

# Spherical harmonic modes of 5.5 post-Newtonian gravitational wave polarizations and associated factorized resummed waveforms for a particle in circular orbit around a Schwarzschild black hole

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Recent breakthroughs in numerical relativity enable one to examine the validity of the post-Newtonian expansion in the late stages of inspiral. For the comparison between post-Newtonian (PN) expansion and numerical simulations, the waveforms in terms of the spin-weighted spherical harmonics are more useful than the plus and cross polarizations, which are used for data analysis of gravitational waves. Factorized resummed waveforms achieve better agreement with numerical results than the conventional Taylor expanded post-Newtonian waveforms. In this paper, we revisit the post-Newtonian expansion of gravitational waves for a test particle of mass  $\mu$  in circular orbit of radius  $r_0$  around a Schwarzschild black hole of mass  $M$  and derive the spherical harmonic components associated with the gravitational wave polarizations up to order  $v^{11}$  beyond Newtonian. Using the more accurate  $h_{\ell m}$ 's computed in this work, we provide the more complete set of associated  $\rho_{\ell m}$ 's and  $\delta_{\ell m}$ 's that form important bricks in the factorized resummation of waveforms with potential applications for the construction of further improved waveforms for prototypical compact binary sources in the future. We also provide ready-to-use expressions of the 5.5PN gravitational waves polarizations  $h_+$  and  $h_\times$  in the test-particle limit for gravitational waves data analysis applications. Additionally, we provide closed analytical expressions for 2.5PN  $h_{\ell m}$ , 2PN  $\rho_{\ell m}$ , and 3PN  $\delta_{\ell m}$ , for general multipolar orders  $\ell$  and  $m$  in the test-particle limit. Finally, we also examine the implications of the present analysis for compact binary sources in Laser Interferometer Space Antenna.

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## I. INTRODUCTION

One of the most important sources of gravitational waves (GW) for the laser interferometer detectors is the inspiral and merger of a compact binary systems. To extract physical information of the source, accurate and efficient theoretical templates are needed to be matched with observed data. The early inspiral phase is accurately described by the analytic post-Newtonian (PN) approximation [1,2], while the late inspiral and the subsequent merger phases are described by a full numerical solution of the Einstein equations.

Since the recent breakthroughs in numerical relativity (NR) [3–6], a number of the simulations have computed gravitational waves through inspiral, merger, and ringdown phases. Among them, comparisons between the PN and NR waveforms have been done very accurately (See e.g. recent reviews Refs. [7,8]). One can use the comparison to investigate the region of validity of the post-Newtonian approximation in the inspiral phase. Additionally, it is also important to investigate whether higher post-Newtonian terms broaden the region of validity because computational cost of NR simulations is very high. The comparisons show that we need to include high PN corrections [9,10].

The PN approximation is not expected to model merger and ringdown due to the break down of the adiabatic approximation or also in some cases due to the break down of the monotonicity of frequency evolution.

*Resummation methods* like Padé approximants [11] can be used to extend the numerical validity of PN expansions (at least) up to the last stable orbit (LSO). The effective-one-body (EOB) approach [12] is a new resummation to extend validity of suitably resummed PN results beyond the LSO, and up to the merger. The EOB analytically provides the complete GW signal emitted by inspiralling, plunging, merging, and ringing binary black holes. By *flexing* it in the parameters carefully chosen to characterize physical effects beyond what is currently analytically computed, the EOB can be further improved and calibrated to Numerical Relativity simulations. Improved EOB models [9,10,13–15] are based on a *multiplicative* decomposition of the multipolar waveform  $h_{lm}$  into a *product* of the Newtonian waveform  $h_{\ell m}^{(N)}$  and a PN-correction factor  $\hat{h}_{\ell m}^{(\epsilon)}$  which is a *product of four factors*  $\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\text{eff}}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$ , with structure  $1 + \mathcal{O}(x)$ . The choice of factors, based on a physical understanding of the main effects influencing the final waveform, facilitates a graded improvement of the analytical waveform and possibility of refinement by match to improved numerical relativity results. The comparison of NR waveforms with the analytical EOB waveforms is currently a very active area of research. In particular, the comparison via the factorized resummation of  $h_{\ell m}$  has been very successful and benefits from inclusion of higher order multipoles (higher  $\ell$ ) and *hybridisation* using test-particle results for higher PN orders and this

provides the dominant motivation for the present investigation.

In this paper, we derive the post-Newtonian expansion of gravitational waves for a test particle of mass  $\mu$  in circular orbit around a Schwarzschild black hole of mass  $M$ . Gravitational waves can be computed using the black hole perturbation formalism. The perturbation of the Schwarzschild black hole can be treated by two different methods. The first deals with metric perturbations of the Schwarzschild black hole and the second with the perturbation of curvature tensors. For the Schwarzschild case, the master equation for the metric perturbations was derived by Regge and Wheeler for the odd parity mode, and Zerilli for the even parity mode [16,17]. For the Kerr case, however, we do not have such a master equation for the metric perturbations. For the curvature perturbations the master equation is known for both the cases and was derived by Bardeen and Press for a Schwarzschild black hole, and Teukolsky for a Kerr black hole [18,19]. Since in the future we would like to extend the present results to the case of a Kerr black hole, in the current work we employ the Teukolsky equation to compute the gravitational waves for the Schwarzschild case also.

The Teukolsky equation is the fundamental equation in black hole perturbation formalism. Although it is limited to the test-particle limit, black hole perturbation formalism has the big advantage that one can go to higher post-Newtonian orders systematically. For a particle in circular orbit around a Schwarzschild black hole [20,21], the gravitational waveforms and energy flux to infinity are known up to  $v^8$  and  $v^{11}$  respectively. For a particle in circular and equatorial orbit around a Kerr black hole [22,23], however, (see for e.g. review Ref. [24]) gravitational waveforms (energy flux to infinity) are known up to  $v^3$  ( $v^8$ ). For the general mass ratio nonspinning compact binaries in quasicircular orbits the amplitude (orbital phase) of gravitational waves are known up to  $v^6$  ( $v^7$ ) [25–31] (see e.g. review Ref. [32]). In this case for spinning precessing compact binaries in quasicircular orbits, the amplitude (orbital phase) of gravitational waves are known up to  $v^3$  ( $v^5$ ) [33–37]. Lastly, for the nonprecessing case, however, the gravitational waveforms in amplitude are known through  $v^3$  and  $v^4$  for spin orbit and spin(1)-spin(2) effects, respectively. In the case of the test-particle limit, there is a rather large gap of post-Newtonian order between waveforms and energy flux mainly because it has not been needed until recently in connection with the comparison and matching of PN analytical waveforms with waveforms from high accuracy numerical simulations.

In this work, taking account of the necessity for the comparison of waveforms between post-Newtonian approximation and numerical relativity, we improve on the accuracy for gravitational waveforms and consider gravitational waveforms also up to order  $v^{11}$  beyond Newtonian, i.e. 5.5PN. We derive 5.5PN waveforms projected onto

spin-weighted spherical harmonics since they form the basis for the comparison of analytical computations with the results of numerical simulations. The central object in this treatment is the function  $Z_{\ell m \omega}$ , and we provide its 2.5PN accurate analytical expression for *arbitrary multipolar orders*  $\ell$  and  $m$ . To facilitate and improve existing works on the factorized resummation of the gravitational waveform, we also provide the  $\rho_{\ell m}$  and  $\delta_{\ell m}$  (see Sec. V) to orders consistent with our new improved 5.5PN GW polarizations. En route to the above results, our present work extends the 1PN results for even  $h_{\ell m}$ 's [29] and odd  $h_{\ell m}$ 's [15] by providing *closed analytical expressions* for  $h_{\ell m}$ ,  $\rho_{\ell m}$ , and  $\delta_{\ell m}$  up to  $O(v^5)$ ,  $O(v^4)$ , and  $O(v^6)$  respectively for *general multipolar orders*  $\ell$  and  $m$ . Finally, we also examine the implications of the present analysis for compact binary sources in Laser Interferometer Space Antenna (LISA).

This paper is organized as follows. In Sec. II, we describe the general formalism and relevant formulas that underlie the present work. In Sec. III, we describe the Mano, Suzuki, and Takasugi method [38,39] which is used in this paper to solve the Teukolsky equation. In particular, employing this formalism, in Sec. III C, we derive the 2.5PN accurate solution of the Teukolsky solution  $Z_{\ell m \omega}$  for arbitrary multipolar orders  $\ell$  and  $m$ . In Sec. IV, we compute the gravitational waveforms expressed as spherical harmonic modes at 5.5PN order. In Sec. V, the  $\rho_{\ell m}$  and  $\delta_{\ell m}$  needed for the construction of factorized resummed waveforms at 5PN are computed. For general multipolar orders  $\ell$  and  $m$ , in Sec. IV, we exhibit closed analytical forms for  $h_{\ell m}$  at 2.5PN, while in Sec. VA we provide ready-to-use expressions for  $\rho_{\ell m}$  at 2PN and  $\delta_{\ell m}$  at 3PN. In Sec. VI, we provide general formulas to compute 5.5PN polarization modes starting from the explicit expressions of spherical harmonic modes at 5.5PN in Secs. IV and V. In Sec. VII, we compare the results from our 5.5PN approximation with the results from a numerical calculation, obtained by solving the Teukolsky equation [40,41]. Section VIII is devoted to a summary of the paper. The paper ends with three Appendices. In Appendices A and B, we list  $\hat{H}_{\ell m}$  and  $\rho_{\ell m}$  for higher values of  $\ell$  (consistent with 5.5PN GW polarizations) than listed in the main text. And finally, in Appendix C we list the complete 5.5PN GW polarizations  $H_{+, \times}$  in the test-particle limit. Throughout this paper, we use the units of  $c = G = 1$ .

## II. GENERAL FORMULATION

In the Teukolsky formalism, the gravitational perturbation of a Kerr black hole is described in terms of the Newman-Penrose variables  $\Psi_0$  and  $\Psi_4$  which satisfy a master equation. In this section we recall the relevant equations needed in this work following the notation in [24]. The Weyl scalar  $\Psi_4$  is related to the amplitude of the gravitational wave at infinity as

$$\Psi_4 \rightarrow \frac{1}{2}(\ddot{h}_+ - i\dot{h}_\times), \quad \text{for } r \rightarrow \infty, \quad (2.1)$$

where dot  $\cdot$  denotes time derivative  $d/dt$ . The master equation for  $\Psi_4$  can be separated into radial and angular parts if we expand  $\Psi_4$  in Fourier harmonic modes as

$$\rho^{-4}\Psi_4 = \sum_{\ell, m} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} R_{\ell m \omega}(r) {}_{-2}S_{\ell m}^{a\omega}(\theta, \varphi), \quad (2.2)$$

where  $\rho = (r - ia \cos\theta)^{-1}$ ,  $M$  and  $aM$  are the mass and angular momentum of the black hole, respectively, and the angular function  ${}_{-2}S_{\ell m}^{a\omega}(\theta, \varphi)$  is the spin-weighted spheroidal harmonic with spin  $s = -2$ ,

$${}_{-2}S_{\ell m}^{a\omega}(\theta, \varphi) = \frac{1}{\sqrt{2\pi}} {}_{-2}\tilde{S}_{\ell m}^{a\omega}(\theta) e^{im\varphi}, \quad (2.3)$$

where  ${}_{-2}\tilde{S}_{\ell m}^{a\omega}(\theta)$  satisfies the angular Teukolsky equation and which is normalized as

$$\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta |{}_{-2}S_{\ell m}^{a\omega}(\theta, \varphi)|^2 = 1. \quad (2.4)$$

In the following, we focus on the Schwarzschild black hole case. The spin-weighted spheroidal harmonic  ${}_{-2}S_{\ell m}^{a\omega}(\theta, \varphi)$  is then reduced to the spin-weighted spherical harmonic  ${}_{-2}Y_{\ell m}(\theta, \varphi)$ , whose definition is given by [29]

$$\begin{aligned} {}_s Y_{\ell m}(\theta, \varphi) &= (-1)^m e^{im\varphi} \sqrt{\frac{(2\ell+1)(\ell+m)!(\ell-m)!}{4\pi(\ell+s)!(\ell-s)!}} \\ &\times \sin^{2\ell}\left(\frac{\theta}{2}\right) \sum_{k=k_1}^{k_2} \binom{\ell-s}{k} \binom{\ell+s}{k+s-m} \\ &\times (-1)^{\ell-k-s} \cot^{2k+s-m}\left(\frac{\theta}{2}\right), \end{aligned} \quad (2.5)$$

where  $k_1 = \max(0, m-s)$  and  $k_2 = \min(\ell+m, \ell-s)$ .

It is straightforward to compute the spin-weighted spherical harmonic, so we need to focus only on how to solve the radial Teukolsky equation, which in Schwarzschild coordinates with  $s = -2$  reads [21],

$$\left[ \Delta^2 \frac{d}{dr} \left( \frac{1}{\Delta} \frac{d}{dr} \right) + U(r) \right] R_{\ell m \omega}(r) = T_{\ell m \omega}(r), \quad (2.6)$$

with  $\Delta = r(r - 2M)$  and

$$U(r) = \frac{r^2}{\Delta} [\omega^2 r^2 - 4i\omega(r - 3M)] - (\ell - 1)(\ell + 2), \quad (2.7)$$

where  $T_{\ell m \omega}$  is the source term which is a contraction of the energy momentum tensor of the small particle and the null tetrad chosen.

We solve Eq. (2.6) using the Green function method. For this purpose, we need a homogeneous solution  $R_{\ell m \omega}^{\text{in}}$  of Eq. (2.6) which satisfies the boundary conditions

$$R_{\ell m \omega}^{\text{in}} = \begin{cases} B_{\ell m \omega}^{\text{trans}} \Delta^2 e^{-i\omega r^*} & \text{for } r^* \rightarrow -\infty, \\ r^3 B_{\ell m \omega}^{\text{ref}} e^{i\omega r^*} + r^{-1} B_{\ell m \omega}^{\text{inc}} e^{-i\omega r^*} & \text{for } r^* \rightarrow +\infty, \end{cases} \quad (2.8)$$

where  $r^* = r + 2M \ln(r/2M - 1)$ . Then the outgoing-wave solution of Eq. (2.6) at infinity with appropriate boundary conditions at horizon is given by

$$R_{\ell m \omega}(r \rightarrow \infty) = \frac{r^3 e^{i\omega r^*}}{2i\omega B_{\ell m \omega}^{\text{inc}}} \int_{2M}^{\infty} dr \frac{R_{\ell m \omega}^{\text{in}} T_{\ell m \omega}(r)}{\Delta^2}, \quad (2.9a)$$

$$\equiv r^3 e^{i\omega r^*} \tilde{Z}_{\ell m \omega}. \quad (2.9b)$$

In the case of a circular orbit, the frequency spectrum of  $T_{\ell m \omega}$  becomes discrete. Then  $\tilde{Z}_{\ell m \omega}$  in Eq. (2.9b) takes the form

$$\tilde{Z}_{\ell m \omega} = Z_{\ell m \omega} \delta(\omega - m\Omega), \quad (2.10)$$

where

$$\begin{aligned} Z_{\ell m \omega} &= \frac{\mu\pi}{i\omega(r_0/M)^2 B_{\ell m \omega}^{\text{inc}}} \left\{ \left[ -{}_0b_{\ell m} - 2i{}_{-1}b_{\ell m} \left( 1 + \frac{i}{2} \right. \right. \right. \\ &\times \left. \left. \frac{\omega r_0^2}{(r_0 - 2M)} \right) + i{}_{-2}b_{\ell m} \frac{\omega r_0}{(1 - 2M/r_0)^2} \right. \\ &\times \left. \left( 1 - \frac{M}{r_0} + \frac{1}{2} i\omega r_0 \right) \right] R_{\ell m \omega}^{\text{in}} + \left[ i{}_{-1}b_{\ell m} - {}_{-2}b_{\ell m} \right. \\ &\times \left. \left( 1 + i \frac{\omega r_0^2}{r_0 - 2M} \right) \right] r_0 R_{\ell m \omega}^{\text{in} \prime}(r_0) \\ &\left. + \frac{1}{2} {}_{-2}b_{\ell m} r_0^2 R_{\ell m \omega}^{\text{in} \prime \prime}(r_0) \right\}, \end{aligned} \quad (2.11)$$

and prime  $\prime$  denotes  $d/dr$  with  ${}_s b_{\ell m}$  are defined by

$$\begin{aligned} {}_0b_{\ell m} &= \frac{1}{2} [(\ell - 1)\ell(\ell + 1)(\ell + 2)]^{1/2} {}_0Y_{\ell m}\left(\frac{\pi}{2}, 0\right) \\ &\times \frac{\tilde{E}r_0}{r_0 - 2M}, \end{aligned} \quad (2.12a)$$

$${}_{-1}b_{\ell m} = [(\ell - 1)(\ell + 2)]^{1/2} {}_{-1}Y_{\ell m}\left(\frac{\pi}{2}, 0\right) \frac{\tilde{L}}{r_0}, \quad (2.12b)$$

$${}_{-2}b_{\ell m} = {}_{-2}Y_{\ell m}\left(\frac{\pi}{2}, 0\right) \tilde{L}\Omega. \quad (2.12c)$$

Here,  $\Omega$ ,  $\tilde{E}$ , and  $\tilde{L}$  are the angular frequency, the specific energy, and the angular momentum of the particle, respectively, which are given by

$$\Omega = \sqrt{\frac{M}{r_0^3}}, \quad \tilde{E} = \frac{r_0 - 2M}{\sqrt{r_0(r_0 - 3M)}}, \quad \tilde{L} = \frac{\sqrt{Mr_0}}{\sqrt{1 - 3M/r_0}}, \quad (2.13)$$

where  $r_0$  is the orbital radius.

In terms of the amplitudes  $Z_{\ell m \omega}$ , the gravitational wave luminosity and the gravitational waveforms are, respectively, given by

$$\frac{dE}{dt} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{|Z_{\ell m \omega}|^2}{4\pi\omega^2}, \quad (2.14)$$

and

$$h_+ - ih_{\times} = -\frac{2}{r} \sum_{\ell, m} \frac{Z_{\ell m \omega}}{\omega^2} {}_{-2}Y_{\ell m}(\theta, \varphi) e^{i\omega(r^* - t)}, \quad (2.15a)$$

$$\equiv \sum_{\ell, m} (h_{\ell m}^+ - ih_{\ell m}^{\times}), \quad (2.15b)$$

where  $\omega = m\Omega$  is the frequency of gravitational waves,<sup>1</sup>  $(\theta, \varphi)$  are the angles defining the location of the observer relative to the source, and  $h_{\ell m}^+$  and  $h_{\ell m}^{\times}$  are real. We calculate the gravitational waveforms in the post-Newtonian expansion, that is, in the expansion with respect to  $\nu = (M/r_0)^{1/2}$ . In order to compute  $Z_{\ell m \omega}$ , we need the series expansion of the ingoing-wave Teukolsky function  $R_{\ell m \omega}^{\text{in}}$  in terms of  $\epsilon = 2M\omega = 2Mm\Omega = O(\nu^3)$  and  $z = \omega r = O(\nu)$  and the asymptotic amplitudes  $B_{\ell m \omega}^{\text{inc}}$  in terms of  $\epsilon$ . We use the formalism developed by Mano, Suzuki, and Takasugi [38] to compute them and give a brief review of the same for the convenience of the reader in the following Sec. III.

### III. THE MANO, SUZUKI, AND TAKASUGI METHOD FOR ANALYTIC SOLUTIONS OF THE HOMOGENEOUS TEUKOLSKY EQUATION

In the formalism developed by Mano, Suzuki, and Takasugi, the homogeneous solutions of the Teukolsky equation are expressed in terms of two kinds of series of special functions: hypergeometric functions and Coulomb wave functions [38,39]. The series of hypergeometric functions is convergent at the horizon, and the series of Coulomb wave functions at infinity. The matching of the two kinds of solutions is done analytically in the overlapping region of convergence. One can thus obtain analytic expressions of the asymptotic amplitudes of the homogeneous solutions without numerical integration. This enables one to compute the gravitational wave flux outward at infinity and into the horizon very accurately [40–42]. Furthermore, the formalism is very powerful for the calculation of the post-Newtonian expansion of the Teukolsky equation since the series expansion is closely related to the low frequency expansion. Using the formalism, the energy flux absorbed into the horizon was calculated up to relative 4PN (i.e. absolute 6.5PN) order for a particle in circular and equatorial orbit around a Kerr black hole in Ref. [43]. Gravitational wave flux to infinity was also computed up to 2.5PN order for slightly eccentric and inclined orbit around a Kerr black hole in Refs. [44,45]. Although the Mano, Suzuki, Takasugi formalism can be

<sup>1</sup>Beware that  $\omega$  is not the orbital frequency used in standard post-Newtonian approximation (e.g. Ref. [30]). Here, following test-particle literature, we have  $\phi = \Omega$ .

applied to the case of a Kerr black hole, we assume that  $q = 0$  since we consider the case of the Schwarzschild black hole in the present paper. For more details of the formalism, we refer the reader to the recent review Ref. [24].

#### A. Ingoing-wave solution of the radial Teukolsky equation

A homogeneous solution of the Teukolsky equation which is expressed as a series of Coulomb wave functions  $R_C^{\nu}$  is given by

$$R_C^{\nu} = z \left(1 - \frac{\epsilon}{z}\right)^{2-i\epsilon} f_{\nu}(z), \quad (3.1)$$

where the function  $f_{\nu}(z)$  is expressed in a series of Coulomb wave functions as

$$f_{\nu}(z) = \sum_{n=-\infty}^{\infty} (-i)^n \frac{(\nu - 1 - i\epsilon)_n}{(\nu + 3 + i\epsilon)_n} a_n^{\nu} F_{n+\nu}(2i - \epsilon, z), \quad (3.2)$$

with  $z = \omega r$ , our notation  $(a)_n = \Gamma(a+n)/\Gamma(a)$ , and where  $F_N(\eta, z)$  is a Coulomb wave function defined by

$$F_N(\eta, z) = e^{-iz} 2^N z^{N+1} \frac{\Gamma(N+1-i\eta)}{\Gamma(2N+2)} \times \Phi(N+1-i\eta, 2N+2; 2iz). \quad (3.3)$$

Here  $\Phi(\alpha, \beta; z)$  is the confluent hypergeometric function, which is regular at  $z = 0$  (see § 13 of Ref. [46]).

The expansion coefficients  $a_n^{\nu}$  satisfy the three-term recurrence relation

$$\alpha_n^{\nu} a_{n+1}^{\nu} + \beta_n^{\nu} a_n^{\nu} + \gamma_n^{\nu} a_{n-1}^{\nu} = 0, \quad (3.4)$$

where

$$\alpha_n^{\nu} = \frac{i\epsilon(n+\nu-1+i\epsilon)(n+\nu-1-i\epsilon)(n+\nu+1+i\epsilon)}{(n+\nu+1)(2n+2\nu+3)}, \quad (3.5a)$$

$$\beta_n^{\nu} = -\ell(\ell+1) + (n+\nu)(n+\nu+1) + 2\epsilon^2 + \frac{\epsilon^2(4+\epsilon^2)}{(n+\nu)(n+\nu+1)}, \quad (3.5b)$$

$$\gamma_n^{\nu} = -\frac{i\epsilon(n+\nu+2+i\epsilon)(n+\nu+2-i\epsilon)(n+\nu-i\epsilon)}{(n+\nu)(2n+2\nu-1)}. \quad (3.5c)$$

We note that the parameter  $\nu$ , called the renormalized angular momentum, introduced in the above formulas does not exist in the Teukolsky equation. This parameter is determined so that the series converges and actually represents a solution of the Teukolsky equation.

The series converges if  $\nu$  satisfies the equation

$$R_n L_{n-1} = 1, \quad (3.6)$$

where  $R_n$  and  $L_n$  are defined in terms of continued frac-



tions (in the second lines of equations below) as

$$\begin{aligned} R_n &\equiv \frac{a_n^\nu}{a_{n-1}^\nu} = -\frac{\gamma_n^\nu}{\beta_n^\nu + \alpha_n^\nu R_{n+1}}, \\ &= -\frac{\gamma_n^\nu}{\beta_n^{\nu-}} \frac{\alpha_n^\nu \gamma_{n+1}^\nu}{\beta_{n+1}^{\nu-}} \frac{\alpha_{n+1}^\nu \gamma_{n+2}^\nu}{\beta_{n+2}^{\nu-}} \dots, \end{aligned} \quad (3.7)$$

$$\begin{aligned} L_n &\equiv \frac{a_n^\nu}{a_{n+1}^\nu} = -\frac{\alpha_n^\nu}{\beta_n^\nu + \gamma_n^\nu L_{n-1}}, \\ &= -\frac{\alpha_n^\nu}{\beta_n^{\nu-}} \frac{\alpha_{n-1}^\nu \gamma_n^\nu}{\beta_{n-1}^{\nu-}} \frac{\alpha_{n-2}^\nu \gamma_{n-1}^\nu}{\beta_{n-2}^{\nu-}} \dots. \end{aligned} \quad (3.8)$$

We can obtain two kinds of the expansion coefficients  $a_n$  by the continued fractions  $R_n$  and  $L_n$ . If we choose  $\nu$  such that it satisfies Eq. (3.6), for a certain  $n$ , the two types of the expansion coefficients coincide, and the series of Coulomb wave function Eq. (3.2) converges for  $r > r_+$ . From Eq. (3.5), we can show  $\alpha_{-n}^{\nu-1} = \gamma_n^\nu$  and  $\beta_{-n}^{\nu-1} = \beta_n^\nu$ . Accordingly, we find that  $a_{-n}^{\nu-1}$  satisfies the same recurrence relation Eq. (3.4), and  $R_C^{\nu-1}$  is also a homogeneous solution of the Teukolsky equation that converges for  $r > r_+$ .

Matching the solution in series of Coulomb wave functions, which converges at infinity, with the one in series of hypergeometric functions, which converges at the horizon, we can obtain the ingoing-wave solution  $R_{lm\omega}^{\text{in}}$ , which converges in the entire region as

$$R_{lm\omega}^{\text{in}} = K_\nu R_C^\nu + K_{-\nu-1} R_C^{\nu-1}, \quad (3.9)$$

where

$$\begin{aligned} K_\nu &= \frac{e^{i\epsilon}(2\epsilon)^{-2-\nu-N} 2^2 i^N \Gamma(3-2i\epsilon) \Gamma(N+2\nu+2)}{\Gamma(N+\nu+3+i\epsilon) \Gamma(N+\nu+1+i\epsilon) \Gamma(N+\nu-1+i\epsilon)} \\ &\times \left( \sum_{n=N}^{\infty} (-1)^n \frac{\Gamma(n+N+2\nu+1)}{(n-N)!} \right) \\ &\times \frac{\Gamma(n+\nu-1+i\epsilon) \Gamma(n+\nu+1+i\epsilon)}{\Gamma(n+\nu+3-i\epsilon) \Gamma(n+\nu+1-i\epsilon)} a_n^\nu \\ &\times \left( \sum_{n=-\infty}^N \frac{(-1)^n}{(N-n)!(N+2\nu+2)_n} \right) \\ &\times \frac{(\nu-1-i\epsilon)_n}{(\nu+3+i\epsilon)_n} a_n^\nu)^{-1}, \end{aligned} \quad (3.10)$$

and  $N$  can be any integer. The factor  $K_\nu$  is a constant which is introduced to match the solutions in the overlap region of convergence. It should be independent of the choice of  $N$ .

