Chapter 4

Quantum motion of a particle on a continuous ring linking Aharonov-Bohm Flux in the presence of dissipative coupling to a bath of harmonic oscillators – non-suppression of orbital diamagnetism (microscopic approach)

4.1 Abstract

In this Chapter we consider the orbital(diamagnetic) moment associated with the quantum motion of a charged particle (the system) moving on a ring threaded by an Aharonov-Bohm(A-B) magnetic flux (+) coupled dissipatively to a continuum of barmonic oscillators (the heat bath). Inasmuch as the purely gauge flux + here enters the

quantum mechanical motion as a *topological/geometric* phase factor, it was our conjecture that its quantum aspects (orbital diamagnetism) may not be suppressed by the dissipative coupling to the bath. In order to decide this conclusively, we have calculated the partition function of the system (coupled to the bath oscillators) in terms of the *Euclidean(imaginary* time) path integral that incorporates the effects of the dissipative coupling to the environment. We *find* from the resulting *partion* function that the *effect* of the bath harmonic oscillators turns out to be essentially a *renormalization* of the inertia (mass) of the quantum particle moving on the ring. Thus, unlike the case of a particle moving in a simply connected region (the plane) under the *influence* of the Lorentz force, where the diamagnetic moment is known to decrease monotonically with increasing dissipation *(resistivity[29, 30])*, in this case for the motion on the non-simply connected region (the ring) the orbital diamagnetic moment remains essentially unaffected by the dissipative coupling to the bath. Thus, the normally expected classicalization does not occur. The same conclusion is also derived from yet another calculation based on the reduced density matrix evolving under the quantum master equation, known to be valid in the high-temperature limit.

We begin this Chapter by recalling once again the important fact that the orbital diamagnetism is known to be a phenomenon of direct physical interest in the molecular and the condensed matter physics, and is of purely quantum origin–it has no classical analogue. This vanishing of the orbital diamagnetism in the classical limit is the celebrated Bohr-van Leeuven theorem[24, 25]. Here, in the presence of an external magnetic field (and the associated local Lorentz force $\frac{e}{c}(\mathbf{v} \times B)$),the induced cyclic currents in the bulk (Maxwell cycles) and the skipping orbits (edge currents) at the boundary (internal as well as external) contribute equal and opposite magnetic moments. This classically exact cancellation may be visualized from the schematic in FIG. 3.1. This makes the study of orbital diamagnetism in a quantum dissipative system interesting *fundamentally*. The vanishing of the orbital diamagnetic moment follows at once from the fact that the equilibrium partition function $Z(B,\beta)$ for the classical system is independent of the magnetic field B (or equivalently, of the vector potential $\mathbf{A}(\mathbf{r})$) inasmuch as the latter enters minimally through the replacement $\mathrm{p} \to \mathrm{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})$, and thus the classical trace over the canonical momentum p makes the partition function independent of $\mathbf{A}(\mathbf{r})$. This, of course, is not permitted quantum mechanically in that the operators $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$ do not commute then. We address here the general question: how does the diamagnetic behaviour (a property of the purely quantum system) change under the influence of the environment-induced decoherence–a case of quantum to classical crossover!. Thusly motivated, we study below some model quantum systems coupled (dissipatively) to the bath. Let us first recall in some detail the vanishing of the classical orbital diamagnetism in some detail. This equilibrium-statistical mechanical result can be re-derived more explicitly from a real space-time-dependent approach in which the classical motion is treated as a stochastic process (the Einstein Brownian, or the Langevin, approach to statistical mechanics), where the equilibrium orbital diamagnetism is obtained in the time $t \to \infty$ limit. This treatment also clarifies the role of the boundary(e.g., a harmonic confinement) through the ordering of the two limits, namely, that the $t \to \infty$ limit is to precede the limit of the confining length-scale $\to \infty$ (the so-called Darwin limiting procedure[27]). Thus, it has been shown explicity in a classical stochastic model system[28] that the orbital diamagnetism obeys Bohr-van Leeuven's theorem. Quantum mechanically, however, this cancellation is incomplete, and this has indeed been demonstrated for a charged quantum particle moving in a plane normal to the external magnetic field, in the presence of a harmonic confinement and dissipative coupling to a harmonic oscillators heat bath[29] treated through a stochastic quantum Langevin equation. In fact, the orbital diamagnetism turns out to be a decreasing function of friction (dissipative coupling)[30]. Interestingly, this implies that an equilibrium thermodynamic property(orbital diamagnetism) is controlled by friction (a transport property)! Indeed, for the simple model above, the orbital diamagnetism was found to be a monotonically decreasing function of the properly scaled electrical resistance of the sample[30].

