Chapter 4

The third post-Newtonian gravitational wave polarisations and associated spherical harmonic modes for inspiralling compact binaries in quasi-circular orbits

4.1 Introduction

Compact binary stars are one of the most important sources of gravitational radiation for the laser interferometric detectors LIGO, VIRGO [12, 13] and the proposed LISA [51]. Until the late inspiral stage of the binary evolution , prior to merger, the gravitational waves emitted by them are accurately described by the post-Newtonian (PN) approximation to general relativity [30], while the late inspiral and subsequent merger and ringdown phases are computed by a full-fledged numerical integration of the Einstein field equations [37, 38, 39, 40]. A new field has emerged recently consisting of high-accuracy comparisons between the PN predictions and the numerically-generated waveforms. Such comparisons and matching to the PN results have proved currently to be very successful [41, 42, 43, 44]. They clearly show the need to include high PN corrections not only for the evolution of the binary's orbital phase but also for the modulation of the gravitational amplitude.

The aim of this chapter is to compute the full gravitational waveform generated by inspiralling compact binaries moving in quasi-circular orbits at the third post-Newtonian (3PN) order[1]. By the full waveform (FWF) at a certain PN order, we mean the waveform including all higher-order amplitude corrections and hence all higher-order harmonics of the orbital frequency consistent with that PN order. The FWF is to be contrasted with the so-called restricted waveform (RWF) which retains only the leading-order harmonic at twice the orbital frequency. In applications to data analysis both the FWF and RWF should incorporate the orbital phase evolution up to the maximum available post-Newtonian order which is currently 3.5PN [116, 32, 71]. Previous investigations[84, 77, 96] have obtained the FWF up to 2.5PN

[1]As usual, we refer to nPN as the order equivalent to terms $\sim (v/c)^{2n}$ in the asymptotic waveform (beyond the Einstein quadrupole formula), where v denotes the binary's orbital velocity and c is the speed of light.

order[2]. Recently Kidder [117] pointed out that there is already enough information in the existing PN results [77] to control the *dominant* mode of the waveform, in a spin-weighted spherical harmonic decomposition, at the 3PN order. This mode, having $(\ell, m) = (2, 2)$, is the one which is computed in most numerical simulations, and which is therefore primarily needed for comparison with the PN waveforms. In the present chapter we shall extend the works [84, 77, 96, 117] by computing all the spin-weighted spherical harmonic modes (ℓ, m) consistent with the 3PN gravitational polarisations.

The data analysis of ground-based and space-based detectors has traditionally been based on the RWF approximation [73, 74, 118, 92, 119, 33, 120]. However, the need to consider the FWF as a more powerful template has been emphasized, not only for performing a more accurate parameter estimation [80, 94, 78, 79], but also for improving the mass reach and the detection rate [81, 82, 97]. Another motivation for considering the FWF instead of the RWF is to perform cosmological measurements of the Hubble parameter and dark energy using supermassive inspiralling black-hole binaries which are known to constitute standard gravitational-wave candles (or sirens) in cosmology [69, 70]. Indeed it has been shown that using the FWF in the data analysis of LISA will yield substantial improvements (with respect to the RWF) of the angular resolution and the estimation of the luminosity distance of gravitational-wave sirens [121, 89]. This means that LISA may be able to uniquely identify the galaxy cluster in which the supermassive black-hole coalescence took place, and thereby permit the measurement of the red-shift of the source which is crucially needed for investigating the equation of state of dark energy [121].

It turns out that in order to control the FWF at the 3PN order we need to further develop the multipolar post-Minkowskian (MPM) wave generation formalism [122, 123, 124, 125, 76, 126]. The MPM formalism describes the radiation field of any isolated post-Newtonian source and constitutes the basis of current PN calculations[3]. In this formalism, the radiation field is first of all parametrized by means of two sets of radiative multipole moments [130]. These moments are then related (by means of an algorithm for solving the non-linearities of the field equations) to the so-called canonical moments which constitute some useful intermediaries for describing the external field of the source. Finally, the canonical moments are expressed in terms of the operational source moments which are given by explicit integrals extending over the matter source and gravitational field. In previous studies [131, 116, 77, 132] most of the required source moments in the case of compact binaries were computed, or techniques were developed to compute them. The important step which remains here is to refine, by applying the MPM framework, the relationships between the radiative and canonical moments — this means taking into account more non-linear interactions between multipole moments — and between the canonical and source moments. The latter relationship involves controlling the coordinate transformation between two MPM algorithms respectively defined from the sets of canonical and source moments.

The plan of this chapter is as follows. In Section 5.2, we outline the post-Newtonian generation formalism, based on multipolar post-Minkowskian (MPM) expansions and matching to a general post-Newtonian source and also brief upon applications to compact binaries, modelled as point particles. In Section 5.3 we recall the basic formulas for defining the FWF

[2] The computation of the FWF is more demanding than that of the phase because it not only requires multipole moments with higher multipolarity but also higher PN accuracy in many of these multipole moments. This is why the FWF is known to a lower PN order than the phase.

[3] An alternative formalism called DIRE has been developed by Will and collaborators [127, 128, 129].

in terms of radiative multipole moments. Section 5.4 summarises the results for all the relevant moments parametrizing the FWF at 3PN order. The time derivatives of source moments are investigated in Section 5.5 and the various hereditary contributions are computed in Section 5.6. The complete polarization waveforms at 3PN order are given in Section 5.7 for data analysis applications. In Section 5.8, we provide the spin-weighted spherical harmonic modes of the 3PN waveform for use in numerical relativity. Finally, section 5.9 provides some applications to the data-analysis of super-massive black hole binaries using the 3PN waveform in the context of LISA.

4.2 The post-Newtonian wave generation formalism

The wave generation formalism relates the gravitational waves observed at a detector in the far-zone of the source to the stress-energy tensor of the source. Successful wave-generation formalisms mix and match approximation techniques from currently available collections. These include post-Minkowskian (PM) methods, post-Newtonian (PN) methods, multipole (M) expansions and perturbations around curved backgrounds. A recent review [30] discusses in detail the formalism we follow in the computation of the gravitational field; we summarise below the main features of this approach. This formalism has two independent aspects addressing two different problems. The first aspect, is the general method applicable to extended or fluid sources with compact support, based on the mixed PM and multipole expansion (we call it a MPM expansion), and matching to some PN source. The second aspect, is the application to point particle binaries modelling ICB.

4.2.1 The MPM expansion and matching to a post-Newtonian source

We follow Refs. [122, 123, 133, 124, 134, 135] to obtain the solution in the exterior of the source within the complete non-linear theory. The above referred works were built on earlier seminal papers of Bonnor [136] and Thorne [130] to set up the multipolar post-Minkowskian expansion. Starting from the general solution to the linearized Einstein's equations in the form of a multipolar expansion (valid in the exterior region), a PM iteration is performed and each multipolar piece is treated individually at any PM order. In addition to terms evaluated at one retarded time, the gravitational field contains terms integrated over the entire past "history" of the source. These are called the hereditary terms. For the external field, the general method is not limited *a priori* to PN sources. But, closed form expressions for the multipole moments can presently be obtained only for PN sources, because the exterior field may be connected to the inner field only if there exists an "overlapping" region where both the MPM and PN expansions are valid and can be matched together. For PN sources, this region always exists and is the exterior ($r > a$) near ($r \ll \lambda$) zone. After matching, it is found that the multipole moments have a non-compact support owing to the gravitational field stress-energy distributed everywhere up to spatial infinity. To include correctly these contributions, the definition of the multipole moments involves a finite part operation, based on analytic continuation. This process is equivalent to a Hadamard "partie finie" of the integrals at the bound at infinity.

The formalism of asymptotic matching procedure has been explored thoroughly and extended in a systematic way to higher PN orders [137, 138, 125, 126]. The final result of this analysis is that, the physical post-Newtonian (slowly moving) source is characterized by six symmetric and trace free (STF) time-varying multipole moments, denoted $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$,[4] which are specified for each source in the form of functionals of the formal PN expansion, up to any PN order, of the stress-energy pseudo-tensor $\tau^{\mu\nu}$ of the material and gravitational fields [126]. These moments parametrize the linear approximation to the vacuum metric outside the source, which is the first approximation in the MPM algorithm. In the linearized gravity case $\tau^{\mu\nu}$ reduces to the compact-support matter stress-energy tensor $T^{\mu\nu}$ and the expressions match perfectly with those derived in Ref. [139].

Starting from the complete set of six STF *source moments* $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$, for which general expressions can be given valid to any PN order, we define a different set of only two "*canonical*" source moments, denoted $\{M_L, S_L\}$, such that the two sets of moments $\{I_L, \cdots, Z_L\}$ and $\{M_L, S_L\}$ are physically equivalent. By this it is meant that they describe the same physical source, *i.e.* the two metrics, constructed respectively out of $\{I_L, \cdots, Z_L\}$ and $\{M_L, S_L\}$, differ by a mere coordinate transformation (are isometric). However, the six general source moments $\{I_L, \cdots, Z_L\}$ are closer rooted to the source because we know their expressions as integrals over $\tau^{\mu\nu}$. On the other hand, the canonical source moments $\{M_L, S_L\}$ are also necessary because their use simplifies the calculation of the external non-linearities. In addition, their existence shows that any radiating isolated source is characterized by two and only two sets of time-varying multipole moments [130, 122].

The MPM formalism is valid all over the weak field region outside the source including the wave zone (up to future null infinity). It is defined in harmonic coordinates. The far zone expansion at Minkowskian future null infinity contains logarithms in the distance which are artifacts of the harmonic coordinates. One can define, step by step in the PM expansion, some *radiative* coordinates by a coordinate transformation so that the log-terms are eliminated [123] and one recovers the standard (Bondi-type) radiative form of the metric, from which the *radiative moments*, denoted $\{U_L, V_L\}$, can be extracted in the usual way [130]. The wave generation formalism resulting from the exterior MPM field and matching to the PN source is able to take into account, in principle, any PN correction in both the source and radiative multipole moments. Nonlinearities in the external field are computed by a post-Minkowskian algorithm. This allows one to obtain the radiative multipole moments $\{U_L, V_L\}$, as some non-linear functional of the canonical moments $\{M_L, S_L\}$, and then of the actual source moments $\{I_L, \cdots, Z_L\}$. These relations between radiative and source moments include many non-linear multipole interactions as the source moments mix with each other as the waves propagate from the source to the detector. The dominant non-linear effect is due to the tails of wave, made of coupling between non-static moments and the total mass of the source, occurring at 1.5PN order ($\sim 1/c^3$) relative to the leading quadrupole radiation [124]. There is a corresponding tail effect in the equations of motion of the source, occurring at 1.5PN order relative to the leading 2.5PN radiation reaction, hence at 4PN order ($\sim 1/c^8$) beyond the Newtonian acceleration [133]. At higher PN orders, there are different types of non-linear multipole interactions, that are responsible for the presence of some important hereditary (*i.e.* past-history dependent) contributions to the waveform and energy flux.

A different wave-generation formalism from isolated sources, based on direct retarded integration of Einstein's equations in harmonic coordinates, is due to Will and Wiseman [127], and provided major improvement and elucidation of earlier investigations in the same

[4] As usual $L = i_1 i_2 \cdots i_\ell$ denotes a multi-index made of ℓ spatial indices (ranging from 1 to 3). The integer ℓ is referred to as the multipolar order. See the footnote in the beginning of the next section for more details

line [140, 130]. This formalism is based on different source multipole moments (defined by integrals extending over the near zone only), together with a different scheme for computing the non-linearities in the external field. It has currently been completed up to the 2PN order. At the most general level, *i.e.* for any PN extended source and in principle at any PN order, the Will-Wiseman formalism is completely equivalent to the present formalism based on MPM expansions with asymptotic matching (see Section 5.3 in [30] for the proof).

4.2.2 Application to compact binary systems

This application represents the second aspect of our approach. To this end, in the first instance, the compact objects (neutron stars or black holes) are modelled as point particles represented by Dirac δ-functions. Indeed for compact objects the effects of finite size and quadrupole distortion induced by tidal interactions are higher order in the PN approximation. However, the general formalism outlined in Section 4.2.1, is set up for a continuous (smooth) matter distribution, with continuous $T^{\mu\nu}$, and cannot be directly applied to point particles, since they lead to divergent integrals at the location of the particles, when $T^{\mu\nu}_{\text{point-particle}}$ is substituted into the source moments $\{I_L, \cdots, Z_L\}$. The calculation needs to be supplemented by a prescription for removing the infinite part of the integrals. Hadamard regularisation, based on Hadamard's notion of partie finie, is what we employ. This is our ansatz for applying a well-defined general "fluid" formalism to an initially ill-defined point-particle source.

To summarise: A systematic analytical approximation scheme has been set up for the calculation of waveforms and associated quantities from point particles to the PN order required (or permitted by given resources). A technical cost is the need to handle δ-functions in a non-linear theory, which is dealt with the Hadamard regularisation scheme or a variant of it. However, we already mentioned that at the 3PN order, subtleties arise due notably to the so called non-distributivity of the Hadamard partie finie, which resulted, as shown in [116], in some "ambiguities" when computing the 3PN mass-type quadrupole moment, which could entirely be encoded into three undetermined numerical coefficients ξ, κ, ζ. These combined into the unique quantity $\theta = \xi + 2\kappa + \zeta$ in the 3PN energy flux for circular orbits.

The latter ambiguities are the analogues of the undetermined parameters found in the binary's EOM at 3PN order, namely ω_k and ω_s in the canonical ADM approach [141, 142], and λ in the harmonic-coordinates formalism [143, 144]. The parameter λ is related to the "static" ambiguity ω_s by $\lambda = -\frac{3}{11}\omega_s - \frac{1987}{3080}$, while the "kinetic" ambiguity ω_k has been determined [143] to the value $\frac{41}{24}$ (see [145, 146] for details). The presence of the static ambiguity ω_s or, equivalently, λ, is a consequence of the Hadamard regularisation scheme which happens to become physically incomplete at the 3PN order. Recently, Damour, Jaranowski and Schäfer [147] proposed to use a better regularisation: *dimensional* regularisation. This led them to a unique determination of ω_s, namely $\omega_s = 0$. More recently [148], the application of dimensional regularisation to the computation of the EOM in harmonic coordinates has led to the equivalent result for λ, which is $\lambda = -\frac{1987}{3080}$. The EOM are thus completely determined to the 3PN order within both the ADM and harmonic-coordinates approaches using Hadamard regularisation supplemented by a crucial argument from dimensional regularisation in order to fix the last parameter.[5] All the 3PN conserved quantities are determined in

[5]Note that both calculations [147, 148] are performed in the limit $\varepsilon \to 0$, where the dimension of space is $d = 3 + \varepsilon$. The principle is to add to the end results given by the Hadamard regularisation [141, 142, 143, 144] the *difference* between the dimensional and Hadamard regularisations, which is specifically due to the poles

Refs. [145, 146]. A 3PN accurate center-of-mass has been constructed and used to reduce the conserved energy and angular momentum [149].

The numerical values of the radiation-field-related ambiguity coefficients ξ, κ and ζ introduced in [116] have also been determined [71] using dimensional regularization, so that the PN corrections to phasing are completely determined to 3.5PN accuracy. However, as we shall work in the present chapter on the 2.5PN waveform, *i.e.* one half order before 3PN, these ambiguities will not concern us here, and we shall find that no ambiguity shows up in any of our calculations based on Hadamard's regularisation. In fact, it can be shown that up to the 2.5PN order Hadamard's regularisation as we shall employ here gives the same result as would dimensional regularisation. The reason is that, in the source multipole moments up to this order, there are no logarithmic divergences occurring at the particles' locations, which correspond in d dimensions to poles in $\varepsilon = d - 3$.

We now give the explicit expressions of the GW polarisations and the multipole moments required to compute them and thereafter proceed directly to the computation of the final 3PN polarisations. We refer the reader to more details of the 3PN extensions of the MPM wave generational formalism and asymptotic matching algorithm to Ref. [150].

