Chapter 5

Tail effects in the 3PN gravitational wave angular momentum flux of compact binaries in quasi-elliptical orbits

5.1 Introduction

5.1.1 Astrophysical compact binaries in quasi-elliptical orbits

In the previous three chapters we have studied the data-analysis and computation of gravitational wave (GW) polarisations from inspiralling compact binaries, but with orbits which were *quasi-circular*. Indeed, these are the most plausible sources for detecting GW with the help of laser interferometers like LIGO, VIRGO and the planned LISA. It has been established long ago, in a seminal work by Peters [47], that compact binaries moving in elliptical orbits with large relative separation circularise under radiation reaction due to loss of energy through emission of GW. Motivated by this, the 3.5PN phasing of inspiralling compact binaries moving in quasi-circular orbits is now complete and available for use in GW data analysis [32, 163]. Recent progress in increasing the sensitivity of LIGO by an order of magnitude to reach Advanced LIGO sensitivity renders the completion of the highly accurate phasing formula very timely. This is because prototype binary sources of GW for ground-based interferometric GW detectors are neutron star (NS) or black hole (BH) binaries close to their merger phase and consequently have lost all their eccentricity by the time the GW from them enter the detector bandwidth.

However, astrophysical investigations indicates the possibility of detecting binaries with eccentricity in the sensitive bandwidth of both terrestrial and space-based gravitational wave detectors. One such scenario is the Kozai oscillations [48]. This occurs in the cores of dense globular clusters where gravitational interactions between *pairs* of binary BH systems are likely. These interactions can lead to an eventual ejection of one of the BH resulting in the formation of a stable *hierarchical triplet*. This is a three-body configuration, where two closely bound BH orbit each other while the third orbits the center-of-mass of the first two. When the two orbital planes have a large inclination angle between them, tidal forces from the outer body increases the eccentricity of the inner stable binary by causing an orbital resonance. The Kozai mechanism described above can lead to eccentricities greater than 0.1 at the time GW from the inner binary enters the bandwidth of Advanced LIGO [49]. Binaries comprising stellar mass BH are estimated to possess a thermal distribution of eccentricities [164]. Eccentric binaries with higher masses situated in globular clusters are also potential sources for LISA. Intermediate mass ($\sim 10^3 M_\odot$) BH binaries with eccentricities between 0.1 and 0.2 are also expected to be generated by the Kozai mechanism [165]. These systems will lie in the bandwidth of LISA. Supermassive BH binaries, which are the most promising sources for LISA, can also merge within Hubble time with high eccentricities if the Kozai mechanism is in operation [166].

Another astrophysical situation where GW from eccentric binaries can be observable has been described by Davis, Levan and King [167]. A NS-BH binary can become eccentric in the late stages of its inspiral. During the first phase of mass transfer, numerical studies [167] show that the NS is not disrupted and orbits the BH in a high eccentricity orbit. The resulting system loses angular momentum via emission of GW which fall in the bandwidth of detectors like Advanced LIGO. Evolution under radiation reaction drives the two bodies to contact which is followed by further phases of mass-transfer. Recently, this mechanism was successful in explaining the light curve of the short gamma-ray-burst GRB 050911 [168].

Recently, Grindlay *et. al* [169] have proposed that short GRBs are produced by NS-NS binary mergers which are formed in globular clusters. These systems have the distinct feature that they possess high eccentricities at short orbital separations (see Figure 2 of [169]).

Apart from globular clusters, galaxies can also host compact binaries which have residual eccentricity in their late-inspiral phase. Asymmetric kicks imparted to NS at the time of their birth will result in highly eccentric NS-NS binaries [170]. The same conclusion also applies to NS-BH and BH-BH binaries [170].