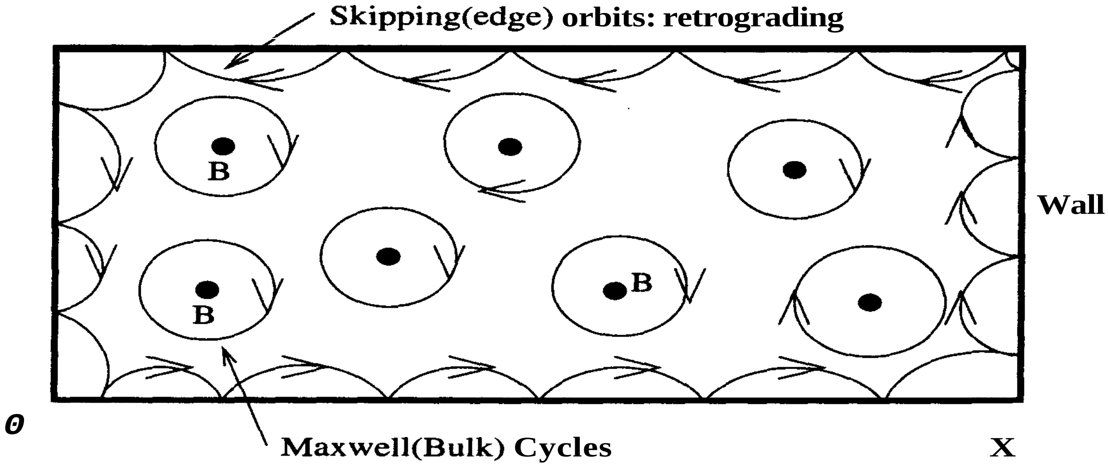

Figure 4.1: Skipping orbits at the boundary (the edge currents) contribute oppositely to the Maxwell cycles (the bulk currents) shown for the particle carrying a charge (assumed positive in the figure). The magnetic field B($\bullet$) is comming out of the plane of the paper in this figure.

There is, however, no other analytically solvable model known that addresses this problem. This has motivated us to explore the orbital diamagnetism in the presence of decoherence (due to coupling to the bath degrees of freedom) for a model of a charged particle moving on a *ring* with the *A-B* flux ϕ threading the ring (FIG. 4b). Here, the flux enters the wave function as a geometric (or topological) phase factor. The decoherence will be introduced through a strictly linear coordinate-coordinate coupling to the harmonic bath oscillators. This is *essentially* different from the earlier case for a particle moving in a plane (simply connected region) in the presence of a magnetic field perpendicular to the plane[29] where the particle experiences the local classical Lorentz force $\frac{e}{c}$(v x B).

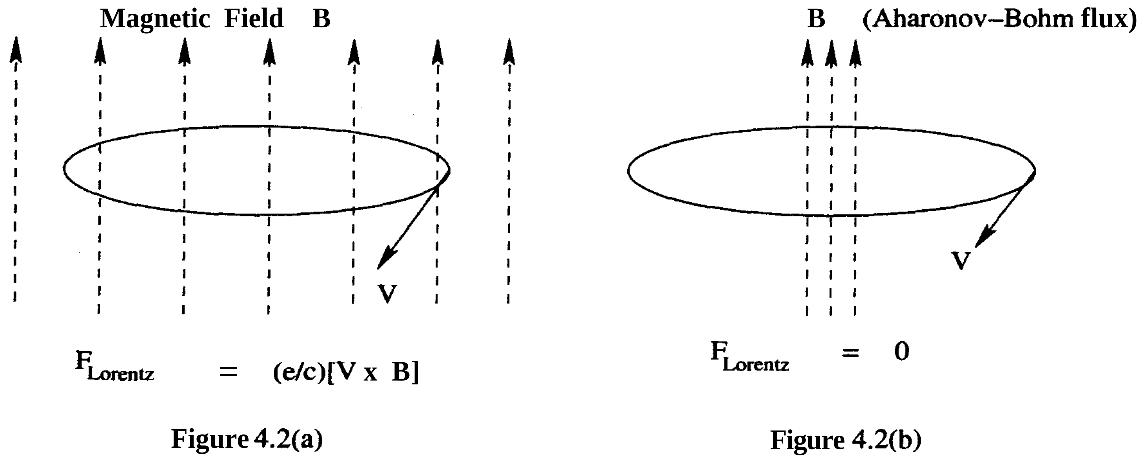

Figure 4.2: Motion of a particle carrying a charge (assumed +ve in the figure) in a plane normal to the magnetic field B under the Lorentz force, Figure 4.2(a); and no Lorentz force for the motion on a *ring* with an Aharonov-Bohm flux ϕ threading through the ring, Figure 4.2(b).

4.2 A charged particle with its coordinate coupled linearly to the co-ordinates of a bath of harmonic oscillators and moving on a ring with an A-B flux threading the ring: The Euclidean Path-Integral Approach.

4.2.1 The Lagrangian and the Euclidean Action

Our model now consists of a free particle moving on a ring(the system) which is threaded by an Aharonov-Bohm flux. The particle is coupled to a continuum of harmonic oscillators(the heat bath). The coupling is strictly linear in the system coordinate $x(t)$ and also in the bath co-ordinates $q_j(t)$, with the coupling, coefficient c_j for the j^{th} harmonic oscillator. The total Lagrangian (L) for the system (S) and the bath (B) is

$$L = L_S + L_B + L_{SB} \ ,$$

$$L_S = \frac{1}{2}m\dot{x}^2(t) + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A} \ ,$$

$$L_B = \frac{1}{2}\sum_j [M_j \dot{q_j}^2(t) - M_j\omega_j^2 q_j^2(t)]$$
$$L_{SB} = \sum_j c_j q_j(t)x(t). \tag{4.1}$$

We also have to include a counter term to cancel out a certain unphysical potential term(generated by the dissipative coupling) involving the system coordinate x(t) as discussed in Chapter 1, and will be treated later below. The corresponding Euclidean action can now be written by introducing the imaginarry time $\tau = it$

$$S_S = i\int_0^{\beta\hbar} \frac{1}{2}m\dot{x}^2(\tau)d\tau - (\frac{e}{c})\int_0^{\beta\hbar} \mathbf{A}\cdot\frac{dx}{d\tau}d\tau \ ;$$
$$S_B = i\int_0^{\beta\hbar}[\frac{1}{2}M_j\dot{q}_j^2(\tau) + \frac{1}{2}M_j\omega_j^2 q_j^2(\tau)]d\tau \ ;$$
$$S_{SB} = -i\int_0^{\beta\hbar}\sum_j c_j q_j(\tau)x(\tau)d\tau \ . \tag{4.2}$$