4.3 The polarization waveforms

The full waveform (FWF) propagating in the asymptotic regions of an isolated source, h_{ij}^{TT}, is the transverse-traceless (TT) projection of the metric deviation at the leading-order $1/R$ in the distance $R = |\mathbf{X}|$ to the source, in a radiative-type coordinate system $X^\mu = (c\,T, \mathbf{X})$. The FWF can be uniquely decomposed [130] into radiative multipole components parametrized by symmetric-trace-free (STF) mass-type moments U_L and current-type ones V_L.[6] The radiative moments are functions of the retarded time $T_R = T - R/c$ in radiative coordinates. By definition we have, up to any multipolar order ℓ,

$$h_{ij}^{\mathrm{TT}} = \frac{4G}{c^2R}\,\mathcal{P}_{ijkl}^{\mathrm{TT}}(\mathbf{N}) \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!}\left\{N_{L-2}\,U_{klL-2}(T_R) - \frac{2\ell}{c(\ell+1)}\,N_{aL-2}\,\varepsilon_{ab(k}\,V_{l)bL-2}(T_R)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right). \tag{4.1}$$

Here $\mathbf{N} = \mathbf{X}/R = (N_i)$ is the unit vector pointing from the source to the far away detector. The TT projection operator in (4.1) reads $\mathcal{P}_{ijkl}^{\mathrm{TT}} = \mathcal{P}_{ik}\mathcal{P}_{jl} - \frac{1}{2}\mathcal{P}_{ij}\mathcal{P}_{kl}$ where $\mathcal{P}_{ij} = \delta_{ij} - N_iN_j$ is the projector orthogonal to the unit direction $\mathbf{N}$. We introduce two unit polarisation vectors $\mathbf{P}$ and $\mathbf{Q}$, orthogonal and transverse to the direction of propagation $\mathbf{N}$ (hence $\mathcal{P}_{ij} = P_iP_j + Q_iQ_j$). Our convention for the choice of $\mathbf{P}$ and $\mathbf{Q}$ will be clarified in Section 5.9. Then the two

$\sim 1/\varepsilon$ and their associated finite part.

[6] The notation is: $L = i_1 \cdots i_\ell$ for a multi-index composed of ℓ multipolar spatial indices $i_1, \cdots, i_\ell$ (ranging from 1 to 3); similarly $L-1 = i_1 \cdots i_{\ell-1}$ and $aL-2 = ai_1 \cdots i_{\ell-2}$; $N_L = N_{i_1} \cdots N_{i_\ell}$ is the product of ℓ spatial vectors N_i (similarly for $x_L = x_{i_1} \cdots x_{i_\ell}$); $\partial_L = \partial_{i_1} \cdots \partial_{i_\ell}$ and say $\partial_{aL-2} = \partial_a\partial_{i_1} \cdots \partial_{i_{\ell-2}}$ denote the product of partial derivatives $\partial_i = \partial/\partial x^i$; in the case of summed-up (dummy) multi-indices L, we do not write the ℓ summations from 1 to 3 over their indices; the STF projection is indicated using brackets, $T_{\langle L\rangle} = \mathrm{STF}[T_L]$; thus $U_L = U_{\langle L\rangle}$ and $V_L = V_{\langle L\rangle}$ for STF moments; for instance we write $x_{\langle i}v_{j\rangle} = \frac{1}{2}(x_iv_j + x_jv_i) - \frac{1}{3}\delta_{ij}\mathbf{x}\cdot\mathbf{v}$; ε_{abc} is the Levi-Civita antisymmetric symbol such that $\varepsilon_{123} = 1$; time derivatives are denoted with a superscript (n).

"plus" and "cross" polarisation states of the FWF are defined by

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} = \frac{4G}{c^2R} \begin{pmatrix} \frac{P_iP_j - Q_iQ_j}{2} \\ \frac{P_iQ_j + P_jQ_i}{2} \end{pmatrix} \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!} \left\{ N_{L-2} U_{ijL-2}(T_R) - \frac{2\ell}{c(\ell+1)} N_{aL-2} \varepsilon_{ab(i} V_{j)bL-2}(T_R) \right\} + \mathcal{O}\left(\frac{1}{R^2}\right) . \tag{4.2}$$

Although the multipole decompositions (4.1) and (4.2) are all what we need for our purpose, it will also be important, having in view the ongoing comparisons between the PN and numerical results [41, 42, 43, 44], to consider separately the various modes (ℓ, m) of the FWF as defined with respect to a basis of spin-weighted spherical harmonics. To this end we decompose h_+ and $h_\times$ in the standard way as (see *e.g.* [41, 117])

$$h_+ - i h_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} \, Y^{\ell m}_{-2}(\Theta, \Phi) \, , \tag{4.3}$$

where the spin-weighted spherical harmonics of weight -2 is function of the spherical angles (Θ, Φ) defining the direction of propagation $\mathbf{N}$,[7] and is given by

$$Y^{\ell m}_{-2} = \sqrt{\frac{2\ell+1}{4\pi}} \, d_2^{\,\ell m}(\Theta) \, e^{i\, m\, \Phi} \, , \tag{4.4a}$$

$$d_2^{\,\ell m} = \sum_{k=k_1}^{k_2} \frac{(-)^k}{k!} \frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+2)!(\ell-2)!}}{(k-m+2)!(\ell+m-k)!(\ell-k-2)!} \left(\cos\frac{\Theta}{2}\right)^{2\ell+m-2k-2} \left(\sin\frac{\Theta}{2}\right)^{2k-m+2} . \tag{4.4b}$$

Here $k_1 = \max(0, m-2)$ and $k_2 = \min(\ell+m, \ell-2)$. Using the orthonormality properties of these harmonics we obtain the separate modes $h^{\ell m}$ from the surface integral

$$h^{\ell m} = \int d\Omega \left[h_+ - i h_\times \right] \overline{Y}^{\,\ell m}_{-2}(\Theta, \Phi) \, , \tag{4.5}$$

where the bar or overline denotes the complex conjugate. On the other hand, we can also, following [117], relate $h^{\ell m}$ directly to the multipole moments U_L and V_L. The result is[8]

$$h^{\ell m} = -\frac{G}{\sqrt{2}\, R\, c^{\ell+2}} \left[U^{\ell m} - \frac{i}{c} V^{\ell m} \right] , \tag{4.6}$$

where $U^{\ell m}$ and $V^{\ell m}$ are the radiative mass and current moments in standard (non-STF) guise [117]. These are related to the STF moments by

$$U^{\ell m} = \frac{4}{\ell!} \sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}} \, \alpha_L^{\ell m} \, U_L \, , \tag{4.7a}$$

[7] For the data analysis of compact binaries in Section 5.9 the direction of propagation will be defined by the angles $(\Theta, \Phi) = (i, \frac{\pi}{2})$ where i is the inclination angle of the orbit over the plane of the sky.

[8] We have an overall sign difference with [117] due to a different choice for the polarization triad $(\mathbf{N}, \mathbf{P}, \mathbf{Q})$.

$$V^{\ell m} = -\frac{8}{\ell!}\sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}}\,\alpha_L^{\ell m}\,V_L\,. \tag{4.7b}$$

Here $\alpha_L^{\ell m}$ denotes the STF tensor connecting together the usual basis of spherical harmonics $Y^{\ell m}$ to the set of STF tensors $N_{\langle L\rangle} = N_{\langle i_1}\cdots N_{i_\ell\rangle}$ (where the brackets indicate the STF projection). Indeed both $Y^{\ell m}$ and $N_{\langle L\rangle}$ are basis of an irreducible representation of weight ℓ of the rotation group. They are related by

$$N_{\langle L\rangle}(\Theta,\Phi) = \sum_{m=-\ell}^{\ell}\alpha_L^{\ell m}\,Y^{\ell m}(\Theta,\Phi)\,, \tag{4.8a}$$

$$Y^{\ell m}(\Theta,\Phi) = \frac{(2\ell+1)!!}{4\pi\ell!}\,\overline{\alpha}_L^{\ell m}\,N_{\langle L\rangle}(\Theta,\Phi)\,, \tag{4.8b}$$

with the STF tensorial coefficient being[9]

$$\alpha_L^{\ell m} = \int d\Omega\, N_{\langle L\rangle}\,\overline{Y}^{\ell m}\,. \tag{4.9}$$

As observed in [117] this is especially useful if some of the radiative moments are known to higher PN order than others. In this case the comparison with the numerical calculation for these individual modes can be made at higher PN accuracy.

4.4 The moments for 3PN waveform

Using the MPM algorithm as mentioned in Section 5.2 the radiative moments $\{U_L, V_L\}$ are related to the canonical moments $\{M_L, S_L\}$, and the canonical moments are in turn expressed in terms of the source moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$. In the current Section we present the results of the computation of all the moments needed for controlling the FWF in the case of compact binary systems up to 3PN order.

4.4.1 The radiative moments for 3PN polarisations

The result concerning the 3PN mass quadrupole moment U_{ij} is already known [124, 134, 135] and we simply report it here. Actually, at 3PN order U_{ij} involves a cubically non-linear term, composed of the so-called tails of tails, whose computation necessitates an extension of the MPM algorithm to cubic order G^3 [135]. We have

$$\begin{aligned}
U_{ij}(T_R) = {} & M_{ij}^{(2)}(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R} d\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{11}{12}\right]M_{ij}^{(4)}(\tau) \\
& + \frac{G}{c^5}\left\{-\frac{2}{7}\int_{-\infty}^{T_R} d\tau M_{a\langle i}^{(3)}(\tau)M_{j\rangle a}^{(3)}(\tau)\right. \\
& \qquad \left. +\frac{1}{7}M_{a\langle i}^{(5)}M_{j\rangle a} - \frac{5}{7}M_{a\langle i}^{(4)}M_{j\rangle a}^{(1)} - \frac{2}{7}M_{a\langle i}^{(3)}M_{j\rangle a}^{(2)} + \frac{1}{3}\varepsilon_{ab\langle i}M_{j\rangle a}^{(4)}S_b\right\}
\end{aligned}$$

[9]The notation used in [130, 117] is related to ours by $\mathcal{Y}_L^{\ell m} = \frac{(2\ell+1)!!}{4\pi\ell!}\,\overline{\alpha}_L^{\ell m}$.

$$
+2\left(\frac{GM}{c^3}\right)^2\int_{-\infty}^{T_R}d\tau\left[\ln^2\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{57}{70}\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{124627}{44100}\right]M_{ij}^{(5)}(\tau)
+\ \mathcal{O}\left(\frac{1}{c^7}\right). \tag{4.10}
$$

Notice the tail integral at 1.5PN order, the tail-of-tail integral at 3PN order, and the non-linear memory integral at 2.5PN. In the tail and tail-of-tail integrals, M represents the mass monopole moment or total mass of the binary system. The constant τ_0 in the tail integrals is given by $\tau_0 = r_0/c$, where r_0 is the arbitrary length scale originally introduced in the MPM formalism, and appearing also in the relation between the radiative and harmonic coordinates.

The moments required at 2.5PN order are new with this chapter (apart from the tails) and involve some interactions between the mass quadrupole moment and the mass octupole or current quadrupole moments. These moments are given by[10]

$$
\begin{aligned}
U_{ijk}(T_R) = {} & M_{ijk}^{(3)}(T_R)+\frac{2GM}{c^3}\int_{-\infty}^{T_R}d\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{97}{60}\right]M_{ijk}^{(5)}(\tau)\\
& +\frac{G}{c^5}\left\{\int_{-\infty}^{T_R}d\tau\left[-\frac{1}{3}M_{a\langle i}^{(3)}(\tau)M_{jk\rangle a}^{(4)}(\tau)-\frac{4}{5}\varepsilon_{ab\langle i}M_{ja}^{(3)}(\tau)S_{k\rangle b}^{(3)}(\tau)\right]\right.\\
& -\frac{4}{3}M_{a\langle i}^{(3)}M_{jk\rangle a}^{(3)}-\frac{9}{4}M_{a\langle i}^{(4)}M_{jk\rangle a}^{(2)}+\frac{1}{4}M_{a\langle i}^{(2)}M_{jk\rangle a}^{(4)}-\frac{3}{4}M_{a\langle i}^{(5)}M_{jk\rangle a}^{(1)}+\frac{1}{4}M_{a\langle i}^{(1)}M_{jk\rangle a}^{(5)}\\
& +\frac{1}{12}M_{a\langle i}^{(6)}M_{jk\rangle a}+\frac{1}{4}M_{a\langle i}M_{jk\rangle a}^{(6)}+\frac{1}{5}\varepsilon_{ab\langle i}\left[-12S_{ja}^{(2)}M_{k\rangle b}^{(3)}-8M_{ja}^{(2)}S_{k\rangle b}^{(3)}-3S_{ja}^{(1)}M_{k\rangle b}^{(4)}\right.\\
& \left.\left.-27M_{ja}^{(1)}S_{k\rangle b}^{(4)}-S_{ja}M_{k\rangle b}^{(5)}-9M_{ja}S_{k\rangle b}^{(5)}-\frac{9}{4}S_{a}M_{jk\rangle b}^{(5)}\right]+\frac{12}{5}S_{\langle i}S_{jk\rangle}^{(4)}\right\}\\
& +\mathcal{O}\left(\frac{1}{c^6}\right),
\end{aligned} \tag{4.11a}
$$

$$
\begin{aligned}
V_{ij}(T_R) = {} & S_{ij}^{(2)}(T_R)+\frac{2GM}{c^3}\int_{-\infty}^{T_R}d\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{7}{6}\right]S_{ij}^{(4)}(\tau)\\
& +\frac{G}{7\,c^5}\left\{4S_{a\langle i}^{(2)}M_{j\rangle a}^{(3)}+8M_{a\langle i}^{(2)}S_{j\rangle a}^{(3)}+17S_{a\langle i}^{(1)}M_{j\rangle a}^{(4)}-3M_{a\langle i}^{(1)}S_{j\rangle a}^{(4)}+9S_{a\langle i}M_{j\rangle a}^{(5)}\right.\\
& -3M_{a\langle i}S_{j\rangle a}^{(5)}-\frac{1}{4}S_{a}M_{ija}^{(5)}-7\varepsilon_{ab\langle i}S_{a}S_{j\rangle b}^{(4)}+\frac{1}{2}\varepsilon_{ac\langle i}\left[3M_{ab}^{(3)}M_{j\rangle bc}^{(3)}+\frac{353}{24}M_{j\rangle bc}^{(2)}M_{ab}^{(4)}\right.\\
& \left.\left.-\frac{5}{12}M_{ab}^{(2)}M_{j\rangle bc}^{(4)}+\frac{113}{8}M_{j\rangle bc}^{(1)}M_{ab}^{(5)}-\frac{3}{8}M_{ab}^{(1)}M_{j\rangle bc}^{(5)}+\frac{15}{4}M_{j\rangle bc}M_{ab}^{(6)}+\frac{3}{8}M_{ab}M_{j\rangle bc}^{(6)}\right]\right\}\\
& +\mathcal{O}\left(\frac{1}{c^6}\right).
\end{aligned} \tag{4.11b}
$$

At 2PN order we have the standard tails and some previously known interactions of the mass

[10]In all formulas below the STF projection $\langle\rangle$ applies only to the "free" indices denoted $ijkl\cdots$ carried by the moments themselves. Thus the dummy indices such as $abc\cdots$ are excluded from the STF projection.

quadrupole with itself [134], namely

$$\begin{aligned}U_{ijkl}(T_R) =\ & M^{(4)}_{ijkl}(T_R) + \frac{G}{c^3}\left\{2M\int_{-\infty}^{T_R} d\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{59}{30}\right]M^{(6)}_{ijkl}(\tau)\right.\\ &\left. +\frac{2}{5}\int_{-\infty}^{T_R} d\tau M^{(3)}_{\langle ij}(\tau)M^{(3)}_{kl\rangle}(\tau) - \frac{21}{5}M^{(5)}_{\langle ij}M_{kl\rangle} - \frac{63}{5}M^{(4)}_{\langle ij}M^{(1)}_{kl\rangle} - \frac{102}{5}M^{(3)}_{\langle ij}M^{(2)}_{kl\rangle}\right\}\\ &+ \mathcal{O}\left(\frac{1}{c^5}\right), \end{aligned} \tag{4.12a}$$

$$\begin{aligned}V_{ijk}(T_R) =\ & S^{(3)}_{ijk}(T_R) + \frac{G}{c^3}\left\{2M\int_{-\infty}^{T_R} d\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{5}{3}\right]S^{(5)}_{ijk}(\tau)\right.\\ &\left. +\frac{1}{10}\varepsilon_{ab\langle i}M^{(5)}_{ja}M_{k\rangle b} - \frac{1}{2}\varepsilon_{ab\langle i}M^{(4)}_{ja}M^{(1)}_{k\rangle b} - 2S_{\langle i}M^{(4)}_{jk\rangle}\right\}\\ &+ \mathcal{O}\left(\frac{1}{c^5}\right). \end{aligned} \tag{4.12b}$$

