed stars must be very far away since their parallax is very small, and as they are distributed homogeneously, their gravitational effect on the solar system bodies would mutually cancel. In 1692, on being asked by Richard Bentley about the fate of matter spread uniformly over finite space, Newton replied that as a result of gravitational attraction, one great spherical mass will finally result. On the other hand, he said, if the matter were spread through infinite space, it would congeal into many masses like the Sun and the fixed stars. Bentley asked why should truly homogeneous and infinite distribution of matter not remain motionless in equilibrium. To this Newton replied that this was unlikely, but remained puzzled over this question. He was clearly far ahead of his time – modern cosmology and theory of structure formation had to wait for another three centuries.

Suggested Reading

[1] R Westfall, *Never at Rest, a Biography of Isaac Newton,* Cambridge University Press, 1980.

[2] S Chandrasekhar, *Newton's Principia for the Common Reader*, Clarendon Press, Oxford, 1995.

[3] I Bernard Cohen and Anne Whitman, *Isaac Newton: The Principia – a New Translation*, University of California Press, 1999.

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