It is be noted here that we have retained "i" in the expression for the Euclidean action. Our objective now is to calculate the partition function $Z(\beta, \mathrm{B})$. To this end, we carry out the following steps: (a) Wick rotation as in Eq. (4.2)to the negative imaginary time, i.e.,$t = -i\tau$, with $0 \leq \tau \leq \hbar\beta$;(2) integrate out the system coordinate $x(\tau)$ with $x(0) = x_a$ $x(\hbar\beta) = x_b$; **(3)** trace out the bath variables $q_j(\tau)$ with $q_j(0) = q_j(\hbar\beta)$,(4) integrate out $q_j(0)(= q_j(\hbar\beta))$; and then compare the resulting Euclidean path integral with a reference-system path integral in order to discern the effects on the system of the system-bath dissipative interaction. Finally, we must integrate over the system variable $x(0)(= x(\hbar\beta))$, to obtain the partition function. It is this last step that introduces the (geometric) phase associated with the A-B flux and the ring topology, namely, that $x(0) = x(\hbar\beta)$ will imply $x(\hbar\beta) = x(0)$ modulo $n2\pi a$; that is to say that we introduce the winding number 'n' in the trace operation. Thus, we will proceed till the last step as if we are working on a one-dimensional unbounded space (infinite line), and finally identify the points which are multiples n of $2\pi a$ apart (a = ring radius), where n $= 0, \pm 1, \pm 2 \ldots$. The winding number n will give the phase factor $e^{in\phi/\phi_0}$ in tracing over $x(0) = (x(\hbar\beta))$. Here $\phi_0 = hc/e$ is the flux quantum, with e > 0

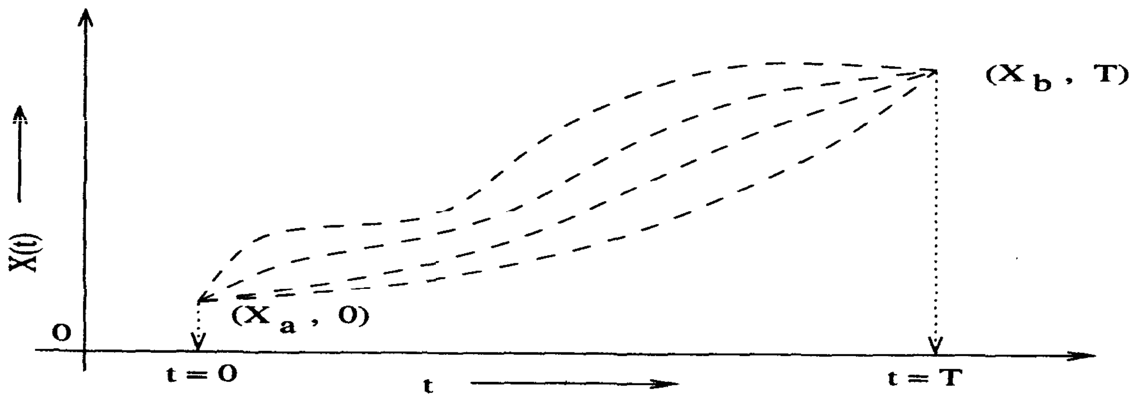

Figure 4.3: A few possible paths that the particle may take in real time evolution from initial space-time point $(x_a, 0)$ to final space-time point(x_b, T).

4.2.2 The path integral

The partial path integral for the system variable $x(\tau)$, with $x(\tau = 0) = x_a$ and $x(\tau = \beta\hbar) = x_b$ involves the Euclidian actions $S_S + S_{SB}$ representing the free particle interacting linearly with the bath degrees of freedom (FIG.4.3). Accordingly, we introduce the kernel

$$K(x_b, \hbar\beta; x_a, 0|\ldots) = e^{\frac{ie}{\hbar c}\int_{x_a}^{x_b} \mathbf{A}\cdot\mathbf{dx}} \int_{x_a,0}^{x_b,\hbar\beta} D[x(\tau)] e^{\frac{-1}{\hbar}\int_0^{\hbar\beta}(\frac{1}{2}m\dot{x}^2(\tau) - f(\tau)x(\tau))d\tau}, \tag{4.3}$$

with

$$f(\tau) = \sum_j c_j q_j(\tau). \tag{4.4}$$

Here the ellipsis $(\ldots)$ denotes the un-integrated bath varibles $(q_j(\tau))$.

Using the standard results from the path integral for a free particle coupled linearly to harmonic oscillators, we obtain

$$\begin{aligned} K(x_b, \hbar\beta; x_a, 0|\ldots) &= \sqrt{\frac{m}{2\pi\hbar^2\beta}} e^{\frac{ie}{\hbar c}\int_{x_a}^{x_b} \mathbf{A}\cdot\mathbf{dx}} \\ &\times e^{\frac{-m}{2\hbar^2\beta}[(x_b - x_a)^2 + \frac{2x_b}{m}\int_0^{\hbar\beta}\tau f(\tau)d\tau + \frac{2x_a}{m}\int_0^{\hbar\beta}(\hbar\beta-\tau)f(\tau)d\tau - \frac{2}{m^2}\int_0^{\hbar\beta}d\tau\int_0^{\tau}ds f(\tau)f(s)(\hbar\beta-\tau)s]} \\ &\times(\ldots)\ , \end{aligned} \tag{4.5}$$