At 1.5PN we again have some non-linear interactions (new with this chapter) involving the mass octupole and current quadrupole and given by

$$\begin{aligned}U_{ijklm}(T_R) =\ & M^{(5)}_{ijklm}(T_R) + \frac{G}{c^3}\left\{2M\int_{-\infty}^{T_R} d\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{232}{105}\right]M^{(7)}_{ijklm}(\tau)\right.\\ & + \frac{20}{21}\int_{-\infty}^{T_R} d\tau M^{(3)}_{\langle ij}(\tau)M^{(4)}_{klm\rangle}(\tau) - \frac{710}{21}M^{(3)}_{\langle ij}M^{(3)}_{klm\rangle} - \frac{265}{7}M^{(2)}_{\langle ijk}M^{(4)}_{lm\rangle} - \frac{120}{7}M^{(2)}_{\langle ij}M^{(4)}_{klm\rangle}\\ &\left. -\frac{155}{7}M^{(1)}_{\langle ijk}M^{(5)}_{lm\rangle} - \frac{41}{7}M^{(1)}_{\langle ij}M^{(5)}_{klm\rangle} - \frac{34}{7}M_{\langle ijk}M^{(6)}_{lm\rangle} - \frac{15}{7}M_{\langle ij}M^{(6)}_{klm\rangle}\right\}\\ &+ \mathcal{O}\left(\frac{1}{c^4}\right), \end{aligned} \tag{4.13a}$$

$$\begin{aligned}V_{ijkl}(T_R) =\ & S^{(4)}_{ijkl}(T_R) + \frac{G}{c^3}\left\{2M\int_{-\infty}^{T_R} d\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{119}{60}\right]S^{(6)}_{ijkl}(\tau)\right.\\ & -\frac{35}{3}S^{(2)}_{\langle ij}M^{(3)}_{kl\rangle} - \frac{25}{3}M^{(2)}_{\langle ij}S^{(3)}_{kl\rangle} - \frac{65}{6}S^{(1)}_{\langle ij}M^{(4)}_{kl\rangle} - \frac{25}{6}M^{(1)}_{\langle ij}S^{(4)}_{kl\rangle} - \frac{19}{6}S_{\langle ij}M^{(5)}_{kl\rangle}\\ & - \frac{11}{6}M_{\langle ij}S^{(5)}_{kl\rangle} - \frac{11}{12}S_{\langle i}M^{(5)}_{jkl\rangle} + \frac{1}{6}\varepsilon_{ab\langle i}\left[-5M^{(3)}_{ja}M^{(3)}_{kl\rangle b} - \frac{11}{2}M^{(4)}_{ja}M^{(2)}_{kl\rangle b} - \frac{5}{2}M^{(2)}_{ja}M^{(4)}_{kl\rangle b}\right.\\ &\left.\left. -\frac{1}{2}M^{(5)}_{ja}M^{(1)}_{kl\rangle b} + \frac{37}{10}M^{(1)}_{ja}M^{(5)}_{kl\rangle b} + \frac{3}{10}M^{(6)}_{ja}M_{kl\rangle b} + \frac{1}{2}M_{ja}M^{(6)}_{kl\rangle b}\right]\right\}\\ &+ \mathcal{O}\left(\frac{1}{c^4}\right). \end{aligned} \tag{4.13b}$$

For all the other higher order moments that are required, it is sufficient to assume the relation between the radiative and canonical moments,

$$U_L(T_R) = M^{(\ell)}_L(T_R) + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{4.14a}$$

$$V_L(T_R) = S_L^{(\ell)}(T_R) + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{4.14b}$$

4.4.2 The canonical moments for 3PN polarisations

We now give the canonical moments in terms of source-rooted multipole moments. It turns out that the difference between these two types of moments (which is due to the presence of the gauge moments) arises only at the small 2.5PN order. The consequence is that we have to worry about this difference only for the 3PN canonical mass quadrupole moment M_{ij}, the 2.5PN mass octupole moment M_{ijk}, and the 2.5PN current quadrupole moment S_{ij}. For the mass quadrupole moment, the requisite correction has already been used in [77] and is given by[11]

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.15}$$

where I_{ij} denotes the source mass quadrupole, and where W is the monopole corresponding to the gauge moments W_L (*i.e.* W is the moment having $\ell = 0$). At the PN order we are working, W is needed only at Newtonian order and will be provided in Section 4.4.3. Notice that the remainder in (4.15) is at order 3.5PN — consistently with the accuracy we aim here. The expression (4.15) is valid in a mass-centred frame defined by the vanishing of the mass dipole moment: $I_i = 0$. Note that a formula generalizing (4.15) to all PN orders (and all multipole interactions) is not possible at present and needs to be investigated anew for specific cases. Thus it is convenient in the present approach to use systematically the source moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ as the fundamental variables describing the source.

Similarly, the other moments M_{ijk} and S_{ij} will admit some correction terms starting at the 2.5PN order. Details of the computation of these new corrections can be found in Ref. [150].

Our explicit results for M_{ijk} and S_{ij} are

$$M_{ijk} = I_{ijk} + \frac{4G}{c^5}\left[W^{(2)} I_{ijk} - W^{(1)} I_{ijk}^{(1)} + 3\, I_{\langle ij} Y_{k\rangle}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.16a}$$

$$\begin{aligned} S_{ij} = J_{ij} + \frac{2G}{c^5}\Big[\varepsilon_{ab\langle i}\left(-I_{j\rangle b}^{(3)} W_a - 2 I_{j\rangle b} Y_a^{(2)} + I_{j\rangle b}^{(1)} Y_a^{(1)}\right) + 3 J_{\langle i} Y_{j\rangle}^{(1)} - 2 J_{ij}^{(1)} W^{(1)}\Big] \\ + \mathcal{O}\left(\frac{1}{c^6}\right), \end{aligned} \tag{4.16b}$$

where W_i and Y_i are the dipole moments corresponding to the moments W_L and Y_L. The remainders in (4.16) are consistent with our approximation 3PN for the FWF. Besides the mass quadrupole moment (4.15), and mass octupole and current quadrupole moments (4.16), we can state that, with the required 3PN precision, all the other moments M_L agree with their corresponding I_L, and similarly the S_L agree with J_L, namely

$$M_L = I_L + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{4.17a}$$

$$S_L = J_L + \mathcal{O}\left(\frac{1}{c^5}\right). \tag{4.17b}$$

[11] The equation (11.7a) in [116] contains a sign error with respect to the original result [76] (with no consequence for any of the results in [116]). The correct sign is reproduced here.

4.4.3 The source moments for 3PN polarisations

We have finally succeeded in parametrizing the FWF entirely in terms of the source moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ up to 3PN order. The interest of this construction lies in the fact that the source moments are known for general PN matter systems. They were obtained by matching the external MPM field of the source to the internal PN field valid in the source's near zone [125, 76, 126]. The source moments have been worked out in the case of compact binary systems with increasing PN precision [131, 116, 132, 77]. Here we list all the required I_L's and J_L's (and also the few needed gauge moments) for non-spinning compact objects and for circular orbits. We do not enter the details because the derivation of these moments follows exactly the same techniques as in [116, 132].

The only moment needed at the 3PN order is the mass quadrupole moment I_{ij}, first computed for circular orbits in [116] and subsequently extended to general orbits in [132]. We write it as

$$I_{ij} = \nu\, m \left(A\, x_{\langle ij\rangle} + B\, \frac{r^3}{Gm}\, v_{\langle ij\rangle} + C\, \sqrt{\frac{r^3}{Gm}}\, x_{\langle i} v_{j\rangle} \right) + \mathcal{O}\left(\frac{1}{c^7}\right) . \tag{4.18}$$

The relative position and velocity of the two bodies in harmonic coordinates are denoted by $x^i = y_1^i - y_2^i$ and $v^i = dx^i/dt = v_1^i - v_2^i$ (spatial indices are lowered and raised with the Kronecker metric so that $x_i = x^i$ and $v_i = v^i$). The distance between the two particles in harmonic coordinates is denoted $r = |\mathbf{x}|$. The two masses are m_1 and m_2, the total mass is $m = m_1 + m_2$ (not to be confused with the mass monopole moment M), the symmetric mass ratio $\nu = m_1 m_2/m^2$ satisfies $0 < \nu \leq 1/4$, and the mass difference ratio is $\Delta = (m_1 - m_2)/m$ which reads also $\Delta = \pm\sqrt{1-4\nu}$ (according to the sign of $m_1 - m_2$). To express the coefficients A, B and C in (4.18) as PN series we introduce the small post-Newtonian parameter

$$\gamma = \frac{Gm}{rc^2} . \tag{4.19}$$

With these notations we have (in the frame of the 'center-of-mass' and for circular orbits)

$$\begin{aligned} A &= 1 + \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right) \\ &\quad + \gamma^3\left(\frac{395899}{13200} - \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \left[\frac{3304319}{166320} - \frac{44}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu \right. \\ &\qquad \left. + \frac{162539}{16632}\nu^2 + \frac{2351}{33264}\nu^3\right) , \end{aligned} \tag{4.20a}$$

$$\begin{aligned} B &= \gamma\left(\frac{11}{21} - \frac{11}{7}\nu\right) + \gamma^2\left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2\right) \\ &\quad + \gamma^3\left(-\frac{357761}{19800} + \frac{428}{105}\ln\left(\frac{r}{r_0}\right) - \frac{92339}{5544}\nu + \frac{35759}{924}\nu^2 + \frac{457}{5544}\nu^3\right) , \end{aligned} \tag{4.20b}$$

$$C = \frac{48}{7}\,\gamma^{5/2}\,\nu . \tag{4.20c}$$

The coefficients A and B correspond to conservative PN orders (which are even), while the coefficient C involves a single term at the odd 2.5PN order due to radiation reaction.

Notice the appearance of logarithms in both A and B at the 3PN order. These logarithms have two distinct origins, depending on whether they are scaled with the constant r_0 associated with the finite part prescription , or with an alternative constant denoted r'_0. The logarithms with r_0 will combine later with other contributions due to tails and tails-of-tails, and the constant r_0 will be absorbed into some unobservable shift of the binary's orbital phase, as can already be seen from the fact that r_0 is associated with the difference of origin of time between harmonic and radiative coordinates.

The other constant r'_0 is defined by $m \ln r'_0 = m_1 \ln r'_1 + m_2 \ln r'_2$, where r'_1 and r'_2 are two *regularization constants* appearing in a Hadamard self-field regularization scheme for the 3PN equations of motion of point masses in harmonic coordinates [143, 144]. The constant r'_0 is therefore present in the 3PN equations of motion and we shall thus also meet this constant in the 3PN orbital frequency given by (4.35) below. The regularization constant r'_0 is unobservable, since it can be removed by a coordinate transformation at 3PN order — r'_0 can rightly be called a *gauge constant*. In practice this means that r'_0 will cancel out when using the 3PN equations of motion to compute the time derivatives of the 3PN quadrupole moment, as will be explicitly verified in Section 5.6.[12]

The list of required moments continues with the 2.5PN order at which we need the mass octupole and current quadrupole given by (with $\Delta = \frac{m_1-m_2}{m}$)

$$\begin{aligned}
I_{ijk} = -\nu\, m\, \Delta \Bigg\{ & x_{\langle ijk \rangle} \left[1 - \gamma\nu - \gamma^2 \left(\frac{139}{330} + \frac{11923}{660}\nu + \frac{29}{110}\nu^2 \right) \right] \\
& + \frac{r^2}{c^2}\, x_{\langle i} v_{jk \rangle} \left[1 - 2\nu - \gamma \left(-\frac{1066}{165} + \frac{1433}{330}\nu - \frac{21}{55}\nu^2 \right) \right] \\
& + \frac{196}{15}\, \frac{r}{c}\, \gamma^2\, \nu\, x_{\langle ij} v_{k \rangle} \Bigg\} + \mathcal{O}\left(\frac{1}{c^6} \right), \qquad (4.21a)
\end{aligned}$$

$$\begin{aligned}
J_{ij} = -\nu\, m\, \Delta \Bigg\{ & \varepsilon_{ab\langle i} x_{j\rangle a} v_b \left[1 + \gamma \left(\frac{67}{28} - \frac{2}{7}\nu \right) + \gamma^2 \left(\frac{13}{9} - \frac{4651}{252}\nu - \frac{1}{168}\nu^2 \right) \right] \\
& - \frac{484}{105}\, \frac{r}{c}\, \gamma^2\, \nu\, \varepsilon_{ab\langle i} v_{j\rangle a} x_b \Bigg\} + \mathcal{O}\left(\frac{1}{c^6} \right). \qquad (4.21b)
\end{aligned}$$

At 2PN order we require:

$$\begin{aligned}
I_{ijkl} = \nu\, m \Bigg\{ & x_{\langle ijkl \rangle} \Bigg[1 - 3\nu + \gamma \left(\frac{3}{110} - \frac{25}{22}\nu + \frac{69}{22}\nu^2 \right) \\
& + \gamma^2 \left(-\frac{126901}{200200} - \frac{58101}{2600}\nu + \frac{204153}{2860}\nu^2 + \frac{1149}{1144}\nu^3 \right) \Bigg] \\
& + \frac{r^2}{c^2}\, x_{\langle ij} v_{kl \rangle} \Bigg[\frac{78}{55}(1 - 5\nu + 5\nu^2) \\
& + \gamma \left(\frac{30583}{3575} - \frac{107039}{3575}\nu + \frac{8792}{715}\nu^2 - \frac{639}{715}\nu^3 \right) \Bigg]
\end{aligned}$$

[12] Note also that the 3PN quadrupole moment [116, 132] depended originally on three constants ξ, κ, ζ (called ambiguity parameters) reflecting some incompleteness of the Hadamard self-field regularization. These constants have been computed by means of the powerful dimensional regularization [148, 71], and we have replaced the result, which was $\xi = -\frac{9871}{9240}$, $\kappa = 0$ and $\zeta = -\frac{7}{33}$, back into (4.20).

$$+\frac{71}{715}\frac{r^4}{c^4}\,v_{\langle ijkl\rangle}\left(1-7\nu+14\nu^2-7\nu^3\right)\Bigg\}+\mathcal{O}\left(\frac{1}{c^5}\right), \tag{4.22a}$$

$$\begin{aligned}J_{ijk} = \nu\, m\, \Bigg\{ & \varepsilon_{ab\langle i}x_{jk\rangle a}v_b\left[1-3\nu+\gamma\left(\frac{181}{90}-\frac{109}{18}\nu+\frac{13}{18}\nu^2\right)\right.\\ & +\left.\gamma^2\left(\frac{1469}{3960}-\frac{5681}{264}\nu+\frac{48403}{660}\nu^2-\frac{559}{3960}\nu^3\right)\right]\\ & +\frac{r^2}{c^2}\,\varepsilon_{ab\langle i}x_a v_{jk\rangle b}\left[\frac{7}{45}\left(1-5\nu+5\nu^2\right)+\gamma\left(\frac{1621}{990}-\frac{4879}{990}\nu+\frac{1084}{495}\nu^2-\frac{259}{990}\nu^3\right)\right]\Bigg\}\\ & +\mathcal{O}\left(\frac{1}{c^5}\right).\end{aligned} \tag{4.22b}$$

At 1.5PN order:

$$\begin{aligned}I_{ijklm} = -\nu\, m\,\Delta\,\Bigg\{ & x_{\langle ijklm\rangle}\left[1-2\nu+\gamma\left(\frac{2}{39}-\frac{47}{39}\nu+\frac{28}{13}\nu^2\right)\right]\\ & +\frac{70}{39}\frac{r^2}{c^2}x_{\langle ijk}v_{lm\rangle}\left(1-4\nu+3\nu^2\right)\Bigg\}+\mathcal{O}\left(\frac{1}{c^4}\right),\end{aligned} \tag{4.23a}$$

$$\begin{aligned}J_{ijkl} = -\nu\, m\,\Delta\,\Bigg\{ & \varepsilon_{ab\langle i}x_{jkl\rangle a}v_b\left[1-2\nu+\gamma\left(\frac{20}{11}-\frac{155}{44}\nu+\frac{5}{11}\nu^2\right)\right]\\ & +\frac{4}{11}\frac{r^2}{c^2}\,\varepsilon_{ab\langle i}x_{ja}v_{kl\rangle b}\left(1-4\nu+3\nu^2\right)\Bigg\}+\mathcal{O}\left(\frac{1}{c^4}\right).\end{aligned} \tag{4.23b}$$