Comparing  $R_{lm\omega}^{\text{in}}$  in Eq. (2.8) with Eq. (3.9) in the limit of  $r^* \rightarrow \pm\infty$ , we can obtain analytic expressions for the asymptotic amplitudes  $B_{lm\omega}^{\text{trans}}$ ,  $B_{lm\omega}^{\text{inc}}$ , and  $B_{lm\omega}^{\text{ref}}$  defined in Eq. (2.8) as

$$B_{lm\omega}^{\text{trans}} = \left(\frac{\epsilon}{\omega}\right)^{-4} \sum_{n=-\infty}^{\infty} a_n^\nu, \quad (3.11a)$$

$$\begin{aligned} B_{lm\omega}^{\text{inc}} &= \omega^{-1} \left[ K_\nu - ie^{-i\pi\nu} \frac{\sin\pi(\nu+i\epsilon)}{\sin\pi(\nu-i\epsilon)} K_{-\nu-1} \right] \\ &\times A_+^\nu e^{-i\epsilon \ln \epsilon}, \end{aligned} \quad (3.11b)$$

$$B_{lm\omega}^{\text{ref}} = \omega^3 [K_\nu + ie^{i\pi\nu} K_{-\nu-1}] A_+^\nu e^{i\epsilon \ln \epsilon}, \quad (3.11c)$$

where

$$\begin{aligned} A_+^\nu &= 2^{-3-i\epsilon} e^{-((\pi\epsilon)/2)} e^{(\pi/2)i(\nu+3)} \frac{\Gamma(\nu+3+i\epsilon)}{\Gamma(\nu-1-i\epsilon)} \\ &\times \sum_{n=-\infty}^{+\infty} a_n^\nu, \end{aligned} \quad (3.12a)$$

$$\begin{aligned} A_-^\nu &= 2^{1+i\epsilon} e^{-((\pi\epsilon)/2)} e^{-(\pi/2)i(\nu-1)} \\ &\times \sum_{n=-\infty}^{+\infty} (-1)^n \frac{(\nu-1-i\epsilon)_n}{(\nu+3+i\epsilon)_n} a_n^\nu. \end{aligned} \quad (3.12b)$$

## B. Low frequency expansions of solutions

In this section, we show the relation of Mano, Suzuki, and Takasugi formalism with the post-Newtonian expansion. In their formalism, we first solve Eq. (3.6) to determine  $\nu$ . Next, we derive the expansion coefficients  $a_n^\nu$  using the continued fractions Eq. (3.7) for  $n > 0$  and Eq. (3.8) for  $n < 0$  with the condition  $a_0^\nu = a_0^{\nu-1} = 1$ . Then we can derive the ingoing-wave solution of the radial Teukolsky equation Eq. (3.9) and the asymptotic amplitudes Eq. (3.11).

When we determine  $\nu$  in the practical calculation, we solve the alternative equation which is equivalent to Eq. (3.6) for  $n = 1$

$$\beta_0^\nu + \alpha_0^\nu R_1 + \gamma_0^\nu L_{-1} = 0, \quad (3.13)$$

where  $R_1$  and  $L_{-1}$  are given by the continued fractions Eqs. (3.7) and (3.8) respectively.

In the limit of low frequency, the orders of  $\alpha_n^\nu$ ,  $\gamma_n^\nu$ , and  $\beta_n^\nu$  in  $\epsilon$  are  $O(\epsilon)$ ,  $O(\epsilon)$ , and  $O(1)$ , respectively, except for certain values of  $n < 0$  (see Refs. [24,38]). However, it is straightforward to derive the low frequency expansion of  $\nu$  by solving Eq. (3.13) order by order in  $\epsilon$ . The low frequency expansion of  $\nu$  up to  $O(\epsilon^2)$  is given by [38].

$$\nu = \ell + \nu^{(2)}(\ell)\epsilon^2 + O(\epsilon^3), \quad (3.14a)$$

$$\begin{aligned} \nu^{(2)}(\ell) &\equiv \frac{1}{2\ell+1} \left[ -2 - \frac{4}{\ell(\ell+1)} \right. \\ &+ \frac{[(\ell+1)^2-4]^2}{(2\ell+1)(2\ell+2)(2\ell+3)} \\ &\left. - \frac{(\ell^2-4)^2}{(2\ell-1)2\ell(2\ell+1)} \right]. \end{aligned} \quad (3.14b)$$

We note that the above expression of  $\nu$  up to  $O(\epsilon^2)$  is independent of  $m$ .

Combining Eq. (3.14) with Eq. (3.7) for  $n > 0$  and Eq. (3.8) for  $n < 0$ , we can derive the expansion coefficients  $a_n^\nu$  up to  $O(\epsilon^2)$  (which is valid for  $l \geq \frac{3}{2}$ ) as [38]

$$a_1^\nu = i \frac{(\ell+3)^2}{2(\ell+1)(2\ell+1)} \epsilon + \frac{(\ell+3)^2}{2(\ell+1)^2(2\ell+1)} \epsilon^2 + O(\epsilon^3), \quad (3.15a)$$

$$a_2^\nu = -\frac{(\ell+3)^2(\ell+4)^2}{4(\ell+1)(2\ell+1)(2\ell+3)^2} \epsilon^2 + O(\epsilon^3), \quad (3.15b)$$

$$a_{-1}^\nu = i \frac{(\ell-2)^2}{2\ell(2\ell+1)} \epsilon - \frac{(\ell-2)^2}{2\ell^2(2\ell+1)} \epsilon^2 + O(\epsilon^3), \quad (3.15c)$$

$$a_{-2}^\nu = -\frac{(\ell-3)^2(\ell-2)^2}{4\ell(2\ell+1)(2\ell-1)^2} \epsilon^2 + O(\epsilon^3). \quad (3.15d)$$

As one can find from Eq. (3.15), the leading order of  $a_n^\nu$  in  $\epsilon$  increases with  $|n|$  for  $|n| \leq 2$  because  $R_{|n|} \sim O(\epsilon)$  and  $L_{-|n|} \sim O(\epsilon)$ . Basically, this property of  $a_n^\nu$  holds for  $|n| \geq 2$  although we have to be careful for  $n < 0$  [24,38]. Thus, we can derive the low frequency expansion of the asymptotic amplitudes Eq. (3.11) using that of  $a_n^\nu$ . Moreover, the Coulomb wave function, defined in Eq. (3.3), is a function of  $z$ . Since each term of the expansion depends on  $\epsilon \sim O(v^3)$  and  $z \sim O(v)$ , the ingoing-wave solution  $R_{lm\omega}^{\text{in}}$  in Eq. (3.9) is very useful for the post-Newtonian expansion.

### C. 2.5PN formulas for $Z_{\ell m \omega}$

In this section, we derive 2.5PN formulas for  $Z_{\ell m \omega}$  Eq. (2.11) using  $O(\epsilon^2)$  results in Sec. III B. In the computation of  $Z_{\ell m \omega}$  Eq. (2.11), we need to estimate the homogeneous Teukolsky solution  $R_{lm\omega}^{\text{in}}$  in Eq. (3.9) and asymptotic amplitude  $B_{lm\omega}^{\text{inc}}$  in Eq. (3.11). In  $O(\epsilon^2)$  calculation, we can neglect  $K_{-\nu-1}$  terms in Eqs. (3.9) and (3.11) since  $K_{-\nu-1}/K_\nu = O(\epsilon^{2\ell})$  when we restrict  $\ell \geq 3/2$  for the Schwarzschild case [38]. Then we can approximate them as  $R_{lm\omega}^{\text{in}} = K_\nu R_C^\nu$  and  $B_{lm\omega}^{\text{inc}} = K_\nu A_+^\nu e^{-i\epsilon \ln \epsilon / \omega}$  in  $O(\epsilon^2)$  calculation. However, we note that one cannot derive 3PN formulas but 2.5PN formulas for  $Z_{\ell m \omega}$  using  $O(\epsilon^2)$  results, Eqs. (3.14) and (3.15), in Sec. III B. This is because the Coulomb wave function Eq. (3.3), which is needed to compute  $R_C^\nu$  in Eq. (3.1), is proportional to  $z^{N+1} = O(v^{N+1})$ . Thus, the post-Newtonian order of a series of Coulomb wave functions for  $n < 0$  grows slower than that for  $n > 0$ . One can estimate the post-Newtonian order of  $a_n^\nu F_{n+\nu}(2i - \epsilon, z)$ , which appears in the homogeneous Teukolsky solution in terms of a series of Coulomb wave functions  $R_C^\nu$  Eq. (3.1), as

$$\frac{a_1^\nu F_{1+\nu}(2i - \epsilon, z)}{a_0^\nu F_\nu(2i - \epsilon, z)} = O(v^4), \quad (3.16a)$$

$$\frac{a_2^\nu F_{2+\nu}(2i - \epsilon, z)}{a_0^\nu F_\nu(2i - \epsilon, z)} = O(v^8), \quad (3.16b)$$

$$\frac{a_{-1}^\nu F_{-1+\nu}(2i - \epsilon, z)}{a_0^\nu F_\nu(2i - \epsilon, z)} = O(v^2), \quad (3.16c)$$

$$\frac{a_{-2}^\nu F_{-2+\nu}(2i - \epsilon, z)}{a_0^\nu F_\nu(2i - \epsilon, z)} = O(v^4). \quad (3.16d)$$

Then one finds that summation over  $n$  from  $-2$  to  $2$  in Eq. (3.2) produces 2.5PN results.

From the expression of asymptotic amplitude  $B_{lm\omega}^{\text{inc}}$  in Eq. (3.11), one finds  $Z_{\ell m \omega} \propto e^{i\epsilon \ln 2\epsilon} e^{-i\pi\nu^{(2)}(\ell)\epsilon^2/2} e^{\pi\epsilon/2}$ , where  $\nu^{(2)}(\ell)$  is defined in Eq. (3.14). If we factor out the phase that arises from the integration in Eq. (2.9) in addition to phase from the asymptotic amplitude  $B_{lm\omega}^{\text{inc}}$ , we can derive  $Z_{\ell m \omega}$  up to 2.5PN as

$$Z_{\ell m \omega} = e^{i\epsilon \ln 2\epsilon} e^{-i\pi\nu^{(2)}(\ell)\epsilon^2/2} e^{(\pi\epsilon/2)} Z_{\ell m \omega}^{(0)} [1 + Z_{\ell m}^{(2)} v^2 + im Z_{\ell m}^{(3)} v^3 + Z_{\ell m}^{(4)} v^4 + im Z_{\ell m}^{(2)} Z_{\ell m}^{(3)} v^5 + O(v^6)], \quad (3.17a)$$

$$= e^{-im\psi_{\ell m}^{(3\text{PN})}} e^{((\pi\epsilon)/2)} Z_{\ell m \omega}^{(0)} [1 + Z_{\ell m}^{(2)} v^2 + Z_{\ell m}^{(4)} v^4 + O(v^6)], \quad (3.17b)$$

$$= e^{-im\psi_{\ell m}^{(3\text{PN})}} Z_{\ell m \omega}^{(0)} [1 + Z_{\ell m}^{(2)} v^2 + m\pi v^3 + Z_{\ell m}^{(4)} v^4 + m\pi Z_{\ell m}^{(2)} v^5 + O(v^6)], \quad (3.17c)$$

where  $Z_{\ell m \omega}^{(0)}$ ,  $Z_{\ell m}^{(2)}$ ,  $Z_{\ell m}^{(3)}$ , and  $Z_{\ell m}^{(4)}$  are real (see Secs. III C 1 and III C 2), and

$$\psi_{\ell m}^{(3\text{PN})} = -2 \ln(4mv^3) v^3 - Z_{\ell m}^{(3)} v^3 + 2m\pi\nu^{(2)}(\ell) v^6, \quad (3.18a)$$

$$= 2v^3 \left( \Psi^{(0)}(\ell) - \frac{1}{\ell} + \frac{1}{2} + \frac{2}{\ell+1} - \ln(4mv^3) - 3 \frac{1 + (-1)^{\ell+m}}{(\ell-1)\ell(\ell+1)(\ell+2)} \right) + 2m\pi\nu^{(2)}(\ell) v^6, \quad (3.18b)$$

where  $\Psi^{(n)}(z)$  is the polygamma function and  $\nu^{(2)}(\ell)$  calculated from Eq. (3.14b).  $\Psi^{(0)}(\ell)$  is related to the digamma function whose explicit value can be calculated using

$$\Psi^{(0)}(\ell) = \sum_{k=1}^{\ell-1} \frac{1}{k} - \gamma, \quad (3.19)$$

where  $\gamma$  is the Euler constant. To go from Eq. (3.17a) to Eq. (3.17b), we move the imaginary terms  $imZ_{\ell m}^{(3)}v^3$  from the amplitude of the post-Newtonian expansion to the phase. The  $e^{\pi\epsilon/2}$  in Eq. (3.17b) is the motivation for the chosen dependence in tail terms in the factorized resummed waveforms in Ref [15] (See Eq. (5.6) in Sec. V). With such overall factorization the remaining series in Eq. (3.17b) is expected to achieve improved convergence to numerical results since coefficients in this series have smaller post-Newtonian coefficients. To compare to spherical harmonic modes in literature, we expand  $e^{\pi\epsilon/2}$  in Eq. (3.17b) and obtain the alternative form Eq. (3.17c). Finally, to go from Eq. (3.18a) to Eq. (3.18b), we have used the general formula of  $Z_{\ell m}^{(3)}$  given in Secs. III C 1 and III C 2. Though this is a 2.5PN calculation, one may notice that we have included the 3PN phase term in Eq. (3.18) derived from the asymptotic amplitude  $B_{lm\omega}^{\text{inc}}$  for completeness. This is sufficient because no other 3PN phase terms were generated from the integration in Eq. (2.9) in our 5.5PN calculation in Secs. IV, V, and VI. Thus, in the case of Schwarzschild black hole, there may not exist any further 3PN phase terms arising from the integration in Eq. (2.9).

Since the leading order of  $Z_{\ell m\omega}$  depends on whether  ${}_0b_{\ell m} \sim O(1)$ , i.e.  ${}_0Y_{\ell m}(\frac{\pi}{2}, 0)$ , vanishes or not [47], we treat the case for  $\ell + m = \text{even}$  in Sec. III C 1 and  $\ell + m = \text{odd}$  in Sec. III C 2. Also observe that the 1PN term of  $Z_{\ell m\omega}$ , i.e.  $Z_{\ell m}^{(2)}$ , contains a linear term of  $\ell$ , which reinforces the suggestion in Ref. [15] to introduce the  $\ell$ th root of the amplitude for factorized resummation (See Sec. V and VA).

### 1. $\ell + m = \text{even case}$

$$Z_{\ell m}^{(0)} = -\frac{\mu m^{\ell+2} \pi^{3/2}}{i^\ell 2^{\ell-1} \Gamma(\ell + 3/2)} \sqrt{\frac{(\ell + 2)(\ell + 1)}{\ell(\ell - 1)}} \times {}_0Y_{\ell m}\left(\frac{\pi}{2}, 0\right) \frac{v^{\ell+2}}{(r_0/M)^2}, \quad (3.20a)$$

$$Z_{\ell m}^{(2)} = -\ell + \frac{1}{2} - \frac{m^2(\ell + 9)}{2(2\ell + 3)(\ell + 1)}, \quad (3.20b)$$

$$Z_{\ell m}^{(3)} = -2\left(\Psi^{(0)}(\ell) - \frac{1}{\ell} + \frac{1}{2} + \frac{2}{\ell + 1} - \frac{6}{(\ell - 1)\ell(\ell + 1)(\ell + 2)}\right), \quad (3.20c)$$

$$Z_{\ell m}^{(4)} = \frac{\ell^2}{2} - \frac{5\ell}{4} + 2 + \frac{17\ell - 1}{8(2\ell - 1)(\ell - 1)} + m^2\left(\frac{1}{4} - \frac{2(3\ell^3 + 6\ell^2 - \ell + 4)}{(\ell + 1)^2(\ell - 1)(\ell + 2)\ell}\right) + \frac{m^4(\ell^2 + 19\ell + 50)}{8(2\ell + 5)(2\ell + 3)(\ell + 2)(\ell + 1)}. \quad (3.20d)$$

### 2. $\ell + m = \text{odd case}$

$$Z_{\ell m}^{(0)} = -\frac{\mu m^{\ell+2} \pi^{3/2}}{i^{\ell+1} 2^{\ell-2} \Gamma(\ell + 3/2)} \sqrt{\frac{\ell + 2}{\ell - 1}} {}_1Y_{\ell m}\left(\frac{\pi}{2}, 0\right) \times \frac{v^{\ell+3}}{(r_0/M)^2}, \quad (3.21a)$$

$$Z_{\ell m}^{(2)} = -\ell + \frac{1}{2} + \frac{2}{\ell} - \frac{m^2(\ell + 4)}{2(\ell + 2)(2\ell + 3)}, \quad (3.21b)$$

$$Z_{\ell m}^{(3)} = -2\left(\Psi^{(0)}(\ell) - \frac{1}{\ell} + \frac{1}{2} + \frac{2}{\ell + 1}\right), \quad (3.21c)$$

$$Z_{\ell m}^{(4)} = \frac{\ell^2}{2} - \frac{5\ell}{4} - \frac{1}{2} + \frac{29\ell + 8}{8\ell(2\ell - 1)} + m^2\left(\frac{1}{4} - \frac{6\ell^3 + 33\ell^2 + 55\ell + 36}{2(\ell + 1)\ell(\ell + 2)(2\ell + 3)}\right) + \frac{m^4(\ell + 6)}{8(2\ell + 5)(2\ell + 3)(\ell + 2)}. \quad (3.21d)$$

Using both the 2.5PN formulas for  $Z_{\ell m\omega}$  in this section and Eq. (2.15), one has a general formula for computation of 5.5PN waveforms for  $\ell \geq 8$ . However, for completeness, we list those modes in Appendix A.

We conclude this rather technical section by recapitulating the main steps in the formalism developed by Mano, Suzuki, and Takasugi, to analytically compute homogeneous solutions of the Teukolsky equation. The most important task in the formalism is to determine the renormalized angular momentum  $\nu$ , which is introduced so that the series of two types of special function, hypergeometric functions and Coulomb wave functions, converge.  $\nu$  is determined by solving the continued fraction equation Eq. (3.13). We then compute the expansion coefficients  $a_n^\nu$  using Eq. (3.7) for  $n > 0$  and Eq. (3.8) for  $n < 0$  with the condition  $a_0^\nu = a_0^{-\nu-1} = 1$ . The asymptotic amplitude of the homogeneous solution  $B_{lm\omega}^{\text{inc}}$  is subsequently calculated and the homogeneous solution  $R_{lm\omega}^{\text{in}}$  constructed using Eqs. (3.11) and (3.9) respectively. Finally, we compute  $Z_{\ell m\omega}$  using Eq. (2.11), which enables one to compute gravitational wave flux to infinity and gravitational waveforms by Eqs. (2.14) and (2.15) respectively.

In the coming sections, Secs. IV, V, and VI, we derive the 5.5PN waveforms computing  $Z_{\ell m\omega}$  following the above steps. These  $Z_{\ell m\omega}$  not only lead to the 5.5PN energy flux obtained in Ref. [21] as required but also contain new terms which are needed for the calculation of 5.5PN waveforms.

## IV. SPHERICAL HARMONIC MODES

In this section, we project the waveforms onto spin-weighted spherical harmonics and compute  $h_{\ell m}$  up to  $O(v^{11})$  which are useful for the comparison between the post-Newtonian and numerical results. For the comparisons between the post-Newtonian expansion and numerical

simulations, we decompose  $h_+$  and  $h_\times$  into the modes of spin-weighted spherical harmonics as

$$h_+ - ih_\times = \sum_{\ell, m} h_{\ell m - 2} Y_{\ell m}(\Theta, \Phi), \quad (4.1)$$

where  $(\Theta, \Phi)$  are the angles defining the direction of propagation of gravitational waves. Using the orthonormality condition of spin-weighted spherical harmonics,  $h_{\ell m}$  can be derived as

$$h_{\ell m} = \int \sin\Theta d\Theta d\Phi (h_+ - ih_\times)_{-2} \bar{Y}_{\ell m}(\Theta, \Phi). \quad (4.2)$$

Recall that the polarizations in Eq. (2.15) are the functions of both the orbital phase  $\Omega t$  and the angles  $(\theta, \varphi)$  defining the observer relative to the source. Therefore, to obtain the polarizations corresponding to the direction of propagation of gravitational waves  $(\Theta, \Phi)$ , in Eq. (4.2) we have to replace  $(\theta, \Omega t + \varphi)$  in  $h_+$  and  $h_\times$  by  $(\Theta, \Omega t + \varphi - \Phi)$  [30]. (In Ref. [30],  $(\theta, \varphi)$  is defined as  $(i, \pi/2)$ .) Then we obtain  $h_{\ell m}$  as

$$h_{\ell m} = -\frac{2}{r} \sum_{\ell', m'} \frac{Z_{\ell' m' \omega'} e^{im' \Omega t'}}{(m' \Omega)^2} \int \sin\Theta d\Theta d\Phi e^{-im'(\Omega t - \Phi)} \times {}_{-2}Y_{\ell' m'}(\Theta, \varphi) {}_{-2}\bar{Y}_{\ell m}(\Theta, \Phi), \quad (4.3a)$$

$$= -\frac{2}{r} \frac{Z_{\ell m \omega} e^{im\Omega(r^* - t)} e^{im\varphi}}{(m\Omega)^2}, \quad (4.3b)$$

where  $Z_{\ell m \omega}$  is given in Sec. II. To go from Eq. (4.3a) to

Eq. (4.3b), we have used the orthonormality condition of spin-weighted spherical harmonics.

Following in spirit but generalizing suitably the notation defined in Ref. [30] we write<sup>2</sup>

$$h_{\ell m} = -\frac{2\mu v^2}{r} H_{\ell m}, \quad (4.4a)$$

$$H_{\ell m} = \sqrt{\frac{16\pi}{5}} \hat{H}_{\ell m} e^{-im\psi_{\ell m}}. \quad (4.4b)$$

Note that the phase in Eq. (4.4b) is more general in that it is multipole—i.e.  $(\ell, m)$ —dependent, while the phase in Ref. [30] is independent of  $(\ell, m)$  equivalent to the 1.5PN-accurate  $\psi_{2,2}^{1.5\text{PN}}$  of this section.

Using the 3PN phase of  $Z_{\ell m \omega}$  given in Eq. (3.18) for any multipolar order  $\ell$  and  $m$  as

$$\psi_{\ell m}^{(3\text{PN})} = 2v^3 \left( \Psi^{(0)}(\ell) - \frac{1}{\ell} + \frac{1}{2} + \frac{2}{\ell+1} - \ln(4mv^3) - 3 \frac{1 + (-1)^{\ell+m}}{(\ell-1)\ell(\ell+1)(\ell+2)} \right) + 2m\pi v^{(2)}(\ell) v^6, \quad (4.5)$$

where  $v^{(2)}(\ell)$  is given in Eq. (3.14), the phase of the waveforms up to 3PN is given by

$$\psi_{\ell m} = \Omega(t - r^*) - \varphi + \psi_{\ell m}^{(3\text{PN})}, \quad (4.6)$$

Using Eq. (4.6), we derive  $H_{2,2}$  to be

$$H_{2,2} = \sqrt{\frac{16\pi}{5}} \exp[-2i\psi_{2,2}^{(3\text{PN})}] \left[ 1 - \frac{107}{42} v^2 + 2\pi v^3 - \frac{2173}{1512} v^4 - \frac{107}{21} \pi v^5 + v^6 \left\{ -\frac{856}{105} \text{eulerlog}(2, v) + \frac{27027409}{646800} + \frac{2}{3} \pi^2 \right\} - \frac{2173}{756} \pi v^7 + v^8 \left\{ -\frac{846557506853}{12713500800} - \frac{107}{63} \pi^2 + \frac{45796}{2205} \text{eulerlog}(2, v) \right\} + v^9 \left\{ \frac{27027409}{323400} \pi - \frac{4}{3} \pi^3 - \frac{1712}{105} \pi \text{eulerlog}(2, v) + i \left( \frac{1712}{315} \pi^2 - \frac{64}{3} \zeta(3) - \frac{259}{81} \right) \right\} + v^{10} \left\{ -\frac{866305477369}{9153720576} - \frac{2173}{2268} \pi^2 + \frac{232511}{19845} \text{eulerlog}(2, v) \right\} + v^{11} \left\{ -\frac{846557506853}{6356750400} \pi + \frac{214}{63} \pi^3 + \frac{91592}{2205} \pi \text{eulerlog}(2, v) + i \left( \frac{3424}{63} \zeta(3) + \frac{3959}{486} - \frac{91592}{6615} \pi^2 \right) \right\} \right] \quad (4.7a)$$

$$= \sqrt{\frac{16\pi}{5}} \exp \left[ -2i \left\{ \psi_{2,2}^{(3\text{PN})} + v^9 \left( \frac{32}{3} \zeta(3) - \frac{856}{315} \pi^2 + \frac{259}{162} \right) \right\} \right] \left[ 1 - \frac{107}{42} v^2 + 2\pi v^3 - \frac{2173}{1512} v^4 - \frac{107}{21} \pi v^5 + v^6 \left( -\frac{856}{105} \text{eulerlog}(2, v) + \frac{27027409}{646800} + \frac{2}{3} \pi^2 \right) - \frac{2173}{756} \pi v^7 + v^8 \left( -\frac{846557506853}{12713500800} - \frac{107}{63} \pi^2 + \frac{45796}{2205} \text{eulerlog}(2, v) \right) + v^9 \left( \frac{27027409}{323400} \pi - \frac{4}{3} \pi^3 - \frac{1712}{105} \pi \text{eulerlog}(2, v) \right) + v^{10} \left( -\frac{866305477369}{9153720576} - \frac{2173}{2268} \pi^2 + \frac{232511}{19845} \text{eulerlog}(2, v) \right) + v^{11} \left( -\frac{846557506853}{6356750400} \pi + \frac{214}{63} \pi^3 + \frac{91592}{2205} \pi \text{eulerlog}(2, v) \right) \right], \quad (4.7b)$$

where  $\text{eulerlog}(2, v) \equiv \gamma + \ln(4v)$ .

<sup>2</sup>Recall that there is an overall difference of sign in  $h_{\ell m}$  between Refs. [29,30] due to a different choice of the polarization triad [29]. The sign of  $h_{\ell m}$  in Eq. (4.3b) matches with that in Ref. [29] and is consistent with Eq. (4.4a). In the test-particle limit,  $m_1 = \mu$ ,  $m_2 = M$ ,  $\Delta = (m_1 - m_2)/(m_1 + m_2)$  in Ref. [30] reduces to  $-1$ . For comparison between black hole perturbation theory and standard post-Newtonian theory, we have to substitute  $\ln x_0 = 17/18 - 2\ln 2 - 2\gamma/3$  into the phase in Ref. [30] in order to match the phase up to 1.5PN since Schwarzschild coordinates are used in the test-particle limit and harmonic coordinates in the generic mass ratio case as pointed out in Ref. [29]. Lastly, unlike in PN works beyond 2.5PN where  $\dot{\Omega}$  due to the radiation reaction must be included [30,48], here in the test-particle case such terms are absent since they are higher order in mass ratio  $\mu/M$ .



Note from Eq. (4.7a) that even after factoring out the 3PN-accurate phase  $\psi_{2,2}^{(3\text{PN})}$ , the remaining part of  $H_{2,2}$  is still complex and not real. However, with a more accurate choice of  $\psi_{2,2}$  involving a  $O(v^9)$  term all the imaginary terms (including that at  $O(v^{11})$ ) in the rest of  $H_{2,2}$  can be absorbed into this phase as can be seen in Eq. (4.7b). Thus, with this improved phase, the remaining Taylor expansion of  $H_{2,2}$  becomes real. Thus, it is useful to introduce  $O(v^9)$  correction to the phase for  $(\ell, m) = (2, 2)$  mode. From the 2.5PN formulas for  $Z_{\ell m \omega}$  in Sec. III C, we find that the leading order of  $\hat{H}_{\ell m}$  is  $O(v^{\ell-2})$  for  $\ell + m$  is even and  $O(v^{\ell-1})$  for  $\ell + m$  is odd. Thus, we find that it is also useful to introduce  $O(v^9)$  correction to the phase for  $\ell = 2, 3$ ,  $(\ell, m) = (4, 4)$ , and  $(\ell, m) = (4, 2)$  modes in the computation of 5.5PN waveforms. The treatment to go from Eq. (4.7a) to Eq. (4.7b) is quite general, and the phase  $\psi_{\ell m}$  for  $2 \leq \ell \leq 4$  up to  $O(v^9)$  can be similarly derived. We thus have

$$\psi_{2,2} = \Omega(t - r^*) - \varphi + \left(\frac{17}{6} - 2\gamma - 2\ln(8v^3)\right)v^3 - \frac{214}{105}\pi v^6 + \left(\frac{32}{3}\zeta(3) - \frac{856}{315}\pi^2 + \frac{259}{162}\right)v^9, \quad (4.8a)$$

$$\psi_{2,1} = \Omega(t - r^*) - \varphi + \left(\frac{10}{3} - 2\gamma - 2\ln(4v^3)\right)v^3 - \frac{107}{105}\pi v^6 + \left(-\frac{214}{315}\pi^2 + \frac{8}{3}\zeta(3) + \frac{29}{81}\right)v^9, \quad (4.8b)$$

$$\psi_{3,3} = \Omega(t - r^*) - \varphi + \left(\frac{127}{30} - 2\gamma - 2\ln(12v^3)\right)v^3 - \frac{13}{7}\pi v^6 + \left(-\frac{26}{7}\pi^2 + 24\zeta(3) - \frac{23173}{9000}\right)v^9, \quad (4.8c)$$

$$\psi_{3,2} = \Omega(t - r^*) - \varphi + \left(\frac{13}{3} - 2\gamma - 2\ln(8v^3)\right)v^3 - \frac{26}{21}\pi v^6 + \left(-\frac{464}{405} - \frac{104}{63}\pi^2 + \frac{32}{3}\zeta(3)\right)v^9, \quad (4.8d)$$

$$\psi_{3,1} = \Omega(t - r^*) - \varphi + \left(\frac{127}{30} - 2\gamma - 2\ln(4v^3)\right)v^3 - \frac{13}{21}\pi v^6 + \left(\frac{8}{3}\zeta(3) - \frac{26}{63}\pi^2 - \frac{23173}{81000}\right)v^9, \quad (4.8e)$$

$$\psi_{4,4} = \Omega(t - r^*) - \varphi + \left(\frac{74}{15} - 2\gamma - 2\ln(16v^3)\right)v^3 - \frac{6284}{3465}\pi v^6 + \left(\frac{128}{3}\zeta(3) - \frac{50272}{10395}\pi^2 - \frac{136528}{10125}\right)v^9, \quad (4.8f)$$

$$\psi_{4,3} = \Omega(t - r^*) - \varphi + \left(\frac{149}{30} - 2\gamma - 2\ln(12v^3)\right)v^3 - \frac{1571}{1155}\pi v^6, \quad (4.8g)$$

$$\psi_{4,2} = \Omega(t - r^*) - \varphi + \left(\frac{74}{15} - 2\gamma - 2\ln(8v^3)\right)v^3 - \frac{3142}{3465}\pi v^6 + \left(-\frac{12568}{10395}\pi^2 + \frac{32}{3}\zeta(3) - \frac{34132}{10125}\right)v^9, \quad (4.8h)$$

$$\psi_{4,1} = \Omega(t - r^*) - \varphi + \left(\frac{149}{30} - 2\gamma - 2\ln(4v^3)\right)v^3 - \frac{1571}{3465}\pi v^6, \quad (4.8i)$$

where  $\zeta(n)$  is the zeta function. The  $O(v^9)$  terms in the above equations are one of the new results derived in this paper while the  $O(v^6)$  terms are consistent with Ref. [20] as required. Note that we show the phase  $\psi_{\ell m}$  up to  $O(v^6)$  for  $(\ell, m) = (4, 3)$  and  $(4, 1)$  modes. These corrections to the phase  $\psi_{\ell m}$  up to  $O(v^3)$ ,  $O(v^6)$ , and  $O(v^9)$  represents phase shift in the waveforms due to the tail effects.