where again the ellipsis $(\ldots)$ denotes the factors that involve the purely bath action S_B. Let us now consider the various terms in the exponent. The last term in the exponent on the RHS of the above equation (Eq.(4.5)) involves only the bath variables and represents the effect of the single degree of freedom $x(\tau)$ of the system of interest(i.e., the particle on the ring) on the very (infinitely) large number of the bath variables $q_j's$. As discussed in Chapter 1 (Introduction), due to the (infinitely) large number of degrees of freedom of the bath, this renormalization effect is negligible. We, therefore, neglect the last term $\frac{2}{m^2}\int_0^{\hbar\beta} d\tau \int_0^{\tau} ds f(\tau) f(s)(\hbar\beta-\tau)s$ in the exponent. The middle two terms, which involve both the system variable $x(\tau)$ and the bath variables $q_j(\tau)$, can be re-written as

$$\begin{aligned}
&\frac{x_b}{\hbar\beta}\int_0^{\hbar\beta}\tau f(\tau)d\tau + \frac{x_a}{\hbar\beta}\int_0^{\hbar\beta}(\hbar\beta-\tau)f(\tau)d\tau \\
&= \sum_j \int_0^{\hbar\beta}\left[\frac{x_b}{\hbar\beta}\tau c_j q_j(\tau) + \frac{x_a}{\hbar\beta}(\hbar\beta-\tau)c_j q_j(\tau)\right]d\tau \\
&= -\sum_j q_j(\tau)\int_0^{\hbar\beta}F_j(\tau)d\tau \; with, F_j(\tau) = c_j[\frac{x_b}{\hbar\beta}\tau + \frac{x_a}{\hbar\beta}(\hbar\beta-\tau)].
\end{aligned} \tag{4.6}$$

With this definition of $F_j(\tau)$, we now path-integrate out the bath variables, leading to the kernel

$$\begin{aligned}
K(x_b, q_j(0), \hbar\beta; x_a, q_j(0), 0) &= \int_{q_j(0)}^{q_j(\hbar\beta)=q_j(0)} \prod_j D[q_j(\tau)] \\
e^{-\frac{1}{\hbar}\sum_j \int_0^{\hbar\beta}[\frac{1}{2}M_j\dot{q}_j^2 - \frac{1}{2}M_j\omega_j^2 q_j^2 + q_j(\tau)F_j(\tau)]d\tau} &\times (\ldots).
\end{aligned} \tag{4.7}$$

Here now the ellipsis $(\ldots)$ denotes the factors from the Kernel above that do not involve the bath degrees of freedom. On performing the path integration[32], we get

$$\begin{aligned}
K(x_b, q_j(0), \beta\hbar; x_a, q_j(0), 0) &= \prod_j \sqrt{\frac{M_j\omega_j}{2\pi\hbar^2 \sinh\omega_j\beta\hbar}} \\
&\times e^{\frac{-M_j\omega_j}{2\hbar\sinh\omega_j\hbar\beta}[-4q_j^2(0)\sinh^2(w_j\hbar\beta/2) + \frac{2q_j(0)}{M_j\omega_j}I_a - \frac{2}{M_j^2\omega_j^2}I_b]} \quad with \\
I_a &= -\int_0^{\hbar\beta}F_j(\tau)\sin\omega_j\hbar\beta d\tau - \int_0^{\hbar\beta}F_j(\tau)\sinh\omega_j(\hbar\beta-\tau)d\tau \,, \\
I_b &= \int_0^{\hbar\beta}d\tau\int_0^{\tau}ds F_j(\tau)F_j(s)\sinh\omega_j(\hbar\beta-\tau)\sinh\omega_j s \,.
\end{aligned} \tag{4.8}$$

The values of the integrals I_a and I_b above are given by

$$I_a = \alpha(1-\cosh\omega_j\hbar\beta)(x_a+x_b) \;\; , \alpha = c_j/\omega_j \;,$$

and

$$I_b = \alpha^2[x_a x_b - \frac{1}{2}\cosh\omega_j\hbar\beta(x_a^2+x_b^2)+\frac{1}{6}\omega_j\hbar\beta\sinh\omega_j\hbar\beta(x_a^2+x_b^2+x_a x_b)+\frac{\sinh\omega_j\hbar\beta}{2\omega_j\hbar\beta}(x_a-x_b)^2].$$

Next, we integrate out the bath variable $q_j(0)$ from $-\infty$ to $+\infty$, which is equivalent to taking a trace over the bath degrees of freedom:

$$\int_{-\infty}^{+\infty}\prod_j dq_j(0)\sqrt{\frac{M_j\omega_j}{2\pi\hbar\sin\omega_j\hbar\beta}}e^{\frac{-M_j\omega_j}{2\hbar\sinh\omega_j\hbar\beta}[-4q_j^2(0)\sinh^2(\omega_j\hbar\beta/2)+\frac{2q_j(0)}{M_j\omega_j}I_a-\frac{2}{M_j^2\omega_j^2}I_b]}. \tag{4.9}$$

The above Gaussian integral gives the full effective kernel of the system

$$K(x_b,\beta\hbar;x_a,0)=\prod_j\frac{1}{2\sinh(\omega_j\hbar\beta/2)}e^{\alpha_0 I_a^2-\beta_0 I_b}\ \text{ with }\alpha_0=\frac{\coth(\omega_j\hbar\beta/2)}{4\hbar M_J\omega_j\sinh^2\omega_j\hbar\beta}$$
$$\text{and }\beta_0=\frac{-1}{\hbar M_J\omega_j\sinh\omega_j\hbar\beta}. \tag{4.10}$$