5.1.2 Template construction for eccentric binaries and data analysis

All the above astrophysical paradigms clearly shows that inspiralling compact binaries in quasi-elliptic orbits are also quite plausible sources for both ground and space based interferometric GW detectors. Construction of templates for eccentric binaries require an accurate knowledge of the secular evolution of the GW phase and the evolution of the orbital elements (like semi-major axis, eccentricity) under radiation reaction. The first attempt in this direction was the classic works of Peters and Mathews [171, 47]. After computing the time-averaged (over an orbit) far-zone energy and angular momentum flux in the Newtonian limit, Peters and Mathews balanced them with the loss of binding energy and angular momentum of the Keplerian orbit. This allowed them to obtain the rate of decay of the orbital elements and showed that eccentricity decays roughly by a factor of three when the semi-major axis of the orbit is halved. After Peters and Mathews the evolution of the orbital elements by this procedure was progressively extended by Blanchet and Schäfer to 1PN in [172] and 1.5PN in [172, 173] and finally to 2PN by Gopakumar and Iyer [174]. While [175, 173] require the 1PN accurate orbital description of Damour and Deruelle [176], [174] crucially employs the generalised 2PN quasi-Keplerian parametrization of the binary's orbital motion in ADM coordinates as given in [177, 178, 179].

More recently, Damour, Gopakumar and Iyer [180] discussed an analytic method for constructing high accuracy templates for the GW signals from compact binaries in quasi-elliptical orbits in their inspiral phase. They go beyond the computation of the slow secular effects by the standard averaging over the orbital period and compute the additional fast oscillatory contributions beyond the average secular contributions. Using an improved "method of variation of constants" and working up to the leading radiation reaction order of 2.5PN, they combine the three time scales involved in the elliptical orbit case -the orbital period, periastron precession and radiation reaction time scales - without making the usual approximation of treating the radiative time scale as an adiabatic process. This was extended to 3.5PN order in Ref [181].

There have been relatively few exercises in data-analysis aspects of GW from eccentric binaries. This is primarily because inclusion of eccentricity in the parameter space leads to a large increase in the number of templates required to search for signals [182], which is consequently accompanied by higher computational costs. To circumvent this issue, Martel and Poisson [183] computed the loss in event rate if eccentric binaries are searched with circular templates. They found that even though the circular templates are not optimal filters, they will be efficient in detecting eccentric binaries. Further, the loss in signal-to-noise-ratio incurred increases with increasing eccentricity for a given total mass of the binary.

Recently Tessmer and Gopakumar [184] revisited the same problem, but used a 2.5PN accurate orbital evolution and adopted the phasing formalism developed in [180] mentioned above. The result of their analysis (with detector Initial LIGO) was that templates which modelled GW from binaries evolving under quadrupolar radiation reaction and whose orbits are 2PN accurate circular orbits are very efficient in searching for eccentric binaries.

5.1.3 3PN angular momentum flux: hereditary terms

The generation problem of gravitational waves for inspiralling compact binaries has been completed at the third post-Newtonian (3PN) order both for the equation of motion of the binary and for its far-zone radiation field. Recently, in a series of two related papers [185, 186], the computation of the energy flux of gravitational waves (GW) from inspiralling compact binaries moving in general non-circular orbits up to 3PN order was discussed. For non-circular orbits, in addition to the conserved energy and gravitational wave energy flux, the angular momentum flux needs to be known to determine the phasing of quasi-eccentric binaries. As mentioned before, a knowledge of the angular momentum flux of the system averaged over an orbit is mandatory to calculate the evolution of the orbital elements of non-circular, in particular, elliptic orbits under GW radiation reaction.