At 1PN order:

$$\begin{aligned}I_{ijklmn} = \nu\, m\,\Bigg\{ & x_{\langle ijklmn\rangle}\left[1-5\nu+5\nu^2+\gamma\left(\frac{1}{14}-\frac{3}{2}\nu+6\nu^2-\frac{11}{2}\nu^3\right)\right]\\ & +\frac{15}{7}\frac{r^2}{c^2}\,x_{\langle ijkl}v_{mn\rangle}\left(1-7\nu+14\nu^2-7\nu^3\right)\Bigg\}+\mathcal{O}\left(\frac{1}{c^4}\right),\end{aligned} \tag{4.24a}$$

$$\begin{aligned}J_{ijklm} = \nu\, m\,\Bigg\{ & \varepsilon_{ab\langle i}x_{jklm\rangle a}v_b\left[1-5\nu+5\nu^2+\gamma\left(\frac{1549}{910}-\frac{1081}{130}\nu+\frac{107}{13}\nu^2-\frac{29}{26}\nu^3\right)\right]\\ & +\frac{54}{91}\frac{r^2}{c^2}\varepsilon_{ab\langle i}x_{jka}v_{lm\rangle b}\left(1-7\nu+14\nu^2-7\nu^3\right)\Bigg\}+\mathcal{O}\left(\frac{1}{c^4}\right).\end{aligned} \tag{4.24b}$$

At 0.5PN order:

$$I_{ijklmno} = -\nu\, m\,\Delta\,(1-4\nu+3\nu^2)\,x_{\langle ijklmno\rangle}+\mathcal{O}\left(\frac{1}{c^2}\right), \tag{4.25a}$$

$$J_{ijklmn} = -\nu\, m\,\Delta\,(1-4\nu+3\nu^2)\,\varepsilon_{ab\langle i}x_{jklmn\rangle a}v_b+\mathcal{O}\left(\frac{1}{c^2}\right). \tag{4.25b}$$

At Newtonian order:

$$I_{ijklmnop} = \nu\, m\,\left(1-7\nu+14\nu^2-7\nu^3\right)\,x_{\langle ijklmnop\rangle}+\mathcal{O}\left(\frac{1}{c^2}\right), \tag{4.26a}$$

$$J_{ijklmno} = \nu\, m\, \left(1 - 7\nu + 14\nu^2 - 7\nu^3\right)\, \varepsilon_{ab\langle i} x_{jklmno\rangle a} v_b + \mathcal{O}\left(\frac{1}{c^2}\right) . \tag{4.26b}$$

The 2.5PN correction terms in I_{ijk} and J_{ij}, the 2PN terms in I_{ijkl} and J_{ijk}, and the 1PN terms in I_{ijklm} and J_{ijkl} are new with this chapter. The higher-order Newtonian moments $I_{ijklmno}$ and J_{ijklmn} were also not needed before, but Newtonian moments are trivial and are given for general ℓ by

$$I_L = \nu\, m\, s_\ell(\nu)\, x_{\langle L\rangle} + \mathcal{O}\left(\frac{1}{c^2}\right) , \tag{4.27a}$$

$$J_{L-1} = \nu\, m\, s_\ell(\nu)\, \varepsilon_{ab\langle i_{\ell-1}} x_{L-2\rangle a} v_b + \mathcal{O}\left(\frac{1}{c^2}\right) , \tag{4.27b}$$

in which we pose

$$s_\ell(\nu) = X_2^{\ell-1} + (-)^\ell X_1^{\ell-1} . \tag{4.28}$$

Here we define $X_1 = \frac{m_1}{m} = \frac{1}{2}(1+\Delta)$ and $X_2 = \frac{m_2}{m} = \frac{1}{2}(1-\Delta)$ with $\Delta = \frac{m_1 - m_2}{m} = \pm\sqrt{1-4\nu}$, so that $X_1 + X_2 = 1$ and $X_1 X_2 = \nu$.

In addition we shall need the mass monopole I agreeing with its canonical counterpart M which parametrizes the various tail terms in Section 5.4.1. Since the tails arise at 1.5PN order we need M only at the 1.5PN relative order. It is given by

$$I = M = m\left(1 - \frac{\nu}{2}\gamma\right) + \mathcal{O}\left(\frac{1}{c^4}\right) . \tag{4.29}$$

We require also the current dipole moment or angular momentum J_i (agreeing with its canonical counterpart S_i) since it appears in some non-linear terms, for instance in (4.10). It is needed only at Newtonian order,

$$J_i = S_i = \nu\, m\, \varepsilon_{iab} x_a v_b + \mathcal{O}\left(\frac{1}{c^2}\right) . \tag{4.30}$$

Finally, we have to provide the few gauge moments that enter the relations between canonical and source moments found in (4.15) and (4.16). They are readily computed from the general expressions of all the gauge moments $\{W_L, X_L, Y_L, Z_L\}$ given in (5.15)–(5.20) of [126]. The calculation is quite simple because these moments, namely the monopolar moment W and the two dipole moments W_i and Y_i, are Newtonian. For circular orbits we find

$$W = \mathcal{O}\left(\frac{1}{c^2}\right) , \tag{4.31a}$$

$$W_i = \frac{1}{10}\, \nu\, m\, \Delta\, r^2\, v^i + \mathcal{O}\left(\frac{1}{c^2}\right) , \tag{4.31b}$$

$$Y_i = \frac{1}{5}\, \frac{G\, m^2\, \nu}{r}\, \Delta\, x^i + \mathcal{O}\left(\frac{1}{c^2}\right) . \tag{4.31c}$$

We are done with all the source multipole moments needed to control the 3PN accurate FWF generated by compact binary sources in quasi-circular orbits.

4.5 Time derivatives of the source multipole moments

For the purpose of computing the time derivatives of the source moments we require the 3PN accurate equations of motion of compact binary sources. Like in the computation of the moments we have to take into account both the conservative effects at 1PN, 2PN and 3PN orders, and the effect of radiation reaction at 2.5PN order.

We consider non-spinning objects so the motion takes place in a fixed plane, say the x-y plane. The relative position $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$, velocity $\mathbf{v} = d\mathbf{x}/dt$, and acceleration $\mathbf{a} = d\mathbf{v}/dt$ are given by

$$\mathbf{x} = r\,\mathbf{n}\,, \tag{4.32a}$$

$$\mathbf{v} = \dot{r}\,\mathbf{n} + r\,\omega\,\boldsymbol{\lambda}\,, \tag{4.32b}$$

$$\mathbf{a} = (\ddot{r} - r\,\omega^2)\,\mathbf{n} + (r\,\dot{\omega} + 2\dot{r}\,\omega)\,\boldsymbol{\lambda}\,. \tag{4.32c}$$

For a while the time derivative will be denoted using an over dot. Here $\boldsymbol{\lambda} = \hat{\mathbf{z}} \times \mathbf{n}$ is perpendicular to the unit vector $\hat{\mathbf{z}}$ along the z-direction orthogonal to the orbital plane, and to the binary's separation direction $\mathbf{n}$. The orbital frequency ω is related in the usual way to the orbital phase ϕ by $\omega = \dot{\phi}$.

Through 3PN order, it is possible to model the motion of the binary as a *quasi-circular* orbit decaying by the effect of radiation reaction at the 2.5PN order. This effect is computed by balancing the change in the orbital energy with the total energy flux radiated by the gravitational waves. At 2.5PN order this yields (see *e.g.* [96])

$$\dot{r} = -\frac{64}{5}\sqrt{\frac{Gm}{r}}\,\nu\,\gamma^{5/2} + O\left(\frac{1}{c^7}\right), \tag{4.33a}$$

$$\dot{\omega} = \frac{96}{5}\,\frac{Gm}{r^3}\,\nu\,\gamma^{5/2} + O\left(\frac{1}{c^7}\right), \tag{4.33b}$$

where γ is given by (4.19). By substituting those expressions into (4.32),[13] we obtain the expressions for the inspiral velocity and acceleration,

$$\mathbf{v} = r\,\omega\,\boldsymbol{\lambda} - \frac{64}{5}\sqrt{\frac{Gm}{r}}\,\nu\,\gamma^{5/2}\,\mathbf{n} + O\left(\frac{1}{c^7}\right), \tag{4.34a}$$

$$\mathbf{a} = -\omega^2\,\mathbf{x} - \frac{32}{5}\sqrt{\frac{Gm}{r^3}}\,\nu\,\gamma^{5/2}\,\mathbf{v} + O\left(\frac{1}{c^7}\right). \tag{4.34b}$$

A central result of PN calculations of the equations of motion is the expression of the orbital frequency ω in terms of the binary's separation r up to 3PN order. This result has been obtained in harmonic coordinates in [143, 144, 148] and independently in [151, 152, 153], and in ADM coordinates in [141, 142, 147]. In the present work r is given in harmonic coordinates and the expression of the 3PN orbital frequency is

$$\omega^2 = \frac{Gm}{r^3}\Big\{1 + \gamma\Big(-3 + \nu\Big) + \gamma^2\left(6 + \frac{41}{4}\nu + \nu^2\right) \tag{4.35}$$

[13] We notice that $\ddot{r} = O(c^{-10})$ is of the order of the *square* of radiation-reaction effects and is therefore zero with this approximation.

$$+\gamma^3\left(-10+\left[-\frac{75707}{840}+\frac{41}{64}\pi^2+22\ln\left(\frac{r}{r'_0}\right)\right]\nu+\frac{19}{2}\nu^2+\nu^3\right)+O\left(\frac{1}{c^8}\right)\Bigg\}.$$

Note that the logarithm at 3PN order involves the same constant r'_0 as in the source quadrupole moment (4.18)–(4.20). This logarithm comes from a Hadamard self-field regularization scheme and its appearance is specific to harmonic coordinates.

As often convenient we shall use in place of the parameter γ given by (4.19) an alternative parameter x directly linked to the orbital frequency (4.35), namely

$$x=\left(\frac{G\,m\,\omega}{c^3}\right)^{2/3}. \tag{4.36}$$

The interest in this parameter stems from its invariant meaning in a large class of coordinate systems including the harmonic and ADM coordinate systems. At 3PN order it is given in terms of x by

$$\begin{aligned}\gamma = x\Bigg\{&1+x\left(1-\frac{\nu}{3}\right)+x^2\left(1-\frac{65}{12}\nu\right) \\ &+x^3\left(1+\left[-\frac{2203}{2520}-\frac{41}{192}\pi^2-\frac{22}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu+\frac{229}{36}\nu^2+\frac{\nu^3}{81}\right)+O\left(\frac{1}{c^8}\right)\Bigg\}.\end{aligned} \tag{4.37}$$

Combining (4.35) with (4.37) we find that the velocity squared $v^2=r^2\omega^2+\dot{r}^2=r^2\omega^2+O(c^{-10})$ is related to x by

$$\begin{aligned}\left(\frac{v}{c}\right)^2 = x\Bigg\{&1+x\left(-2+\frac{2}{3}\nu\right)+x^2\left(1+\frac{53}{6}\nu+\frac{\nu^2}{3}\right) \\ &+x^3\left(\left[-\frac{36227}{1260}+\frac{41}{96}\pi^2+\frac{44}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu-\frac{29}{9}\nu^2+\frac{10}{81}\nu^3\right)+O\left(\frac{1}{c^8}\right)\Bigg\}.\end{aligned} \tag{4.38}$$

During the computation of the time derivatives of the source moments, each time an acceleration is produced the result is consistently *order reduced*, *i.e.* the acceleration is replaced with (4.34b) at the right PN order. Such an order reduction will generate in particular some 2.5PN radiation-reaction terms which are to be taken into account in the 3PN waveform. This occurs when computing the time derivatives of the moments I_{ij}, I_{ijk} and J_{ij} that appear in the FWF at Newtonian and 0.5PN orders. On the other hand, when computing the polarization states following (4.2) we shall meet some scalar products of the polarization vectors **P** and **Q** with the relative velocity **v**. If those scalar products occur at Newtonian and 0.5PN orders (*i.e.* in multipolar pieces corresponding to the moments I_{ij}, I_{ijk} and J_{ij}) we shall have to take into account the 2.5PN radiation-reaction term coming from the expression of **v** given by (4.34a).[14] However it was shown in [96] that the radiation-reaction terms in the FWF at the 2.5PN order can be absorbed into a modification of the orbital phase, where they appear to constitute in fact a very small phase modulation, comparable with unknown contributions in the phase being at least of order 5PN — negligible here since the phase is known only to 3.5PN order. In the present chapter, we have chosen[15] to include all the radiation-

[14] Not considering the radiation-reaction contribution in **v** given by (4.34a) has been the source of an error in [77] which has been pointed out and corrected in [96].

[15] As usual there are many different ways of presenting PN results at a given order of approximation, and

reaction terms coming from both (4.34a) and (4.34b), and to present them as 2.5PN and 3PN amplitude corrections in our final results which will be presented in (4.63)–(4.64) and (4.71), (4.72) ... (4.77) below.

Let us next check that the Hadamard self-field regularization constant ${r'}_0$ appearing both in the 3PN orbital frequency (4.35) and in the 3PN quadrupole moment (4.20),[16] is actually a gauge constant. To this end we simply verify that ${r'}_0$ will be eliminated when expressing the FWF in terms of the gauge invariant parameter (5.97). From (4.20) we see that the dependence on ${r'}_0$ of the 3PN quadrupole moment is

$$I_{ij} = \nu\, m \left[1 - \frac{44}{3}\gamma^3\, \nu\, \ln\left(\frac{r}{{r'}_0}\right)\right] x_{\langle ij\rangle} + \cdots + O\left(\frac{1}{c^7}\right) . \tag{4.39}$$

We indicate by dots all the terms that are independent of ${r'}_0$ (for convenience we also show the Newtonian term). Now the FWF depends on the second time derivative of the quadrupole moment. For circular orbits this reads [coming back to the superscript notation (n) for time derivatives]

$$I^{(2)}_{ij} = 2\nu\, m \left[1 - \frac{44}{3}\gamma^3\, \nu\, \ln\left(\frac{r}{{r'}_0}\right)\right]\left(v_{\langle ij\rangle} + x_{\langle i}a_{j\rangle}\right) + \cdots + O\left(\frac{1}{c^7}\right) . \tag{4.40}$$

Replacing v_i and a_i by their values (4.34) we get with the required approximation (still being interested only in the fate of the constant ${r'}_0$)

$$I^{(2)}_{ij} = 2\nu\, m\, v^2 \left[1 - \frac{44}{3}\gamma^3\, \nu\, \ln\left(\frac{r}{{r'}_0}\right)\right]\left(\lambda_{\langle ij\rangle} - n_{\langle ij\rangle}\right) + \cdots + O\left(\frac{1}{c^7}\right) . \tag{4.41}$$

The squared velocity $v^2 = r^2\omega^2 + O(c^{-10})$ appears in factor. It is now clear that replacing v^2 by its expression in terms of the parameter x following (4.38), we produce another logarithmic term containing ${r'}_0$, namely

$$v^2 = c^2\, x \left[1 + \frac{44}{3}x^3\, \nu\, \ln\left(\frac{r}{{r'}_0}\right)\right] + \cdots + O\left(\frac{1}{c^7}\right) , \tag{4.42}$$

which will cancel out the dependence of the quadrupole moment on ${r'}_0$ at 3PN order (using the fact that γ can be replaced by x in a small 3PN term). Thus, finally,

$$I^{(2)}_{ij} = 2\nu\, m\, c^2\, x\left(\lambda_{\langle ij\rangle} - n_{\langle ij\rangle}\right) + \cdots + O\left(\frac{1}{c^7}\right) , \tag{4.43}$$

is independent on ${r'}_0$, which means that this constant cannot affect any physical result at the 3PN order.

choosing one or another is often a matter of convenience.

[16] The other moments are given at 2.5PN order at most; they do not depend on ${r'}_0$ since the appearance of regularization constants is a feature of the 3PN approximation.

4.6 Computation of the tail and memory integrals

The results of Sections 5.4–5.5 yield the complete control of the *instantaneous* part of the FWF. We now tackle the computation of the *hereditary* part, which is composed of tails (and tails-of-tails and squared-tails) and non-linear memory terms. The hereditary integrals have been explicitly provided in Section 5.4 as contributions to the various radiative moments U_L and V_L given by (4.10)–(4.13). Our computation will basically be a straightforward extension of the computation performed at 2.5PN order in Section 4 of [77]. Since we employ exactly the same techniques, we skip most of the details and rely on [77] for justification of the method and proofs.

We first consider the non-linear memory terms. Up to 3PN order we have the 2.5PN memory integrals in the radiative mass quadrupole moment U_{ij} given by (4.10) and the radiative mass hexadecapole moment U_{ijkl} given by (4.12a) — these are the memory terms contributing to the FWF at 2.5PN order [77] — and, in addition, we have the memory integral in the mass octupole moment U_{ijk} given by (4.11a) and the one in U_{ijklm} given by (4.13a) — these contribute specifically at 3PN order.[17] Like in [77] we obtain the corresponding integrands (*i.e.* the terms under the integral sign) and compute directly their contributions to the two wave polarizations h_+ and $h_\times$. Indeed it is convenient to perform the relevant contractions of the integrands with the polarization vectors **P** and **Q** (see Section 5.8 for the conventions we adopt) so as to only deal with *scalar* quantities.