Now, on substituting the values of $\mathbf{I}_a$, I_b, α_0, and β_0, the full effective kernel for the system can be written as

$$K(x_b,\beta\hbar;x_a,0)=\sqrt{\frac{m}{2\pi\hbar^2\beta}}\prod_j\left[\frac{1}{2\sinh(\omega_j\hbar\beta/2)}\right]e^{\frac{ie}{\hbar c}\int_{x_a}^{x_b}\mathbf{A}\cdot\mathbf{dx}}$$
$$\times e^{\frac{c_j^2}{4\hbar M_j\omega_j^3}[(\tanh(\omega_j\hbar\beta/2)-\coth(\omega_j\hbar\beta)+1/\sinh\omega_j\hbar\beta+\omega_j\hbar\beta/2)(x_a+x_b)^2]}$$
$$\times e^{\frac{c_j^2}{4\hbar M_j\omega_j^3}\left[-\frac{2mM_j\omega_j^3}{\hbar\beta c_j^2}-\coth(\omega_j\hbar\beta)-1/\sinh\omega_j\hbar\beta+\omega_j\hbar\beta/6+2/\omega_j\hbar\beta\right](x_a-x_b)^2}. \tag{4.11}$$

4.2.3 The winding number and the partition function

Now we have to consider the factor invoving the sum $(x_a + x_b)/2$ in the exponent on the RHS of Eq.(4.11). This is where the counter terms come in. Indeed, for $x_a = x_b$, this term is readily seen to be a potential(harmonic), quadratic in the system coordinate, generated by the integration over the bath oscillator coordinates. This term is unphysical in that it destroys the translational symmetry of the system—it tends to specially treat the particle position at x $= 0$. This unphysical term must be considered as cacelled out by a counter term assumed to have been introduced in the system Lagrangian as discussed in the introduction (Chapter 1). We assume this to have been done and ignore this term for the time being, and return to it later (see Appendix at the end of this Chapter). Thus, we are finally left only with integration over the relative coordinate $(x_b - x_a)$, before setting $x_a = x_b$ modulo $n.2\pi a$. Thus , we consider the second part of the exponent (in Eq.(4.11)) which involves the system mass m in it and the system co-ordinate difference $(x_a - x_b)$, and consider now the effect of the Aharonov-Bohm magnetic flux when setting $x_b = x_a$ to get the partion function $Z(\beta, \mathrm{B})$. As dissused above, this means $x_b - x_a = 2\pi a n$ where n $= 0, \pm 1, \pm 2, \ldots$, the winding number, and, accordingly, the $\int_{x_a}^{x_b} \mathbf{A} \,.\, \mathbf{dx} = n\phi$. Hence, the effective system partition function is

$$Z(B,\beta) \equiv K(x_b, \hbar\beta; x_a, 0)|_{x_b = x_a \ modulo\ n.2\pi a} = \sqrt{\frac{m}{2\pi\hbar^2\beta}} \prod_j \left[\frac{1}{2\sinh(\omega_j\hbar\beta/2)}\right]$$
$$\times \sum_{n=-\infty}^{n=+\infty} e^{\alpha_1 (2\pi r n)^2} e^{\frac{ie}{\hbar c}\phi n} \quad with$$
$$\alpha_1 = \frac{-c_j^2}{4\hbar M_j \omega_j^3}\left[\frac{2mM_j\omega_j^3}{\hbar\beta c_j^2} + \coth(\omega_j\hbar\beta) + 1/\sinh\omega_j\hbar\beta - \omega_j\hbar\beta/6 - 2/\omega_j\hbar\beta)\right]$$
$$+ \text{ corrections from the counter term.} \tag{4.12}$$

It is to be noted here that what is really important for analyzing the orbiatl moment is the term —$alpha_1$ in the exponet of Eq. (4.12) that involves the magnetic flux ϕ. Thus the magnetic moment M $= \frac{1}{\beta}\frac{\partial}{\partial B} lnZ(B,\beta)$, with $\phi =$?ia^{2}B, and , therefore, all other factors independent of B simply drop out. Thus, finally we have

$$Z(B,\beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}} \prod_j \left[\sinh(\frac{\omega_j\hbar\beta}{2})\right]^{-1} \sum_{n=-\infty}^{n=+\infty} e^{-\frac{m_{eff}}{2\hbar^2\beta}(2\pi r n)^2} e^{\frac{ie}{\hbar c}\phi_B n}, \tag{4.13}$$

where m_{eff} is given by (see Appendex 4.5)

$$\frac{m_{eff}}{m} = 1 + \frac{1}{6\pi}\hbar^2 \left[\frac{\eta}{m}\omega_c\right]\beta^2 \tag{4.14}$$

with ω_c the bath high-frequency cut-off, and η the frictional coefficient as introduced in the Appendix 4.5 to be in line with the convention[39]. The RHS of Eq.4.13 (apart from the pre-factors involving the bath parameters) can be readily identified with the path integral for the charged particle on the ring threaded by the A-B flux, except for bare mass m replaced by the effective mass m_{eff}. Thus the latter is the only effect of the dissipative coupling on the orbital magnetic moment. This essentialy vindicates our conjure.

4.2.4 Discussion

The Eqs. (4.13) and (4.14) are the important final results of the present study. These clearly show that the effect of the dissipative coupling to the bath oscillators on the motion of the charged particle moving on a ring threaded by the Aharonov-Bohm flux ϕ is only to change its effective mass. Thus, the effect of bath on the system can be incarporated as a renormalized mass of the particle.