Using  $\psi_{\ell m}$  in the above Eq. (4.8) and  $Z_{\ell m \omega}$  which can be derived by the method in Sec. III, the amplitudes  $\hat{H}_{\ell m}$  up to  $O(v^{11})$  for  $2 \leq \ell \leq 4$  are derived as

$$\begin{aligned} \hat{H}_{2,2} = & 1 - \frac{107}{42}v^2 + 2\pi v^3 - \frac{2173}{1512}v^4 - \frac{107}{21}\pi v^5 + v^6 \left( -\frac{856}{105}\text{eulerlog}(2, v) + \frac{27\,027\,409}{646\,800} + \frac{2}{3}\pi^2 \right) - \frac{2173}{756}\pi v^7 \\ & + v^8 \left( -\frac{846\,557\,506\,853}{12\,713\,500\,800} - \frac{107}{63}\pi^2 + \frac{45\,796}{2205}\text{eulerlog}(2, v) \right) + v^9 \left( \frac{27\,027\,409}{323\,400}\pi - \frac{4}{3}\pi^3 - \frac{1712}{105}\pi\text{eulerlog}(2, v) \right) \\ & + v^{10} \left( -\frac{866\,305\,477\,369}{915\,3720\,576} - \frac{2173}{2268}\pi^2 + \frac{232\,511}{19\,845}\text{eulerlog}(2, v) \right) \\ & + v^{11} \left( -\frac{846\,557\,506\,853}{635\,6750\,400}\pi + \frac{214}{63}\pi^3 + \frac{91\,592}{2205}\pi\text{eulerlog}(2, v) \right), \end{aligned} \quad (4.9a)$$

$$\begin{aligned} \hat{H}_{2,1} = & -\frac{1}{3}i \left( v - \frac{17}{28}v^3 + \pi v^4 - \frac{43}{126}v^5 - \frac{17}{28}\pi v^6 + v^7 \left( \frac{30\,811\,367}{2\,910\,600} - \frac{214}{105}\text{eulerlog}(1, v) + \frac{1}{6}\pi^2 \right) - \frac{43}{126}\pi v^8 \right. \\ & + v^9 \left( \frac{46\,023\,611}{10\,594\,584} - \frac{17}{168}\pi^2 + \frac{1819}{1470}\text{eulerlog}(1, v) \right) + v^{10} \left( \frac{30\,811\,367}{2\,910\,600}\pi - \frac{214}{105}\pi\text{eulerlog}(1, v) - \frac{1}{6}\pi^3 \right) \\ & \left. + v^{11} \left( \frac{4601}{6615}\text{eulerlog}(1, v) + \frac{13\,504\,725\,881}{1\,191\,890\,700} - \frac{43}{756}\pi^2 \right) \right), \end{aligned} \quad (4.9b)$$

$$\begin{aligned} \hat{H}_{3,3} = & \frac{3}{56} \sqrt{210} i \left( v - 4v^3 + 3\pi v^4 + \frac{123}{110} v^5 - 12\pi v^6 + v^7 \left( \frac{3}{2} \pi^2 + \frac{109301083}{1401400} - \frac{78}{7} \text{eulerlog}(3, v) \right) + \frac{369}{110} \pi v^8 \right. \\ & + v^9 \left( \frac{312}{7} \text{eulerlog}(3, v) - \frac{84974767}{350350} - 6\pi^2 \right) + v^{10} \left( \frac{327903249}{1401400} \pi - \frac{234}{7} \pi \text{eulerlog}(3, v) - \frac{9}{2} \pi^3 \right) \\ & \left. + v^{11} \left( -\frac{96055340127}{2620618000} - \frac{4797}{385} \text{eulerlog}(3, v) + \frac{369}{220} \pi^2 \right) \right), \end{aligned} \tag{4.9c}$$

$$\begin{aligned} \hat{H}_{3,2} = & \frac{\sqrt{35}}{21} \left( v^2 - \frac{193}{90} v^4 + 2\pi v^5 - \frac{1451}{3960} v^6 - \frac{193}{45} \pi v^7 + v^8 \left( \frac{2501041027}{75675600} + \frac{2}{3} \pi^2 - \frac{104}{21} \text{eulerlog}(2, v) \right) \right. \\ & - \frac{1451}{1980} \pi v^9 + v^{10} \left( -\frac{23891243939}{495331200} - \frac{193}{135} \pi^2 + \frac{10036}{945} \text{eulerlog}(2, v) \right) \\ & \left. + v^{11} \left( \frac{2501041027}{37837800} \pi - \frac{4}{3} \pi^3 - \frac{208}{21} \pi \text{eulerlog}(2, v) \right) \right), \end{aligned} \tag{4.9d}$$

$$\begin{aligned} \hat{H}_{3,1} = & -\frac{\sqrt{14}}{168} i \left( v - \frac{8}{3} v^3 + \pi v^4 + \frac{607}{198} v^5 - \frac{8}{3} \pi v^6 + v^7 \left( \frac{305915969}{37837800} + \frac{1}{6} \pi^2 - \frac{26}{21} \text{eulerlog}(1, v) \right) + \frac{607}{198} \pi v^8 \right. \\ & + v^9 \left( -\frac{4}{9} \pi^2 - \frac{15638341}{1091475} + \frac{208}{63} \text{eulerlog}(1, v) \right) + v^{10} \left( \frac{305915969}{37837800} \pi - \frac{26}{21} \pi \text{eulerlog}(1, v) - \frac{1}{6} \pi^3 \right) \\ & \left. + v^{11} \left( \frac{3262398379463}{127362034800} - \frac{7891}{2079} \text{eulerlog}(1, v) + \frac{607}{1188} \pi^2 \right) \right), \end{aligned} \tag{4.9e}$$

$$\begin{aligned} \hat{H}_{4,4} = & -\frac{8}{63} \sqrt{35} \left( v^2 - \frac{593}{110} v^4 + 4\pi v^5 + \frac{1068671}{200200} v^6 - \frac{1186}{55} \pi v^7 + v^8 \left( \frac{302024067749}{2497294800} + \frac{8}{3} \pi^2 - \frac{50272}{3465} \text{eulerlog}(4, v) \right) \right. \\ & + \frac{1068671}{50050} \pi v^9 + v^{10} \left( -\frac{796354151819507}{1436905008000} - \frac{2372}{165} \pi^2 + \frac{14905648}{190575} \text{eulerlog}(4, v) \right) \\ & \left. + v^{11} \left( \frac{302024067749}{624323700} \pi - \frac{201088}{3465} \pi \text{eulerlog}(4, v) - \frac{32}{3} \pi^3 \right) \right), \end{aligned} \tag{4.9f}$$

$$\begin{aligned} \hat{H}_{4,3} = & \frac{9}{280} \sqrt{70} i \left( v^3 - \frac{39}{11} v^5 + 3\pi v^6 + \frac{7206}{5005} v^7 - \frac{117}{11} \pi v^8 + v^9 \left( \frac{9204203473}{138738600} + \frac{3}{2} \pi^2 - \frac{3142}{385} \text{eulerlog}(3, v) \right) \right. \\ & \left. + \frac{21618}{5005} \pi v^{10} + v^{11} \left( -\frac{1658233837937}{8648039400} - \frac{117}{22} \pi^2 + \frac{122538}{4235} \text{eulerlog}(3, v) \right) \right), \end{aligned} \tag{4.9g}$$

$$\begin{aligned} \hat{H}_{4,2} = & \frac{\sqrt{5}}{63} \left( v^2 - \frac{437}{110} v^4 + 2\pi v^5 + \frac{1038039}{200200} v^6 - \frac{437}{55} \pi v^7 + v^8 \left( \frac{67008495809}{2497294800} + \frac{2}{3} \pi^2 - \frac{12568}{3465} \text{eulerlog}(2, v) \right) \right. \\ & + \frac{1038039}{100100} \pi v^9 + v^{10} \left( -\frac{1826104347159431}{18679765104000} + \frac{2746108}{190575} \text{eulerlog}(2, v) - \frac{437}{165} \pi^2 \right) \\ & \left. + v^{11} \left( \frac{67008495809}{1248647400} \pi - \frac{25136}{3465} \pi \text{eulerlog}(2, v) - \frac{4}{3} \pi^3 \right) \right), \end{aligned} \tag{4.9h}$$

$$\begin{aligned} \hat{H}_{4,1} = & -\frac{\sqrt{10}}{840} i \left( v^3 - \frac{101}{33} v^5 + \pi v^6 + \frac{42982}{15015} v^7 - \frac{101}{33} \pi v^8 + v^9 \left( \frac{9092103793}{1248647400} - \frac{3142}{3465} \text{eulerlog}(1, v) + \frac{1}{6} \pi^2 \right) \right. \\ & \left. + \frac{42982}{15015} \pi v^{10} + v^{11} \left( \frac{317342}{114345} \text{eulerlog}(1, v) - \frac{11770664091577}{700491191400} - \frac{101}{198} \pi^2 \right) \right), \end{aligned} \tag{4.9i}$$

$$\text{where } \text{eulerlog}(m, \nu) = \gamma + \ln(2m\nu). \quad (4.10)$$

The results in this paper are consistent with 3PN results in the test-particle limit in Ref. [30]. The  $O(v^7)$  to  $O(v^{11})$  terms in the above equations are one of the new results derived in this paper. We also note that the spherical harmonic modes in this paper are simpler than Ref. [30] since we completely factored out the phase. We show 5.5PN expressions of  $\hat{H}_{\ell m}$  for  $5 \leq \ell \leq 13$  in Appendix A.

### 2.5PN formulas for $h_{\ell m}$

In this section, we derive 2.5PN formulas for spherical harmonic modes  $h_{\ell m}$  in the test-particle limit using 2.5PN formula of  $Z_{\ell m \omega}$  in Eq. (3.17) in Sec. III C. Once we have the 2.5PN formula of  $Z_{\ell m \omega}$  in Eq. (3.17), we can derive 2.5PN  $h_{\ell m}$  from Eq. (4.3b) as

$$h_{\ell m} = -\frac{2}{r} \frac{Z_{\ell m \omega} e^{im\Omega(r^*-t)} e^{im\phi}}{\omega^2}, \quad (4.11a)$$

$$= -\frac{2Z_{\ell m \omega}^{(0)}}{r\omega^2} e^{-im[\Omega(t-r^*)-\phi+\psi_{\ell m}^{(3PN)}]} [1 + Z_{\ell m}^{(2)} v^2 + m\pi v^3 + Z_{\ell m}^{(4)} v^4 + m\pi Z_{\ell m}^{(2)} v^5 + O(v^6)], \quad (4.11b)$$

where  $Z_{\ell m \omega}^{(0)}$ ,  $Z_{\ell m}^{(2)}$ , and  $Z_{\ell m}^{(4)}$  are given in Eq. (3.20) for  $\ell + m = \text{even}$  case and Eq. (3.21) for  $\ell + m = \text{odd}$  case. If we define  $h_{\ell m}^{(0)} \equiv -2Z_{\ell m \omega}^{(0)}/(r\omega^2)$ ,  $h_{\ell m}^{(0)}$  is given as

$$h_{\ell m}^{(0)} = \frac{2}{r} \frac{\mu m^\ell \pi^{3/2}}{i^\ell 2^{\ell-1} \Gamma(\ell + 3/2)} \sqrt{\frac{(\ell+2)(\ell+1)}{\ell(\ell-1)}} {}_0Y_{\ell m}\left(\frac{\pi}{2}, 0\right) v^\ell, \quad (\ell + m = \text{even}), \quad (4.12a)$$

$$h_{\ell m}^{(0)} = \frac{2}{r} \frac{\mu m^\ell \pi^{3/2}}{i^{\ell+1} 2^{\ell-2} \Gamma(\ell + 3/2)} \sqrt{\frac{\ell+2}{\ell-1}} {}_{-1}Y_{\ell m}\left(\frac{\pi}{2}, 0\right) v^{\ell+1}, \quad (\ell + m = \text{odd}). \quad (4.12b)$$

## V. RESUMMED WAVEFORMS

Recently, Damour, Iyer, and Nagar [15] suggested a factorized resummed waveform which improves agreement with the results of numerical simulations. They decomposed the waveforms into five factors as

$$h_{\ell m} = h_{\ell m}^{(N, \epsilon_p)} \hat{S}_{\text{eff}}^{(\epsilon_p)} T_{\ell m} e^{i\delta_{\ell m}} (\rho_{\ell m})^\ell. \quad (5.1)$$

Here,  $\epsilon_p$  denotes the parity of the multipolar waveforms. In the case of circular orbits,  $\epsilon_p = 0$  when  $\ell + m$  is even and  $\epsilon_p = 1$  when  $\ell + m$  is odd.

The first factor  $h_{\ell m}^{(N, \epsilon_p)}$  represents the Newtonian contribution to waveforms.

$$h_{\ell m}^{(N, \epsilon_p)} = \frac{GM\nu}{c^2 r} n_{\ell m}^{(\epsilon_p)} c_{\ell+\epsilon_p}(\nu) v^{\ell+\epsilon_p} Y^{\ell-\epsilon_p, -m}\left(\frac{\pi}{2}, \phi\right), \quad (5.2)$$

where  $\phi$  is the orbital phase and  $n_{\ell m}^{(\epsilon_p)}$  are

$$n_{\ell m}^{(0)} = (im)^\ell \frac{8\pi}{(2\ell+1)!!} \sqrt{\frac{(\ell+1)(\ell+2)}{\ell(\ell-1)}}, \quad (5.3a)$$

$$n_{\ell m}^{(1)} = -(im)^\ell \frac{16\pi i}{(2\ell+1)!!} \sqrt{\frac{(2\ell+1)(\ell+2)(\ell^2-m^2)}{(2\ell-1)(\ell+1)\ell(\ell-1)}}, \quad (5.3b)$$

and  $c_{\ell+\epsilon_p}(\nu)$  are functions of the symmetric mass ratio  $\nu \equiv \mu M/(M+\mu)^2$ , defined by

$$c_{\ell+\epsilon_p}(\nu) = \left(\frac{1}{2} - \frac{1}{2}\sqrt{1-4\nu}\right)^{\ell+\epsilon_p-1} + (-)^{\ell+\epsilon_p} \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-4\nu}\right)^{\ell+\epsilon_p-1}. \quad (5.4)$$

The second factor in Eq. (5.1),  $\hat{S}_{\text{eff}}^{(\epsilon_p)}$ , is motivated by the effective source term for partial waves in the perturbation formalism and its replacement by analogous quantities that characterize the effective-one-body (EOB) dynamics [15]

$$\hat{S}_{\text{eff}}^{(\epsilon_p)} = \begin{cases} \tilde{E}, & \text{for } \epsilon_p = 0, \\ \nu \tilde{L}/M, & \text{for } \epsilon_p = 1. \end{cases} \quad (5.5)$$

The third factor in Eq. (5.1),  $T_{\ell m}$ , is the resummed tail factor which resums the leading logarithms of the tail effects [14,15,49]

$$T_{\ell m} = \frac{\Gamma(\ell+1-2i\hat{k})}{\Gamma(\ell+1)} e^{\pi\hat{k}} e^{2i\hat{k}\ln(2kr_{0s})}, \quad (5.6)$$

where  $k = m\Omega$  and  $\hat{k} = Mk$ . Here we denote by  $r_{0s}$  what was denoted by  $r_0$  in [15]. As pointed out in Sec. III C,  $Z_{\ell m \omega} \propto e^{\pi\epsilon/2} e^{i\epsilon\ln 2\epsilon}$  and this motivates the idea to introduce  $e^{\pi\hat{k}}$  in addition to  $e^{2i\hat{k}\ln(2kr_{0s})}$  in the resummed tail

factor to improve the convergence of the residual PN series.

The fourth factor in Eq. (5.1),  $\delta_{\ell m}$ , is a supplementary phase of the resummed tail factor  $T_{\ell m}$ . If we decompose  $T_{\ell m}$  as  $T_{\ell m} = |T_{\ell m}| e^{i\tau_{\ell m}}$ , the phase of  $T_{\ell m}$ , i.e.  $\tau_{\ell m}$ , can be derived by the post-Newtonian expansion of  $T_{\ell m}$  up to the required order. Using Eqs. (4.4) and (5.1), the difference between the phase  $\psi_{\ell m}$  and the phase of the resummed tail factor  $T_{\ell m}$  is included into  $\delta_{\ell m}$  up to 4.5PN as

$$\delta_{\ell m}(r_{0s}) = -m\psi_{\ell m} + m\phi - \tau_{\ell m}(r_{0s}) \quad (5.7a)$$

$$\begin{aligned} &= -m\psi_{\ell m} + m\Omega(t - r^*) - mv^3 \left( 2 \ln \left( \frac{2mr_{0s}v^3}{M} \right) \right. \\ &\quad \left. - 2\Psi^{(0)}(\ell) - \frac{2}{\ell} \right) - (mv^3)^3 \left( \frac{4}{3}\Psi^{(2)}(\ell) + \frac{8}{3\ell^3} \right) \end{aligned} \quad (5.7b)$$

$$\begin{aligned} &= 2mv^3 \left( \frac{2}{\ell} - \frac{2}{\ell+1} + 3 \frac{1 + (-1)^{\ell+m}}{(\ell-1)\ell(\ell+1)(\ell+2)} \right. \\ &\quad \left. - 2 \ln \left( \frac{r_{0s}}{2Me^{-1/2}} \right) \right) - 2\pi\nu^{(2)}(\ell)(mv^3)^2 \\ &\quad - m\psi_{\ell m}^{(9)}v^9 - (mv^3)^3 \left( \frac{4}{3}\Psi^{(2)}(\ell) + \frac{8}{3\ell^3} \right), \end{aligned} \quad (5.7c)$$

where  $\psi_{\ell m}^{(9)}$  is  $O(v^9)$  term of  $\psi_{\ell m}$  in Eq. (4.4), and  $\phi = \Omega(t - r^*)$  comes from the spherical harmonics  $Y^{\ell-\epsilon_p, -m}(\frac{\pi}{2}, \phi) \propto e^{-im\phi}$  in the Newtonian contribution to  $h_{\ell m}$  in Eq. (5.2). Note that  $-\Omega r^*$  in  $\phi$  is introduced to cancel out  $-\Omega r^*$  in  $\psi_{\ell m}$  and may be interpreted as the initial value of  $\phi$ .

The choice<sup>3</sup> of  $r_{0s} = 2M/\sqrt{e}$  in Schwarzschild coordinate will reproduce the phase  $\delta_{\ell m}$  in Ref. [15]. In this paper, we choose  $r_{0s} = 2M/\sqrt{e}$  to reproduce  $\delta_{\ell m}$  in Ref. [15] and define as  $\delta_{\ell m}(r_{0s}) \equiv \delta_{\ell m}$ . Using Eqs. (4.8) and (5.7), we can derive 4.5PN expression of  $\delta_{\ell m}$  for  $2 \leq \ell \leq 4$  as

$$\delta_{2,2} = \frac{7}{3}v^3 + \frac{428}{105}\pi v^6 + \left( -\frac{2203}{81} + \frac{1712}{315}\pi^2 \right)v^9, \quad (5.8a)$$

$$\delta_{2,1} = \frac{2}{3}v^3 + \frac{107}{105}\pi v^6 + \left( -\frac{272}{81} + \frac{214}{315}\pi^2 \right)v^9, \quad (5.8b)$$

$$\delta_{3,3} = \frac{13}{10}v^3 + \frac{39}{7}\pi v^6 + \left( -\frac{227827}{3000} + \frac{78}{7}\pi^2 \right)v^9, \quad (5.8c)$$

$$\delta_{3,2} = \frac{2}{3}v^3 + \frac{52}{21}\pi v^6 + \left( -\frac{9112}{405} + \frac{208}{63}\pi^2 \right)v^9, \quad (5.8d)$$

$$\delta_{3,1} = \frac{13}{30}v^3 + \frac{13}{21}\pi v^6 + \left( -\frac{227827}{81000} + \frac{26}{63}\pi^2 \right)v^9, \quad (5.8e)$$

$$\delta_{4,4} = \frac{14}{15}v^3 + \frac{25136}{3465}\pi v^6 + \left( -\frac{55144}{375} + \frac{201088}{10395}\pi^2 \right)v^9, \quad (5.8f)$$

$$\delta_{4,3} = \frac{3}{5}v^3 + \frac{1571}{385}\pi v^6, \quad (5.8g)$$

$$\delta_{4,2} = \frac{7}{15}v^3 + \frac{6284}{3465}\pi v^6 + \left( -\frac{6893}{375} + \frac{25136}{10395}\pi^2 \right)v^9, \quad (5.8h)$$

$$\delta_{4,1} = \frac{1}{5}v^3 + \frac{1571}{3465}\pi v^6. \quad (5.8i)$$

Note the new  $O(v^6)$  corrections for  $(\ell, m) = (2, 1)$ ,  $\ell = 3$ , and  $\ell = 4$  modes and the new  $O(v^9)$  corrections in Eq. (5.8) beyond those available from Ref. [15]. This is because, in the case of the test-particle limit, Ref. [15] used the results in Ref. [21], that provided only the 5.5PN energy flux but not the 5.5PN GW polarizations. Having computed the 5.5PN waveforms in this work we are able to improve on this accuracy.

According to Ref. [15], the decomposition of the post-Newtonian waveforms into five factors improves the convergence of the waveforms since the coefficients of the post-Newtonian expansion become smaller. However, the convergence of the amplitude,  $|h_{\ell m}/(h_{\ell m}^{(N, \epsilon_p)} \hat{S}_{\text{eff}}^{(\epsilon_p)} T_{\ell m} e^{i\delta_{\ell m}})|$ , was not good enough around the innermost stable circular orbit (ISCO). To alleviate this problem, Ref. [15] introduced the  $\ell$ th root of the amplitude,  $\rho_{\ell m}$ , to deal with the linear dependence on  $\ell$  in the 1PN terms of the amplitude (Notice the related linear dependence on  $\ell$  in the 1PN terms of  $Z_{\ell m \omega}$  derived in Sec. III C). Then, the coefficients of the post-Newtonian expansion become smaller and give much better improvement even around ISCO. As shown in Sec. VII, factorized resummed waveforms achieve about 5 times better agreement with numerical calculation than Taylor expanded waveforms.

Using Eqs. (4.3) and (5.1),  $\rho_{\ell m}$  can be derived as  $\rho_{\ell m} = (h_{\ell m}/(h_{\ell m}^{(N, \epsilon_p)} \hat{S}_{\text{eff}}^{(\epsilon_p)} T_{\ell m} e^{i\delta_{\ell m}}))^{1/\ell}$ . Then, we can derive 5PN expressions of  $\rho_{\ell m}$  for  $2 \leq \ell \leq 4$  as

<sup>3</sup>Note that in the generic mass case, where one works in harmonic coordinates [15,29,30]  $\ln x_0$  is chosen to be  $\ln x_0 = 11/18 - 2/3\gamma - 4/3 \ln 2 + 2/3 \ln[G(m_1 + m_2)/c^2/r_0^3]$ , where  $r_0^h$  is a freely-specifiable constant related to the choice of the origin of the retarded time in radiative coordinates relative to harmonic coordinates. We denote by  $r_0^h$  the  $r_0$  in Refs. [29,30] to avoid conflict with the notation used in perturbation works where  $r_0$  as in this paper denotes the orbital radius. With  $r_0^h = 2M/\sqrt{e}$  the above expression reduces to  $\ln x_0 = 17/18 - 2 \ln 2 - 2\gamma/3$ , the relation used to match PN results in terms of  $\ln x_0$  to black hole perturbation results in Schwarzschild coordinates.



$$\begin{aligned} \rho_{2,2} = & 1 - \frac{43}{42}v^2 - \frac{20555}{10584}v^4 + \left(\frac{1556919113}{122245200} - \frac{428}{105} \text{eulerlog}(2, v)\right)v^6 + \left(\frac{9202}{2205} \text{eulerlog}(2, v) - \frac{387216563023}{160190110080}\right)v^8 \\ & + \left(\frac{439877}{55566} \text{eulerlog}(2, v) - \frac{16094530514677}{533967033600}\right)v^{10}, \end{aligned} \quad (5.9a)$$

$$\begin{aligned} \rho_{2,1} = & 1 - \frac{59}{56}v^2 - \frac{47009}{56448}v^4 + \left(\frac{7613184941}{2607897600} - \frac{107}{105} \text{eulerlog}(1, v)\right)v^6 + \left(-\frac{1168617463883}{911303737344} + \frac{6313}{5880} \text{eulerlog}(1, v)\right)v^8 \\ & + \left(-\frac{63735873771463}{16569158860800} + \frac{5029963}{5927040} \text{eulerlog}(1, v)\right)v^{10}, \end{aligned} \quad (5.9b)$$

$$\begin{aligned} \rho_{3,3} = & 1 - \frac{7}{6}v^2 - \frac{6719}{3960}v^4 + \left(\frac{3203101567}{227026800} - \frac{26}{7} \text{eulerlog}(3, v)\right)v^6 + \left(\frac{13}{3} \text{eulerlog}(3, v) - \frac{57566572157}{8562153600}\right)v^8 \\ & + \left(-\frac{903823148417327}{30566888352000} + \frac{87347}{13860} \text{eulerlog}(3, v)\right)v^{10}, \end{aligned} \quad (5.9c)$$

$$\begin{aligned} \rho_{3,2} = & 1 - \frac{164}{135}v^2 - \frac{180566}{200475}v^4 + \left(\frac{5849948554}{940355325} - \frac{104}{63} \text{eulerlog}(2, v)\right)v^6 \\ & + \left(-\frac{10607269449358}{3072140846775} + \frac{17056}{8505} \text{eulerlog}(2, v)\right)v^8, \end{aligned} \quad (5.9d)$$

$$\begin{aligned} \rho_{3,1} = & 1 - \frac{13}{18}v^2 + \frac{101}{7128}v^4 + \left(\frac{11706720301}{6129723600} - \frac{26}{63} \text{eulerlog}(1, v)\right)v^6 + \left(\frac{169}{567} \text{eulerlog}(1, v) + \frac{2606097992581}{4854741091200}\right)v^8 \\ & + \left(\frac{430750057673539}{297110154781440} - \frac{1313}{224532} \text{eulerlog}(1, v)\right)v^{10}, \end{aligned} \quad (5.9e)$$

$$\begin{aligned} \rho_{4,4} = & 1 - \frac{269}{220}v^2 - \frac{14210377}{8808800}v^4 + \left(-\frac{12568}{3465} \text{eulerlog}(4, v) + \frac{16600939332793}{1098809712000}\right)v^6 \\ & + \left(\frac{845198}{190575} \text{eulerlog}(4, v) - \frac{172066910136202271}{19426955708160000}\right)v^8, \end{aligned} \quad (5.9f)$$

$$\begin{aligned} \rho_{4,3} = & 1 - \frac{111}{88}v^2 - \frac{6894273}{7047040}v^4 + \left(-\frac{1571}{770} \text{eulerlog}(3, v) + \frac{1664224207351}{195343948800}\right)v^6 \\ & + \left(-\frac{2465107182496333}{460490801971200} + \frac{174381}{67760} \text{eulerlog}(3, v)\right)v^8, \end{aligned} \quad (5.9g)$$

$$\begin{aligned} \rho_{4,2} = & 1 - \frac{191}{220}v^2 - \frac{3190529}{8808800}v^4 + \left(-\frac{3142}{3465} \text{eulerlog}(2, v) + \frac{848238724511}{219761942400}\right)v^6 \\ & + \left(\frac{300061}{381150} \text{eulerlog}(2, v) - \frac{12864377174485679}{19426955708160000}\right)v^8, \end{aligned} \quad (5.9h)$$

$$\begin{aligned} \rho_{4,1} = & 1 - \frac{301}{264}v^2 - \frac{7775491}{21141120}v^4 + \left(-\frac{1571}{6930} \text{eulerlog}(1, v) + \frac{1227423222031}{1758095539200}\right)v^6 \\ & + \left(-\frac{29584392078751453}{37299754959667200} + \frac{67553}{261360} \text{eulerlog}(1, v)\right)v^8. \end{aligned} \quad (5.9i)$$

We note that in the factorized resummed waveforms, all the  $\rho_{\ell m}$ 's contain only even powers of  $v$  [15]. Thus, 5.5PN waveforms produce 5PN expressions of  $\rho_{\ell m}$ . We show 5PN expressions of  $\rho_{\ell m}$  for  $5 \leq \ell \leq 7$  in Appendix B, but not for  $8 \leq \ell \leq 13$  since one can derive them using 2PN expression of  $\rho_{\ell m}$  in Sec. VA. 5PN expressions of  $\rho_{\ell m}$  in this paper are consistent at lower PN orders to expressions derived in Ref. [15].