We find that the memory integrals in h_+ and $h_\times$ are composed of two types of terms. First there is a term, only present in the plus polarization h_+, which does not depend on the orbital phase and can thus be viewed as a *zero-frequency* (DC) term. Actually, because of the steady inspiral, this term is a steadily varying function of time, with an amplitude increasing like some power law of the time remaining till the coalescence. Strictly speaking, this term is to be regarded as *the* memory contribution because it does depend on the behaviour of the system in the remote past, and therefore must be computed using some model for the evolution of the binary system in the past. In the present chapter we find that the only zero-frequency term up to 3PN order is the one which appeared already at 2.5PN order and was evaluated in [77] — interestingly there are no other terms of this type at the 3PN order. Because of the cumulative effect of integration over the whole past we know that this term, though originating from 2.5PN order, finally contributes in the FWF at the Newtonian level [154, 155, 156]. In practice the computation of this DC term reduces (in the circular orbit case) to the evaluation of the single elementary integral

$$\mathrm{I}(T_R) = \frac{(G\,m)^{p-1}}{c^{2p-3}} \int_{-\infty}^{T_R} \frac{d\tau}{r^p(\tau)}\,. \tag{4.44}$$

Here $r(\tau)$ denotes the binary's separation at any time $\tau \leq T_R$ (where $T_R = T - R/c$ is the current time). The coefficient in front of (4.44) is chosen for convenience to make the integral dimensionless. The integral (4.44) is easily computed using a simplified model of binary evolution in the past in which the orbit is assumed to remain circular apart from the gradual inspiral at any time. In this model the binary separation evolves like $r(\tau) \propto (T_c - \tau)^{1/4}$ where T_c denotes the instant of coalescence (see [77] for more details). In the remote past we thus have $r(\tau) \sim (-\tau)^{1/4}$ so the integral (4.44) converges when $p > 4$ (actually we shall

[17] Recall that the non-linear memory terms occur only in the *mass-type* radiative multipole moments U_L.

only need the case $p = 5$ like in [77]). The result reads

$$\mathrm{I}(T_R) = \frac{5}{64(p-4)}\,\frac{x^{p-4}(T_R)}{\nu}\,, \tag{4.45}$$

where $x(T_R)$ denotes the *current* value (*i.e.* at the current retarded time T_R) of the parameter x defined by (5.97). Witness the memory effect: the end result (4.45) is of order $x^{p-4} = O(c^{-2p+8})$ which is a factor c^5 *larger* than the original formal PN order $O(c^{-2p+3})$ as shown in (4.44). Hence, although the memory term is formally of order 2.5PN, its actual contribution to the waveform is comparable to a Newtonian term. As mentioned above we do not find memory (zero-frequency) contributions originating from the next 3PN order, and therefore finally no DC term at 0.5PN order.

Second there are other terms, present in both polarizations, which depend on the orbital phase, and oscillate like some harmonics of the orbital phase (say $n\,\phi$). Such phase-dependent, oscillating terms do not exhibit the memory effect, essentially because the oscillations, due to the sequence of orbital cycles in the entire life of the binary system, more or less compensate each other. As a result these terms, in contrast with (4.44)–(4.45), keep on their formal PN order. We recover the 2.5PN terms investigated in [77] and in addition we obtain several other terms at 3PN order. The latter are computed by a slight generalization of the method followed in [77]: instead of (4.18) in [77] we need to consider the integral

$$\mathrm{J}(T_R) = \frac{(G\,m)^{p-1}}{c^{2p-3}} \int_{-\infty}^{T_R} d\tau\,\frac{e^{i\,n\,\phi(\tau)}}{r^p(\tau)}\,, \tag{4.46}$$

where $\phi(\tau)$ is the orbital phase at any time, where n and p range over integer or half-integer values (*e.g.* $n = 1, 3, 5$ and $p = 11/2$ at 3PN order), and where the coefficient is chosen to make the integral dimensionless. Following the steps (4.18)–(4.23) in [77] we compute this integral using our model of binary's past evolution, and in the adiabatic limit, which means that the *current* value of the adiabatic parameter ξ associated with the binary inspiral is considered to be small and of PN order $\xi(T_R) = O(c^{-5})$. We then find

$$\mathrm{J}(T_R) = x^{p-\frac{3}{2}}(T_R)\,\frac{e^{i\,n\,\phi(T_R)}}{i\,n}\left[1 + O\left(\frac{1}{c^5}\right)\right]. \tag{4.47}$$

This result (valid only if $n \neq 0$) permits to handle all the phase-dependent oscillating terms coming from the memory integrals.

We next turn to the computation of the tails and tails-of-tails present in the radiative moments (4.10)–(4.13). Again we closely follow the previous investigation [77] to which we refer for more details. The computation of tails reduces to the evaluation of an elementary integral involving a logarithmic kernel,

$$\mathrm{K}(T_R) = \frac{(G\,m)^{p-1}}{c^{2p-3}} \int_{-\infty}^{T_R} d\tau\,\frac{e^{i\,n\,\phi(\tau)}}{r^p(\tau)}\ln\left(\frac{T_R-\tau}{T_c-T_R}\right), \tag{4.48}$$

in which the logarithm has been scaled with the constant time $T_c - T_R$, instead of the previous normalization by $2\tau_0$, where T_c is the instant of coalescence in the model of [77]. Such scaling can always be done at the price of adding another term proportional to some integral of the type $\mathrm{J}(T_R)$ computed previously. Following the derivation of this integral in [77], we find that, at dominant order in the adiabatic approximation,

$$\mathrm{K}(T_R) = x^{p-\frac{3}{2}}(T_R)\,\frac{e^{i\,n\,\phi(T_R)}}{i\,n}\left[\frac{\pi}{2i} - \ln\left(\frac{n}{\xi(T_R)}\right) - C + O\left(\frac{\ln c}{c^5}\right)\right]. \tag{4.49}$$

Here $C = 0.577\cdots$ is the Euler constant, and $\xi(T_R)$ denotes the current value of the adiabatic parameter associated with the inspiral, which is defined by $\xi(T_R) = [(T_c - T_R)\omega(T_R)]^{-1}$ in the model of [77]. The adiabatic parameter is related to the PN parameter x by

$$\xi(T_R) = \frac{256\nu}{5}\,x^{5/2}(T_R)\,. \tag{4.50}$$

The squared-tails are computed using the same integral (4.48)–(4.49). Concerning the tails-of-tails we simply have to consider an integral involving a logarithm squared,

$$\mathrm{L}(T_R) = \frac{(G\,m)^{p-1}}{c^{2p-3}}\int_{-\infty}^{T_R} d\tau\,\frac{e^{i\,n\,\phi(\tau)}}{r^p(\tau)}\ln^2\left(\frac{T_R-\tau}{T_c-T_R}\right), \tag{4.51}$$

which is computed using the same technique with the result

$$\mathrm{L}(T_R) = x^{p-\frac{3}{2}}(T_R)\,\frac{e^{i\,n\,\phi(T_R)}}{i\,n}\left[\frac{\pi^2}{6} + \left(C + \ln\left(\frac{n}{\xi(T_R)}\right) + \frac{i\pi}{2}\right)^2 + O\left(\frac{\ln c}{c^5}\right)\right]. \tag{4.52}$$

We are done with the computation of all tails and tails-of-tails in the 3PN waveform.

For completeness let us give also the two technical formulas which enables one to arrive at the results (4.49) and (4.52). Posing $y = (T_R-\tau)/(T_c-T_R)$ and $\lambda = n/\xi$, and working at the leading order in the adiabatic limit $\xi \to 0$ or equivalently when $\lambda \to +\infty$, the formulas express that, for any positive or negative λ (see *e.g.* [157] p. 573 and 574),

$$\int_0^1 dy\,\ln y\,e^{-i\lambda y} = \frac{1}{\lambda}\left[-\frac{\pi}{2}\mathrm{sign}(\lambda) + i(\ln|\lambda| + C)\right] + O\left(\frac{1}{\lambda^2}\right), \tag{4.53a}$$

$$\int_0^1 dy\,\ln^2 y\,e^{-i\lambda y} = \frac{i}{\lambda}\left(-\frac{\pi^2}{6} + \left[-\frac{\pi}{2}\mathrm{sign}(\lambda) + i(\ln|\lambda| + C)\right]^2\right) + O\left(\frac{1}{\lambda^3}\right). \tag{4.53b}$$

Notice that we are only interested in the *recent past* contribution to the integrals (4.53), corresponding to the interval $0 \leq y \leq 1$ equivalent to the time interval $2T_R - T_c \leq \tau \leq T_R$. The reason is that the *remote past* contribution, given by $1 < y < +\infty$ or equivalently $-\infty < \tau < 2T_R - T_c$, is small in the adiabatic limit. This is a characteristic feature of tails: they die out very rapidly, therefore they depend essentially on the recent past evolution of the matter source [124, 158]. In the case at hand this technically means that the remote-past contributions to the integrals are of order

$$\int_1^{+\infty} dy\,\ln y\,e^{-i\lambda y} = O\left(\frac{1}{\lambda^2}\right), \tag{4.54a}$$

$$\int_1^{+\infty} dy\,\ln^2 y\,e^{-i\lambda y} = O\left(\frac{1}{\lambda^3}\right), \tag{4.54b}$$

as can easily be verified by using integration by parts.

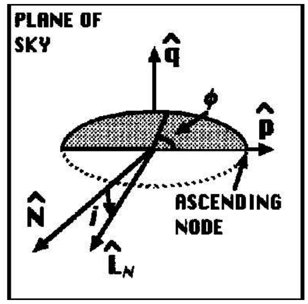

Figure 4.1: The geometry of the binary system. The vectors **P**, **Q** and **N** are illustrated. Adapted from Ref. [127].

4.7 3PN polarization waveforms for data analysis

We specify our conventions for the orbital phase and polarization vectors defining the polarization waveforms (4.2) in the case of quasi-circular binary systems of non-spinning compact objects. If the orbital plane is chosen to be the x-y plane (like in Section 5.6), with the orbital phase ϕ measuring the direction of the unit vector $\mathbf{n} = \mathbf{x}/r$ along the relative separation vector, then

$$\mathbf{n} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi \,, \tag{4.55}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit directions along x and y. Following [84, 77] we choose the polarization vector **P** to lie along the x-axis and the observer to be in the y-z plane with

$$\mathbf{N} = s_i\,\hat{\mathbf{y}} + c_i\,\hat{\mathbf{z}} \,, \tag{4.56}$$

where we pose $c_i = \cos i$ and $s_i = \sin i$, with i being the orbit's inclination angle ($0 \leq i \leq \pi$). With this choice **P** lies along the intersection of the orbital plane with the plane of the sky in the direction of the *ascending node* $\mathcal{N}$, *i.e.* that point at which the bodies cross the plane of the sky moving toward the observer. The orbital phase ϕ is the angle between the ascending node $\mathcal{N}$ and the direction of body one (say). The rotating orthonormal triad $(\mathbf{n}, \boldsymbol{\lambda}, \hat{\mathbf{z}})$ describing the motion of the binary [see (4.32)] is then related to the fixed polarization triad $(\mathbf{N}, \mathbf{P}, \mathbf{Q})$ by

$$\mathbf{n} = \mathbf{P} \cos\phi + (c_i\,\mathbf{Q} + s_i\,\mathbf{N}) \sin\phi \,, \tag{4.57a}$$

$$\boldsymbol{\lambda} = -\mathbf{P} \sin\phi + (c_i\,\mathbf{Q} + s_i\,\mathbf{N}) \cos\phi \,, \tag{4.57b}$$

$$\hat{\mathbf{z}} = -s_i\,\mathbf{Q} + c_i\,\mathbf{N} \,. \tag{4.57c}$$

As in previous works [84, 77] we shall present the wave polarizations (4.2) as a series expansion in powers of the gauge-invariant PN parameter x defined by (5.97). With a convenient overall factorization we write them as

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} = \frac{2\,G\,m\,\nu\,x}{c^2\,R} \begin{pmatrix} H_+ \\ H_\times \end{pmatrix} + O\left(\frac{1}{R^2}\right), \tag{4.58}$$

with the following PN series expansion

$$H_{+,\times} = \sum_{n=0}^{+\infty} x^{n/2}\, H_{+,\times}^{(n/2)}\,. \tag{4.59}$$

The PN coefficients $H_{+,\times}^{(n/2)}$ will be given as functions of the orbital phase ϕ, and will also be polynomials in the symmetric mass ratio ν and depend on the inclination angle i. In addition they will involve, at high PN order, the logarithm of x as we shall discuss below.

Following [84, 77] it is convenient to perform a change of phase variable, from the actual orbital phase ϕ satisfying $\dot{\phi} = \omega$, to some new variable denoted ψ. Recall that the orbital phase ϕ evolves by gravitational radiation reaction and its expression as a function of time is known from previous work [116, 32, 71] up to 3.5PN order. We then pose[18]

$$\psi = \phi - \frac{2GM\omega}{c^3} \ln\left(\frac{\omega}{\omega_0}\right), \tag{4.60}$$

where M is the binary's total mass given by (4.29), and where ω_0 denotes the constant

$$\omega_0 = \frac{e^{\frac{11}{12}-C}}{4\tau_0}\,. \tag{4.61}$$

Here $\tau_0 = r_0/c$ is the normalization of logarithms in the tail integrals of the radiative moments (4.10)–(4.13). Like τ_0 the constant ω_0 is arbitrary and we choose $\omega_0 = \pi f_{\text{seismic}}$ where f_{seismic} is the entry frequency of some ground-based interferometric detector. Using (4.29) and the notation (5.97) the new phase variable reads

$$\psi = \phi - 3x^{3/2}\left[1 - \frac{\nu}{2}x\right] \ln\left(\frac{x}{x_0}\right), \tag{4.62}$$

where $x_0 = (\frac{Gm\omega_0}{c^3})^{2/3}$.[19] Our modified phase variable (4.60)–(4.62) will be valid up to 3PN order but in fact it turns out to be the same as at the previous 2.5PN order [77].

The logarithmic term in ψ corresponds to some spreading of the different frequency components of the wave along the line of sight from the source to the far-away detector, and expresses physically the tail effect as a small delay in the arrival time of gravitational waves. However, practically speaking, the main interest of this term is to minimize the occurrence of logarithms in the FWF. Indeed we notice that the logarithmic term in (4.60), although of formal PN order $O(c^{-3})$, represents in fact a very small modulation of the orbital phase: compared with the dominant phase evolution whose order is that of the inverse of radiation reaction, *i.e.* $\phi = O(\xi^{-1}) = O(c^5)$, this term is of order $O(c^{-8})$ namely 4PN in the phase

[18] A similar phase variable is also introduced in black-hole perturbation theory [159, 160, 161].

[19] We have $\ln x_0 = \frac{11}{18} - \frac{2}{3}C - \frac{4}{3}\ln 2 + \frac{2}{3}\ln\left(\frac{Gm}{c^2 r_0}\right)$ in agreement with the equation (68) of [117].

evolution, which can be regarded as negligible to the present accuracy. Thus the logarithms associated with the phase modulation in (4.60) will be "eliminated" from the FWF at 3PN order. This does not mean that we should ignore them but that the formulation in terms of the small phase modulation (4.60) is quite natural (for the data analysis it is probably better to keep the logarithm as it stands in the definition of the phase variable ψ). However all the logarithms will not be "removed" by this process, and we shall find that some "true" logarithms remain starting at the 3PN order.