Now, using the identity

$$e^{-\eta^2/4\chi} = \sqrt{\frac{\chi}{\pi}} \int_{-\infty}^{\infty} e^{-\chi x^2 + i\eta x} dx,$$

for the factor with n^2 in the exponent of Eq.(4.13), we get

$$Z(B,\beta) = \prod_j \sqrt{\frac{m}{m_{eff}}} \left[\sinh(\frac{\omega_j \hbar\beta}{2})\right]^{-1} \sum_{n=-\infty}^{n=+\infty} e^{-\beta \frac{\hbar^2}{2m_{eff}r^2}[n - \frac{\phi_B}{\phi_0}]^2} \ , \ \phi_0 = \frac{hc}{e}. \qquad (4.15)$$

This is essentially (to within trivial factors independent of ϕ) the partition function for the charged particle moving on a ring threaded by the A-B flux, known from equilibrium statistical mechanics. Hence, the orbital diamagnetism (dependence of $Z(B,\beta)$ on ϕ) persists, with a change only of the effective mass, despite the frictional coupling to the bath. The change, of course, vanishes rapidly(quadratically in β) at higher temperatures. This is in contrast to the results known[30] for the case of motion of a charged particle on a plane perpendicular to the magnetic field, where the particle is subject to the Lorentz force unlike our case for the ring.

4.3 Orbital diamagnetism of a charged particle moving on a ring with Aharonov-Bohm Flux: Density matix treatment based on quantum brownian motion master Equation.

4.3.1 The density matrix and its equation of motion

We reconsider our model consisting of a free particle on a ring(the system), threaded by an Aharonov-Bohm flux. The particle is coupled dissipativelyto a system of harmonic oscillators(the bath). In the following we use the quantum Brownian motion master equation to study the effect of this dissipative coupling on the orbital dimagnetism. The Quantum Brownian Motion master equation for the density matrix operator $\hat{\rho}$ is[18](in the absence of the vector potential)

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[H_{sys}, \hat{\rho}] - \frac{i\gamma}{2\hbar}[x, [\dot{x}, \hat{\rho}]_+] - \frac{\gamma kT}{\hbar^2}[x, [x, \hat{\rho}]]. \tag{4.16}$$

For the motion on the ring threaded by the A-B flux ϕ, we introduce the density matrix in the 8-representation $\langle\theta_1|\hat{\rho}|\theta_2\rangle \equiv \hat{\rho}_{\theta_1,\theta_2} = \hat{\rho}^{\star}_{\theta_2,\theta_1}$, with x $= a\theta$, and $\hat{\dot{x}}$ on the R.H.S. of Eq. (4.16) replaced by $\frac{1}{ma}(\hat{p}_\theta - \frac{e}{c}A_\theta)$. The system Hamiltonian is

$$H_{sys} = \frac{1}{2m}\left(\hat{p} - \frac{e}{c}\hat{A}_\theta\right)^2 \ , \ \hat{p}_\theta = -i\hbar\frac{1}{a}\frac{\partial}{\partial\theta} \ , \ A_\theta = \phi/2\pi a \ , \tag{4.17}$$

with a $=$ ring radius. The unitary evolution part, i.e., the first term on the R.H.S in the master equation (Eq.(4.16)), gives

$$\frac{i\hbar}{2ma^2}\left[\frac{\partial^2}{\partial\theta_1^2} - \frac{\partial^2}{\partial\theta_2^2}\right]\langle\theta_1|\hat{\rho}|\theta_2\rangle + \frac{e\phi_{\theta}}{2\pi ma^2 c}\left[\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right]\langle\theta_1|\hat{\rho}|\theta_2\rangle \ . \tag{4.18}$$

The non-unitary part, i.e. the second and the third terms on the RHS in Eq.(4.16), give

$$(\theta_1 - \theta_2)\left[-\frac{\gamma}{2m}\left[\frac{\partial}{\partial\theta_1} - \frac{\partial}{\partial\theta_2}\right] + \frac{i\gamma e\phi}{2\pi m\hbar c} - \frac{\gamma kTa^2}{\hbar^2}(\theta_1 - \theta_2)\right]\rho_{\theta_1,\theta_2}. \tag{4.19}$$

Thus, we have

$$\frac{d\rho_{\theta_1,\theta_2}}{dt} = \left\{\frac{i\hbar}{2ma^2}\left[\frac{\partial^2}{\partial\theta_1^2} - \frac{\partial^2}{\partial\theta_2^2}\right] - \frac{\gamma}{2m}(\theta_1 - \theta_2)\left[\frac{\partial}{\partial\theta_1} - \frac{\partial}{\partial\theta_2}\right]\right\}\rho_{\theta_1,\theta_2}$$
$$-\left\{\frac{\gamma kTa^2}{\hbar^2}(\theta_1-\theta_2)^2 - \frac{i\gamma e\phi}{m\hbar c}(\theta_1-\theta_2) - \frac{e\phi}{2\pi ma^2c}\left[\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right]\right\}\rho_{\theta_1,\theta_2}. \quad (4.20)$$

It is conveniwnt at this stage to introduce a gauge transformation; $\rho_{\theta_1,\theta_2} = e^{\frac{i\phi\theta}{\phi_0}}\tilde{\rho}_{\theta_1,\theta_2}e^{\frac{-i\phi\theta}{\phi_0}}$. Substituting this in Eq.(4.20), we obtain

$$\frac{d\tilde{\rho}_{\theta_1,\theta_2}}{dt} = \left\{\frac{i\hbar}{2ma^2}\left[\frac{\partial^2}{\partial\theta_1^2} - \frac{\partial^2}{\partial\theta_2^2}\right] - \frac{\gamma}{2m}(\theta_1-\theta_2)\left[\frac{\partial}{\partial\theta_1} - \frac{\partial}{\partial\theta_2}\right] - \frac{\gamma kTa^2}{\hbar^2}(\theta_1-\theta_2)^2\right\}\tilde{\rho}_{\theta_1,\theta_2} \quad (4.21)$$