### A. 2PN formulas for $\rho_{\ell m}$

In this section, we derive 2PN formulas for resummed waveforms  $\rho_{\ell m}$  Eq. (5.1) using 2.5PN formula of  $Z_{\ell m \omega}$  Eq. (3.17) in Sec. III C. As explained in Sec. III C, we do

not have to derive 5PN  $\rho_{\ell m}$  for  $\ell \geq 8$  since one can derive them using the general formulas in this section.

Once we have the 2.5PN formula of  $Z_{\ell m \omega}$  Eq. (3.17), we can derive 2PN  $\rho_{\ell m}$  and 3PN  $\delta_{\ell m}$  from Eqs. (4.3) and (5.1) as

$$\rho_{\ell m} = \left( \frac{h_{\ell m}}{h_{\ell m}^{(N, \epsilon_p)} \hat{S}_{\text{eff}}^{(\epsilon_p)} T_{\ell m} e^{i\delta_{\ell m}}} \right)^{1/\ell}, \quad (5.10a)$$

$$= 1 + \rho_{\ell m}^{(2)} v^2 + \rho_{\ell m}^{(4)} v^4 + O(v^6), \quad (5.10b)$$

and

$$\delta_{\ell m} = -m\psi_{\ell m} + m\Omega(t - r^*) - \tau_{\ell m}, \quad (5.11a)$$

$$= -m\psi_{\ell m} + m\Omega(t - r^*) - mv^3 \left( 2 \ln(4mv^3) - 1 - 2\Psi(\ell) - \frac{2}{\ell} \right), \quad (5.11b)$$

$$= 2mv^3 \left( \frac{2}{\ell} - \frac{2}{\ell+1} + 3 \frac{1 + (-1)^{\ell+m}}{(\ell-1)\ell(\ell+1)(\ell+2)} \right) - 2(mv^3)^2 \pi\nu^{(2)}(\ell), \quad (5.11c)$$

where  $\tau_{\ell m}$  is the phase of the tail term  $T_{\ell m}$ , defined as  $T_{\ell m} = |T_{\ell m}| e^{i\tau_{\ell m}}$ . As noted in Sec. V, factorized resummed waveforms  $\rho_{\ell m}$  have only even powers of  $\nu$  [15]. As mentioned earlier in Sec. III C, although this is a 2.5PN calculation, we give the 3PN formula for  $\delta_{\ell m}$  since there may not exist any further 3PN phase contributions.

Observe that, in the following subsections, 1PN terms of  $\rho_{\ell m}$ , i.e.  $\rho_{\ell m}^{(2)}$ , are consistent with that of Ref. [15] and 2PN terms  $\rho_{\ell m}^{(4)}$  are our new results.

### 1. $\ell + m = \text{even case}$

$$\rho_{\ell m}^{(2)} = -1 + \frac{1}{\ell} - \frac{m^2(\ell+9)}{2\ell(\ell+1)(2\ell+3)}, \quad (5.12a)$$

$$\rho_{\ell m}^{(4)} = -\frac{1}{4} + \frac{5\ell^3 + 5\ell^2 - 4\ell + 2}{4(2\ell-1)(\ell-1)\ell^2} - \frac{m^2(5\ell^5 + 20\ell^4 + 60\ell^3 + 86\ell^2 + 39\ell + 30)}{2\ell^2(2\ell+3)(\ell+1)^2(\ell+2)(\ell-1)} - \frac{m^4(91\ell^3 + 268\ell^2 - 249\ell - 810)}{8\ell^2(2\ell+3)^2(\ell+1)^2(\ell+2)(2\ell+5)}. \quad (5.12b)$$

### 2. $\ell + m = \text{odd case}$

$$\rho_{\ell m}^{(2)} = -1 - \frac{1}{\ell} + \frac{2}{\ell^2} - \frac{m^2(\ell+4)}{2\ell(\ell+2)(2\ell+3)}, \quad (5.13a)$$

$$\rho_{\ell m}^{(4)} = -\frac{1}{4} + \frac{\ell^4 - 6\ell^3 - 18\ell^2 + 32\ell - 8}{4(2\ell-1)\ell^4} - \frac{m^2(5\ell^4 + 20\ell^3 + 33\ell^2 + 34\ell + 8)}{2\ell^3(2\ell+3)(\ell+2)(\ell+1)} - \frac{m^4(3\ell^2 - 28\ell - 80)}{8\ell^2(2\ell+3)^2(\ell+2)^2(2\ell+5)}. \quad (5.13b)$$

## VI. + AND $\times$ POLARIZATIONS

In the previous sections we have explicitly listed the 5.5PN  $h_{\ell m}$  and 5PN  $\rho_{\ell m}$  that are directly useful for a comparison of analytical PN results with NR simulations. In this section, we present the general formulas using which the + and  $\times$  polarizations can be obtained from the formulas for  $h_{\ell m}$  listed earlier. To compare the results

in literature in the test-particle limit, we use the same notation as in Refs. [20,47] to derive plus and cross polarizations of gravitational waveforms. With  $h_{\ell m}^{+, \times}$  defined as in Eq. (2.15), we have

$$h_{\ell m}^{+, \times} + h_{\ell, -m}^{+, \times} = -\left(\frac{\mu}{r}\right)\left(\frac{M}{r_0}\right)\zeta_{\ell m}^{+, \times}. \quad (6.1)$$

For gravitational waves propagating towards the observer located at  $(\theta, \varphi)$  relative to the source, we have  $(\Theta, \Phi) = (\theta, \varphi)$ . In this case, from Eqs. (2.15) and (4.1) it follows that  $h_{\ell m}^+ - ih_{\ell m}^\times = h_{\ell m-2} Y_{\ell m}(\theta, \varphi)$ . Then we find

$$-\left(\frac{\mu}{r}\right)\left(\frac{M}{r_0}\right)\zeta_{\ell m}^+ = \text{Re}[h_{\ell m-2} Y_{\ell m}(\theta, \varphi) + h_{\ell, -m-2} Y_{\ell, -m}(\theta, \varphi)], \quad (6.2a)$$

$$-\left(\frac{\mu}{r}\right)\left(\frac{M}{r_0}\right)\zeta_{\ell m}^\times = -\text{Im}[h_{\ell m-2} Y_{\ell m}(\theta, \varphi) + h_{\ell, -m-2} Y_{\ell, -m}(\theta, \varphi)]. \quad (6.2b)$$

Using spherical harmonic modes of gravitational waveforms  $\hat{H}_{\ell m}$  and phase factor  $\psi_{\ell m}$  introduced in Sec. IV, we derive  $\zeta_{\ell m}^{+, \times}$  for the even  $m$  case as

$$\zeta_{\ell m}^+ = 8\sqrt{\frac{\pi}{5}}[-{}_2Y_{\ell m}(\theta, 0) + {}_{-2}Y_{\ell m}(\pi - \theta, 0)] \times \cos(m\psi_{\ell m})\hat{H}_{\ell m}, \quad (6.3a)$$

$$\zeta_{\ell m}^\times = 8i\sqrt{\frac{\pi}{5}}[-{}_2Y_{\ell m}(\theta, 0) - {}_{-2}Y_{\ell m}(\pi - \theta, 0)] \times [-i \sin(m\psi_{\ell m})]\hat{H}_{\ell m}, \quad (6.3b)$$

and for the odd  $m$  case as

$$\zeta_{\ell m}^+ = 8\sqrt{\frac{\pi}{5}}[-{}_2Y_{\ell m}(\theta, 0) + {}_{-2}Y_{\ell m}(\pi - \theta, 0)] \times [-i \sin(m\psi_{\ell m})]\hat{H}_{\ell m}, \quad (6.4a)$$

$$\zeta_{\ell m}^\times = 8i\sqrt{\frac{\pi}{5}}[-{}_2Y_{\ell m}(\theta, 0) - {}_{-2}Y_{\ell m}(\pi - \theta, 0)] \times \cos(m\psi_{\ell m})\hat{H}_{\ell m}. \quad (6.4b)$$

Here we used the known properties that  $Z_{\ell, -m, -\omega} = (-1)^\ell \bar{Z}_{\ell m \omega}$  and  ${}_s Y_{\ell, -m}(\theta, \varphi) = (-1)^{\ell+s} {}_s \bar{Y}_{\ell m}(\pi - \theta, \varphi)$ , where the bar denotes complex conjugate. We note that the signs of  $\zeta_{\ell m}^+$  in Refs [20,47] are opposite to each other, as pointed out in Ref. [29]. The signs of resulting  $\zeta_{\ell m}^+$  in this paper are same as the ones in Ref. [47].

Using the above relations, we compared  $\zeta_{\ell m}^{+, \times}$  to the 4PN expressions in Ref. [20]. We find agreement in almost all the terms except four corresponding to  $\zeta_{8,7}^+$ ,  $\zeta_{8,7}^\times$ , and  $\zeta_{10,6}^+$  in addition to a misprint of the sign of  $\zeta_{7,3}^\times$  in Ref. [20], pointed out in Refs. [30,50]. Our resulting expressions for  $\zeta_{8,7}^+$ ,  $\zeta_{8,7}^\times$ , and  $\zeta_{10,6}^+$  are given by

$$\zeta_{8,7}^+ = -\frac{823543}{829440}(5 + 7 \cos(2\theta)) \sin(\theta)^5 \sin(7\psi_{8,7}) \times \left( v^7 - \frac{3343}{380} v^9 + 7\pi v^{10} + \frac{42607}{1710} v^{11} \right), \quad (6.5)$$

$$\zeta_{8,7}^\times = \frac{823543}{414720}(\cos(3\theta) + 5 \cos(\theta)) \sin(\theta)^5 \cos(7\psi_{8,7}) \times \left( v^7 - \frac{3343}{380} v^9 + 7\pi v^{10} + \frac{42607}{1710} v^{11} \right), \quad (6.6)$$

$$\zeta_{10,6}^+ = -\frac{2187}{115763200}(19318 + 31299 \cos(2\theta) + 16218 \cos(4\theta) + 4845 \cos(6\theta)) \sin(\theta)^4 \times \cos(6\psi_{10,6}) \left( v^8 - \frac{5491}{506} v^{10} + 6\pi v^{11} \right). \quad (6.7)$$

The differences from Ref. [20] are as follows: the sign of  $\zeta_{8,7}^+$  is changed,  $\zeta_{8,7}^\times$  is multiplied by  $-[207360(\cos(3\theta) + 5 \cos(\theta))]/[414720(2 + \cos(2\theta))]$ , and  $\zeta_{10,6}^+$  is multiplied by  $-4087/4096$ .

Once we obtain the 5.5PN  $\zeta_{\ell m}^{+, \times}$ , the 5.5PN plus and cross polarizations measured by a general observer located at  $(\theta, \varphi)$  are derived using

$$h_+ = -\left(\frac{\mu}{r}\right)\left(\frac{M}{r_0}\right) \sum_{\ell=2}^{13} \sum_{m=1}^{\ell} \zeta_{\ell m}^+, \quad (6.8a)$$

$$h_\times = -\left(\frac{\mu}{r}\right)\left(\frac{M}{r_0}\right) \sum_{\ell=2}^{13} \sum_{m=1}^{\ell} \zeta_{\ell m}^\times. \quad (6.8b)$$

In standard terminology used for GW polarizations [29], the above results for the polarization corresponds to the choice  $(\vec{N} = \vec{e}_R, \vec{P} = \vec{e}_\Theta, \vec{Q} = \vec{e}_\Phi)$ . The standard PN expressions [30] in the test-particle limit corresponds to  $(\vec{P} = -\vec{e}_\Phi, \vec{Q} = \vec{e}_\Theta)$  evaluated at  $(\theta, \varphi) = (i, \pi/2)$  leading to an overall sign difference. This also shows up in the overall sign difference in  $h_{\ell m}$ . Ready-to-use 5.5PN expressions of GW polarization modes  $h_+$  and  $h_\times$  in the test-particle limit are listed in Appendix C for possible use in GW data analysis applications.

## VII. COMPARISON WITH NUMERICAL RESULTS

To assess quantitative implications of our present work, in this section we perform two different types of comparisons. First, we compare the formulas of our post-Newtonian expansion with results obtained by the numerical solution of the Teukolsky equation. Secondly, for the case of LISA we investigate the adequacy of the present  $\mathcal{O}(v^{11})$  waveform for dephasing accuracy of about a fraction of a cycle.

The numerical calculation is based on the high precision code which deals with the gravitational waves from a particle in a circular orbit around a black hole [40,41]. The numerical method uses the formalism developed by Mano, Suzuki, and Takasugi, which is the same formalism used in the post-Newtonian expansion in this paper, to compute the homogeneous solution of the Teukolsky equation. The precision of the energy flux of each mode  $(\ell, m)$  can be achieved to about 14 significant figures in the double precision calculation. Thus, the dominant errors for the energy flux in the code are due to truncation of summation over  $(\ell, m)$  mode in Eq. (2.14). In the numerical calculation, we set maximum value of the summation over  $\ell$  in Eq. (2.14) as  $\ell = 20$ . Then we find the relative error of the energy flux at  $r_0 = 6M$ , ISCO of Schwarzschild black hole, is less than  $10^{-10}$ . Here we define the relative error as  $F[\ell = 20]/F[2 \leq \ell \leq 20]$ , where  $F = dE/dt$ . References [40,41] did not compute the energy absorption into the black hole horizon, but we include it in the computation of the energy flux, using in this paper the more general code for eccentric and inclined orbits developed in Ref. [42].

In the left panel of Fig. 1, we show the absolute values of the relative error of energy flux between the 5.5PN approximation and numerical results as a function of the orbital velocity  $v$ . We compare three different post-Newtonian schemes, referred to as Taylor-flux 5.5PN,  $\rho$ -flux 5.5PN, and  $\rho$ -waveform 5.5PN. Taylor-flux 5.5PN uses the result of Taylor expanded post-Newtonian energy flux, which is shown in Eq. (3.1) in Ref. [21].  $\rho$ -flux 5.5PN uses resummed waveforms  $\rho_{\ell m}$  in Ref. [15], which computed  $\rho_{\ell m}$  using the results of 5.5PN energy flux, while  $\rho$ -waveform 5.5PN uses resummed waveforms  $\rho_{\ell m}$  obtained in Sec. V. Thus,  $\rho_{\ell m}$  in  $\rho$ -flux 5.5PN is computed up to  $\mathcal{O}[v^{10-2(\ell-2+\epsilon_p)}]$  relative to the lowest order of itself, while  $\rho_{\ell m}$  in  $\rho$ -waveform 5.5PN is computed up to  $\mathcal{O}[v^{10-(\ell-2)}]$  for  $\ell$  is even and  $\mathcal{O}[v^{10-(\ell-3+2\epsilon_p)}]$  for  $\ell$  is odd. However, we do not find any visible difference between  $\rho$ -flux 5.5PN and  $\rho$ -waveform 5.5PN in the left panel of Fig. 1. This may be because they do not have any difference in the dominant mode  $(\ell, m) = (2, 2)$  but a few correction terms in the nondominant mode  $(\ell, m) = (2, 1)$  and  $\ell \geq 3$ . In the right panel of Fig. 1, we show the absolute values of the relative error of energy flux between  $\rho$ -flux 5.5PN and  $\rho$ -waveform 5.5PN. The relative error of energy flux between  $\rho$ -flux 5.5PN and  $\rho$ -waveform 5.5PN is less than  $10^{-3}$  even in the region around ISCO. Thus, we do not compare the results from  $\rho$ -flux 5.5PN to the ones from  $\rho$ -waveform 5.5PN in the following calculations.

In order to estimate the validity of the 5.5PN expansion, we compare the phase after two years evolution with the numerical calculation. Here, we examine two systems, which were also studied in Ref. [51] for the comparison of the phase between EOB waveforms and numerical ones. Both systems sweep different frequency regions in the

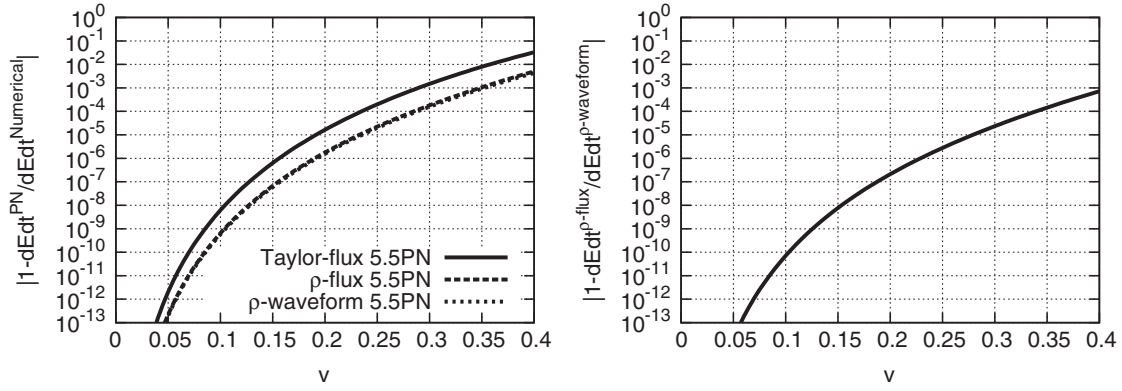


FIG. 1. (Left) Absolute values of difference of energy flux between numerical and post-Newtonian results as a function of the orbital velocity  $v$ . The numerical calculation is performed using the high precision code for energy flux in Refs. [40,41]. For the numerical calculation, we set the maximum value of  $\ell$  to 20 in Eq. (2.14) that leads to a relative accuracy less than  $10^{-10}$  in the numerical calculation. Taylor-flux 5.5PN computes energy flux using Taylor expanded post-Newtonian approximation shown in Eq. (3.1) in Ref. [21].  $\rho$ -flux 5.5PN uses resummed waveforms  $\rho_{\ell m}$  in Ref. [15], which derived  $\rho_{\ell m}$  using the results of the 5.5PN energy flux, while  $\rho$ -waveform 5.5PN uses  $\rho_{\ell m}$  obtained in Sec. V using the new 5.5PN waveform. The difference for the case of  $\rho$ -waveform 5.5PN and  $\rho$ -flux 5.5PN is not visible in this figure. (Right) Absolute values of difference of flux between  $\rho$ -waveform 5.5PN and  $\rho$ -flux 5.5PN results.

LISA band during their two years quasicircular inspiral. One of the systems, named system-I, has masses  $(M, \mu) = (10^5, 10)M_\odot$  and starts inspiral from  $r_0 \approx 29.34M$  to  $r_0 \approx 16.1M$ , whose frequency corresponds to from  $f_{\text{GW}} \approx 4 \times 10^{-3}$  Hz to  $f_{\text{GW}} \approx 10^{-2}$  Hz. The other system, named system-II, has masses  $(M, \mu) = (10^6, 10)M_\odot$  and starts inspiral from  $r_0 \approx 10.6M$  to  $r_0 \approx 6.0M$ , whose frequency corresponds to from  $f_{\text{GW}} \approx 1.8 \times 10^{-3}$  Hz to  $f_{\text{GW}} \approx 4.4 \times 10^{-3}$  Hz. System-I (II) represents the early (late)

inspiral phase of an extreme mass ratio inspiral in the frequency band of LISA.

For the numerical calculation of the phase, we adopt the one described in Ref. [52], which is also used in Ref. [51]. The numerical calculation is implemented as follows. First, we prepare  $10^3$  points data over the range from  $v = 0.01$  to  $v = 0.408$ . In this work, the data contains  $v$ ,  $dr_0/dt$ , and  $\Psi_{\ell m}$ , where  $\Psi_{\ell m}$  is the phase of  $Z_{\ell m \omega}$ ,  $dr_0/dt = (\partial r_0 / \partial \tilde{E}) d\tilde{E}/dt = (r_0 - 6M) / (2\sqrt{r_0(r_0 - 3M)}) d\tilde{E}/dt$ , and

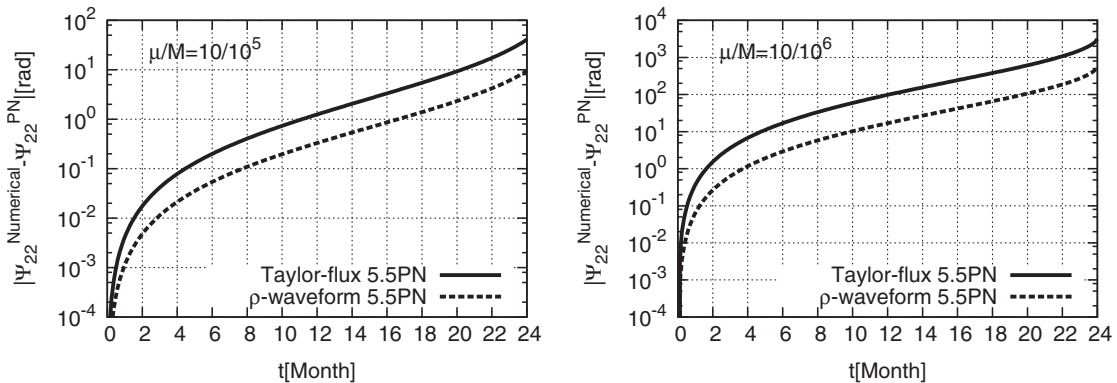


FIG. 2. Absolute values of the phase difference between the post-Newtonian expansion and the numerical results for  $\ell = m = 2$  mode as a function of the time in the month. The left panel shows the dephase due to two years inspiral for  $(M, \mu) = (10^5, 10)M_\odot$ . The inspiral is considered between  $r_0 \approx 29.34M$  and  $r_0 \approx 16.1M$  and corresponds to frequencies from  $f_{\text{GW}} \approx 4 \times 10^{-3}$  Hz to  $f_{\text{GW}} \approx 10^{-2}$  Hz. The right panel shows the dephase due to two years inspiral for  $(M, \mu) = (10^6, 10)M_\odot$  for inspiral from  $r_0 \approx 10.6M$  to  $r_0 \approx 6.0M$ , with associated frequencies from  $f_{\text{GW}} \approx 1.8 \times 10^{-3}$  Hz to  $f_{\text{GW}} \approx 4.4 \times 10^{-3}$  Hz. The left (right) panel represents the early (late) inspiral phase of an extreme mass ratio inspiral for LISA band. The details of each post-Newtonian approximation is the same as in Fig. 1.



$d\tilde{E}/dt$  is derived from the energy balance equation  $d\tilde{E}/dt = -dE/dt$  and Eq. (2.14). Then, from the  $10^3$  points data of  $(v, dr_0/dt, \Psi_{\ell m})$ , we compute  $(r_0(t), \Psi_{\ell m}(t))$  using cubic spline interpolation [53]. Though one can use stepping algorithm such as Runge-Kutta method, we use the interpolation for the computation of  $(r_0(t), \Psi_{\ell m}(t))$  to save computational time. Then one can compute the phase of the waveforms by  $m \int_0^t \Omega(t') dt' - \Psi_{\ell m}(t)$ , where  $\Omega(t) = \sqrt{M/r_0^3(t)}$ .

In Fig. 2, we show the absolute values of the phase difference of the dominant mode  $(\ell, m) = (2, 2)$  between 5.5 post-Newtonian approximations and numerical results. We find that after two years evolution the dephasing is  $\sim 40(3000)$  rads for system-I (system-II) when using the Taylor-flux 5.5PN [21], and  $\sim 10(530)$  rads for system-I (system-II) when using  $\rho$ -waveform 5.5PN. These results are consistent with Ref. [51]. Though  $\rho$ -waveform 5.5PN achieves about 5 times better phasing than Taylor-flux 5.5PN, the accuracy of the phasing is not enough to extract physical parameters of the source of gravitational waves by the data analysis of LISA because we have to reduce the dephase to within 1 rad during the observation [54]. In Ref. [51], the EOB model with 6PN factorized resummation, which is calibrated to numerical results, reduced the phase errors to  $\sim$  rad. The EOB model with the calibrated 6.5PN Padé approximation reduced the phase errors to less than 0.1 rad. Thus, for parameter estimation in LISA, we need post-Newtonian terms higher than 5.5PN and other resummation techniques like the EOB.

## VIII. SUMMARY AND CONCLUSION

In this work we have revisited the post-Newtonian expansion of gravitational waves for a test particle of mass  $\mu$  in circular orbit of radius  $r_0$  around a Schwarzschild black hole of mass  $M$  and provided 5.5PN GW polarizations ready for use in GW data analysis applications. Taking account of the need to compare post-Newtonian waveforms with numerical relativity waveforms, we have derived the spherical harmonic components associated with the gravitational wave polarizations up to order  $v^{11}$  beyond Newtonian. We have derived more accurate factorized post-Newtonian waveforms at higher multipolar orders, extending work in Ref. [15] to obtain better agreement with numerical results than conventional Taylor expanded post-Newtonian waveforms. In addition to  $h_{\ell m}$  modes corresponding to 5.5 post-Newtonian waveforms, we have derived a general expression for 2.5PN accurate  $Z_{\ell m \omega}$ , needed to obtain spherical modes and polarization modes, for general multipolar orders  $\ell$  and  $m$ . We also provide general analytical results for spherical harmonic modes at 2.5PN, 2PN factorized waveforms  $\rho_{\ell m}$ , and 3PN phase  $\delta_{\ell m}$  for arbitrary multipolar orders  $\ell$  and  $m$ . Thanks

to these 2.5PN or 2PN expressions of waveforms, we do not have to provide explicit expressions of waveforms for  $\ell \geq 8$  modes in the 5.5PN calculation since we can compute them using their general 2.5PN or 2PN formulas.

To investigate the validity of post-Newtonian approximation up to  $v^{11}$ , we have compared the phase with numerical calculation of black hole perturbation for two years inspiral. We have found that the phase difference became larger than 10 rad, though the resummed waveforms achieve better agreement with numerical results than conventional Taylor expanded post-Newtonian waveforms. Thus, we need higher post-Newtonian order corrections than 5.5PN in order to extract physical information from LISA data analysis. In Ref. [51], the EOB model with 6PN and 6.5PN approximation, which are calibrated to numerical results, reduced the phase to  $\lesssim$  rad. These results motivate one to derive the factorized waveforms at post-Newtonian orders higher than  $v^{12}$ .