With those conventions and notation we find for the plus polarization[20]

$$
\begin{aligned}
H_+^{(0)} &= -(1+c_i^2)\cos 2\psi - \frac{1}{96}\, s_i^2\,(17+c_i^2)\,, && (4.63a)\\
H_+^{(0.5)} &= -\,s_i\,\Delta\left[\cos\psi\left(\frac{5}{8}+\frac{1}{8}c_i^2\right) - \cos 3\psi\left(\frac{9}{8}+\frac{9}{8}c_i^2\right)\right], && (4.63b)\\
H_+^{(1)} &= \cos 2\psi\left[\frac{19}{6}+\frac{3}{2}c_i^2-\frac{1}{3}c_i^4+\nu\left(-\frac{19}{6}+\frac{11}{6}c_i^2+c_i^4\right)\right]\\
&\quad -\cos 4\psi\left[\frac{4}{3}s_i^2(1+c_i^2)(1-3\nu)\right], && (4.63c)\\
H_+^{(1.5)} &= s_i\,\Delta\cos\psi\left[\frac{19}{64}+\frac{5}{16}c_i^2-\frac{1}{192}c_i^4+\nu\left(-\frac{49}{96}+\frac{1}{8}c_i^2+\frac{1}{96}c_i^4\right)\right]\\
&\quad +\cos 2\psi\left[-2\pi(1+c_i^2)\right]\\
&\quad + s_i\,\Delta\cos 3\psi\left[-\frac{657}{128}-\frac{45}{16}c_i^2+\frac{81}{128}c_i^4\right.\\
&\qquad \left.+\nu\left(\frac{225}{64}-\frac{9}{8}c_i^2-\frac{81}{64}c_i^4\right)\right]\\
&\quad + s_i\,\Delta\cos 5\psi\left[\frac{625}{384}s_i^2(1+c_i^2)(1-2\nu)\right], && (4.63d)\\
H_+^{(2)} &= \pi\, s_i\,\Delta\cos\psi\left[-\frac{5}{8}-\frac{1}{8}c_i^2\right]\\
&\quad +\cos 2\psi\left[\frac{11}{60}+\frac{33}{10}c_i^2+\frac{29}{24}c_i^4-\frac{1}{24}c_i^6\right.\\
&\qquad +\nu\left(\frac{353}{36}-3c_i^2-\frac{251}{72}c_i^4+\frac{5}{24}c_i^6\right)\\
&\qquad \left.+\nu^2\left(-\frac{49}{12}+\frac{9}{2}c_i^2-\frac{7}{24}c_i^4-\frac{5}{24}c_i^6\right)\right]\\
&\quad +\pi\, s_i\,\Delta\cos 3\psi\left[\frac{27}{8}(1+c_i^2)\right]\\
&\quad +\frac{2}{15}s_i^2\cos 4\psi\left[59+35c_i^2-8c_i^4-\frac{5}{3}\nu\left(131+59c_i^2-24c_i^4\right)\right.\\
&\qquad \left.+5\nu^2\left(21-3c_i^2-8c_i^4\right)\right]
\end{aligned}
$$

[20] We also requote the previous 2.5PN results [77] taking into account the published Erratum [77] and the correcting term associated with radiation reaction and pointed out in [96].

$$
\begin{aligned}
&+\cos 6\psi\left[-\frac{81}{40}\, s_i^4(1+\,c_i^2)\left(1-5\nu+5\nu^2\right)\right]\\
&+\,s_i\,\Delta\sin\psi\left[\frac{11}{40}+\frac{5\ln 2}{4}+\,c_i^2\left(\frac{7}{40}+\frac{\ln 2}{4}\right)\right]\\
&+\,s_i\,\Delta\sin 3\psi\left[\left(-\frac{189}{40}+\frac{27}{4}\ln(3/2)\right)(1+\,c_i^2)\right],
\end{aligned}
\tag{4.63e}
$$

$$
\begin{aligned}
H_+^{(2.5)} =\;& s_i\,\Delta\cos\psi\left[\frac{1771}{5120}-\frac{1667}{5120}\,c_i^2+\frac{217}{9216}\,c_i^4-\frac{1}{9216}\,c_i^6\right.\\
&+\nu\left(\frac{681}{256}+\frac{13}{768}\,c_i^2-\frac{35}{768}\,c_i^4+\frac{1}{2304}\,c_i^6\right)\\
&\left.+\,\nu^2\left(-\frac{3451}{9216}+\frac{673}{3072}\,c_i^2-\frac{5}{9216}\,c_i^4-\frac{1}{3072}\,c_i^6\right)\right]\\
&+\pi\cos 2\psi\left[\frac{19}{3}+3\,c_i^2-\frac{2}{3}\,c_i^4+\nu\left(-\frac{16}{3}+\frac{14}{3}\,c_i^2+2\,c_i^4\right)\right]\\
&+s_i\,\Delta\cos 3\psi\left[\frac{3537}{1024}-\frac{22977}{5120}\,c_i^2-\frac{15309}{5120}\,c_i^4+\frac{729}{5120}\,c_i^6\right.\\
&+\nu\left(-\frac{23829}{1280}+\frac{5529}{1280}\,c_i^2+\frac{7749}{1280}\,c_i^4-\frac{729}{1280}\,c_i^6\right)\\
&\left.+\,\nu^2\left(\frac{29127}{5120}-\frac{27267}{5120}\,c_i^2-\frac{1647}{5120}\,c_i^4+\frac{2187}{5120}\,c_i^6\right)\right]\\
&+\cos 4\psi\left[-\frac{16\pi}{3}\,(1+\,c_i^2)\,s_i^2(1-3\nu)\right]\\
&+s_i\,\Delta\cos 5\psi\left[-\frac{108125}{9216}+\frac{40625}{9216}\,c_i^2+\frac{83125}{9216}\,c_i^4-\frac{15625}{9216}\,c_i^6\right.\\
&+\nu\left(\frac{8125}{256}-\frac{40625}{2304}\,c_i^2-\frac{48125}{2304}\,c_i^4+\frac{15625}{2304}\,c_i^6\right)\\
&\left.+\,\nu^2\left(-\frac{119375}{9216}+\frac{40625}{3072}\,c_i^2+\frac{44375}{9216}\,c_i^4-\frac{15625}{3072}\,c_i^6\right)\right]\\
&+\Delta\cos 7\psi\left[\frac{117649}{46080}\,s_i^5(1+\,c_i^2)(1-4\nu+3\nu^2)\right]\\
&+\sin 2\psi\left[-\frac{9}{5}+\frac{14}{5}\,c_i^2+\frac{7}{5}\,c_i^4+\nu\left(32+\frac{56}{5}\,c_i^2-\frac{28}{5}\,c_i^4\right)\right]\\
&+s_i^2(1+\,c_i^2)\sin 4\psi\left[\frac{56}{5}-\frac{32\ln 2}{3}+\nu\left(-\frac{1193}{30}+32\ln 2\right)\right],
\end{aligned}
\tag{4.63f}
$$

$$
\begin{aligned}
H_+^{(3)} =\;& \pi\,\Delta\,s_i\cos\psi\left[\frac{19}{64}+\frac{5}{16}\,c_i^2-\frac{1}{192}\,c_i^4+\nu\left(-\frac{19}{96}+\frac{3}{16}\,c_i^2+\frac{1}{96}\,c_i^4\right)\right]\\
&+\,\cos 2\psi\left[-\frac{465497}{11025}+\left(\frac{856\,C}{105}-\frac{2\,\pi^2}{3}+\frac{428}{105}\,\ln(16\,x)\right)(1+c_i^2)\right.\\
&-\frac{3561541}{88200}\,c_i^2-\frac{943}{720}\,c_i^4+\frac{169}{720}\,c_i^6-\frac{1}{360}\,c_i^8
\end{aligned}
$$

$$
\begin{aligned}
&+\nu\left(\frac{2209}{360}-\frac{41\pi^2}{96}(1+c_i^2)+\frac{2039}{180}\,c_i^2+\frac{3311}{720}\,c_i^4-\frac{853}{720}\,c_i^6+\frac{7}{360}\,c_i^8\right)\\
&+\nu^2\left(\frac{12871}{540}-\frac{1583}{60}\,c_i^2-\frac{145}{108}\,c_i^4+\frac{56}{45}\,c_i^6-\frac{7}{180}\,c_i^8\right)\\
&+\nu^3\left(-\frac{3277}{810}+\frac{19661}{3240}\,c_i^2-\frac{281}{144}\,c_i^4-\frac{73}{720}\,c_i^6+\frac{7}{360}\,c_i^8\right)\Bigg]\\
+\;&\pi\,\Delta\,s_i\cos 3\psi\left[-\frac{1971}{128}-\frac{135}{16}\,c_i^2+\frac{243}{128}\,c_i^4+\nu\left(\frac{567}{64}-\frac{81}{16}\,c_i^2-\frac{243}{64}\,c_i^4\right)\right]\\
+\;&s_i^2\cos 4\psi\Bigg[-\frac{2189}{210}+\frac{1123}{210}\,c_i^2+\frac{56}{9}\,c_i^4-\frac{16}{45}\,c_i^6\\
&+\nu\left(\frac{6271}{90}-\frac{1969}{90}\,c_i^2-\frac{1432}{45}\,c_i^4+\frac{112}{45}\,c_i^6\right)\\
&+\nu^2\left(-\frac{3007}{27}+\frac{3493}{135}\,c_i^2+\frac{1568}{45}\,c_i^4-\frac{224}{45}\,c_i^6\right)\\
&+\nu^3\left(\frac{161}{6}-\frac{1921}{90}\,c_i^2-\frac{184}{45}\,c_i^4+\frac{112}{45}\,c_i^6\right)\Bigg]\\
+\;&\Delta\cos 5\psi\left[\frac{3125\,\pi}{384}\,s_i^3(1+c_i^2)(1-2\nu)\right]\\
+\;&s_i^4\cos 6\psi\Bigg[\frac{1377}{80}+\frac{891}{80}\,c_i^2-\frac{729}{280}\,c_i^4\\
&+\nu\left(-\frac{7857}{80}-\frac{891}{16}\,c_i^2+\frac{729}{40}\,c_i^4\right)\\
&+\nu^2\left(\frac{567}{4}+\frac{567}{10}\,c_i^2-\frac{729}{20}\,c_i^4\right)\\
&+\nu^3\left(-\frac{729}{16}-\frac{243}{80}\,c_i^2+\frac{729}{40}\,c_i^4\right)\Bigg]\\
+\;&\cos 8\psi\left[-\frac{1024}{315}\,s_i^6\,(1+c_i^2)(1-7\nu+14\nu^2-7\nu^3)\right]\\
+\;&\Delta\,s_i\sin\psi\Bigg[-\frac{2159}{40320}-\frac{19\,\ln 2}{32}+\left(-\frac{95}{224}-\frac{5\,\ln 2}{8}\right)c_i^2+\left(\frac{181}{13440}+\frac{\ln 2}{96}\right)c_i^4\\
&+\nu\left(\frac{81127}{10080}+\frac{19\,\ln 2}{48}+\left(-\frac{41}{48}-\frac{3\,\ln 2}{8}\right)c_i^2+\left(-\frac{313}{480}-\frac{\ln 2}{48}\right)c_i^4\right)\Bigg]\\
+\;&\sin 2\psi\left[-\frac{428\,\pi}{105}\,(1+c_i^2)\right]\\
+\;&\Delta\,s_i\sin 3\psi\Bigg[\frac{205119}{8960}-\frac{1971}{64}\,\ln(3/2)+\left(\frac{1917}{224}-\frac{135}{8}\,\ln(3/2)\right)c_i^2\\
&+\left(-\frac{43983}{8960}+\frac{243}{64}\ln(3/2)\right)c_i^4\\
&+\nu\Bigg(-\frac{54869}{960}+\frac{567}{32}\,\ln(3/2)+\left(-\frac{923}{80}-\frac{81}{8}\,\ln(3/2)\right)c_i^2
\end{aligned}
$$

$$+\left(\frac{41851}{2880}-\frac{243}{32}\ \ln(3/2)\right)c_i^4\Bigg)\Bigg]$$

$$+\ \Delta\, s_i^3(1+c_i^2)\sin 5\psi\left[-\frac{113125}{5376}+\frac{3125}{192}\ \ln(5/2)+\nu\left(\frac{17639}{320}-\frac{3125}{96}\ \ln(5/2)\right)\right]. \tag{4.63g}$$

For the cross polarizations we obtain

$$H_\times^{(0)} = -2c_i\sin 2\psi\,, \tag{4.64a}$$

$$H_\times^{(0.5)} = s_i c_i\,\Delta\left[-\frac{3}{4}\sin\psi+\frac{9}{4}\sin 3\psi\right], \tag{4.64b}$$

$$\begin{aligned}H_\times^{(1)} &= c_i\sin 2\psi\left[\frac{17}{3}-\frac{4}{3}\,c_i^2+\nu\left(-\frac{13}{3}+4\,c_i^2\right)\right]\\&\quad+c_i\,s_i^2\sin 4\psi\left[-\frac{8}{3}(1-3\nu)\right],\end{aligned} \tag{4.64c}$$

$$\begin{aligned}H_\times^{(1.5)} &= s_i c_i\,\Delta\sin\psi\left[\frac{21}{32}-\frac{5}{96}\,c_i^2+\nu\left(-\frac{23}{48}+\frac{5}{48}\,c_i^2\right)\right]\\&\quad-4\pi\,c_i\sin 2\psi\\&\quad+\ s_i c_i\,\Delta\ \sin 3\psi\left[-\frac{603}{64}+\frac{135}{64}\,c_i^2+\nu\left(\frac{171}{32}-\frac{135}{32}\,c_i^2\right)\right]\\&\quad+\ s_i c_i\,\Delta\ \sin 5\psi\left[\frac{625}{192}(1-2\nu)\,s_i^2\right],\end{aligned} \tag{4.64d}$$

$$\begin{aligned}H_\times^{(2)} &= s_i c_i\,\Delta\cos\psi\left[-\frac{9}{20}-\frac{3}{2}\ln 2\right]\\&\quad+\ s_i c_i\,\Delta\cos 3\psi\left[\frac{189}{20}-\frac{27}{2}\ln(3/2)\right]\\&\quad-\ s_i c_i\,\Delta\left[\frac{3\,\pi}{4}\right]\sin\psi\\&\quad+c_i\sin 2\psi\left[\frac{17}{15}+\frac{113}{30}\,c_i^2-\frac{1}{4}\,c_i^4\right.\\&\qquad+\nu\left(\frac{143}{9}-\frac{245}{18}\,c_i^2+\frac{5}{4}\,c_i^4\right)\\&\qquad\left.+\nu^2\left(-\frac{14}{3}+\frac{35}{6}\,c_i^2-\frac{5}{4}c_i^4\right)\right]\\&\quad+\ s_i c_i\,\Delta\sin 3\psi\left[\frac{27\pi}{4}\right]\\&\quad+\frac{4}{15}\,c_i\,s_i^2\ \sin 4\psi\left[55-12\,c_i^2-\frac{5}{3}\,\nu\left(119-36\,c_i^2\right)+5\,\nu^2\left(17-12\,c_i^2\right)\right]\\&\quad+c_i\sin 6\psi\left[-\frac{81}{20}\,s_i^4(1-5\nu+5\nu^2)\right],\end{aligned} \tag{4.64e}$$

$$H_\times^{(2.5)} = \frac{6}{5}\,s_i^2\,c_i\,\nu$$

$$
\begin{aligned}
&+\ c_i \cos 2\psi \left[2 - \frac{22}{5}\, c_i^2 + \nu\left(-\frac{282}{5} + \frac{94}{5}\, c_i^2\right)\right] \\
&+\ c_i\, s_i^2 \cos 4\psi \left[-\frac{112}{5} + \frac{64}{3} \ln 2 + \nu\left(\frac{1193}{15} - 64 \ln 2\right)\right] \\
&+\ s_i\, c_i\, \Delta \sin\psi \left[-\frac{913}{7680} + \frac{1891}{11520}\, c_i^2 - \frac{7}{4608}\, c_i^4 \right. \\
&\quad + \nu\left(\frac{1165}{384} - \frac{235}{576}\, c_i^2 + \frac{7}{1152}\, c_i^4\right) \\
&\quad \left. + \nu^2\left(-\frac{1301}{4608} + \frac{301}{2304}\, c_i^2 - \frac{7}{1536}\, c_i^4\right)\right] \\
&+ \pi\, c_i \sin 2\psi \left[\frac{34}{3} - \frac{8}{3}\, c_i^2 + \nu\left(-\frac{20}{3} + 8\, c_i^2\right)\right] \\
&+\ s_i\, c_i\, \Delta \sin 3\psi \left[\frac{12501}{2560} - \frac{12069}{1280}\, c_i^2 + \frac{1701}{2560}\, c_i^4 \right. \\
&\quad + \nu\left(-\frac{19581}{640} + \frac{7821}{320}\, c_i^2 - \frac{1701}{640}\, c_i^4\right) \\
&\quad \left. + \nu^2\left(\frac{18903}{2560} - \frac{11403}{1280}\, c_i^2 + \frac{5103}{2560}\, c_i^4\right)\right] \\
&+\ s_i^2\, c_i \sin 4\psi \left[-\frac{32\pi}{3}(1 - 3\nu)\right] \\
&+ \Delta\, s_i\, c_i \sin 5\psi \left[-\frac{101875}{4608} + \frac{6875}{256}\, c_i^2 - \frac{21875}{4608}\, c_i^4 \right. \\
&\quad + \nu\left(\frac{66875}{1152} - \frac{44375}{576}\, c_i^2 + \frac{21875}{1152}\, c_i^4\right) \\
&\quad \left. + \nu^2\left(-\frac{100625}{4608} + \frac{83125}{2304}\, c_i^2 - \frac{21875}{1536}\, c_i^4\right)\right] \\
&+ \Delta\, s_i^5\, c_i \sin 7\psi \left[\frac{117649}{23040}\left(1 - 4\nu + 3\nu^2\right)\right], \qquad (4.64\text{f})
\end{aligned}
$$