4.3.2 The $t \to \infty$ *limit.* Steady-State Solution for the density matrix

Next, we transform the above equation by defining the center-of-mass $u = (\theta_1 + \theta_2)/2$ and the relative co-ordinates $\theta = (\theta_1 - \theta_2)$. The transformed equation is

$$\frac{d\tilde{\rho}(u,\theta)}{dt} = \left\{\frac{i\hbar}{ma^2}\frac{\partial^2}{\partial u\partial\theta} - \frac{\gamma}{m}\theta\frac{\partial}{\partial\theta} - \frac{\gamma k_B T a^2\theta^2}{\hbar^2}\right\}\tilde{\rho}(u,\theta). \quad (4.22)$$

As our particle is a free particle(no confining potential) and we are interested in the steady-state condition, we set $\frac{d\tilde{\rho}(u,z)}{dt} = 0$, *i.e., in the steady state (which is the equilibrium state in the time* $t \to \infty$ *limit) the density matrix* $\tilde{\rho}$ *can not depend upon time. Also, because of the uniformity in the 0-space it can not depend on the center-of-mass co-ordinate* $u = (\theta_1 + \theta_2)/2$. *The density matrix will then be a function of the relative co-ordinate* $\theta = \theta_1 - \theta_2$ *only. Thus, we get*

$$\frac{d\tilde{\rho}(\theta)}{d\theta} = -\left\{\frac{mK_BTa^2\theta}{\hbar^2}\right\}\tilde{\rho}(\theta). \quad (4.23)$$

It is now seen from Eq.(4.23) that in the steady state of the system, the friction coefficient γ drops out !. This already means that the orbital magnetization, to be obtained from the density matrix, will be independent of the frictional coupling y. For completeness, however, we derive below the expression for the partition function obtained from $\tilde{\rho}$, and thus show explicitly its dependence on the external magnetic field, (or flux ϕ).

Equation (4.23) gives the solution for $\tilde{\rho}(\theta)$:

$$\tilde{\rho}(\theta) = e^{-\left\{\frac{mk_B T a^2 \theta^2}{2\hbar^2}\right\}}, \tag{4.24}$$

or for $\rho(\theta) \equiv \rho(\theta_1 - \theta_2)$:

$$\rho(\theta_1 - \theta_2) = e^{-\left\{\frac{mk_B T a^2 (\theta_1 - \theta_2)^2}{2\hbar^2}\right\}} e^{i\overset{2\pi n}{\wedge}\phi/\phi_0}, \phi_0 = \frac{\mathrm{hc}}{\mathrm{e}} \text{ , the } flux \text{ quantum .} \tag{4.25}$$

4.3.3 The partition function

We now make a transition from the unbounded line $-\infty < \theta < +\infty$ to the bounded ring $0 \leq \theta \leq 2\pi$. This is realized by formally identifying the points θ and $\theta + 2\pi n$, for integer n (the winding number) that gives the topologically distinct paths (differing in the winding number) connecting θ_1 and θ_2 and must be counted as such in calculating the density matrix for the ring. (Distinct here means that the paths differing in n can not be deformed continuously into each other). Thus, we get for the density matrix on the ring $\hat{\rho}_{ring}$.

$$\langle \theta_1 | \hat{\rho}_{ring} | \theta_2 \rangle = \sum_{n=-\infty}^{n=+\infty} e^{-\left\{\frac{mk_B T a^2 (\theta_1 - \theta_2 + 2\pi n)^2}{2\hbar^2}\right\}} e^{in\frac{\phi}{\phi_0}}. \tag{4.26}$$

Here we have introduced the suffix 'ring' to emphasize the winding-number aspect, and made use of the fact that a winding number n links the flux(ϕ) n times. The ring current I_{ring} is then obtained from $\hat{\rho}_{ring}$ as

$$I_{ring} = -\left[\frac{ie\hbar}{ma^2}\frac{\partial}{\partial\theta_1}\langle\theta_1|\hat{\rho}_{ring}|\theta_2\rangle\right]_{\theta_1=\theta_2}, \tag{4.27}$$

giving the ring magnetic moment $M_{ring} = \pi a^2 I_{ring}$. Clearly, the Ring Magnetic Moment does not involve the dissipative coupling γ in that $\hat{\rho}_{ring}$ is independent of γ, as already demonstrated. The partition function for the ring $Z_{ring}(B, \beta)$ can now be expressed in terms of the density matrix $\hat{\rho}_{ring}$ as

$$Z_{ring}(B,\beta) = \sum_{n=-\infty}^{n=+\infty} \hat{\rho}_{ring}(\theta_1 = \theta_2 + 2\pi n) = \sum_{n=-\infty}^{n=+\infty} e^{-\frac{mk_BTa^2}{2\hbar^2}(2\pi n)^2} e^{in\frac{\phi}{\phi_0}}. \tag{4.28}$$

This can be cast in a more familiar form by use of the identity for the gaussion factor occuring in Eq.(4.28)

$$e^{-\frac{mk_BTa^2}{2\hbar^2}(2\pi n)^2} = \sqrt{\frac{2\pi ma^2k_BT}{\hbar^2}} \int_{-\infty}^{+\infty} e^{-\frac{\hbar^2}{2ma^2k_BT}y^2} e^{-i(2\pi n)y} dy\ , \tag{4.29}$$

followed by an identity for the Shah function (the Dirac comb) $III(y)$

$$III(y) \equiv \sum_{n=-\infty}^{n=+\infty} \delta(y-n) = \sum_{n=-\infty}^{n=+\infty} e^{i2\pi ny}. \tag{4.30}$$

We obtain

$$Z_{ring}(B,\beta) = \sum_{n=-\infty}^{n=+\infty} e^{-\frac{\beta\hbar^2}{2m^2}(n+\frac{\phi}{\phi_0})^2}\ , with\ \ , \beta = \frac{1}{k_BT}. \tag{4.31}$$

This is, of course, the elementary expression for the partition function at temperature T for a charged particle on a ring of radius 'a' threaded by a magnetic flux ϕ. This is independent of the friction coefficient γ. But it depends upon the the Aharonov-Bohm magnetic flux ϕ, and hence gives the non-zero diamagnetism.