Another extension of the present investigation is the computation of gravitational waveforms for a particle in a circular orbit around a Kerr black hole. Unlike a Schwarzschild black hole, the waveforms in the Teukolsky formalism are expressed in terms of spin-weighted spheroidal harmonics in the case of a Kerr black hole. Thus, the calculation of the spin-weighted spherical harmonic components from the plus and cross polarizations for the Kerr case involves an additional transformation between the spin-weighted spheroidal harmonics and the spin-weighted spherical harmonics. It is also important to extend the factorized resummation in Ref. [15] to the case of a Kerr black hole [55] and also noncircular orbits. These and similar related issues are left to future investigations.

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## APPENDIX A: 5.5PN FORMULAS FOR $\hat{H}_{\ell m}$ FOR $5 \leq \ell \leq 13$

In this appendix, we give 5.5PN gravitational waveforms of spherical harmonic modes for  $5 \leq \ell \leq 13$ , which are not shown completely in Sec. IV. We do not give the phase factor  $\psi_{\ell m}$  Eq. (4.6) since one can derive it from the general 3PN formula. As we noted in Sec. III C, we can also derive  $\hat{H}_{\ell m}$  for  $\ell \geq 8$  using the general 2.5PN formula of  $Z_{\ell m \omega}$  in Sec. III C, but we show them explicitly for ready reference.

$$\begin{aligned} \hat{H}_{5,5} = & -\frac{625}{6336}\sqrt{66}i\left(v^3 - \frac{263}{39}v^5 + 5\pi v^6 + \frac{9185}{819}v^7 - \frac{1315}{39}\pi v^8 + v^9\left(\frac{25}{6}\pi^2 + \frac{154743717347}{919836918} - \frac{7730}{429}\text{eulerlog}(5, v)\right)\right) \\ & + \frac{45925}{819}\pi v^{10} + v^{11}\left(-\frac{20684766695623}{20047034007} + \frac{2032990}{16731}\text{eulerlog}(5, v) - \frac{6575}{234}\pi^2\right), \end{aligned} \quad (\text{A1a})$$

$$\begin{aligned} \hat{H}_{5,4} = & -\frac{32}{1485}\sqrt{165}\left(v^4 - \frac{4451}{910}v^6 + 4\pi v^7 + \frac{10715}{2184}v^8 - \frac{8902}{455}\pi v^9\right) \\ & + v^{10}\left(\frac{19767269662573}{183967383600} + \frac{8}{3}\pi^2 - \frac{24736}{2145}\text{eulerlog}(4, v)\right) + \frac{10715}{546}\pi v^{11}, \end{aligned} \quad (\text{A1b})$$

$$\begin{aligned} \hat{H}_{5,3} = & \frac{9}{3520}\sqrt{330}i\left(v^3 - \frac{69}{13}v^5 + 3\pi v^6 + \frac{12463}{1365}v^7 - \frac{207}{13}\pi v^8 + v^9\left(\frac{45961635793}{851700850} - \frac{4638}{715}\text{eulerlog}(3, v) + \frac{3}{2}\pi^2\right)\right) \\ & + \frac{12463}{455}\pi v^{10} + v^{11}\left(-\frac{29896692069163}{105185054975} - \frac{207}{26}\pi^2 + \frac{320022}{9295}\text{eulerlog}(3, v)\right), \end{aligned} \quad (\text{A1c})$$

$$\begin{aligned} \hat{H}_{5,2} = & \frac{2}{1485}\sqrt{55}\left(v^4 - \frac{3911}{910}v^6 + 2\pi v^7 + \frac{63439}{10920}v^8 - \frac{3911}{455}\pi v^9\right) \\ & + v^{10}\left(\frac{4450644105337}{183967383600} + \frac{2}{3}\pi^2 - \frac{6184}{2145}\text{eulerlog}(2, v)\right) + \frac{63439}{5460}\pi v^{11}, \end{aligned} \quad (\text{A1d})$$

$$\begin{aligned} \hat{H}_{5,1} = & -\frac{\sqrt{385}}{110880}i\left(v^3 - \frac{179}{39}v^5 + \pi v^6 + \frac{5023}{585}v^7 - \frac{179}{39}\pi v^8 + v^9\left(-\frac{12574241461}{22995922950} + \frac{1}{6}\pi^2 - \frac{1546}{2145}\text{eulerlog}(1, v)\right)\right) \\ & + \frac{5023}{585}\pi v^{10} + v^{11}\left(-\frac{197070394320953}{8519989452975} + \frac{276734}{83655}\text{eulerlog}(1, v) - \frac{179}{234}\pi^2\right), \end{aligned} \quad (\text{A1e})$$

$$\begin{aligned} \hat{H}_{6,6} = & \frac{54}{715}\sqrt{143}\left(v^4 - \frac{113}{14}v^6 + 6\pi v^7 + \frac{1372317}{73304}v^8 - \frac{339}{7}\pi v^9\right) \\ & + v^{10}\left(\frac{993676618159}{4569124560} - \frac{21624}{1001}\text{eulerlog}(6, v) + 6\pi^2\right) + \frac{4116951}{36652}\pi v^{11}, \end{aligned} \quad (\text{A2a})$$

$$\begin{aligned} \hat{H}_{6,5} = & -\frac{3125}{216216}\sqrt{429}i\left(v^5 - \frac{149}{24}v^7 + 5\pi v^8 + \frac{156857}{15708}v^9 - \frac{745}{24}\pi v^{10}\right) \\ & + v^{11}\left(\frac{43028942343011}{279630423072} - \frac{45050}{3003}\text{eulerlog}(5, v) + \frac{25}{6}\pi^2\right), \end{aligned} \quad (\text{A2b})$$

$$\begin{aligned} \hat{H}_{6,4} = & -\frac{128}{19305}\sqrt{78}\left(v^4 - \frac{93}{14}v^6 + 4\pi v^7 + \frac{3261767}{219912}v^8 - \frac{186}{7}\pi v^9\right) \\ & + v^{10}\left(\frac{60337719431417}{699076057680} - \frac{28832}{3003}\text{eulerlog}(4, v) + \frac{8}{3}\pi^2\right) + \frac{3261767}{54978}\pi v^{11}, \end{aligned} \quad (\text{A2c})$$

$$\begin{aligned} \hat{H}_{6,3} = & \frac{81}{40040}\sqrt{65}i\left(v^5 - \frac{133}{24}v^7 + 3\pi v^8 + \frac{794009}{78540}v^9 - \frac{133}{8}\pi v^{10}\right) \\ & + v^{11}\left(\frac{3}{2}\pi^2 + \frac{37970972674259}{776751175200} - \frac{5406}{1001}\text{eulerlog}(3, v)\right), \end{aligned} \quad (\text{A2d})$$

$$\begin{aligned} \hat{H}_{6,2} = & \frac{2}{19305}\sqrt{65}\left(v^4 - \frac{81}{14}v^6 + 2\pi v^7 + \frac{14482483}{1099560}v^8 - \frac{81}{7}\pi v^9\right) \\ & + v^{10}\left(\frac{1019185435937}{99868008240} - \frac{7208}{3003}\text{eulerlog}(2, v) + \frac{2}{3}\pi^2\right) + \frac{14482483}{549780}\pi v^{11}, \end{aligned} \quad (\text{A2e})$$

$$\begin{aligned} \hat{H}_{6,1} = & -\frac{\sqrt{26}}{216216}i\left(v^5 - \frac{125}{24}v^7 + \pi v^8 + \frac{809959}{78540}v^9 - \frac{125}{24}\pi v^{10}\right) \\ & + v^{11}\left(-\frac{1063981210373}{411221210400} + \frac{1}{6}\pi^2 - \frac{1802}{3003}\text{eulerlog}(1, v)\right), \end{aligned} \quad (\text{A2f})$$

$$\hat{H}_{7,7} = \frac{16807}{1235520} \sqrt{6006} i \left( v^5 - \frac{319}{34} v^7 + 7\pi v^8 + \frac{2805191}{100776} v^9 - \frac{2233}{34} \pi v^{10} + v^{11} \left( \frac{49}{6} \pi^2 - \frac{83636}{3315} \text{eulerlog}(7, v) + \frac{35250285887269}{132292686240} \right) \right), \quad (\text{A3a})$$

$$\hat{H}_{7,6} = \frac{81}{5005} \sqrt{429} \left( v^6 - \frac{1787}{238} v^8 + 6\pi v^9 + \frac{3917323}{235144} v^{10} - \frac{5361}{119} \pi v^{11} \right), \quad (\text{A3b})$$

$$\hat{H}_{7,5} = -\frac{15625}{1729728} \sqrt{66} i \left( v^5 - \frac{271}{34} v^7 + 5\pi v^8 + \frac{15689017}{705432} v^9 - \frac{1355}{34} \pi v^{10} + v^{11} \left( \frac{157114390081769}{1296468325152} - \frac{59740}{4641} \text{eulerlog}(5, v) + \frac{25}{6} \pi^2 \right) \right), \quad (\text{A3c})$$

$$\hat{H}_{7,4} = -\frac{128}{45045} \sqrt{66} \left( v^6 - \frac{14543}{2142} v^8 + 4\pi v^9 + \frac{4794461}{302328} v^{10} - \frac{29086}{1071} \pi v^{11} \right), \quad (\text{A3d})$$

$$\hat{H}_{7,3} = \frac{243}{320320} \sqrt{6} i \left( v^5 - \frac{239}{34} v^7 + 3\pi v^8 + \frac{13620457}{705432} v^9 - \frac{717}{34} \pi v^{10} + v^{11} \left( -\frac{35844}{7735} \text{eulerlog}(3, v) + \frac{18990037711669}{720260180640} + \frac{3}{2} \pi^2 \right) \right), \quad (\text{A3e})$$

$$\hat{H}_{7,2} = \frac{\sqrt{3}}{9009} \left( v^6 - \frac{13619}{2142} v^8 + 2\pi v^9 + \frac{32998691}{2116296} v^{10} - \frac{13619}{1071} \pi v^{11} \right), \quad (\text{A3f})$$

$$\hat{H}_{7,1} = -\frac{\sqrt{2}}{1729728} i \left( v^5 - \frac{223}{34} v^7 + \pi v^8 + \frac{4251691}{235144} v^9 - \frac{223}{34} \pi v^{10} + v^{11} \left( -\frac{131624253235451}{6482341625760} + \frac{1}{6} \pi^2 - \frac{11948}{23205} \text{eulerlog}(1, v) \right) \right), \quad (\text{A3g})$$

$$\hat{H}_{8,8} = -\frac{16384}{5360355} \sqrt{170170} \left( v^6 - \frac{3653}{342} v^8 + 8\pi v^9 + \frac{124891}{3240} v^{10} - \frac{14612}{171} \pi v^{11} \right), \quad (\text{A4a})$$

$$\hat{H}_{8,7} = \frac{117649}{126023040} \sqrt{170170} i \left( v^7 - \frac{3343}{380} v^9 + 7\pi v^{10} + \frac{42607}{1710} v^{11} \right), \quad (\text{A4b})$$

$$\hat{H}_{8,6} = \frac{243}{595595} \sqrt{51051} \left( v^6 - \frac{353}{38} v^8 + 6\pi v^9 + \frac{2255}{72} v^{10} - \frac{1059}{19} \pi v^{11} \right), \quad (\text{A4c})$$

$$\hat{H}_{8,5} = -\frac{78125}{176432256} \sqrt{4862} i \left( v^7 - \frac{611}{76} v^9 + 5\pi v^{10} + \frac{7901}{342} v^{11} \right), \quad (\text{A4d})$$

$$\hat{H}_{8,4} = -\frac{128}{765765} \sqrt{374} \left( v^6 - \frac{2837}{342} v^8 + 4\pi v^9 + \frac{122681}{4536} v^{10} - \frac{5674}{171} \pi v^{11} \right), \quad (\text{A4e})$$

$$\hat{H}_{8,3} = \frac{81}{10890880} \sqrt{5610} i \left( v^7 - \frac{2863}{380} v^9 + 3\pi v^{10} + \frac{4213}{190} v^{11} \right), \quad (\text{A4f})$$

$$\hat{H}_{8,2} = \frac{\sqrt{85}}{765765} \left( v^6 - \frac{2633}{342} v^8 + 2\pi v^9 + \frac{563317}{22680} v^{10} - \frac{2633}{171} \pi v^{11} \right), \quad (\text{A4g})$$

$$\hat{H}_{8,1} = -\frac{\sqrt{238}}{176432256} i \left( v^7 - \frac{2767}{380} v^9 + \pi v^{10} + \frac{37267}{1710} v^{11} \right), \quad (\text{A4h})$$

$$\hat{H}_{9,9} = -\frac{1594323}{150492160}\sqrt{20995}i\left(v^7 - \frac{419}{35}v^9 + 9\pi v^{10} + \frac{38269131}{752675}v^{11}\right), \quad (\text{A5a})$$

$$\hat{H}_{9,8} = -\frac{131072}{59520825}\sqrt{41990}\left(v^8 - \frac{13969}{1386}v^{10} + 8\pi v^{11}\right), \quad (\text{A5b})$$

$$\hat{H}_{9,7} = \frac{5764801}{1741409280}\sqrt{1235}i\left(v^7 - \frac{53}{5}v^9 + 7\pi v^{10} + \frac{40687373}{967725}v^{11}\right), \quad (\text{A5c})$$

$$\hat{H}_{9,6} = \frac{486}{734825}\sqrt{3705}\left(v^8 - \frac{12877}{1386}v^{10} + 6\pi v^{11}\right), \quad (\text{A5d})$$

$$\hat{H}_{9,5} = -\frac{390625}{1218986496}\sqrt{247}i\left(v^7 - \frac{67}{7}v^9 + 5\pi v^{10} + \frac{9862901}{270963}v^{11}\right), \quad (\text{A5e})$$

$$\hat{H}_{9,4} = -\frac{512}{59520825}\sqrt{17290}\left(v^8 - \frac{12097}{1386}v^{10} + 4\pi v^{11}\right), \quad (\text{A5f})$$

$$\hat{H}_{9,3} = \frac{81}{75246080}\sqrt{1995}i\left(v^7 - \frac{311}{35}v^9 + 3\pi v^{10} + \frac{24906689}{752675}v^{11}\right), \quad (\text{A5g})$$

$$\hat{H}_{9,2} = \frac{2}{8502975}\sqrt{95}\left(v^8 - \frac{11629}{1386}v^{10} + 2\pi v^{11}\right), \quad (\text{A5h})$$

$$\hat{H}_{9,1} = -\frac{\sqrt{2090}}{19155502080}i\left(v^7 - \frac{299}{35}v^9 + \pi v^{10} + \frac{19443469}{615825}v^{11}\right), \quad (\text{A5i})$$

$$\hat{H}_{10,10} = \frac{390625}{49997493}\sqrt{58786}\left(v^8 - \frac{6707}{506}v^{10} + 10\pi v^{11}\right), \quad (\text{A6a})$$

$$\hat{H}_{10,9} = -\frac{14348907}{14484870400}\sqrt{293930}i\left(v^9 - \frac{5223}{460}v^{11}\right), \quad (\text{A6b})$$

$$\hat{H}_{10,8} = -\frac{2097152}{1249937325}\sqrt{7735}\left(v^8 - \frac{6023}{506}v^{10} + 8\pi v^{11}\right), \quad (\text{A6c})$$

$$\hat{H}_{10,7} = \frac{5764801}{23944377600}\sqrt{46410}i\left(v^9 - \frac{14549}{1380}v^{11}\right), \quad (\text{A6d})$$

$$\hat{H}_{10,6} = \frac{729}{5143775}\sqrt{2730}\left(v^8 - \frac{5491}{506}v^{10} + 6\pi v^{11}\right), \quad (\text{A6e})$$

$$\hat{H}_{10,5} = -\frac{1953125}{23465490048}\sqrt{546}i\left(v^9 - \frac{13709}{1380}v^{11}\right), \quad (\text{A6f})$$

$$\hat{H}_{10,4} = -\frac{4096}{1249937325}\sqrt{1365}\left(v^8 - \frac{5111}{506}v^{10} + 4\pi v^{11}\right), \quad (\text{A6g})$$

$$\hat{H}_{10,3} = \frac{243}{1034633600}\sqrt{2730}i\left(v^9 - \frac{4383}{460}v^{11}\right), \quad (\text{A6h})$$

$$\hat{H}_{10,2} = \frac{2}{178562475}\sqrt{105}\left(v^8 - \frac{4883}{506}v^{10} + 2\pi v^{11}\right), \quad (\text{A6i})$$

$$\hat{H}_{10,1} = -\frac{\sqrt{35}}{27935107200}i\left(v^9 - \frac{12869}{1380}v^{11}\right), \quad (\text{A6j})$$

$$\hat{H}_{11,11} = \frac{2357947691}{754833945600}\sqrt{572033}i\left(v^9 - \frac{218}{15}v^{11}\right), \quad (\text{A7a})$$

$$\hat{H}_{11,10} = \frac{1953125}{973028133}\sqrt{104006}v^{10}, \quad (\text{A7b})$$

$$\hat{H}_{11,9} = -\frac{129140163}{102508313600}\sqrt{22287}i\left(v^9 - \frac{66}{5}v^{11}\right), \quad (\text{A7c})$$

$$\hat{H}_{11,8} = -\frac{8388608}{24325703325}\sqrt{37145}v^{10}, \quad (\text{A7d})$$

$$\hat{H}_{11,7} = \frac{40353607}{169452518400}\sqrt{1955}i\left(v^9 - \frac{182}{15}v^{11}\right), \quad (\text{A7e})$$

$$\hat{H}_{11,6} = \frac{2187}{20021155}\sqrt{782}v^{10}, \quad (\text{A7f})$$

$$\hat{H}_{11,5} = -\frac{9765625}{332126936064}\sqrt{69}i\left(v^9 - \frac{34}{3}v^{11}\right), \quad (\text{A7g})$$

$$\hat{H}_{11,4} = -\frac{8192}{4865140665}\sqrt{483}v^{10}, \quad (\text{A7h})$$

$$\hat{H}_{11,3} = \frac{2187}{102508313600}\sqrt{1610}i\left(v^9 - \frac{54}{5}v^{11}\right), \quad (\text{A7i})$$

$$\hat{H}_{11,2} = \frac{2}{1158366825}\sqrt{115}v^{10}, \quad (\text{A7j})$$

$$\hat{H}_{11,1} = -\frac{\sqrt{598}}{5140059724800}i\left(v^9 - \frac{158}{15}v^{11}\right), \quad (\text{A7k})$$



$$\hat{H}_{12,12} = -\frac{1492992}{786545375}\sqrt{2451570}v^{10}, \quad (\text{A8a})$$

$$\hat{H}_{12,11} = \frac{25937424601}{21027517056000}\sqrt{408595}iv^{11}, \quad (\text{A8b})$$

$$\hat{H}_{12,10} = \frac{1953125}{1529044209}\sqrt{35530}v^{10}, \quad (\text{A8c})$$

$$\hat{H}_{12,9} = -\frac{1162261467}{951862912000}\sqrt{4845}iv^{11}, \quad (\text{A8d})$$

$$\hat{H}_{12,8} = -\frac{16777216}{121628516625}\sqrt{11305}v^{10}, \quad (\text{A8e})$$

$$\hat{H}_{12,7} = \frac{282475249}{6608648217600}\sqrt{11305}iv^{11}, \quad (\text{A8f})$$

$$\hat{H}_{12,6} = \frac{729}{100105775}\sqrt{3570}v^{10}, \quad (\text{A8g})$$

$$\hat{H}_{12,5} = -\frac{9765625}{1850421500928}\sqrt{255}iv^{11}, \quad (\text{A8h})$$

$$\hat{H}_{12,4} = -\frac{2048}{3475100475}\sqrt{30}v^{10}, \quad (\text{A8i})$$

$$\hat{H}_{12,3} = \frac{6561}{190372582400}\sqrt{30}iv^{11}, \quad (\text{A8j})$$

$$\hat{H}_{12,2} = \frac{2}{5791834125}\sqrt{5}v^{10}, \quad (\text{A8k})$$

$$\hat{H}_{12,1} = -\frac{\sqrt{770}}{77100895872000}iv^{11}, \quad (\text{A8l})$$

$$\hat{H}_{13,13} = -\frac{1792160394037}{256212207820800}\sqrt{289731}iv^{11}, \quad (\text{A9a})$$

$$\hat{H}_{13,11} = \frac{3138428376721}{1513981228032000}\sqrt{22287}iv^{11}, \quad (\text{A9b})$$

$$\hat{H}_{13,9} = -\frac{3486784401}{7614903296000}\sqrt{1938}iv^{11}, \quad (\text{A9c})$$

$$\hat{H}_{13,7} = \frac{1977326743}{2379113358336000}\sqrt{746130}iv^{11}, \quad (\text{A9d})$$

$$\hat{H}_{13,5} = -\frac{48828125}{133230348066816}\sqrt{561}iv^{11}, \quad (\text{A9e})$$

$$\hat{H}_{13,3} = \frac{6561}{7614903296000}\sqrt{165}iv^{11}, \quad (\text{A9f})$$

$$\hat{H}_{13,1} = -\frac{\sqrt{3}}{252330204672000}iv^{11}. \quad (\text{A9g})$$

### APPENDIX B: 5PN FORMULAS FOR $\rho_{\ell m}$ FOR $5 \leq \ell \leq 7$

In this appendix, we give the 5PN resummed gravitational waveforms for  $5 \leq \ell \leq 7$ , which are not shown completely in Sec. V. We do not give the phase factor  $\delta_{\ell m}$  Eq. (5.7) since one can derive it from the general 3PN formula Eq. (5.11). For  $\ell \geq 8$ , we do not show  $\rho_{\ell m}$  since they are explicitly given by the general 2PN formula in Sec. VA, Eqs. (5.10) to (5.12).

$$\begin{aligned} \rho_{5,5} = & 1 - \frac{487}{390}v^2 - \frac{3353747}{2129400}v^4 + \left( -\frac{1546}{429} \text{eulerlog}(5, v) + \frac{190606537999247}{11957879934000} \right) v^6 \\ & + \left( -\frac{1213641959949291437}{118143853747920000} + \frac{376451}{83655} \text{eulerlog}(5, v) \right) v^8, \end{aligned} \quad (\text{B1a})$$

$$\rho_{5,4} = 1 - \frac{2908}{2275}v^2 - \frac{16213384}{15526875}v^4 + \left( -\frac{24736}{10725} \text{eulerlog}(4, v) + \frac{6704294638171892}{653946558890625} \right) v^6, \quad (\text{B1b})$$

$$\begin{aligned} \rho_{5,3} = & 1 - \frac{25}{26}v^2 - \frac{410833}{709800}v^4 + \left( -\frac{4638}{3575} \text{eulerlog}(3, v) + \frac{7618462680967}{1328653326000} \right) v^6 \\ & + \left( -\frac{77082121019870543}{39381284582640000} + \frac{2319}{1859} \text{eulerlog}(3, v) \right) v^8, \end{aligned} \quad (\text{B1c})$$

$$\rho_{5,2} = 1 - \frac{2638}{2275}v^2 - \frac{7187914}{15526875}v^4 + \left( \frac{1539689950126502}{653946558890625} - \frac{6184}{10725} \text{eulerlog}(2, v) \right) v^6, \quad (\text{B1d})$$

$$\begin{aligned} \rho_{5,1} = & 1 - \frac{319}{390}v^2 - \frac{31877}{304200}v^4 + \left( -\frac{1546}{10725} \text{eulerlog}(1, v) + \frac{7685351978519}{11957879934000} \right) v^6 \\ & + \left( -\frac{821807362819271}{10740350340720000} + \frac{22417}{190125} \text{eulerlog}(1, v) \right) v^8, \end{aligned} \quad (\text{B1e})$$

$$\rho_{6,6} = 1 - \frac{53}{42}v^2 - \frac{1025435}{659736}v^4 + \left(-\frac{3604}{1001} \text{eulerlog}(6, v) + \frac{610931247213169}{36701493028200}\right)v^6, \quad (\text{B2a})$$

$$\rho_{6,5} = 1 - \frac{185}{144}v^2 - \frac{59574065}{54286848}v^4 + \left(-\frac{22525}{9009} \text{eulerlog}(5, v) + \frac{67397117912549267}{5798416452820992}\right)v^6, \quad (\text{B2b})$$

$$\rho_{6,4} = 1 - \frac{43}{42}v^2 - \frac{476887}{659736}v^4 + \left(-\frac{14416}{9009} \text{eulerlog}(4, v) + \frac{180067034480351}{24467662018800}\right)v^6, \quad (\text{B2c})$$

$$\rho_{6,3} = 1 - \frac{169}{144}v^2 - \frac{152153941}{271434240}v^4 + \left(-\frac{901}{1001} \text{eulerlog}(3, v) + \frac{116042497264681103}{28992082264104960}\right)v^6, \quad (\text{B2d})$$

$$\rho_{6,2} = 1 - \frac{37}{42}v^2 - \frac{817991}{3298680}v^4 + \left(-\frac{3604}{9009} \text{eulerlog}(2, v) + \frac{812992177581}{453104852200}\right)v^6, \quad (\text{B2e})$$

$$\rho_{6,1} = 1 - \frac{161}{144}v^2 - \frac{79192261}{271434240}v^4 + \left(-\frac{901}{9009} \text{eulerlog}(1, v) + \frac{6277796663889319}{28992082264104960}\right)v^6, \quad (\text{B2f})$$

$$\rho_{7,7} = 1 - \frac{151}{119}v^2 - \frac{32358125}{20986602}v^4 + \left(-\frac{11948}{3315} \text{eulerlog}(7, v) + \frac{66555794049401803}{3856993267327200}\right)v^6, \quad (\text{B3a})$$

$$\rho_{7,6} = 1 - \frac{1072}{833}v^2 - \frac{195441224}{171390583}v^4, \quad (\text{B3b})$$

$$\rho_{7,5} = 1 - \frac{127}{119}v^2 - \frac{17354227}{20986602}v^4 + \left(-\frac{59740}{32487} \text{eulerlog}(5, v) + \frac{192862646381533}{22039961527584}\right)v^6, \quad (\text{B3c})$$

$$\rho_{7,4} = 1 - \frac{8878}{7497}v^2 - \frac{2995755988}{4627545741}v^4, \quad (\text{B3d})$$

$$\rho_{7,3} = 1 - \frac{111}{119}v^2 - \frac{7804375}{20986602}v^4 + \left(-\frac{35844}{54145} \text{eulerlog}(3, v) + \frac{1321461327981547}{428554807480800}\right)v^6, \quad (\text{B3e})$$

$$\rho_{7,2} = 1 - \frac{8416}{7497}v^2 - \frac{1625746984}{4627545741}v^4, \quad (\text{B3f})$$

$$\rho_{7,1} = 1 - \frac{103}{119}v^2 - \frac{1055091}{6995534}v^4 + \left(-\frac{11948}{162435} \text{eulerlog}(1, v) + \frac{142228318411021}{550999038189600}\right)v^6. \quad (\text{B3g})$$

### APPENDIX C: 5.5PN FORMULAS FOR POLARIZATION MODES

In this appendix, we list the complete ready-to-use 5.5PN gravitational wave polarizations in the test-particle limit obtained using Eq. (6.8) in Sec. VI,

$$h_+ = -\left(\frac{\mu}{r}\right)\left(\frac{M}{r_0}\right) \sum_{\ell=2}^{13} \sum_{m=1}^{\ell} \zeta_{\ell m}^+, \quad (\text{C1a})$$

$$h_{\times} = -\left(\frac{\mu}{r}\right)\left(\frac{M}{r_0}\right) \sum_{\ell=2}^{13} \sum_{m=1}^{\ell} \zeta_{\ell m}^{\times}, \quad (\text{C1b})$$

where  $\zeta_{\ell m}^{+, \times}$  Eqs. (6.3) and (6.4), are functions of both the phase  $\psi_{\ell m}$  Eq. (4.8), and the angles  $(\theta, \varphi)$  defining the observer relative to the source.

Using the same notation and the same normalization  $h_{+, \times} = 2\mu x H_{+, \times} / r$  as in Ref. [30], where  $x = v^2$ , we consider the post-Newtonian expansion of  $H_{+, \times}$  defined as

$$H_{+, \times} = \sum_{n=0}^{11} x^{n/2} H_{+, \times}^{(n/2)}. \quad (\text{C2})$$

For the computation of the PN coefficients  $H_{+, \times}^{(n/2)}$ , we

change the phase variable  $\psi_{\ell m}$  in each mode to 1.5PN accurate  $\psi_{2,2}$  as in [30]. Using  $\psi_{2,2}$  at 1.5PN Eq. (4.8), we define the phase  $\psi$  as<sup>4</sup>

$$\psi = \Omega(t - r^*) - \varphi + \left(\frac{17}{6} - 2\gamma - 3 \ln(4x)\right)x^{3/2}. \quad (\text{C3})$$

The above expression for  $\psi$  can be rewritten as

$$\psi = \phi - 3x^{3/2} \ln\left(\frac{x}{x_0}\right) - \varphi \quad (\text{C4})$$

by recalling that in the test-particle case (see Sec. IV)  $\ln x_0 = 17/18 - 2 \ln 2 - 2\gamma/3$ .