$$
\begin{aligned}
H_\times^{(3)} &= \Delta\, s_i\, c_i \cos\psi \left[\frac{11617}{20160} + \frac{21}{16}\ \ln 2 + \left(-\frac{251}{2240} - \frac{5}{48}\ \ln 2\right) c_i^2 \right. \\
&\quad \left. + \nu\left(-\frac{48239}{5040} - \frac{5}{24}\ \ln 2 + \left(\frac{727}{240} + \frac{5}{24}\ \ln 2\right) c_i^2\right)\right] \\
&+\ c_i \cos 2\psi \left[\frac{856\,\pi}{105}\right] \\
&+\ \Delta\, s_i\, c_i \cos 3\psi \left[-\frac{36801}{896} + \frac{1809}{32}\ \ln(3/2) + \left(\frac{65097}{4480} - \frac{405}{32}\ \ln(3/2)\right) c_i^2 \right. \\
&\quad \left. + \nu\left(\frac{28445}{288} - \frac{405}{16}\ \ln(3/2) + \left(-\frac{7137}{160} + \frac{405}{16}\ \ln(3/2)\right) c_i^2\right)\right] \\
&+\ \Delta\, s_i^3\, c_i \cos 5\psi \left[\frac{113125}{2688} - \frac{3125}{96}\ \ln(5/2) + \nu\left(-\frac{17639}{160} + \frac{3125}{48}\ \ln(5/2)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\ \pi\,\Delta\, s_i\, c_i \sin\psi\left[\frac{21}{32} - \frac{5}{96}\,c_i^2 + \nu\left(-\frac{5}{48} + \frac{5}{48}\,c_i^2\right)\right]\\
&+\ c_i \sin 2\psi\left[-\frac{3620761}{44100} + \frac{1712\,C}{105} - \frac{4\,\pi^2}{3} + \frac{856}{105}\,\ln(16\,x)\right.\\
&\quad - \frac{3413}{1260}\,c_i^2 + \frac{2909}{2520}\,c_i^4 - \frac{1}{45}\,c_i^6\\
&\quad + \nu\left(\frac{743}{90} - \frac{41\,\pi^2}{48} + \frac{3391}{180}\,c_i^2 - \frac{2287}{360}\,c_i^4 + \frac{7}{45}\,c_i^6\right)\\
&\quad + \nu^2\left(\frac{7919}{270} - \frac{5426}{135}\,c_i^2 + \frac{382}{45}\,c_i^4 - \frac{14}{45}\,c_i^6\right)\\
&\quad \left. + \nu^3\left(-\frac{6457}{1620} + \frac{1109}{180}\,c_i^2 - \frac{281}{120}\,c_i^4 + \frac{7}{45}\,c_i^6\right)\right]\\
&+\ \pi\,\Delta\, s_i\, c_i \sin 3\psi\left[-\frac{1809}{64} + \frac{405}{64}\,c_i^2 + \nu\left(\frac{405}{32} - \frac{405}{32}\,c_i^2\right)\right]\\
&+\ s_i^2\, c_i \sin 4\psi\left[-\frac{1781}{105} + \frac{1208}{63}\,c_i^2 - \frac{64}{45}\,c_i^4\right.\\
&\quad + \nu\left(\frac{5207}{45} - \frac{536}{5}\,c_i^2 + \frac{448}{45}\,c_i^4\right)\\
&\quad + \nu^2\left(-\frac{24838}{135} + \frac{2224}{15}\,c_i^2 - \frac{896}{45}\,c_i^4\right)\\
&\quad \left. + \nu^3\left(\frac{1703}{45} - \frac{1976}{45}\,c_i^2 + \frac{448}{45}\,c_i^4\right)\right]\\
&+\ \Delta \sin 5\psi\left[\frac{3125\,\pi}{192}\,s_i^3\, c_i (1-2\nu)\right]\\
&+\ s_i^4\, c_i \sin 6\psi\left[\frac{9153}{280} - \frac{243}{35}\,c_i^2 + \nu\left(-\frac{7371}{40} + \frac{243}{5}\,c_i^2\right)\right.\\
&\quad \left. + \nu^2\left(\frac{1296}{5} - \frac{486}{5}\,c_i^2\right) + \nu^3\left(-\frac{3159}{40} + \frac{243}{5}\,c_i^2\right)\right]\\
&+\ \sin 8\psi\left[-\frac{2048}{315}\,s_i^6\, c_i (1 - 7\nu + 14\nu^2 - 7\nu^3)\right]. \qquad (4.64\text{g})
\end{aligned}
$$

Notice the obvious fact that the polarization waveforms remain invariant when we rotate by π the separation direction between the particles and simultaneously exchange the labels of the two particles, *i.e.* when we apply the transformation $(\psi, \Delta) \rightarrow (\psi + \pi, -\Delta)$. Moreover, due to the parity invariance, H_+ is unchanged after the replacement $i \rightarrow \pi - i$, while $H_\times$ being the projection of $h_{ij}^{\rm TT}$ on a tensorial product of two vectors of inverse parity types, is changed into its opposite.

We have performed two important tests on these expressions. First of all we have verified that the perturbative limit $\nu \rightarrow 0$ of the polarization waveforms (4.63)–(4.64) is in full agreement up to 3PN order with the result of black-hole perturbation theory as reported in the Appendix B of [160].[21] Our second test is the verification that the wave polarizations

[21] In [77] a misprint was spotted in the Appendix B of [160]: the sign of the harmonic coefficient $\zeta_{7,3}^{\times}$ should

(4.63)–(4.64) give back the correct energy flux at 3PN order. The asymptotic flux is given in terms of the polarizations by

$$\mathcal{F}^{\mathrm{GW}} = \lim_{R\to+\infty} \frac{R^2 c^3}{4G} \int \frac{d\Omega}{4\pi} \left[(\dot{h}_+)^2 + (\dot{h}_\times)^2 \right], \tag{4.65}$$

where $d\Omega$ is the solid angle element associated with the direction of propagation $\mathbf{N}$. We have $d\Omega = \sin\Theta\, d\Theta\, d\Phi$ where (Θ, Φ) are the angles defining $\mathbf{N}$, following the notation of Section 5.2. To obtain the polarizations corresponding to this general convention for $\mathbf{N}$ we have to make some simple replacements in (4.63)–(4.64) for i and ψ. As is clear from the geometry of the problem we must replace $(i, \psi) \to (\Theta, \psi + \pi/2 - \Phi)$. The time derivative of the polarizations is computed in the adiabatic approximation, using $\dot{\phi} = \omega$ and $\dot{\omega}$ given by (4.33b). Of course one must take into account the difference between ϕ and the variable ψ used in (4.63)–(4.64). Finally, the angular integration in (4.65) is readily performed and the result is in perfect agreement with the 3PN energy flux given by (12.9) of [116].[22]

As already mentioned there are some “true” logarithms which remain in the FWF at the 3PN order — *i.e.* after it has been expressed with the help of the PN parameter x and the phase variable ψ. Inspection of (4.63)–(4.64) shows that these logarithms have the effect of correcting the Newtonian polarizations in the following way:

$$\begin{pmatrix} H_+ \\ \\ H_\times \end{pmatrix} = \begin{pmatrix} -(1+\, c_i^2)\cos 2\psi \\ \\ -2c_i \sin 2\psi \end{pmatrix} \left(1 - \frac{428}{105}\, x^3\, \ln(16\, x)\right) + \cdots + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.66}$$

where the dots represent the terms independent of logarithms. In our previous computation of the 3PN flux using (4.65) we have already checked that these logarithms are consistent with similar logarithms occurring at 3PN order in the flux. Indeed we easily see that they correspond in the 3PN flux to the terms

$$\mathcal{F}^{\mathrm{GW}} = \frac{32c^5}{5G}\nu^2 x^5 \left[1 - \frac{856}{105}\, x^3\, \ln(16\, x) + \cdots + \mathcal{O}\left(\frac{1}{c^7}\right)\right], \tag{4.67}$$

already known from (12.9) in [116]. Technically the logarithm in (4.66) or (4.67) is due to the tails-of-tails at 3PN order. Notice that this logarithm survives in the test-mass limit $\nu \to 0$ and is therefore also seen to appear in linear black hole perturbations [159, 160, 161].

4.8 3PN spherical harmonic modes for numerical relativity

The spin-weighted spherical harmonic modes of the polarization waveforms at 3PN order can now be obtained from using the angular integration (4.5). An alternative route would be to use the relations (4.6)–(4.7) giving the modes directly in terms of separate contributions of the radiative moments U_L and V_L. In the present chapter the two routes are equivalent because all the radiative moments are “uniformly” given with the approximation that is nec-

be changed, so that one should read $\zeta^\times_{7,3} = +\frac{729}{10250240}\cos(\theta)(167 + ...)\sin(\theta)(v^5\cos(3\psi) - ...)$.

[22]The ambiguity parameters therein are now known to be $\lambda = -\frac{1987}{3080}$ [147, 148] and $\theta = \xi + 2\kappa + \zeta = -\frac{11831}{9240}$ [71].

essary and sufficient to control the 3PN waveform.

In this respect one should be careful about what we mean by controlling the modes up to 3PN order. We mean — having in mind the standard PN practice — that the accuracy of the modes is exactly the one which is needed to obtain the 3PN waveform. Thus the dominant mode h^{22} will have full 3PN accuracy, but higher-order modes, which start at some higher PN order, will have a lower *relative* PN accuracy. For instance we shall see that the mode h^{44} starts at 1PN order thus it will be given only with 2PN relative accuracy.

The angular integration in (4.5) is over the angles (Θ, Φ). Like in our previous computation of the flux (4.65), it should be performed after substituting $(i, \psi) \to (\Theta, \psi + \pi/2 - \Phi)$ in the wave polarizations. Denoting $h = h_+ - i h_\times$ the integral we consider is thus

$$h^{\ell m} = \int d\Omega\, h(\Theta, \psi + \pi/2 - \Phi)\, \overline{Y}_{-2}^{\ell m}(\Theta, \Phi)\,. \tag{4.68}$$

Changing Φ into $\psi + \pi/2 - \psi'$ and Θ into $i' = \arccos c_i'$, and using the known dependence of the spherical harmonics on the azimuthal angle Φ [see (4.4)], we obtain

$$h^{\ell m} = (-i)^m\, e^{-im\psi} \int_0^{2\pi} d\psi' \int_{-1}^{1} dc_i'\, h(i', \psi')\, Y_{-2}^{\ell m}(i', \psi')\,, \tag{4.69}$$

exhibiting the azimuthal factor $e^{-im\psi}$ appropriate for each mode. Let us factorize out in all the modes an overall coefficient including $e^{-im\psi}$, and such that the dominant mode with $(\ell, m) = (2, 2)$ starts with one (by pure convention) at the Newtonian order. Remembering also our previous factorization in (4.58) we pose

$$h^{\ell m} = \frac{2G\, m\, \nu\, x}{R\, c^2}\, H^{\ell m}\,, \tag{4.70a}$$

$$H^{\ell m} = \sqrt{\frac{16\pi}{5}}\, \hat{H}^{\ell m}\, e^{-im\psi}\,, \tag{4.70b}$$

and list all the results in terms of $\hat{H}^{\ell m}$,[23]

$$\begin{aligned}
\hat{H}^{22} &= 1 + x\left(-\frac{107}{42} + \frac{55\nu}{42}\right) + 2\pi x^{3/2} + x^2\left(-\frac{2173}{1512} - \frac{1069\nu}{216} + \frac{2047\nu^2}{1512}\right) \\
&+ x^{5/2}\left(-\frac{107\pi}{21} - 24 i\nu + \frac{34\pi\nu}{21}\right) + x^3\Bigg(\frac{27027409}{646800} - \frac{856C}{105} + \frac{428 i\pi}{105} + \frac{2\pi^2}{3} \\
&+ \left(-\frac{278185}{33264} + \frac{41\pi^2}{96}\right)\nu - \frac{20261\nu^2}{2772} + \frac{114635\nu^3}{99792} - \frac{428}{105}\ln(16x)\Bigg) + O\left(\frac{1}{c^7}\right)\,,
\end{aligned} \tag{4.71a}$$

$$\begin{aligned}
\hat{H}^{21} &= \frac{1}{3} i\,\Delta\Bigg[x^{1/2} + x^{3/2}\left(-\frac{17}{28} + \frac{5\nu}{7}\right) + x^2\left(\pi + i\left(-\frac{1}{2} - 2\ln 2\right)\right) \\
&+ x^{5/2}\left(-\frac{43}{126} - \frac{509\nu}{126} + \frac{79\nu^2}{168}\right) + x^3\Bigg(-\frac{17\pi}{28} + \frac{3\pi\nu}{14} \\
&+ i\left(\frac{17}{56} + \nu\left(-\frac{995}{84} - \frac{3\ln 2}{7}\right) + \frac{17\ln 2}{14}\right)\Bigg)\Bigg] + O\left(\frac{1}{c^7}\right)\,,
\end{aligned} \tag{4.71b}$$

---

[23] The modes having $m < 0$ are easily deduced using $\hat{H}^{\ell,-m} = (-)^\ell \overline{\hat{H}}^{\ell m}$.

$$\hat{H}^{20} = -\frac{5}{14\sqrt{6}} + O\left(\frac{1}{c^7}\right), \tag{4.71c}$$

$$\begin{aligned}\hat{H}^{33} = & -\frac{3}{4}i\sqrt{\frac{15}{14}}\,\Delta\Bigg[x^{1/2} + x^{3/2}(-4+2\nu) + x^2\left(3\pi + i\left(-\frac{21}{5} + 6\ln(3/2)\right)\right) \\ & + x^{5/2}\left(\frac{123}{110} - \frac{1838\nu}{165} + \frac{887\nu^2}{330}\right) + x^3\Bigg(-12\pi + \frac{9\pi\nu}{2} \\ & + i\left(\frac{84}{5} - 24\ln(3/2) + \nu\left(-\frac{48103}{1215} + 9\ln(3/2)\right)\right)\Bigg)\Bigg] + O\left(\frac{1}{c^7}\right),\end{aligned} \tag{4.72a}$$

$$\begin{aligned}\hat{H}^{32} = & \frac{1}{3}\sqrt{\frac{5}{7}}\Bigg[x(1-3\nu) + x^2\left(-\frac{193}{90} + \frac{145\nu}{18} - \frac{73\nu^2}{18}\right) + x^{5/2}\left(2\pi - 6\pi\nu + i\left(-3 + \frac{66\nu}{5}\right)\right) \\ & + x^3\left(-\frac{1451}{3960} - \frac{17387\nu}{3960} + \frac{5557\nu^2}{220} - \frac{5341\nu^3}{1320}\right)\Bigg] + O\left(\frac{1}{c^7}\right),\end{aligned} \tag{4.72b}$$

$$\begin{aligned}\hat{H}^{31} = & \frac{i\,\Delta}{12\sqrt{14}}\Bigg[x^{1/2} + x^{3/2}\left(-\frac{8}{3} - \frac{2\nu}{3}\right) + x^2\left(\pi + i\left(-\frac{7}{5} - 2\ln 2\right)\right) \\ & + x^{5/2}\left(\frac{607}{198} - \frac{136\nu}{99} - \frac{247\nu^2}{198}\right) + x^3\Bigg(-\frac{8\pi}{3} - \frac{7\pi\nu}{6} \\ & + i\left(\frac{56}{15} + \frac{16\ln 2}{3} + \nu\left(-\frac{1}{15} + \frac{7\ln 2}{3}\right)\right)\Bigg)\Bigg] + O\left(\frac{1}{c^7}\right),\end{aligned} \tag{4.72c}$$

$$\hat{H}^{30} = -\frac{2}{5}i\sqrt{\frac{6}{7}}x^{5/2}\nu + O\left(\frac{1}{c^7}\right), \tag{4.72d}$$

$$\begin{aligned}\hat{H}^{44} = & -\frac{8}{9}\sqrt{\frac{5}{7}}\Bigg[x(1-3\nu) + x^2\left(-\frac{593}{110} + \frac{1273\nu}{66} - \frac{175\nu^2}{22}\right) \\ & + x^{5/2}\left(4\pi - 12\pi\nu + i\left(-\frac{42}{5} + \nu\left(\frac{1193}{40} - 24\ln 2\right) + 8\ln 2\right)\right) \\ & + x^3\left(\frac{1068671}{200200} - \frac{1088119\nu}{28600} + \frac{146879\nu^2}{2340} - \frac{226097\nu^3}{17160}\right)\Bigg] + O\left(\frac{1}{c^7}\right),\end{aligned} \tag{4.73a}$$