4.4 Discussion

In this Chapter we have derived the orbital magnetic moment for a charged particle moving quantum mechanically on a ring threaded by a given magnetic flux in the presence of a dissipative coupling to the bath of harmonic oscillators. We first treated the problem through the Euclidean path integral and found that the path integral reduces to that of a free particle in the absence of the dissipative coupling, except for a renormalization of the inertia(mass) of the particle. Thus, the frictional coupling to the bath does not suppress(decohere) the quantum orbital diamagnetism, unlike the case known for the motion in a plane. We attribute this qualitative difference to the fact that in the case of the ring, the flux is a purly gauge flux (Aharonov-Bohm flux) and enters the path integral as geometrical/topological phase factor. There is no dynamical (Lorentz) force $\frac{e}{c}(\mathbf{v} \times \mathrm{B})$ acting locally on the particle, as is the case for the motion in a plane (or any simply connected region). This result is also recovered from the density matrix for the ring obtained by solving the Quantum-Brownian master equation. Again, the friction coefficient drops out and we obtain the non-zero orbital diamagnetism, as for the free particle without any dissipative coupling to the bath. The latter treatment, however, is valid only in the high temperature limit(as is the quantum master equation) on which it is based.

4.5 Appendix: Cancellation of the unphysical term in the action (generated by the dissipative coupling to the environmental degrees of freedom eliminated or integrated out) by the counter terms introduced in the system Lagrangian.

We will show analytically how the unphysical terms involving $(x_a+x_b)^2$ in the exponent on the R.H.S. of Eq. (4.11) get cancelled by a term to be introduced in the system Lagrangian L_S as a potential $V_{counter} = \frac{1}{2}m\Omega^2x^2$, quadratic in the system co-ordinate x. For simplicity, we will consider this in the high temperature limit. The introduction of this counter term generates two distinct terms in the exponent on the R.H.S. of Eq. (4.11), one containing $(x_b - x_a)^2$ and the other containing $(x_a + x_b)^2$. The latter can then be made to cancel out the unphysical term in Eq. (4.5) by a proper choice of the parameter Ω. We are then left only with the other (physical) term containing the factor $(x_a - x_b)^2$, which, of course, combines with $(-m/2\hbar^2\beta)(x_b - x_a)^2$ present there in the exponent. More explicitly, consider the effect of introducing the counter term $\frac{1}{2}m\Omega^2x^2$ in L_S on the exponent in the R.H.S. of Eq.(4.5) obtained by integrating over the $[x(t)]$ path intergral. The exponent gets modified additively by a term

$$-\frac{m\beta\Omega^2}{8}(x_b+x_a)^2-\frac{m\Omega^2\beta}{24}(x_b-x_a)^2,$$

(where we have considered the high-temperature limit, $\hbar\Omega\beta \ll 1$). These terms carry over to the exponent on the R.H.S. of Eq.(4.11), and combine there with the (x, $+ x_b)^2$ and the $(x_a - x_b)^2$ terms to give

$$(x_a+x_b)^2\left(-\frac{m\beta\Omega^8}{8}+\sum_j\frac{c_j^2\beta}{4M_j\omega_j^2}\right)$$

and

$$(x_a-x_b)^2\left(-\frac{m\beta\Omega^8}{24}+\sum_j\frac{c_j^2\beta}{24M_j\omega_j^2}\right)$$

Now, with the choice of the counter term parameter

$$\Omega^2=\sum_j\frac{2c_j^2}{mM_j\omega_j^2}\quad, \tag{4.32}$$

the unphysical term containing $(x_a + x_b)^2$ is cancelled out, while the physical term containing $(x_b - x_a)^2$ gets renormalized as

$$\frac{m(x_b-x_a)^2}{2\beta\hbar^2}[1+\sum_j\frac{c_j^2\beta^2\hbar^2}{12mM_j\omega_j^2}] \tag{4.33}$$

This is nothing but a renormalization of the inertia(mass) of the system particle:

$$m\to m_{eff}=m(1+\frac{\hbar^2\beta^2}{12m}\sum_j\frac{c_j^2}{M_j\omega_j^2}). \tag{4.34}$$

Following the convention[39], we introduce the spectral function for the dissipative coupling to the bath-oscillator continuum

$$J(\omega)\equiv\frac{\pi}{2}\sum_j\frac{1}{m_j\omega_j}c_j^2\delta(\omega-\omega_j)\equiv\eta\omega e^{-\omega/\omega_c}, \tag{4.35}$$

with $\omega_c =$ a high frequency cut-off, and η the friction coefficient due to the dissipative coupling to the environment. We can re-write $m_e$ff *as*

$$\frac{m_{eff}}{m}=1+\frac{1}{6\pi}\hbar^2(\frac{\eta}{m}\omega_c)\beta^2. \tag{4.36}$$

The correction is, however, of second order in β, and decreases to zero as $\beta \to 0$, (temperature $\to \infty$).

PART II