We list the 5.5PN  $H_+^{(n/2)}$  and  $H_{\times}^{(n/2)}$  in Appendix subsections C 1 and C 2 respectively. In the following subsections, we use shorthand notations  $c_{\theta} = \cos \theta$  and

<sup>4</sup>Contrary to simple expectations, it is worth noting that a more accurate 4.5PN  $\psi_{2,2}$  Eq. (4.8) does not further simplify the expressions of  $H_{+, \times}^{(n/2)}$  since the 3PN and 4.5PN terms in  $\psi_{\ell m}$  are functions of  $\ell'$  and  $m$  unlike the 1.5PN  $\ln v$  terms of  $\psi_{\ell m}$  which are the same for all modes. Consequently, though 1.5PN  $\psi_{2,2}$  simplifies the expressions of  $H_{+, \times}^{(n/2)}$ , no such simplification is obtained at higher PN orders.

$s_\theta = \sin\theta$ . As mentioned earlier, to compare to standard PN expressions [30] in the test-particle limit, where  $\Delta = -1$ , we need to replace  $(\theta, \varphi)$  by  $(i, \pi/2)$ . With this replacement our phase variable  $\psi$  Eq. (C3) is related to the  $\psi_{\text{BFIS}}$  used in [30] as  $\psi = \psi_{\text{BFIS}} - \pi/2$  and the polarizations agree<sup>5</sup> modulo an overall sign for reasons discussed in Sec. VI.

### 1. Plus modes

$$H_+^{(0)} = -\cos(2\psi)(1 + c_\theta^2), \quad (\text{C5a})$$

$$H_+^{(0.5)} = s_\theta \sin(\psi) \left( \frac{5}{8} + \frac{1}{8} c_\theta^2 \right) + \frac{9}{8} s_\theta (1 + c_\theta^2) \sin(3\psi), \quad (\text{C5b})$$

$$H_+^{(1)} = \cos(2\psi) \left( \frac{19}{6} + \frac{3}{2} c_\theta^2 - \frac{1}{3} c_\theta^4 \right) + \frac{4}{3} s_\theta^2 \cos(4\psi)(1 + c_\theta^2), \quad (\text{C5c})$$

$$H_+^{(1.5)} = -2\pi \cos(2\psi)(1 + c_\theta^2) + s_\theta \sin(\psi) \left( -\frac{19}{64} - \frac{5}{16} c_\theta^2 + \frac{1}{192} c_\theta^4 \right) + s_\theta \sin(3\psi) \left( -\frac{657}{128} - \frac{45}{16} c_\theta^2 + \frac{81}{128} c_\theta^4 \right) - \frac{625}{384} s_\theta^3 (1 + c_\theta^2) \sin(5\psi), \quad (\text{C5d})$$

$$H_+^{(2)} = s_\theta \cos(\psi) \left( \frac{5}{4} \ln(2) + \frac{11}{40} + \left( \frac{1}{4} \ln(2) + \frac{7}{40} \right) c_\theta^2 \right) + \cos(2\psi) \left( \frac{11}{60} + \frac{33}{10} c_\theta^2 + \frac{29}{24} c_\theta^4 - \frac{1}{24} c_\theta^6 \right) + s_\theta \cos(3\psi)(1 + c_\theta^2) \left( -\frac{27}{4} \ln\left(\frac{3}{2}\right) + \frac{189}{40} \right) + s_\theta^2 \cos(4\psi) \left( -\frac{118}{15} - \frac{14}{3} c_\theta^2 + \frac{16}{15} c_\theta^4 \right) - \frac{81}{40} s_\theta^4 \cos(6\psi)(1 + c_\theta^2) + \pi s_\theta \sin(\psi) \left( \frac{5}{8} + \frac{1}{8} c_\theta^2 \right) + \frac{27}{8} \pi s_\theta (1 + c_\theta^2) \sin(3\psi), \quad (\text{C5e})$$

$$H_+^{(2.5)} = \pi \cos(2\psi) \left( \frac{19}{3} + 3c_\theta^2 - \frac{2}{3} c_\theta^4 \right) + \frac{16}{3} \pi s_\theta^2 \cos(4\psi)(1 + c_\theta^2) + s_\theta \sin(\psi) \left( -\frac{1771}{5120} + \frac{1667}{5120} c_\theta^2 - \frac{217}{9216} c_\theta^4 + \frac{1}{9216} c_\theta^6 \right) + \sin(2\psi) \left( -\frac{9}{5} + \frac{14}{5} c_\theta^2 + \frac{7}{5} c_\theta^4 \right) + s_\theta \sin(3\psi) \left( \frac{3537}{1024} - \frac{22977}{5120} c_\theta^2 - \frac{15309}{5120} c_\theta^4 + \frac{729}{5120} c_\theta^6 \right) + s_\theta^2 (1 + c_\theta^2) \sin(4\psi) \left( -\frac{56}{5} + \frac{32}{3} \ln(2) \right) + s_\theta^3 \sin(5\psi) \left( \frac{108125}{9216} + \frac{1875}{256} c_\theta^2 - \frac{15625}{9216} c_\theta^4 \right) + \frac{117649}{46080} s_\theta^5 (1 + c_\theta^2) \sin(7\psi), \quad (\text{C5f})$$

$$H_+^{(3)} = s_\theta \cos(\psi) \left[ -\frac{2159}{40320} - \frac{19}{32} \ln(2) + \left( -\frac{5}{8} \ln(2) - \frac{95}{224} \right) c_\theta^2 + \left( \frac{1}{96} \ln(2) + \frac{181}{13440} \right) c_\theta^4 \right] + \cos(2\psi) \left[ -\frac{465497}{11025} - \frac{2}{3} \pi^2 + \frac{856}{105} \text{eulerlog}(2, \nu) + \left( -\frac{2}{3} \pi^2 + \frac{856}{105} \text{eulerlog}(2, \nu) - \frac{3561541}{88200} \right) c_\theta^2 - \frac{943}{720} c_\theta^4 + \frac{169}{720} c_\theta^6 - \frac{1}{360} c_\theta^8 \right] + s_\theta \cos(3\psi) \left[ \frac{1971}{64} \ln\left(\frac{3}{2}\right) - \frac{205119}{8960} + \left( -\frac{1917}{224} + \frac{135}{8} \ln\left(\frac{3}{2}\right) \right) c_\theta^2 + \left( -\frac{243}{64} \ln\left(\frac{3}{2}\right) + \frac{43983}{8960} \right) c_\theta^4 \right] + s_\theta^2 \cos(4\psi) \left[ \frac{2189}{210} - \frac{1123}{210} c_\theta^2 - \frac{56}{9} c_\theta^4 + \frac{16}{45} c_\theta^6 \right] + s_\theta^3 \cos(5\psi)(1 + c_\theta^2) \times \left[ -\frac{113125}{5376} + \frac{3125}{192} \ln\left(\frac{5}{2}\right) \right] + s_\theta^4 \cos(6\psi) \left[ \frac{1377}{80} + \frac{891}{80} c_\theta^2 - \frac{729}{280} c_\theta^4 \right] + \frac{1024}{315} s_\theta^6 \cos(8\psi)[1 + c_\theta^2] + \pi s_\theta \sin(\psi) \left[ -\frac{19}{64} - \frac{5}{16} c_\theta^2 + \frac{1}{192} c_\theta^4 \right] - \frac{428}{105} \pi [1 + c_\theta^2] \sin(2\psi) + \pi s_\theta \sin(3\psi) \left[ -\frac{1971}{128} - \frac{135}{16} c_\theta^2 + \frac{243}{128} c_\theta^4 \right] - \frac{3125}{384} \pi s_\theta^3 [1 + c_\theta^2] \sin(5\psi), \quad (\text{C5g})$$

<sup>5</sup>Since we do not compute the  $m = 0$  modes in this work, we will not recover the "direct current" terms in [30,31].

$$\begin{aligned}
 H_+^{(3,5)} = & \pi s_\theta \cos(\psi) \left[ \frac{5}{4} \ln(2) - \frac{53}{140} + \left( \frac{1}{4} \ln(2) + \frac{41}{420} \right) c_\theta^2 \right] + \pi \cos(2\psi) \left[ \frac{11}{30} + \frac{33}{5} c_\theta^2 + \frac{29}{12} c_\theta^4 - \frac{1}{12} c_\theta^6 \right] \\
 & + \pi s_\theta \cos(3\psi) (1 + c_\theta^2) \left[ -\frac{81}{4} \ln\left(\frac{3}{2}\right) + \frac{1107}{140} \right] + \pi s_\theta^2 \cos(4\psi) \left[ -\frac{472}{15} - \frac{56}{3} c_\theta^2 + \frac{64}{15} c_\theta^4 \right] \\
 & - \frac{243}{20} \pi s_\theta^4 \cos(6\psi) [1 + c_\theta^2] + s_\theta \sin(\psi) \left[ -\frac{11}{20} \ln(2) + \frac{5}{48} \pi^2 - \frac{5}{4} \ln(2)^2 - \frac{183}{140} \text{eulerlog}(1, \nu) + \frac{3614800949}{541900800} \right. \\
 & \left. + \left( -\frac{13}{84} \text{eulerlog}(1, \nu) - \frac{7}{20} \ln(2) - \frac{1}{4} \ln(2)^2 + \frac{16709489}{18063360} + \frac{1}{48} \pi^2 \right) c_\theta^2 + \frac{6169}{143360} c_\theta^4 - \frac{29}{40960} c_\theta^6 + \frac{1}{737280} c_\theta^8 \right] \\
 & + \sin(2\psi) \left[ \frac{17957}{5040} - \frac{7583}{1680} c_\theta^2 - \frac{1201}{240} c_\theta^4 + \frac{83}{336} c_\theta^6 \right] + s_\theta \sin(3\psi) \left[ \frac{27}{16} \pi^2 - \frac{81}{4} \ln\left(\frac{3}{2}\right)^2 - \frac{351}{28} \text{eulerlog}(3, \nu) \right. \\
 & \left. + \frac{154720053}{2007040} + \frac{567}{20} \ln\left(\frac{3}{2}\right) + \left( \frac{27}{16} \pi^2 - \frac{351}{28} \text{eulerlog}(3, \nu) - \frac{81}{4} \ln\left(\frac{3}{2}\right)^2 + \frac{567}{20} \ln\left(\frac{3}{2}\right) + \frac{79851141}{1003520} \right) c_\theta^2 \right. \\
 & \left. + \frac{562761}{143360} c_\theta^4 - \frac{19683}{20480} c_\theta^6 + \frac{729}{40960} c_\theta^8 \right] + s_\theta^2 \sin(4\psi) \left[ -\frac{944}{15} \ln(2) + \frac{7076}{105} + \left( -\frac{112}{3} \ln(2) + \frac{1212}{35} \right) c_\theta^2 \right. \\
 & \left. + \left( \frac{128}{15} \ln(2) - \frac{1328}{105} \right) c_\theta^4 \right] + s_\theta^3 \sin(5\psi) \left[ -\frac{3953125}{172032} + \frac{2696875}{516096} c_\theta^2 + \frac{6078125}{516096} c_\theta^4 - \frac{390625}{516096} c_\theta^6 \right] \\
 & + s_\theta^4 (1 + c_\theta^2) \sin(6\psi) \left[ \frac{20169}{560} - \frac{243}{10} \ln(3) \right] + s_\theta^5 \sin(7\psi) \left[ -\frac{36824137}{1474560} - \frac{1529437}{92160} c_\theta^2 + \frac{5764801}{1474560} c_\theta^4 \right] \\
 & - \frac{4782969}{1146880} s_\theta^7 [1 + c_\theta^2] \sin(9\psi), \tag{C5h}
 \end{aligned}$$

$$\begin{aligned}
 H_+^{(4)} = & s_\theta \cos(\psi) \left[ -\frac{1771}{2560} \ln(2) - \frac{5290289}{19353600} + \left( \frac{8167913}{19353600} + \frac{1667}{2560} \ln(2) \right) c_\theta^2 + \left( -\frac{217}{4608} \ln(2) - \frac{141173}{2322432} \right) c_\theta^4 \right. \\
 & \left. + \left( \frac{1}{4608} \ln(2) + \frac{4129}{11612160} \right) c_\theta^6 \right] + \cos(2\psi) \left[ \frac{19}{9} \pi^2 - \frac{49684}{2079} \text{eulerlog}(2, \nu) + \frac{390473238599}{4610390400} \right. \\
 & \left. + \left( \frac{52743420953}{1536796800} + \pi^2 - \frac{52676}{3465} \text{eulerlog}(2, \nu) \right) c_\theta^2 + \left( \frac{12568}{10395} \text{eulerlog}(2, \nu) - \frac{2903305411}{461039040} - \frac{2}{9} \pi^2 \right) c_\theta^4 \right. \\
 & \left. - \frac{1028339}{1995840} c_\theta^6 + \frac{61}{2880} c_\theta^8 - \frac{1}{8640} c_\theta^{10} \right] + s_\theta \cos(3\psi) \left[ -\frac{10611}{512} \ln\left(\frac{3}{2}\right) + \frac{13690863}{716800} \right. \\
 & \left. + \left( \frac{68931}{2560} \ln\left(\frac{3}{2}\right) - \frac{612441}{20480} \right) c_\theta^2 + \left( -\frac{16433847}{716800} + \frac{45927}{2560} \ln\left(\frac{3}{2}\right) \right) c_\theta^4 + \left( \frac{1003347}{716800} - \frac{2187}{2560} \ln\left(\frac{3}{2}\right) \right) c_\theta^6 \right] \\
 & + s_\theta^2 \cos(4\psi) \left[ \frac{3196066957}{28814940} - \frac{201088}{10395} \text{eulerlog}(4, \nu) + \frac{32}{9} \pi^2 + \frac{448}{5} \ln(2) - \frac{128}{3} \ln(2)^2 \right. \\
 & \left. + \left( \frac{448}{5} \ln(2) + \frac{3689222389}{28814940} - \frac{201088}{10395} \text{eulerlog}(4, \nu) + \frac{32}{9} \pi^2 - \frac{128}{3} \ln(2)^2 \right) c_\theta^2 + \frac{320458}{31185} c_\theta^4 - \frac{2648}{945} c_\theta^6 + \frac{64}{945} c_\theta^8 \right] \\
 & + s_\theta^3 \cos(5\psi) \left[ -\frac{540625}{4608} \ln\left(\frac{5}{2}\right) + \frac{355593125}{2322432} + \left( \frac{17029375}{193536} - \frac{9375}{128} \ln\left(\frac{5}{2}\right) \right) c_\theta^2 + \left( -\frac{64515625}{2322432} + \frac{78125}{4608} \ln\left(\frac{5}{2}\right) \right) c_\theta^4 \right] \\
 & + s_\theta^4 \cos(6\psi) \left[ -\frac{2193561}{49280} + \frac{153333}{49280} c_\theta^2 + \frac{94041}{4480} c_\theta^4 - \frac{6561}{4480} c_\theta^6 \right] + s_\theta^5 \cos(7\psi) (1 + c_\theta^2) \left[ -\frac{823543}{23040} \ln\left(\frac{7}{2}\right) \right. \\
 & \left. + \frac{485772721}{8294400} \right] + s_\theta^6 \cos(8\psi) \left[ -\frac{101888}{2835} - \frac{512}{21} c_\theta^2 + \frac{16384}{2835} c_\theta^4 \right] - \frac{390625}{72576} s_\theta^8 \cos(10\psi) [1 + c_\theta^2] \\
 & + \pi s_\theta \sin(\psi) \left[ -\frac{1771}{5120} + \frac{1667}{5120} c_\theta^2 - \frac{217}{9216} c_\theta^4 + \frac{1}{9216} c_\theta^6 \right] + \pi \sin(2\psi) \left[ \frac{86788}{10395} + \frac{45742}{3465} c_\theta^2 + \frac{22822}{10395} c_\theta^4 \right] \\
 & + \pi s_\theta \sin(3\psi) \left[ \frac{10611}{1024} - \frac{68931}{5120} c_\theta^2 - \frac{45927}{5120} c_\theta^4 + \frac{2187}{5120} c_\theta^6 \right] + \pi s_\theta^2 (1 + c_\theta^2) \sin(4\psi) \left[ \frac{128}{3} \ln(2) - \frac{365152}{10395} \right] \\
 & + \pi s_\theta^3 \sin(5\psi) \left[ \frac{540625}{9216} + \frac{9375}{256} c_\theta^2 - \frac{78125}{9216} c_\theta^4 \right] + \frac{823543}{46080} \pi s_\theta^5 [1 + c_\theta^2] \sin(7\psi), \tag{C5i}
 \end{aligned}$$



$$\begin{aligned}
 H_+^{(4,5)} = & \pi s_\theta \cos(\psi) \left[ -\frac{19}{32} \ln(2) + \frac{5044511}{17297280} + \left( -\frac{5}{8} \ln(2) - \frac{36451}{160160} \right) c_\theta^2 + \left( \frac{1}{96} \ln(2) + \frac{66827}{5765760} \right) c_\theta^4 \right] \\
 & + \pi \cos(2\psi) \left[ -\frac{930994}{11025} + \frac{1712}{105} \operatorname{eulerlog}(2, \nu) + \frac{4}{3} \pi^2 + \left( \frac{1712}{105} \operatorname{eulerlog}(2, \nu) - \frac{3561541}{44100} + \frac{4}{3} \pi^2 \right) c_\theta^2 \right. \\
 & \left. - \frac{943}{360} c_\theta^4 + \frac{169}{360} c_\theta^6 - \frac{1}{180} c_\theta^8 \right] + \pi s_\theta \cos(3\psi) \left[ -\frac{7502679}{183040} + \frac{5913}{64} \ln\left(\frac{3}{2}\right) + \left( \frac{405}{8} \ln\left(\frac{3}{2}\right) - \frac{1244619}{160160} \right) c_\theta^2 \right. \\
 & \left. + \left( -\frac{729}{64} \ln\left(\frac{3}{2}\right) + \frac{16238961}{1281280} \right) c_\theta^4 \right] + \pi s_\theta^2 \cos(4\psi) \left[ \frac{4378}{105} - \frac{2246}{105} c_\theta^2 - \frac{224}{9} c_\theta^4 + \frac{64}{45} c_\theta^6 \right] + \pi s_\theta^3 \cos(5\psi) (1 + c_\theta^2) \\
 & \times \left[ -\frac{208834375}{2306304} + \frac{15625}{192} \ln\left(\frac{5}{2}\right) \right] + \pi s_\theta^4 \cos(6\psi) \left[ \frac{4131}{40} + \frac{2673}{40} c_\theta^2 - \frac{2187}{140} c_\theta^4 \right] + \frac{8192}{315} \pi s_\theta^6 \cos(8\psi) [1 + c_\theta^2] \\
 & + s_\theta \sin(\psi) \left[ \frac{2985361}{4324320} \operatorname{eulerlog}(1, \nu) + \frac{2725185940926109}{797855721062400} - \frac{19}{384} \pi^2 + \frac{19}{32} \ln(2)^2 + \frac{2159}{20160} \ln(2) \right. \\
 & \left. + \left( \frac{15737}{40040} \operatorname{eulerlog}(1, \nu) - \frac{117462416679523}{88650635673600} + \frac{5}{8} \ln(2)^2 - \frac{5}{96} \pi^2 + \frac{95}{112} \ln(2) \right) c_\theta^2 + \left( -\frac{181}{6720} \ln(2) + \frac{1}{1152} \pi^2 \right. \right. \\
 & \left. \left. - \frac{773}{205920} \operatorname{eulerlog}(1, \nu) - \frac{1}{96} \ln(2)^2 - \frac{66829188883}{3799312957440} \right) c_\theta^4 + \frac{23406169}{12076646400} c_\theta^6 - \frac{341}{29491200} c_\theta^8 \right. \\
 & \left. + \frac{1}{88473600} c_\theta^{10} \right] + \sin(2\psi) \left[ \frac{64}{3} \zeta(3) + \frac{236081}{50400} - \frac{856}{63} \pi^2 + \left( -\frac{856}{63} \pi^2 - \frac{250211}{151200} + \frac{64}{3} \zeta(3) \right) c_\theta^2 + \frac{2416699}{453600} c_\theta^4 \right. \\
 & \left. - \frac{630719}{453600} c_\theta^6 + \frac{319}{16200} c_\theta^8 \right] + s_\theta \sin(3\psi) \left[ \frac{5913}{64} \ln\left(\frac{3}{2}\right)^2 - \frac{615357}{4480} \ln\left(\frac{3}{2}\right) + \frac{17738649}{320320} \operatorname{eulerlog}(3, \nu) - \frac{1971}{256} \pi^2 \right. \\
 & \left. - \frac{863088610282869}{3283356876800} + \left( -\frac{497738577409209}{3283356876800} + \frac{1433673}{40040} \operatorname{eulerlog}(3, \nu) - \frac{5751}{112} \ln\left(\frac{3}{2}\right) - \frac{135}{32} \pi^2 + \frac{405}{8} \ln\left(\frac{3}{2}\right)^2 \right) c_\theta^2 \right. \\
 & \left. + \left( \frac{131949}{4480} \ln\left(\frac{3}{2}\right) - \frac{729}{64} \ln\left(\frac{3}{2}\right)^2 + \frac{831522561921}{46905098240} + \frac{243}{256} \pi^2 - \frac{187839}{45760} \operatorname{eulerlog}(3, \nu) \right) c_\theta^4 + \frac{370480473}{149094400} c_\theta^6 \right. \\
 & \left. - \frac{713691}{4587520} c_\theta^8 + \frac{6561}{4587520} c_\theta^{10} \right] + s_\theta^2 \sin(4\psi) \left[ -\frac{1332067}{14175} + \frac{8756}{105} \ln(2) + \left( -\frac{4492}{105} \ln(2) + \frac{971317}{14175} \right) c_\theta^2 \right. \\
 & \left. + \left( -\frac{448}{9} \ln(2) + \frac{147944}{2025} \right) c_\theta^4 + \left( \frac{128}{45} \ln(2) - \frac{10208}{2025} \right) c_\theta^6 \right] + s_\theta^3 \sin(5\psi) \left[ -\frac{131672845144375}{1013150121984} \right. \\
 & \left. - \frac{565625}{2688} \ln\left(\frac{5}{2}\right) + \frac{15625}{192} \ln\left(\frac{5}{2}\right)^2 + \frac{2415625}{82368} \operatorname{eulerlog}(5, \nu) - \frac{15625}{2304} \pi^2 + \left( \frac{15625}{192} \ln\left(\frac{5}{2}\right)^2 + \frac{2415625}{82368} \operatorname{eulerlog}(5, \nu) \right. \right. \\
 & \left. \left. - \frac{91729414920625}{506575060992} - \frac{565625}{2688} \ln\left(\frac{5}{2}\right) - \frac{15625}{2304} \pi^2 \right) c_\theta^2 - \frac{2916390625}{120766464} c_\theta^4 + \frac{169140625}{24772608} c_\theta^6 - \frac{9765625}{49545216} c_\theta^8 \right] \\
 & + s_\theta^4 \sin(6\psi) \left[ -\frac{1723599}{5600} + \frac{4131}{20} \ln(3) + \left( -\frac{1053891}{5600} + \frac{2673}{20} \ln(3) \right) c_\theta^2 + \left( \frac{77517}{1400} - \frac{2187}{70} \ln(3) \right) c_\theta^4 \right] \\
 & + s_\theta^5 \sin(7\psi) \left[ \frac{184464102431}{2300313600} + \frac{6079747373}{2300313600} c_\theta^2 - \frac{3810533461}{106168320} c_\theta^4 + \frac{282475249}{106168320} c_\theta^6 \right] + s_\theta^6 (1 + c_\theta^2) \sin(8\psi) \\
 & \times \left[ -\frac{1306624}{14175} + \frac{32768}{315} \ln(2) \right] + s_\theta^7 \sin(9\psi) \left[ \frac{2358003717}{45875200} + \frac{81310473}{2293760} c_\theta^2 - \frac{387420489}{45875200} c_\theta^4 \right] \\
 & + \frac{25937424601}{3715891200} s_\theta^9 [1 + c_\theta^2] \sin(11\psi), \tag{C5j}
 \end{aligned}$$

$$\begin{aligned}
 H_+^{(5)} = & s_\theta \cos(\psi) \left[ -\frac{1753}{1680} \pi^2 - \frac{11}{20} \ln(2)^2 + \frac{5}{24} \ln(2) \pi^2 + \frac{3614800949}{270950400} \ln(2) + \frac{5}{3} \zeta(3) - \frac{17}{28} \text{eulerlog}(1, \nu) \right. \\
 & + \frac{2808855689}{851558400} - \frac{5}{6} \ln(2)^3 - \frac{183}{70} \ln(2) \text{eulerlog}(1, \nu) + \left( -\frac{13}{60} \text{eulerlog}(1, \nu) + \frac{1376391589}{993484800} - \frac{7}{20} \ln(2)^2 \right. \\
 & - \frac{13}{42} \ln(2) \text{eulerlog}(1, \nu) + \frac{1}{3} \zeta(3) - \frac{1}{6} \ln(2)^3 - \frac{503}{5040} \pi^2 + \frac{1}{24} \ln(2) \pi^2 + \frac{16709489}{9031680} \ln(2) \Big) c_\theta^2 + \left( \frac{6169}{71680} \ln(2) \right. \\
 & + \frac{247872791}{2235340800} \Big) c_\theta^4 + \left( -\frac{423289}{182476800} - \frac{29}{20480} \ln(2) \Big) c_\theta^6 + \left( \frac{13093}{2554675200} + \frac{1}{368640} \ln(2) \Big) c_\theta^8 \right] \\
 & + \cos(2\psi) \left[ \frac{15970130661967}{222615993600} + \frac{11}{90} \pi^2 - \frac{713942}{135135} \text{eulerlog}(2, \nu) + \left( -\frac{332896}{15015} \text{eulerlog}(2, \nu) + \frac{643219288717}{4947022080} \right. \right. \\
 & + \frac{11}{5} \pi^2 \Big) c_\theta^2 + \left( \frac{1458499764979}{77915597760} + \frac{29}{36} \pi^2 - \frac{599887}{135135} \text{eulerlog}(2, \nu) \Big) c_\theta^4 + \left( \frac{901}{9009} \text{eulerlog}(2, \nu) - \frac{1}{36} \pi^2 \right. \right. \\
 & + \frac{13211394139}{51943731840} \Big) c_\theta^6 - \frac{449833}{6652800} c_\theta^8 + \frac{673}{604800} c_\theta^{10} - \frac{1}{302400} c_\theta^{12} \Big] + s_\theta \cos(3\psi) \left[ \frac{37689199239}{110387200} - \frac{1701}{20} \ln\left(\frac{3}{2}\right) \right. \\
 & - \frac{1053}{20} \text{eulerlog}(3, \nu) + 81 \zeta(3) + \frac{81}{2} \ln\left(\frac{3}{2}\right)^3 - \frac{464160159}{1003520} \ln\left(\frac{3}{2}\right) - \frac{13581}{560} \pi^2 - \frac{81}{8} \pi^2 \ln\left(\frac{3}{2}\right) \\
 & + \frac{1053}{14} \text{eulerlog}(3, \nu) \ln\left(\frac{3}{2}\right) + \left( -\frac{13581}{560} \pi^2 - \frac{239553423}{501760} \ln\left(\frac{3}{2}\right) + \frac{1053}{14} \text{eulerlog}(3, \nu) \ln\left(\frac{3}{2}\right) \right. \\
 & - \frac{1053}{20} \text{eulerlog}(3, \nu) - \frac{81}{8} \pi^2 \ln\left(\frac{3}{2}\right) + \frac{704592069}{1971200} - \frac{1701}{20} \ln\left(\frac{3}{2}\right)^2 + 81 \zeta(3) + \frac{81}{2} \ln\left(\frac{3}{2}\right)^3 \Big) c_\theta^2 \\
 & + \left( -\frac{1688283}{71680} \ln\left(\frac{3}{2}\right) + \frac{32368167}{1103872} \Big) c_\theta^4 + \left( -\frac{37208403}{3942400} + \frac{59049}{10240} \ln\left(\frac{3}{2}\right) \Big) c_\theta^6 + \left( -\frac{2187}{20480} \ln\left(\frac{3}{2}\right) + \frac{3181599}{15769600} \Big) c_\theta^8 \right] \\
 & + s_\theta^2 \cos(4\psi) \left[ \frac{3776}{15} \ln(2)^2 - \frac{944}{45} \pi^2 - \frac{101708443628797}{194788994400} - \frac{56608}{105} \ln(2) + \frac{15158848}{135135} \text{eulerlog}(4, \nu) \right. \\
 & + \left( -\frac{388693669747}{1113079968} + \frac{10079296}{135135} \text{eulerlog}(4, \nu) - \frac{9696}{35} \ln(2) - \frac{112}{9} \pi^2 + \frac{448}{3} \ln(2)^2 \Big) c_\theta^2 + \left( \frac{128}{45} \pi^2 + \frac{58122585749}{2029052025} \right. \right. \\
 & + \frac{10624}{105} \ln(2) - \frac{461312}{45045} \text{eulerlog}(4, \nu) - \frac{512}{15} \ln(2)^2 \Big) c_\theta^4 + \frac{438602}{51975} c_\theta^6 - \frac{632}{945} c_\theta^8 + \frac{8}{945} c_\theta^{10} \Big] \\
 & + s_\theta^3 \cos(5\psi) \left[ -\frac{36498184375}{119218176} + \frac{19765625}{86016} \ln\left(\frac{5}{2}\right) + \left( \frac{4506809375}{39739392} - \frac{13484375}{258048} \ln\left(\frac{5}{2}\right) \right) c_\theta^2 \right. \\
 & + \left( \frac{68434046875}{357654528} - \frac{30390625}{258048} \ln\left(\frac{5}{2}\right) \Big) c_\theta^4 + \left( -\frac{5114453125}{357654528} + \frac{1953125}{258048} \ln\left(\frac{5}{2}\right) \Big) c_\theta^6 \right] \\
 & + s_\theta^4 \cos(6\psi) \left[ \frac{729}{5} \ln(3)^2 - \frac{60507}{140} \ln(3) - \frac{243}{20} \pi^2 - \frac{672377033403}{6412806400} + \frac{218943}{5005} \text{eulerlog}(6, \nu) \right. \\
 & + \left( \frac{729}{5} \ln(3)^2 + \frac{218943}{5005} \text{eulerlog}(6, \nu) - \frac{60507}{140} \ln(3) - \frac{1440739832607}{6412806400} - \frac{243}{20} \pi^2 \Big) c_\theta^2 - \frac{25725681}{492800} c_\theta^4 \right. \\
 & + \frac{133407}{8960} c_\theta^6 - \frac{2187}{4480} c_\theta^8 \Big] + s_\theta^5 \cos(7\psi) \left[ \frac{257768959}{737280} \ln\left(\frac{7}{2}\right) - \frac{139854425207}{243302400} \right. \\
 & + \left( \frac{10706059}{46080} \ln\left(\frac{7}{2}\right) - \frac{11134654307}{30412800} \Big) c_\theta^2 + \left( \frac{75478539493}{729907200} - \frac{40353607}{737280} \ln\left(\frac{7}{2}\right) \Big) c_\theta^4 \right]
 \end{aligned}$$