$$\begin{aligned}\hat{H}^{43} = & -\frac{9i\,\Delta}{4\sqrt{70}}\Bigg[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{39}{11} + \frac{1267\nu}{132} - \frac{131\nu^2}{33}\right) \\ & + x^3\left(3\pi - 6\pi\nu + i\left(-\frac{32}{5} + \nu\left(\frac{16301}{810} - 12\ln(3/2)\right) + 6\ln(3/2)\right)\right)\Bigg] + O\left(\frac{1}{c^7}\right),\end{aligned} \tag{4.73b}$$

$$\begin{aligned}\hat{H}^{42} = & \frac{1}{63}\sqrt{5}\Bigg[x(1-3\nu) + x^2\left(-\frac{437}{110} + \frac{805\nu}{66} - \frac{19\nu^2}{22}\right) + x^{5/2}\Bigg(2\pi - 6\pi\nu \\ & + i\left(-\frac{21}{5} + \frac{84\nu}{5}\right)\Bigg) + x^3\left(\frac{1038039}{200200} - \frac{606751\nu}{28600} + \frac{400453\nu^2}{25740} + \frac{25783\nu^3}{17160}\right)\Bigg] + O\left(\frac{1}{c^7}\right),\end{aligned} \tag{4.73c}$$

$$\hat{H}^{41} = \frac{i\,\Delta}{84\sqrt{10}}\Bigg[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{101}{33} + \frac{337\nu}{44} - \frac{83\nu^2}{33}\right)$$

$$+ x^3\left(\pi - 2\pi\nu + i\left(-\frac{32}{15} - 2\ln 2 + \nu\left(\frac{1661}{30} + 4\ln 2\right)\right)\right)\Bigg] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.73d}$$

$$\hat{H}^{40} = -\frac{1}{504\sqrt{2}} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.73e}$$

$$\hat{H}^{55} = \frac{625i\,\Delta}{96\sqrt{66}}\Bigg[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{263}{39} + \frac{688\nu}{39} - \frac{256\nu^2}{39}\right)$$
$$+ x^3\left(5\pi - 10\pi\nu + i\left(-\frac{181}{14} + \nu\left(\frac{105834}{3125} - 20\ln(5/2)\right) + 10\ln(5/2)\right)\right)\Bigg] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.74a}$$

$$\hat{H}^{54} = -\frac{32}{9\sqrt{165}}\left[x^2\left(1 - 5\nu + 5\nu^2\right) + x^3\left(-\frac{4451}{910} + \frac{3619\nu}{130} - \frac{521\nu^2}{13} + \frac{339\nu^3}{26}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.74b}$$

$$\hat{H}^{53} = -\frac{9}{32}i\sqrt{\frac{3}{110}}\,\Delta\Bigg[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{69}{13} + \frac{464\nu}{39} - \frac{88\nu^2}{39}\right)$$
$$+ x^3\left(3\pi - 6\pi\nu + i\left(-\frac{543}{70} + \nu\left(\frac{83702}{3645} - 12\ln(3/2)\right) + 6\ln(3/2)\right)\right)\Bigg] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.74c}$$

$$\hat{H}^{52} = \frac{2}{27\sqrt{55}}\left[x^2\left(1 - 5\nu + 5\nu^2\right) + x^3\left(-\frac{3911}{910} + \frac{3079\nu}{130} - \frac{413\nu^2}{13} + \frac{231\nu^3}{26}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.74d}$$

$$\hat{H}^{51} = \frac{i\,\Delta}{288\sqrt{385}}\Bigg[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{179}{39} + \frac{352\nu}{39} - \frac{4\nu^2}{39}\right)$$
$$+ x^3\left(\pi - 2\pi\nu + i\left(-\frac{181}{70} - 2\ln 2 + \nu\left(\frac{626}{5} + 4\ln 2\right)\right)\right)\Bigg] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.74e}$$

$$\hat{H}^{50} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.74f}$$

$$\hat{H}^{66} = \frac{54}{5\sqrt{143}}\left[x^2\left(1 - 5\nu + 5\nu^2\right) + x^3\left(-\frac{113}{14} + \frac{91\nu}{2} - 64\nu^2 + \frac{39\nu^3}{2}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.75a}$$

$$\hat{H}^{65} = \frac{3125i\,x^{5/2}\,\Delta}{504\sqrt{429}}\left[1 - 4\nu + 3\nu^2\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.75b}$$

$$\hat{H}^{64} = -\frac{128}{495}\sqrt{\frac{2}{39}}\left[x^2\left(1 - 5\nu + 5\nu^2\right) + x^3\left(-\frac{93}{14} + \frac{71\nu}{2} - 44\nu^2 + \frac{19\nu^3}{2}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.75c}$$

$$\hat{H}^{63} = -\frac{81i\,x^{5/2}\,\Delta}{616\sqrt{65}}\left[1 - 4\nu + 3\nu^2\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.75d}$$

$$\hat{H}^{62} = \frac{2}{297\sqrt{65}}\left[x^2\left(1 - 5\nu + 5\nu^2\right) + x^3\left(-\frac{81}{14} + \frac{59\nu}{2} - 32\nu^2 + \frac{7\nu^3}{2}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.75e}$$

$$\hat{H}^{61} = \frac{i\,x^{5/2}\,\Delta}{8316\sqrt{26}}\left[1 - 4\nu + 3\nu^2\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.75f}$$

$$\hat{H}^{60} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.75g}$$

$$\hat{H}^{77} = -i\,\frac{16807}{1440}\,\sqrt{\frac{7}{858}}\;\Delta\;x^{5/2}\Big[1 - 4\nu + 3\nu^2\Big] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.76a}$$

$$\hat{H}^{76} = \frac{81}{35}\,\sqrt{\frac{3}{143}}\;x^3\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.76b}$$

$$\hat{H}^{75} = i\,\frac{15625}{26208}\,\frac{1}{\sqrt{66}}\,\Delta\;x^{5/2}\left(1 - 4\nu + 3\nu^2\right) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.76c}$$

$$\hat{H}^{74} = -\frac{128}{1365}\,\sqrt{\frac{2}{33}}\;x^3\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.76d}$$

$$\hat{H}^{73} = -i\,\frac{243}{160160}\,\sqrt{\frac{3}{2}}\,\Delta\;x^{5/2}\left[1 - 4\nu + 3\nu^2\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.76e}$$

$$\hat{H}^{72} = \frac{x^3\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right)}{3003\,\sqrt{3}} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.76f}$$

$$\hat{H}^{71} = \frac{i\,x^{5/2}\,\Delta}{864864\,\sqrt{2}}\Big[1 - 4\nu + 3\nu^2\Big] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.76g}$$

$$\hat{H}^{70} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.76h}$$

$$\hat{H}^{88} = -\frac{16384}{63}\,\sqrt{\frac{2}{85085}}\,x^3\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.77a}$$

$$\hat{H}^{87} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.77b}$$

$$\hat{H}^{86} = \frac{243}{35}\,\sqrt{\frac{3}{17017}}\,x^3\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.77c}$$

$$\hat{H}^{85} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.77d}$$

$$\hat{H}^{84} = -\frac{128}{4095}\,\sqrt{\frac{2}{187}}\,x^3\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.77e}$$

$$\hat{H}^{83} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.77f}$$

$$\hat{H}^{82} = \frac{x^3}{9009\,\sqrt{85}}\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.77g}$$

$$\hat{H}^{81} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.77h}$$

$$\hat{H}^{80} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.77i}$$

while all the higher-order modes fall into the PN remainder and are negligible. However, we shall give here for the reader's convenience their leading order expressions for non zero m (see the derivation in [117]). For $\ell + m$ even we find:

$$\hat{H}^{\ell m} = \frac{(-)^{(\ell-m+2)/2}}{2^{\ell+1}(\frac{\ell+m}{2})!(\frac{\ell-m}{2})!(2\ell-1)!!}\left(\frac{5(\ell+1)(\ell+2)(\ell+m)!(\ell-m)!}{\ell(\ell-1)(2\ell+1)}\right)^{1/2} s_\ell(\nu)\,(im)^\ell\, x^{\ell/2-1} + O\left(\frac{1}{c^{\ell-2}}\right), \tag{4.78}$$

where we recall that the function $s_\ell(\nu)$ is defined in (4.28). For $\ell + m$ odd we have:

$$\hat{H}^{\ell m} = \frac{(-)^{(\ell-m-1)/2}}{2^{\ell-1}(\frac{\ell+m-1}{2})!(\frac{\ell-m-1}{2})!(2\ell+1)!!}\left(\frac{5(\ell+2)(2\ell+1)(\ell+m)!(\ell-m)!}{\ell(\ell-1)(\ell+1)}\right)^{1/2} \times s_{\ell+1}(\nu)\, i\,(im)^\ell\, x^{(\ell-1)/2} + O\left(\frac{1}{c^{\ell-2}}\right). \tag{4.79}$$

When $m = 0$, $\hat{H}^{\ell m}$ may not vanish due to DC contributions of the memory integrals. We already know that such an effect arises at Newtonian order [see (4.63a)], hence the non zero values of $\hat{H}^{20}$ and $\hat{H}^{40}$.

We find that the result for $\hat{H}^{22}$ at 3PN order given by (4.71a) is in complete agreement with the result of Kidder [117]. The only difference is our use of the particular phase variable (4.62) which permits to remove most of the logarithmic terms, showing that they are actually negligible modulations of the orbital phase. For the other harmonics we find agreement with the results of [117] up to 2.5PN order, but the results have here been completed by all the 3PN contributions.

4.9 Implications of the 3PN waveform for LISA observations

In the previous two chapters it was shown how the use of a 2.5PN accurate amplitude-corrected waveform (called the FWF in those chapters) resulted in an increased mass-reach and improved estimations of sky-position and luminosity distance for super massive black hole (SMBH) binaries with LISA, when compared to the RWF. In this section we present a short summary of a recent investigation [162] on how the 3PN accurate waveform (FWF) performs in these aspects relative to the RWF.

Recall that the RWF is a waveform whose amplitude is Newtonian and phase is accurate upto 3.5PN. The RWF contains only the harmonic whose frequency is twice the orbital frequency. Higher PN corrections lead to addition of higher harmonics in the waveform. It is these higher harmonics that are solely responsible for the enhanced mass-reach and improved parameter estimation of the FWF. It was noted, in Section II of chapter 2, that the $(2n+2)^{\text{th}}$ harmonic first appears at the n^{th} PN order correction in the amplitude. Hence, the 3PN waveform will have a harmonic which is eight times the fundamental orbital frequency.

This fact obviously leads to an increased mass-reach compared to the 2.5PN waveform for which the highest harmonic is at 7Ψ, Ψ being the orbital phase. The harmonic at 8Ψ will

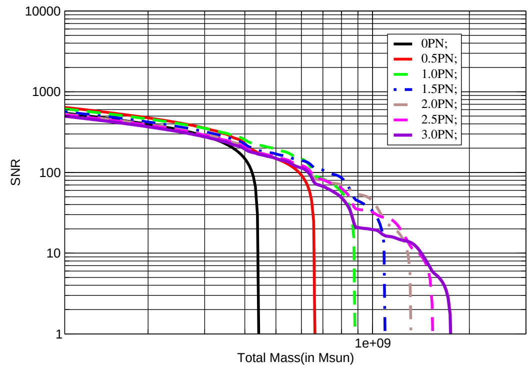

Figure 4.2: SNR versus total mass for successive PN amplitude corrected waveforms and 3.5PN phasing for mass ratio 0.1 with lower cut-off frequency 10^{-5}Hz. The upper cut-off of the k^{th}-harmonic in the FWF is inversely proportional to the total mass. As the mass increases the cut-off for the 2^{nd} harmonic (or the RWF) falls below the lower cut-off of the LISA detector bandwidth. The higher harmonics still enter the sensitive bandwidth of LISA and higher PN order waveforms produce significant SNR. The 3PN waveform has the highest mass reach, being 4 times the mass reach of the RWF. Sources are at a luminosity distance of 3 Gpc with fixed angles given by $\theta_S = \cos^{-1}(-0.6)$, $\phi_S = 1$, $\theta_L = \cos^{-1}(0.2)$, $\phi_L = 3$. The noise curve is the same as used in chapters 2 and 3. Adapted from Ref. [162].

be quantitatively more significant for the equal-mass case as the mass-reach will be better by 33% relative to the 2.5PN waveform as opposed to the unequal mass case where it is only 14%. The step-by-step increase in mass-reach, as measured by the SNR, when one goes from the RWF to the 3PN FWF in steps of 0.5PN is shown in Figure 4.2. Our system of interest, as in chapters 2 and 3, are SMBH binaries, detected with LISA.

The dramatic increase in angular resolution of LISA and improved estimation of luminosity distance for SMBH binaries as discussed in chapter 3, holds true for the 3PN waveform. Because of the increased angular resolution, the number of clusters within the reduced error-box is of the order one, enabling the identification of the host galaxy cluster of the SMBH binary. This helps in a measurement of the source's red-shift and leads to an independent confirmation of the cosmological parameters. Presence of possible EM counterparts can lead to an accurate determination of redshift. In such cases, the improved estimation of luminosity distance will lead to LISA's ability to constrain the dark energy equation of state index w, as shown in chapter 3 . The 3PN waveform, like the 2.5PN one (whose performance is also presented in the table for comparison), can limit the number of clusters $\mathbf{N}_{\text{cluster}}$ within its angular error-box to less than one for most choices of source location and orientation and the error estimates on w are comparable to other dark-energy missions.

μ_S	φ_S rad	μ_L	φ_L rad	Model	SNR	$\Delta \ln D_L$ (10^{-3})	$\Delta\Omega_S$ (10^{-6}str)	$\Delta \ln \mathcal{M}$ (10^{-6})	$\Delta\delta$ (10^{-6})	$\mathbf{N}_{\text{cluster}}$	Δw
$(m_1,\ m_2) = (10^6,\ 10^7)M_\odot$; $f_s = 10^{-5}$Hz;											
0.3	5	0.8	2	RWF	562	8.72	10.3	16.5	53.5	**0.21**	0.049
				2.5PN	510	7.97	8.2	14.9	46.2	**0.17**	0.045
				3.0PN	497	7.96	8.4	15.0	46.7	**0.17**	0.045
−0.1	2	−0.2	4	RWF	870	7.59	59	13.4	38.2	**1.20**	0.043
				2.5PN	779	6.35	22.8	12.3	33.9	**0.46**	0.036
				3.0PN	754	6.34	21.5	12.4	34.5	**0.44**	0.035
−0.8	1	0.5	3	RWF	1943	2.25	84.6	8.9	22.7	**1.70**	0.013
				2.5PN	1661	2.12	39.1	8.2	21.0	**0.79**	0.012
				3.0PN	1589	2.18	39.8	8.3	21.6	**0.81**	0.012
−0.5	3	−0.6	−2	RWF	1345	5.66	77.9	11.3	30.4	**1.60**	0.032
				2.5PN	1172	2.97	26.3	10.1	27.6	**0.54**	0.017
				3.0PN	1124	2.93	25.6	10.3	28.1	**0.52**	0.016
0.9	2	−0.8	5	RWF	2716	88.68	38.8	7.2	15.9	**0.79**	0.499
				2.5PN	2187	1.35	12.4	7.4	17.8	**0.25**	0.008
				3.0PN	2097	1.39	13.3	7.5	18.2	**0.27**	0.008
−0.6	1	0.2	3	RWF	1555	3.27	112.7	10.8	27.9	**2.30**	0.019
				2.5PN	1356	2.76	74.4	9.7	25.2	**1.52**	0.015
				3.0PN	1299	2.74	72.2	9.8	25.8	**1.47**	0.015
−0.1	3	−0.9	6	RWF	1592	4.05	213	11.7	28.3	**4.34**	–
				2.5PN	1402	3.85	199.4	10.6	25.3	**4.06**	–
				3.0PN	1345	3.79	197.1	10.7	25.9	**4.02**	–

Table 4.1: Comparison of accuracy in LISA's measurement of the various parameters at 0PN, 2.5PN and 3.0PN for seven different sets of the angular parameters and for a binary with masses $(10^6 - 10^7)M_\odot$ at a distance of 3 Gpc ($z = 0.55$) with lower cut-off frequency 10^{-5}Hz. The noise curve is the same as used in chapters 2 and 3. When the number of clusters in the error box on the sky is significantly larger than 1, it will not be possible to determine redshift unless the inspiral event has a clear optical counterpart; we have chosen not to quote results for Δw in such cases. Adapted from Ref. [162].