$$\begin{aligned}
 & + s_\theta^6 \cos(8\psi) \left[ \frac{4298368}{31185} + \frac{763264}{51975} c_\theta^2 - \frac{843776}{14175} c_\theta^4 + \frac{65536}{14175} c_\theta^6 \right] + s_\theta^7 \cos(9\psi) (1 + c_\theta^2) \left[ -\frac{62623413117}{441548800} \right. \\
 & + \frac{43046721}{286720} \ln(3) - \frac{43046721}{573440} \ln(2) \left. \right] + s_\theta^8 \cos(10\psi) \left[ \frac{116796875}{1596672} + \frac{7421875}{145152} c_\theta^2 - \frac{9765625}{798336} c_\theta^4 \right] \\
 & + \frac{17496}{1925} s_\theta^{10} \cos(12\psi) [1 + c_\theta^2] + \pi s_\theta \sin(\psi) \left[ \frac{3779306549}{541900800} - \frac{5}{48} \pi^2 - \frac{5}{4} \ln(2)^2 - \frac{183}{140} \text{eulerlog}(1, \nu) + \frac{53}{70} \ln(2) \right. \\
 & + \left( -\frac{41}{210} \ln(2) - \frac{13}{84} \text{eulerlog}(1, \nu) - \frac{1}{4} \ln(2)^2 + \frac{18666353}{18063360} - \frac{1}{48} \pi^2 \right) c_\theta^2 + \frac{6169}{143360} c_\theta^4 - \frac{29}{40960} c_\theta^6 \\
 & + \frac{1}{737280} c_\theta^8 \left. \right] + \pi \sin(2\psi) \left[ \frac{10559321}{1081080} + \frac{16481}{8008} c_\theta^2 - \frac{8420261}{1081080} c_\theta^4 + \frac{32003}{72072} c_\theta^6 \right] \\
 & + \pi s_\theta \sin(3\psi) \left[ -\frac{1053}{28} \text{eulerlog}(3, \nu) - \frac{243}{4} \ln\left(\frac{3}{2}\right)^2 - \frac{81}{16} \pi^2 + \frac{516995487}{2007040} + \frac{3321}{70} \ln\left(\frac{3}{2}\right) + \left(\frac{3321}{70} \ln\left(\frac{3}{2}\right) \right. \right. \\
 & + \frac{265971087}{1003520} - \frac{81}{16} \pi^2 - \frac{1053}{28} \text{eulerlog}(3, \nu) - \frac{243}{4} \ln\left(\frac{3}{2}\right)^2 \left. \right) c_\theta^2 + \frac{1688283}{143360} c_\theta^4 - \frac{59049}{20480} c_\theta^6 + \frac{2187}{40960} c_\theta^8 \left. \right] \\
 & + \pi s_\theta^2 \sin(4\psi) \left[ -\frac{3776}{15} \ln(2) + \frac{28847824}{135135} + \left( -\frac{448}{3} \ln(2) + \frac{2735696}{27027} \right) c_\theta^2 + \left( \frac{512}{15} \ln(2) - \frac{2048192}{45045} \right) c_\theta^4 \right] \\
 & + \pi s_\theta^3 \sin(5\psi) \left[ -\frac{19765625}{172032} + \frac{13484375}{516096} c_\theta^2 + \frac{30390625}{516096} c_\theta^4 - \frac{1953125}{516096} c_\theta^6 \right] + \pi s_\theta^4 (1 + c_\theta^2) \sin(6\psi) \\
 & \times \left[ -\frac{729}{5} \ln(3) + \frac{7776729}{40040} \right] + \pi s_\theta^5 \sin(7\psi) \left[ -\frac{257768959}{1474560} - \frac{10706059}{92160} c_\theta^2 + \frac{40353607}{1474560} c_\theta^4 \right] \\
 & - \frac{43046721}{1146880} \pi s_\theta^7 [1 + c_\theta^2] \sin(9\psi), \tag{C5k}
 \end{aligned}$$

$$\begin{aligned}
 H_+^{(5.5)} & = \pi s_\theta \cos(\psi) \left[ \frac{1561379273}{47048601600} - \frac{1771}{2560} \ln(2) + \left( \frac{370608237}{1742540800} + \frac{1667}{2560} \ln(2) \right) c_\theta^2 \right. \\
 & + \left( -\frac{1475977183}{28229160960} - \frac{217}{4608} \ln(2) \right) c_\theta^4 + \left( \frac{840821}{2566287360} + \frac{1}{4608} \ln(2) \right) c_\theta^6 \left. \right] \\
 & + \pi \cos(2\psi) \left[ -\frac{99368}{2079} \text{eulerlog}(2, \nu) - \frac{38}{9} \pi^2 + \frac{80210663899}{461039040} \right. \\
 & + \left( -\frac{105352}{3465} \text{eulerlog}(2, \nu) + \frac{46804865561}{768398400} - 2\pi^2 \right) c_\theta^2 + \left( -\frac{17442960719}{1152597600} + \frac{25136}{10395} \text{eulerlog}(2, \nu) + \frac{4}{9} \pi^2 \right) c_\theta^4 \\
 & - \frac{1028339}{997920} c_\theta^6 + \frac{61}{1440} c_\theta^8 - \frac{1}{4320} c_\theta^{10} \left. \right] + \pi s_\theta \cos(3\psi) \left[ \frac{71755246683}{1742540800} - \frac{31833}{512} \ln\left(\frac{3}{2}\right) + \left( \frac{206793}{2560} \ln\left(\frac{3}{2}\right) \right. \right. \\
 & - \frac{125895720231}{1742540800} \left. \right) c_\theta^2 + \left( \frac{137781}{2560} \ln\left(\frac{3}{2}\right) - \frac{102720365379}{1742540800} \right) c_\theta^4 + \left( -\frac{6561}{2560} \ln\left(\frac{3}{2}\right) + \frac{612958509}{158412800} \right) c_\theta^6 \left. \right] \\
 & + \pi s_\theta^2 \cos(4\psi) \left[ \frac{18906768449}{36018675} + \frac{2921216}{10395} \ln(2) - \frac{512}{3} \ln(2)^2 - \frac{128}{9} \pi^2 - \frac{804352}{10395} \text{eulerlog}(4, \nu) \right. \\
 & + \left( -\frac{512}{3} \ln(2)^2 - \frac{128}{9} \pi^2 + \frac{21372545609}{36018675} + \frac{2921216}{10395} \ln(2) - \frac{804352}{10395} \text{eulerlog}(4, \nu) \right) c_\theta^2 \\
 & + \frac{1281832}{31185} c_\theta^4 - \frac{10592}{945} c_\theta^6 + \frac{256}{945} c_\theta^8 \left. \right] + \pi s_\theta^3 \cos(5\psi) \left[ \frac{3731798959375}{5645832192} - \frac{2703125}{4608} \ln\left(\frac{5}{2}\right) \right. \\
 & + \left( -\frac{46875}{128} \ln\left(\frac{5}{2}\right) + \frac{173466753125}{470486016} \right) c_\theta^2 + \left( -\frac{65689140625}{513257472} + \frac{390625}{4608} \ln\left(\frac{5}{2}\right) \right) c_\theta^4 \left. \right] \\
 & + \pi s_\theta^4 \cos(6\psi) \left[ -\frac{6580683}{24640} + \frac{459999}{24640} c_\theta^2 + \frac{282123}{2240} c_\theta^4 - \frac{19683}{2240} c_\theta^6 \right] + \pi s_\theta^5 \cos(7\psi) (1 + c_\theta^2) \\
 & \times \left[ \frac{692452248803}{1833062400} - \frac{5764801}{23040} \ln\left(\frac{7}{2}\right) \right] + \pi s_\theta^6 \cos(8\psi) \left[ -\frac{815104}{2835} - \frac{4096}{21} c_\theta^2 + \frac{131072}{2835} c_\theta^4 \right]
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{1953125}{36288} \pi s_\theta^8 \cos(10\psi)[1 + c_\theta^2] + s_\theta \sin(\psi) \left[ \frac{5290289}{9676800} \ln(2) + \frac{61540628400739667719}{9223212135481344000} + \frac{1771}{2560} \ln(2)^2 \right. \\
 & + \frac{257536997}{420076800} \text{eulerlog}(1, \nu) - \frac{1771}{30720} \pi^2 + \left( \frac{1667}{30720} \pi^2 + \frac{376667450178313183}{144112689616896000} - \frac{1231221763}{2940537600} \text{eulerlog}(1, \nu) \right. \\
 & - \frac{8167913}{9676800} \ln(2) - \frac{1667}{2560} \ln(2)^2 \Big) c_\theta^2 + \left( \frac{141173}{1161216} \ln(2) + \frac{9999193}{588107520} \text{eulerlog}(1, \nu) - \frac{74897190722604377}{1844642427096268800} \right. \\
 & - \frac{217}{55296} \pi^2 + \frac{217}{4608} \ln(2)^2 \Big) c_\theta^4 + \left( -\frac{2987}{53464320} \text{eulerlog}(1, \nu) - \frac{57193270974289}{20961845762457600} - \frac{1}{4608} \ln(2)^2 - \frac{4129}{5806080} \ln(2) \right. \\
 & + \frac{1}{55296} \pi^2 \Big) c_\theta^6 + \frac{118483271}{2779486617600} c_\theta^8 - \frac{221}{1857945600} c_\theta^{10} + \frac{1}{14863564800} c_\theta^{12} \Big] + \sin(2\psi) \left[ -\frac{608}{9} \zeta(3) \right. \\
 & + \frac{159032}{17325} \text{eulerlog}(2, \nu) + \frac{1204678}{31185} \pi^2 - \frac{249409850407}{3841992000} + \left( -\frac{12752}{825} \text{eulerlog}(2, \nu) + \frac{54465777787}{640332000} - 32\zeta(3) \right. \\
 & + \frac{282784}{10395} \pi^2 \Big) c_\theta^2 + \left( -\frac{12568}{2475} \text{eulerlog}(2, \nu) - \frac{33734}{31185} \pi^2 + \frac{742766124271}{23051952000} + \frac{64}{9} \zeta(3) \right) c_\theta^4 + \frac{1280598643}{419126400} c_\theta^6 \\
 & - \frac{2997143}{19958400} c_\theta^8 + \frac{18421}{19958400} c_\theta^{10} \Big] + s_\theta \sin(3\psi) \left[ -\frac{3511527147}{108908800} \text{eulerlog}(3, \nu) - \frac{31833}{512} \ln\left(\frac{3}{2}\right)^2 + \frac{10611}{2048} \pi^2 \right. \\
 & + \frac{41072589}{358400} \ln\left(\frac{3}{2}\right) + \frac{3514307823600697989}{151822421983232000} + \left( -\frac{68931}{10240} \pi^2 - \frac{9723416976987988977}{37955605495808000} + \frac{206793}{2560} \ln\left(\frac{3}{2}\right)^2 \right. \\
 & - \frac{1837323}{10240} \ln\left(\frac{3}{2}\right) + \frac{3804113403}{108908800} \text{eulerlog}(3, \nu) \Big) c_\theta^2 + \left( -\frac{10835432671991490567}{151822421983232000} + \frac{137781}{2560} \ln\left(\frac{3}{2}\right)^2 + \frac{2141460099}{108908800} \right. \\
 & \times \text{eulerlog}(3, \nu) - \frac{45927}{10240} \pi^2 - \frac{49301541}{358400} \ln\left(\frac{3}{2}\right) \Big) c_\theta^4 + \left( \frac{3010041}{358400} \ln\left(\frac{3}{2}\right) + \frac{2187}{10240} \pi^2 - \frac{1088541343451439}{431313698816000} \right. \\
 & - \frac{6532569}{9900800} \text{eulerlog}(3, \nu) - \frac{6561}{2560} \ln\left(\frac{3}{2}\right)^2 \Big) c_\theta^6 + \frac{77975093151}{137258598400} c_\theta^8 - \frac{2814669}{183500800} c_\theta^{10} + \frac{59049}{734003200} c_\theta^{12} \Big] \\
 & + s_\theta^2 \sin(4\psi) \left[ \frac{256}{9} \ln(2) \pi^2 - \frac{1024}{9} \ln(2)^3 - \frac{134057079091}{120062250} + \frac{6392133914}{7203735} \ln(2) + \frac{402176}{2475} \text{eulerlog}(4, \nu) - \frac{2048}{9} \zeta(3) \right. \\
 & + \frac{1792}{5} \ln(2)^2 - \frac{1608704}{10395} \ln(2) \text{eulerlog}(4, \nu) + \frac{1079488}{31185} \pi^2 + \left( -\frac{17336400559}{13340250} + \frac{1079488}{31185} \pi^2 + \frac{402176}{2475} \text{eulerlog}(4, \nu) \right. \\
 & - \frac{2048}{9} \zeta(3) - \frac{1024}{9} \ln(2)^3 + \frac{1792}{5} \ln(2)^2 + \frac{256}{9} \ln(2) \pi^2 + \frac{7378444778}{7203735} \ln(2) - \frac{1608704}{10395} \ln(2) \text{eulerlog}(4, \nu) \Big) c_\theta^2 \\
 & + \left( -\frac{34988014}{297675} + \frac{2563664}{31185} \ln(2) \right) c_\theta^4 + \left( \frac{14435216}{363825} - \frac{21184}{945} \ln(2) \right) c_\theta^6 + \left( \frac{512}{945} \ln(2) - \frac{1178944}{1091475} \right) c_\theta^8 \Big] \\
 & + s_\theta^3 \sin(5\psi) \left[ -\frac{3514496875}{16803072} \text{eulerlog}(5, \nu) + \frac{1240914560526518125}{1561602054684672} + \frac{2703125}{55296} \pi^2 + \frac{1777965625}{1161216} \ln\left(\frac{5}{2}\right) \right. \\
 & - \frac{2703125}{4608} \ln\left(\frac{5}{2}\right)^2 + \left( \frac{85146875}{96768} \ln\left(\frac{5}{2}\right) - \frac{46875}{128} \ln\left(\frac{5}{2}\right)^2 - \frac{349221875}{2450448} \text{eulerlog}(5, \nu) + \frac{60761394072551429375}{98380929445134336} \right. \\
 & + \frac{15625}{512} \pi^2 \Big) c_\theta^2 + \left( -\frac{390625}{55296} \pi^2 - \frac{322578125}{1161216} \ln\left(\frac{5}{2}\right) + \frac{233359375}{10692864} \text{eulerlog}(5, \nu) - \frac{202353378607890625}{13415581287972864} \right. \\
 & + \frac{390625}{4608} \ln\left(\frac{5}{2}\right)^2 \Big) c_\theta^4 - \frac{15781558984375}{667076788224} c_\theta^6 + \frac{5166015625}{2378170368} c_\theta^8 - \frac{244140625}{7134511104} c_\theta^{10} \Big] + s_\theta^4 \sin(6\psi) \left[ -\frac{6580683}{12320} \ln(3) \right. \\
 & + \frac{2775122451}{3449600} + \left( -\frac{68404257}{492800} + \frac{459999}{12320} \ln(3) \right) c_\theta^2 + \left( \frac{282123}{1120} \ln(3) - \frac{1528530021}{3449600} \right) c_\theta^4 + \left( -\frac{19683}{1120} \ln(3) \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{120860181}{3449600} c_\theta^6 \Big] + s_\theta^5 \sin(7\psi) \Big[ -\frac{210540970605284389}{11407807217664000} + \frac{5764801}{276480} \pi^2 - \frac{2459922941}{38188800} \text{eulerlog}(7, \nu) \\
& + \frac{3400409047}{4147200} \ln\left(\frac{7}{2}\right) - \frac{5764801}{23040} \ln\left(\frac{7}{2}\right)^2 + \left( -\frac{5764801}{23040} \ln\left(\frac{7}{2}\right)^2 + \frac{3400409047}{4147200} \ln\left(\frac{7}{2}\right) - \frac{2459922941}{38188800} \text{eulerlog}(7, \nu) \right. \\
& + \left. \frac{1312380850473448903}{5703903608832000} + \frac{5764801}{276480} \pi^2 \right) c_\theta^2 + \frac{31513175135281}{297802137600} c_\theta^4 - \frac{191800694071}{6370099200} c_\theta^6 + \frac{13841287201}{12740198400} c_\theta^8 \Big] \\
& + s_\theta^6 \sin(8\psi) \Big[ -\frac{3260416}{2835} \ln(2) + \frac{133789696}{130977} + \left( -\frac{16384}{21} \ln(2) + \frac{146262016}{218295} \right) c_\theta^2 + \left( \frac{524288}{2835} \ln(2) - \frac{603619328}{3274425} \right) c_\theta^4 \Big] \\
& + s_\theta^7 \sin(9\psi) \Big[ -\frac{15729085317609}{68629299200} - \frac{232036175097}{6239027200} c_\theta^2 + \frac{77871518289}{807403520} c_\theta^4 - \frac{31381059609}{4037017600} c_\theta^6 \Big] \\
& + s_\theta^8 (1 + c_\theta^2) \sin(10\psi) \Big[ \frac{7195703125}{33530112} - \frac{1953125}{18144} \ln(5) \Big] + s_\theta^9 \sin(11\psi) \Big[ -\frac{18493383740513}{178362777600} - \frac{25937424601}{353894400} c_\theta^2 \\
& + \frac{3138428376721}{178362777600} c_\theta^4 \Big] - \frac{23298085122481}{1961990553600} s_\theta^{11} [1 + c_\theta^2] \sin(13\psi). \tag{C51}
\end{aligned}$$

## 2. Cross modes

$$H_\times^{(0)} = -2c_\theta \sin(2\psi), \tag{C6a}$$

$$H_\times^{(0.5)} = -\frac{3}{4} \cos(\psi) s_\theta c_\theta - \frac{9}{4} s_\theta c_\theta \cos(3\psi), \tag{C6b}$$

$$H_\times^{(1)} = c_\theta \sin(2\psi) \left( \frac{17}{3} - \frac{4}{3} c_\theta^2 \right) + \frac{8}{3} s_\theta^2 c_\theta \sin(4\psi), \tag{C6c}$$

$$H_\times^{(1.5)} = s_\theta c_\theta \cos(\psi) \left( \frac{21}{32} - \frac{5}{96} c_\theta^2 \right) + s_\theta c_\theta \cos(3\psi) \left( \frac{603}{64} - \frac{135}{64} c_\theta^2 \right) + \frac{625}{192} s_\theta^3 c_\theta \cos(5\psi) - 4\pi c_\theta \sin(2\psi), \tag{C6d}$$

$$\begin{aligned}
H_\times^{(2)} = & -\frac{3}{4} \pi c_\theta s_\theta \cos(\psi) - \frac{27}{4} \pi s_\theta c_\theta \cos(3\psi) + s_\theta c_\theta \sin(\psi) \left( \frac{3}{2} \ln(2) + \frac{9}{20} \right) + c_\theta \sin(2\psi) \left( \frac{17}{15} + \frac{113}{30} c_\theta^2 - \frac{1}{4} c_\theta^4 \right) \\
& + s_\theta c_\theta \sin(3\psi) \left( -\frac{27}{2} \ln\left(\frac{3}{2}\right) + \frac{189}{20} \right) + s_\theta^2 c_\theta \sin(4\psi) \left( -\frac{44}{3} + \frac{16}{5} c_\theta^2 \right) - \frac{81}{20} s_\theta^4 c_\theta \sin(6\psi), \tag{C6e}
\end{aligned}$$

$$\begin{aligned}
H_\times^{(2.5)} = & s_\theta c_\theta \cos(\psi) \left( -\frac{913}{7680} + \frac{1891}{11520} c_\theta^2 - \frac{7}{4608} c_\theta^4 \right) + c_\theta \cos(2\psi) \left( 2 - \frac{22}{5} c_\theta^2 \right) + s_\theta c_\theta \cos(3\psi) \\
& \times \left( -\frac{12501}{2560} + \frac{12069}{1280} c_\theta^2 - \frac{1701}{2560} c_\theta^4 \right) + s_\theta^2 c_\theta \cos(4\psi) \left( \frac{112}{5} - \frac{64}{3} \ln(2) \right) + s_\theta^3 c_\theta \cos(5\psi) \\
& \times \left( -\frac{101875}{4608} + \frac{21875}{4608} c_\theta^2 \right) - \frac{117649}{23040} s_\theta^5 c_\theta \cos(7\psi) + \pi c_\theta \sin(2\psi) \left( \frac{34}{3} - \frac{8}{3} c_\theta^2 \right) + \frac{32}{3} \pi s_\theta^2 c_\theta \sin(4\psi), \tag{C6f}
\end{aligned}$$

$$\begin{aligned}
H_\times^{(3)} = & \pi s_\theta c_\theta \cos(\psi) \left( \frac{21}{32} - \frac{5}{96} c_\theta^2 \right) + \frac{856}{105} \pi c_\theta \cos(2\psi) + \pi s_\theta c_\theta \cos(3\psi) \left( \frac{1809}{64} - \frac{405}{64} c_\theta^2 \right) + \frac{3125}{192} \pi s_\theta^3 c_\theta \cos(5\psi) \\
& + s_\theta c_\theta \sin(\psi) \left( -\frac{21}{16} \ln(2) - \frac{11617}{20160} + \left( \frac{5}{48} \ln(2) + \frac{251}{2240} \right) c_\theta^2 \right) + c_\theta \sin(2\psi) \left( -\frac{3620761}{44100} + \frac{1712}{105} \text{eulerlog}(2, \nu) \right. \\
& - \left. \frac{4}{3} \pi^2 - \frac{3413}{1260} c_\theta^2 + \frac{2909}{2520} c_\theta^4 - \frac{1}{45} c_\theta^6 \right) + s_\theta c_\theta \sin(3\psi) \left( -\frac{36801}{896} + \frac{1809}{32} \ln\left(\frac{3}{2}\right) + \left( -\frac{405}{32} \ln\left(\frac{3}{2}\right) + \frac{65097}{4480} \right) c_\theta^2 \right) \\
& + s_\theta^2 c_\theta \sin(4\psi) \left( \frac{1781}{105} - \frac{1208}{63} c_\theta^2 + \frac{64}{45} c_\theta^4 \right) + s_\theta^3 c_\theta \sin(5\psi) \left( \frac{3125}{96} \ln\left(\frac{5}{2}\right) - \frac{113125}{2688} \right) + s_\theta^4 c_\theta \sin(6\psi) \\
& \times \left( \frac{9153}{280} - \frac{243}{35} c_\theta^2 \right) + \frac{2048}{315} s_\theta^6 c_\theta \sin(8\psi), \tag{C6g}
\end{aligned}$$



$$\begin{aligned}
 H_{\times}^{(3.5)} = & s_{\theta} c_{\theta} \cos(\psi) \left[ \frac{307}{210} \text{eulerlog}(1, \nu) + \frac{9}{10} \ln(2) - \frac{1}{8} \pi^2 + \frac{3}{2} \ln(2)^2 - \frac{2025831067}{270950400} - \frac{436553}{2580480} c_{\theta}^2 + \frac{2291}{286720} c_{\theta}^4 \right. \\
 & - \frac{1}{40960} c_{\theta}^6 \left. \right] + c_{\theta} \cos(2\psi) \left[ -\frac{1831}{280} + \frac{3419}{252} c_{\theta}^2 - \frac{1109}{840} c_{\theta}^4 \right] + s_{\theta} c_{\theta} \cos(3\psi) \left[ -\frac{75787641}{501760} + \frac{351}{14} \text{eulerlog}(3, \nu) \right. \\
 & - \frac{567}{10} \ln\left(\frac{3}{2}\right) + \frac{81}{2} \ln\left(\frac{3}{2}\right)^2 - \frac{27}{8} \pi^2 - \frac{1794069}{143360} c_{\theta}^2 + \frac{7209}{1792} c_{\theta}^4 - \frac{2187}{20480} c_{\theta}^6 \left. \right] + s_{\theta}^2 c_{\theta} \cos(4\psi) \left[ \frac{352}{3} \ln(2) \right. \\
 & - \frac{13096}{105} + \left( -\frac{128}{5} \ln(2) + \frac{3712}{105} \right) c_{\theta}^2 \left. \right] + s_{\theta}^3 c_{\theta} \cos(5\psi) \left[ \frac{90625}{2304} - \frac{9115625}{258048} c_{\theta}^2 + \frac{78125}{28672} c_{\theta}^4 \right] \\
 & + s_{\theta}^4 c_{\theta} \cos(6\psi) \left[ -\frac{20169}{280} + \frac{243}{5} \ln(3) \right] + s_{\theta}^5 c_{\theta} \cos(7\psi) \left[ \frac{35177051}{737280} - \frac{823543}{81920} c_{\theta}^2 \right] + \frac{4782969}{573440} s_{\theta}^7 c_{\theta} \cos(9\psi) \\
 & + \pi s_{\theta} c_{\theta} \sin(\psi) \left[ -\frac{59}{210} + \frac{3}{2} \ln(2) \right] + \pi c_{\theta} \sin(2\psi) \left[ \frac{34}{15} + \frac{113}{15} c_{\theta}^2 - \frac{1}{2} c_{\theta}^4 \right] + \pi s_{\theta} c_{\theta} \sin(3\psi) \left[ -\frac{81}{2} \ln\left(\frac{3}{2}\right) + \frac{1107}{70} \right] \\
 & + \pi s_{\theta}^2 c_{\theta} \sin(4\psi) \left[ -\frac{176}{3} + \frac{64}{5} c_{\theta}^2 \right] - \frac{243}{10} \pi s_{\theta}^4 c_{\theta} \sin(6\psi), \tag{C6h}
 \end{aligned}$$

$$\begin{aligned}
 H_{\times}^{(4)} = & \pi s_{\theta} c_{\theta} \cos(\psi) \left[ -\frac{913}{7680} + \frac{1891}{11520} c_{\theta}^2 - \frac{7}{4608} c_{\theta}^4 \right] + \pi c_{\theta} \cos(2\psi) \left[ -\frac{187384}{10395} - \frac{59452}{10395} c_{\theta}^2 \right] + \pi s_{\theta} c_{\theta} \cos(3\psi) \\
 & \times \left[ -\frac{37503}{2560} + \frac{36207}{1280} c_{\theta}^2 - \frac{5103}{2560} c_{\theta}^4 \right] + \pi s_{\theta}^2 c_{\theta} \cos(4\psi) \left[ -\frac{256}{3} \ln(2) + \frac{730304}{10395} \right] + \pi s_{\theta}^3 c_{\theta} \cos(5\psi) \\
 & \times \left[ -\frac{509375}{4608} + \frac{109375}{4608} c_{\theta}^2 \right] - \frac{823543}{23040} \pi s_{\theta}^5 c_{\theta} \cos(7\psi) + s_{\theta} c_{\theta} \sin(\psi) \left[ \frac{913}{3840} \ln(2) + \frac{4253489}{9676800} + \left( -\frac{5164843}{14515200} \right. \right. \\
 & - \frac{1891}{5760} \ln(2) \left. \right) c_{\theta}^2 + \left( \frac{7}{2304} \ln(2) + \frac{26263}{5806080} \right) c_{\theta}^4 \left. \right] + c_{\theta} \sin(2\psi) \left[ \frac{337599069919}{2305195200} - \frac{457928}{10395} \text{eulerlog}(2, \nu) + \frac{34}{9} \pi^2 \right. \\
 & + \left( \frac{64048}{10395} \text{eulerlog}(2, \nu) - \frac{8}{9} \pi^2 - \frac{935502853}{28814940} \right) c_{\theta}^2 - \frac{634357}{332640} c_{\theta}^4 + \frac{6611}{45360} c_{\theta}^6 - \frac{1}{864} c_{\theta}^8 \left. \right] + s_{\theta} c_{\theta} \sin(3\psi) \\
 & \times \left[ \frac{10428183}{358400} - \frac{37503}{1280} \ln\left(\frac{3}{2}\right) + \left( \frac{36207}{640} \ln\left(\frac{3}{2}\right) - \frac{12089331}{179200} \right) c_{\theta}^2 + \left( \frac{2162943}{358400} - \frac{5103}{1280} \ln\left(\frac{3}{2}\right) \right) c_{\theta}^4 \right] + s_{\theta}^2 c_{\theta} \sin(4\psi) \\
 & \times \left[ \frac{64}{9} \pi^2 - \frac{402176}{10395} \text{eulerlog}(4, \nu) + \frac{896}{5} \ln(2) - \frac{256}{3} \ln(2)^2 + \frac{3160094713}{14407470} + \frac{166802}{4455} c_{\theta}^2 - \frac{4304}{405} c_{\theta}^4 + \frac{64}{189} c_{\theta}^6 \right] \\
 & + s_{\theta}^3 c_{\theta} \sin(5\psi) \left[ \frac{333911875}{1161216} - \frac{509375}{2304} \ln\left(\frac{5}{2}\right) + \left( -\frac{86196875}{1161216} + \frac{109375}{2304} \ln\left(\frac{5}{2}\right) \right) c_{\theta}^2 \right] + s_{\theta}^4 c_{\theta} \sin(6\psi) \\
 & \times \left[ -\frac{1931607}{24640} + \frac{68769}{1120} c_{\theta}^2 - \frac{2187}{448} c_{\theta}^4 \right] + s_{\theta}^5 c_{\theta} \sin(7\psi) \left[ \frac{485772721}{4147200} - \frac{823543}{11520} \ln\left(\frac{7}{2}\right) \right] + s_{\theta}^6 c_{\theta} \sin(8\psi) \\
 & \times \left[ -\frac{195584}{2835} + \frac{8192}{567} c_{\theta}^2 \right] - \frac{390625}{36288} s_{\theta}^8 c_{\theta} \sin(10\psi), \tag{C6i}
 \end{aligned}$$

$$\begin{aligned}
 H_{\times}^{(4.5)} = & s_{\theta} c_{\theta} \cos(\psi) \left[ -\frac{11617}{10080} \ln(2) - \frac{2435639}{2162160} \text{eulerlog}(1, \nu) + \frac{7}{64} \pi^2 - \frac{21}{16} \ln(2)^2 - \frac{731286776414651}{398927860531200} \right. \\
 & + \left( \frac{5}{48} \ln(2)^2 - \frac{5}{576} \pi^2 + \frac{341}{7280} \text{eulerlog}(1, \nu) - \frac{136508869499}{604436152320} + \frac{251}{1120} \ln(2) \right) c_{\theta}^2 - \frac{358093861}{22140518400} c_{\theta}^4 \\
 & + \frac{5927}{33177600} c_{\theta}^6 - \frac{11}{44236800} c_{\theta}^8 \left. \right] + c_{\theta} \cos(2\psi) \left[ \frac{1712}{63} \pi^2 - \frac{128}{3} \zeta(3) + \frac{7787}{25200} - \frac{377687}{28350} c_{\theta}^2 + \frac{200021}{32400} c_{\theta}^4 \right. \\
 & - \frac{16619}{113400} c_{\theta}^6 \left. \right] + s_{\theta} c_{\theta} \cos(3\psi) \left[ \frac{779604098799429}{1641678438400} + \frac{1809}{128} \pi^2 - \frac{16500051}{160160} \text{eulerlog}(3, \nu) + \frac{110403}{448} \ln\left(\frac{3}{2}\right) \right. \\
 & - \frac{5427}{32} \ln\left(\frac{3}{2}\right)^2 + \left( -\frac{5929269115101}{82083921920} - \frac{195291}{2240} \ln\left(\frac{3}{2}\right) + \frac{2553471}{160160} \text{eulerlog}(3, \nu) - \frac{405}{128} \pi^2 + \frac{1215}{32} \ln\left(\frac{3}{2}\right)^2 \right) c_{\theta}^2 \\
 & - \frac{7442184177}{820019200} c_{\theta}^4 + \frac{2413233}{2867200} c_{\theta}^6 - \frac{24057}{2293760} c_{\theta}^8 \left. \right] + s_{\theta}^2 c_{\theta} \cos(4\psi) \left[ -\frac{14248}{105} \ln(2) + \frac{87718}{567} + \left( \frac{9664}{63} \ln(2) \right. \right. \\
 & - \left. \left. \frac{204416}{945} \right) c_{\theta}^2 + \left( \frac{269888}{14175} - \frac{512}{45} \ln(2) \right) c_{\theta}^4 \right] + s_{\theta}^3 c_{\theta} \cos(5\psi) \left[ -\frac{2415625}{41184} \text{eulerlog}(5, \nu) + \frac{565625}{1344} \ln\left(\frac{5}{2}\right) \right. \\
 & + \frac{15625}{1152} \pi^2 - \frac{15625}{96} \ln\left(\frac{5}{2}\right)^2 + \frac{132078703275625}{506575060992} + \frac{968461990625}{10627448832} c_{\theta}^2 - \frac{1795859375}{74317824} c_{\theta}^4 + \frac{21484375}{24772608} c_{\theta}^6 \left. \right] \\
 & + s_{\theta}^4 c_{\theta} \cos(6\psi) \left[ \frac{1633689}{2800} - \frac{27459}{70} \ln(3) + \left( \frac{2916}{35} \ln(3) - \frac{199989}{1400} \right) c_{\theta}^2 \right] + s_{\theta}^5 c_{\theta} \cos(7\psi) \left[ -\frac{165535790119}{1150156800} \right. \\
 & + \frac{13627166021}{132710400} c_{\theta}^2 - \frac{443889677}{53084160} c_{\theta}^4 \left. \right] + s_{\theta}^6 c_{\theta} \cos(8\psi) \left[ \frac{2613248}{14175} - \frac{65536}{315} \ln(2) \right] + s_{\theta}^7 c_{\theta} \cos(9\psi) \left[ -\frac{90876411}{917504} \right. \\
 & + \frac{473513931}{22937600} c_{\theta}^2 \left. \right] - \frac{25937424601}{1857945600} s_{\theta}^9 c_{\theta} \cos(11\psi) + \pi s_{\theta} c_{\theta} \sin(\psi) \left[ -\frac{22483}{1729728} - \frac{21}{16} \ln(2) \right. \\
 & + \left. \left( \frac{2581}{29120} + \frac{5}{48} \ln(2) \right) c_{\theta}^2 \right] + \pi c_{\theta} \sin(2\psi) \left[ \frac{8}{3} \pi^2 + \frac{3424}{105} \text{eulerlog}(2, \nu) - \frac{3620761}{22050} - \frac{3413}{630} c_{\theta}^2 + \frac{2909}{1260} c_{\theta}^4 \right. \\
 & - \left. \frac{2}{45} c_{\theta}^6 \right] + \pi s_{\theta} c_{\theta} \sin(3\psi) \left[ -\frac{45938043}{640640} + \frac{5427}{32} \ln\left(\frac{3}{2}\right) + \left( -\frac{1215}{32} \ln\left(\frac{3}{2}\right) + \frac{3259953}{91520} \right) c_{\theta}^2 \right] \\
 & + \pi s_{\theta}^2 c_{\theta} \sin(4\psi) \left[ \frac{7124}{105} - \frac{4832}{63} c_{\theta}^2 + \frac{256}{45} c_{\theta}^4 \right] + \pi s_{\theta}^3 c_{\theta} \sin(5\psi) \left[ -\frac{208834375}{1153152} + \frac{15625}{96} \ln\left(\frac{5}{2}\right) \right] \\
 & + \pi s_{\theta}^4 c_{\theta} \sin(6\psi) \left[ \frac{27459}{140} - \frac{1458}{35} c_{\theta}^2 \right] + \frac{16384}{315} \pi s_{\theta}^6 c_{\theta} \sin(8\psi), \tag{C6j}
 \end{aligned}$$

$$\begin{aligned}
 H_{\times}^{(5)} = & \pi s_{\theta} c_{\theta} \cos(\psi) \left[ \frac{3}{2} \ln(2)^2 - \frac{2137436827}{270950400} - \frac{59}{105} \ln(2) + \frac{1}{8} \pi^2 + \frac{307}{210} \operatorname{eulerlog}(1, \nu) - \frac{436553}{2580480} c_{\theta}^2 + \frac{2291}{286720} c_{\theta}^4 \right. \\
 & - \frac{1}{40960} c_{\theta}^6 \left. \right] + \pi c_{\theta} \cos(2\psi) \left[ -\frac{11456867}{540540} + \frac{5135479}{270270} c_{\theta}^2 - \frac{412037}{180180} c_{\theta}^4 \right] + \pi s_{\theta} c_{\theta} \cos(3\psi) \left[ \frac{243}{2} \ln\left(\frac{3}{2}\right)^2 \right. \\
 & + \frac{81}{8} \pi^2 - \frac{3321}{35} \ln\left(\frac{3}{2}\right) + \frac{1053}{14} \operatorname{eulerlog}(3, \nu) - \frac{253780587}{501760} - \frac{5382207}{143360} c_{\theta}^2 + \frac{21627}{1792} c_{\theta}^4 - \frac{6561}{20480} c_{\theta}^6 \left. \right] \\
 & + \pi s_{\theta}^2 c_{\theta} \cos(4\psi) \left[ \frac{1408}{3} \ln(2) - \frac{53183264}{135135} + \left( -\frac{512}{5} \ln(2) + \frac{5600512}{45045} \right) c_{\theta}^2 \right] + \pi s_{\theta}^3 c_{\theta} \cos(5\psi) \left[ \frac{453125}{2304} \right. \\
 & - \frac{45578125}{258048} c_{\theta}^2 + \frac{390625}{28672} c_{\theta}^4 \left. \right] + \pi s_{\theta}^4 c_{\theta} \cos(6\psi) \left[ -\frac{7776729}{20020} + \frac{1458}{5} \ln(3) \right] + \pi s_{\theta}^5 c_{\theta} \cos(7\psi) \left[ \frac{246239357}{737280} \right. \\
 & - \frac{5764801}{81920} c_{\theta}^2 \left. \right] + \frac{43046721}{573440} \pi s_{\theta}^7 c_{\theta} \cos(9\psi) + s_{\theta} c_{\theta} \sin(\psi) \left[ 2\zeta(3) - \ln(2)^3 - \frac{2881}{2520} \pi^2 - \frac{307}{105} \ln(2) \operatorname{eulerlog}(1, \nu) \right. \\
 & + \frac{1470466037}{331161600} - \frac{9}{10} \ln(2)^2 + \frac{2025831067}{135475200} \ln(2) - \frac{173}{210} \operatorname{eulerlog}(1, \nu) + \frac{1}{4} \ln(2) \pi^2 + \left( \frac{436553}{1290240} \ln(2) \right. \\
 & + \frac{1120328171}{2980454400} \left. \right) c_{\theta}^2 + \left( -\frac{71012509}{2980454400} - \frac{2291}{143360} \ln(2) \right) c_{\theta}^4 + \left( \frac{36899}{425779200} + \frac{1}{20480} \ln(2) \right) c_{\theta}^6 \left. \right] \\
 & + c_{\theta} \sin(2\psi) \left[ \frac{125273231828323}{779155977600} + \frac{34}{45} \pi^2 - \frac{313384}{19305} \operatorname{eulerlog}(2, \nu) + \left( \frac{12035301795191}{194788994400} + \frac{113}{45} \pi^2 \right. \right. \\
 & - \frac{2198276}{135135} \operatorname{eulerlog}(2, \nu) \left. \right) c_{\theta}^2 + \left( -\frac{1}{6} \pi^2 + \frac{31862}{45045} \operatorname{eulerlog}(2, \nu) - \frac{198525061333}{129859329600} \right) c_{\theta}^4 - \frac{24734341}{64864800} c_{\theta}^6 \\
 & + \frac{98573}{9979200} c_{\theta}^8 - \frac{1}{25200} c_{\theta}^{10} \left. \right] + s_{\theta} c_{\theta} \sin(3\psi) \left[ \frac{18081675159}{27596800} - \frac{1701}{10} \ln\left(\frac{3}{2}\right)^2 + \frac{1053}{7} \operatorname{eulerlog}(3, \nu) \ln\left(\frac{3}{2}\right) \right. \\
 & + 162\zeta(3) - \frac{13581}{280} \pi^2 - \frac{1053}{10} \operatorname{eulerlog}(3, \nu) - \frac{227362923}{250880} \ln\left(\frac{3}{2}\right) + 81 \ln\left(\frac{3}{2}\right)^3 - \frac{81}{4} \pi^2 \ln\left(\frac{3}{2}\right) + \left( \frac{5493794949}{55193600} \right. \\
 & - \frac{5382207}{71680} \ln\left(\frac{3}{2}\right) \left. \right) c_{\theta}^2 + \left( \frac{21627}{896} \ln\left(\frac{3}{2}\right) - \frac{254813931}{6899200} \right) c_{\theta}^4 + \left( \frac{9024291}{7884800} - \frac{6561}{10240} \ln\left(\frac{3}{2}\right) \right) c_{\theta}^6 \left. \right] \\
 & + s_{\theta}^2 c_{\theta} \sin(4\psi) \left[ -\frac{352}{9} \pi^2 + \frac{28469888}{135135} \operatorname{eulerlog}(4, \nu) + \frac{1408}{3} \ln(2)^2 - \frac{2667962875727}{2782699920} - \frac{104768}{105} \ln(2) \right. \\
 & + \left( -\frac{307712}{9009} \operatorname{eulerlog}(4, \nu) - \frac{512}{5} \ln(2)^2 + \frac{29696}{105} \ln(2) + \frac{128}{15} \pi^2 + \frac{195652128227}{2029052025} \right) c_{\theta}^2 + \frac{20597648}{675675} c_{\theta}^4 \\
 & - \frac{69536}{22275} c_{\theta}^6 + \frac{16}{315} c_{\theta}^8 \left. \right] + s_{\theta}^3 c_{\theta} \sin(5\psi) \left[ \frac{453125}{1152} \ln\left(\frac{5}{2}\right) - \frac{23517203125}{44706816} + \left( -\frac{45578125}{129024} \ln\left(\frac{5}{2}\right) \right. \right. \\
 & + \frac{100077521875}{178827264} \left. \right) c_{\theta}^2 + \left( \frac{390625}{14336} \ln\left(\frac{5}{2}\right) - \frac{2938515625}{59609088} \right) c_{\theta}^4 \left. \right] + s_{\theta}^4 c_{\theta} \sin(6\psi) \left[ -\frac{243}{10} \pi^2 + \frac{1458}{5} \ln(3)^2 \right. \\
 & - \frac{140126622651}{641280640} + \frac{437886}{5005} \operatorname{eulerlog}(6, \nu) - \frac{60507}{70} \ln(3) - \frac{45043209}{228800} c_{\theta}^2 + \frac{12323259}{246400} c_{\theta}^4 - \frac{2187}{1120} c_{\theta}^6 \left. \right] \\
 & + s_{\theta}^5 c_{\theta} \sin(7\psi) \left[ \frac{246239357}{368640} \ln\left(\frac{7}{2}\right) - \frac{400350017431}{364953600} + \left( \frac{31563932561}{121651200} - \frac{5764801}{40960} \ln\left(\frac{7}{2}\right) \right) c_{\theta}^2 \right] \\
 & + s_{\theta}^6 c_{\theta} \sin(8\psi) \left[ \frac{39104768}{155925} - \frac{26046464}{155925} c_{\theta}^2 + \frac{65536}{4725} c_{\theta}^4 \right] + s_{\theta}^7 c_{\theta} \sin(9\psi) \left[ -\frac{62623413117}{220774400} + \frac{43046721}{143360} \ln(3) \right. \\
 & - \frac{43046721}{286720} \ln(2) \left. \right] + s_{\theta}^8 c_{\theta} \sin(10\psi) \left[ \frac{112890625}{798336} - \frac{1953125}{66528} c_{\theta}^2 \right] + \frac{34992}{1925} s_{\theta}^{10} c_{\theta} \sin(12\psi), \tag{C6k}
 \end{aligned}$$

$$\begin{aligned}
 H_{\times}^{(5.5)} = & s_{\theta} c_{\theta} \cos(\psi) \left[ -\frac{44966609055228224561}{4611606067740672000} - \frac{913}{46080} \pi^2 + \frac{913}{3840} \ln(2)^2 - \frac{13785043}{210038400} \text{eulerlog}(1, \nu) \right. \\
 & + \frac{4253489}{4838400} \ln(2) + \left( -\frac{5164843}{7257600} \ln(2) - \frac{1891}{5760} \ln(2)^2 + \frac{2289618368599426463}{4611606067740672000} + \frac{1891}{69120} \pi^2 \right. \\
 & \left. - \frac{3991213}{27227200} \text{eulerlog}(1, \nu) \right) c_{\theta}^2 + \left( \frac{267271}{294053760} \text{eulerlog}(1, \nu) - \frac{7}{27648} \pi^2 + \frac{7}{2304} \ln(2)^2 \right. \\
 & \left. + \frac{5542529007807251}{461160606774067200} + \frac{26263}{2903040} \ln(2) \right) c_{\theta}^4 - \frac{374959007}{694871654400} c_{\theta}^6 + \frac{190121}{81749606400} c_{\theta}^8 - \frac{13}{7431782400} c_{\theta}^{10} \left. \right] \\
 & + c_{\theta} \cos(2\psi) \left[ \frac{3195371713}{48024900} + \frac{1088}{9} \zeta(3) - \frac{449612}{6237} \pi^2 - \frac{29728}{3465} \text{eulerlog}(2, \nu) + \left( -\frac{646758699649}{5762988000} \right. \right. \\
 & \left. \left. + \frac{345376}{17325} \text{eulerlog}(2, \nu) - \frac{256}{9} \zeta(3) + \frac{228764}{31185} \pi^2 \right) c_{\theta}^2 - \frac{825921}{78400} c_{\theta}^4 + \frac{10107527}{10478160} c_{\theta}^6 - \frac{29171}{3326400} c_{\theta}^8 \right] \\
 & + s_{\theta} c_{\theta} \cos(3\psi) \left[ -\frac{37503}{5120} \pi^2 + \frac{3926864534929543197}{75911210991616000} - \frac{31284549}{179200} \ln\left(\frac{3}{2}\right) + \frac{112509}{1280} \ln\left(\frac{3}{2}\right)^2 \right. \\
 & \left. + \frac{2354400351}{54454400} \text{eulerlog}(3, \nu) + \left( \frac{36207}{2560} \pi^2 + \frac{19408843768367516823}{75911210991616000} + \frac{36267993}{89600} \ln\left(\frac{3}{2}\right) \right. \right. \\
 & \left. \left. - \frac{372580641}{5445440} \text{eulerlog}(3, \nu) - \frac{108621}{640} \ln\left(\frac{3}{2}\right)^2 \right) c_{\theta}^2 + \left( -\frac{6488829}{179200} \ln\left(\frac{3}{2}\right) - \frac{5103}{5120} \pi^2 + \frac{190312011}{54454400} \text{eulerlog}(3, \nu) \right. \right. \\
 & \left. \left. + \frac{15309}{1280} \ln\left(\frac{3}{2}\right)^2 + \frac{59325223140785601}{37955605495808000} \right) c_{\theta}^4 - \frac{18541319661}{6862929920} c_{\theta}^6 + \frac{414504297}{4037017600} c_{\theta}^8 - \frac{255879}{367001600} c_{\theta}^{10} \right] \\
 & + s_{\theta}^2 c_{\theta} \cos(4\psi) \left[ \frac{4096}{9} \zeta(3) - \frac{804352}{2475} \text{eulerlog}(4, \nu) - \frac{3584}{5} \ln(2)^2 + \frac{3217408}{10395} \ln(2) \text{eulerlog}(4, \nu) \right. \\
 & \left. + \frac{2048}{9} \ln(2)^3 - \frac{2158976}{31185} \pi^2 - \frac{512}{9} \ln(2) \pi^2 - \frac{12640378852}{7203735} \ln(2) + \frac{43812580277}{20010375} + \left( \frac{290708048}{654885} \right. \right. \\
 & \left. \left. - \frac{1334416}{4455} \ln(2) \right) c_{\theta}^2 + \left( -\frac{469895744}{3274425} + \frac{34432}{405} \ln(2) \right) c_{\theta}^4 + \left( -\frac{512}{189} \ln(2) + \frac{5637568}{1091475} \right) c_{\theta}^6 \right] \\
 & + s_{\theta}^3 c_{\theta} \cos(5\psi) \left[ -\frac{2546875}{27648} \pi^2 + \frac{3326215625}{8401536} \text{eulerlog}(5, \nu) - \frac{10344364790975800625}{7027209246081024} - \frac{1669559375}{580608} \ln\left(\frac{5}{2}\right) \right. \\
 & \left. + \frac{2546875}{2304} \ln\left(\frac{5}{2}\right)^2 + \left( -\frac{3884921875}{58810752} \text{eulerlog}(5, \nu) + \frac{430984375}{580608} \ln\left(\frac{5}{2}\right) + \frac{6095596325790625}{288225379233792} \right. \right. \\
 & \left. \left. + \frac{546875}{27648} \pi^2 - \frac{546875}{2304} \ln\left(\frac{5}{2}\right)^2 \right) c_{\theta}^2 + \frac{445893828125}{5294260224} c_{\theta}^4 - \frac{44912890625}{4904976384} c_{\theta}^6 + \frac{634765625}{3567255552} c_{\theta}^8 \right] \\
 & + s_{\theta}^4 c_{\theta} \cos(6\psi) \left[ \frac{5794821}{6160} \ln(3) - \frac{2447860257}{1724800} + \left( -\frac{206307}{280} \ln(3) + \frac{6280821}{4928} \right) c_{\theta}^2 + \left( -\frac{194738499}{1724800} \right. \right. \\
 & \left. \left. + \frac{6561}{112} \ln(3) \right) c_{\theta}^4 \right] + s_{\theta}^5 c_{\theta} \cos(7\psi) \left[ -\frac{5764801}{138240} \pi^2 - \frac{3400409047}{2073600} \ln\left(\frac{7}{2}\right) + \frac{2459922941}{19094400} \text{eulerlog}(7, \nu) \right. \\
 & \left. + \frac{68859193174529009}{5703903608832000} + \frac{5764801}{11520} \ln\left(\frac{7}{2}\right)^2 - \frac{42612989808613}{108291686400} c_{\theta}^2 + \frac{6799703840321}{70071091200} c_{\theta}^4 - \frac{25705247659}{6370099200} c_{\theta}^6 \right] \\
 & + s_{\theta}^6 c_{\theta} \cos(8\psi) \left[ -\frac{6415888384}{3274425} + \frac{6258688}{2835} \ln(2) + \left( \frac{1480835072}{3274425} - \frac{262144}{567} \ln(2) \right) c_{\theta}^2 \right] + s_{\theta}^7 c_{\theta} \cos(9\psi) \\
 & \times \left[ \frac{14460154075971}{34314649600} - \frac{1678822119}{6307840} c_{\theta}^2 + \frac{45328197213}{2018508800} c_{\theta}^4 \right] + s_{\theta}^8 c_{\theta} \cos(10\psi) \left[ \frac{1953125}{9072} \ln(5) - \frac{7195703125}{16765056} \right] \\
 & + s_{\theta}^9 c_{\theta} \cos(11\psi) \left[ \frac{17922760399291}{89181388800} - \frac{3709051717943}{89181388800} c_{\theta}^2 \right] + \frac{23298085122481}{980995276800} s_{\theta}^{11} c_{\theta} \cos(13\psi) + \pi s_{\theta} c_{\theta} \sin(\psi) \\
 & \times \left[ \frac{11112194167}{23524300800} + \frac{913}{3840} \ln(2) + \left( -\frac{9969427309}{35286451200} - \frac{1891}{5760} \ln(2) \right) c_{\theta}^2 + \left( \frac{8204407}{2016368640} + \frac{7}{2304} \ln(2) \right) c_{\theta}^4 \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \pi c_\theta \sin(2\psi) \left[ \frac{342543430879}{1152597600} - \frac{68}{9} \pi^2 - \frac{915856}{10395} \operatorname{eulerlog}(2, \nu) + \left( -\frac{5395550969}{72037350} + \frac{16}{9} \pi^2 \right. \right. \\
 & + \left. \left. \frac{128096}{10395} \operatorname{eulerlog}(2, \nu) \right) c_\theta^2 - \frac{634357}{166320} c_\theta^4 + \frac{6611}{22680} c_\theta^6 - \frac{1}{432} c_\theta^8 \right] + \pi s_\theta c_\theta \sin(3\psi) \left[ \frac{57217535811}{871270400} \right. \\
 & - \left. \frac{112509}{1280} \ln\left(\frac{3}{2}\right) + \left( -\frac{73264265343}{435635200} + \frac{108621}{640} \ln\left(\frac{3}{2}\right) \right) c_\theta^2 + \left( \frac{2035978173}{124467200} - \frac{15309}{1280} \ln\left(\frac{3}{2}\right) \right) c_\theta^4 \right] \\
 & + \pi s_\theta^2 c_\theta \sin(4\psi) \left[ -\frac{1024}{3} \ln(2)^2 + \frac{5842432}{10395} \ln(2) - \frac{256}{9} \pi^2 - \frac{1608704}{10395} \operatorname{eulerlog}(4, \nu) + \frac{37453814458}{36018675} \right. \\
 & + \left. \frac{667208}{4455} c_\theta^2 - \frac{17216}{405} c_\theta^4 + \frac{256}{189} c_\theta^6 \right] + \pi s_\theta^3 c_\theta \sin(5\psi) \left[ -\frac{2546875}{2304} \ln\left(\frac{5}{2}\right) + \frac{3499894615625}{2822916096} \right. \\
 & + \left. \left( \frac{546875}{2304} \ln\left(\frac{5}{2}\right) - \frac{136354984375}{403273728} \right) c_\theta^2 \right] + \pi s_\theta^4 c_\theta \sin(6\psi) \left[ -\frac{5794821}{12320} + \frac{206307}{560} c_\theta^2 - \frac{6561}{224} c_\theta^4 \right] \\
 & + \pi s_\theta^5 c_\theta \sin(7\psi) \left[ -\frac{5764801}{11520} \ln\left(\frac{7}{2}\right) + \frac{692452248803}{916531200} \right] + \pi s_\theta^6 c_\theta \sin(8\psi) \left[ -\frac{1564672}{2835} + \frac{65536}{567} c_\theta^2 \right] \\
 & - \frac{1953125}{18144} \pi s_\theta^8 c_\theta \sin(10\psi). \tag{C61}
 \end{aligned}$$

Note that the  $\cos 5\psi$  term in  $H_+^{(2.5)}$  and the  $\sin 5\psi$  term in  $H_\times^{(2.5)}$  in [30] have been further simplified in our presentation above and are equivalent.

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