In this chapter, we compute all the *hereditary* terms in the angular momentum flux of inspiralling compact binaries moving in non-circular orbits up to 3PN order generalising earlier work at 1.5PN order (tails and spin-orbit) by Schäfer and Rieth [173]. The hereditary terms, unlike the *instantaneous* terms which are functions of the retarded time, depend on the dynamics of the system in its entire past. The 3PN hereditary contribution to angular momentum flux comes, apart from the tail terms, the *tails of tails* and *tail-squared* terms [135, 134]. Unlike the energy flux case, the angular momentum flux also contains an interesting *memory* contribution at 2.5PN. Using the angular momentum flux expression in conjunction with the 3PN accurate hereditary part of the energy flux obtained in Ref. [187, 185], we compute the hereditary part of the evolution of the orbital elements, semi-major axis a_r, eccentricity e_t, mean motion n and the periastron advance parameter k. Evolution of other related parameters such as orbital period P can be derived from these expressions. We also provide the expressions for the fluxes and evolution of orbital elements in the limit of small eccentricity up to second order in e_t. All the results of this chapter are provided in terms of the PN parameter $x = (\frac{Gm\omega}{c^3})^{2/3}$, related to the orbital frequency ω, which helps one to recover the circular orbit limit straightforwardly. The *instantaneous* terms in the 3PN angular momentum flux were computed by Arun in [188].

The rest of this chapter is organised as follows. In Section 5.2, we provide the general expression for the angular momentum flux in terms of radiative moments and relations between the radiative and source moments, keeping only terms relevant for computation of the hereditary terms upto 3PN accuracy. Section 5.3 reviews the solution of the equations of motion of compact binaries upto the accuracy we require for this chapter. In Section 5.4, we provide the Fourier domain representations of the source multipole moments and their use in averaging the flux over the orbital time-scale. Section 5.5 provides explicit expressions of the hereditary contributions in terms of the Fourier amplitudes. In Section 5.6 we give details of the numerical evaluation of these contributions. We provide the complete 3PN hereditary terms in Section 5.7 along with relevant checks. In Section 5.8, evolution of the orbital elements due to hereditary terms are provided. Section 5.9 comprises the conclusion and discussion of the results of this chapter. Finally, in Section 5.10, a list of the Fourier coefficients of the Newtonian moments are given in terms of Bessel functions.

5.2 Structure of the hereditary terms in the angular momentum flux

The complete 3PN accurate angular momentum flux in the source's far-zone, written in terms of the symmetric trace-free (STF) mass and current type radiative multipole moments (U_Ls and V_Ls) [130] can be found in Ref. [188]. Below we provide the far zone angular momentum flux $\mathcal{F}^{\mathcal{J}}{}_i$ upto 1PN order which will be sufficient to control the hereditary part of $\mathcal{F}^{\mathcal{J}}{}_i$ upto 3PN.

$$\mathcal{F}^{\mathcal{J}}{}_i \quad = \quad \frac{G}{c^5}\,\epsilon_{ijk}\left\{\frac{2}{5}U_{ja}U^{(1)}_{ka} + \frac{1}{c^2}\left[\frac{1}{63}U_{jab}U^{(1)}_{kab} + \frac{32}{45}V_{ja}V^{(1)}_{ka}\right]\right\} + \cdots\,. \qquad (5.1)$$

The dots indicate terms which contribute to the 3PN instantaneous terms in $\mathcal{F}^{\mathcal{J}}{}_i$ and we do not write them explicitly. U_L and V_L (with $L = ijk...$ a multi-index composed of l indices, each index running from 1 to 3) are the mass and current type radiative multipole moments respectively and $U^{(l)}_L$ and $V^{(l)}_L$ denote their l^{th} time derivatives.

The moments are functions of retarded time $T_R \equiv T - \frac{R}{c}$ in radiative coordinates. ε_{ipq} is the usual Levi-Civita symbol such that $\varepsilon_{123} = +1$. The shorthand $O(n)$ indicates that the post-Newtonian remainder is of order of $O(c^{-n})$.

Using the multipolar Post-Minkowskian (MPM) formalism outlined in the previous chapter, we re-express the radiative moments in Eq. (5.1) in terms of the source moments to an accuracy sufficient for the computation of the hereditary part of the angular momentum flux up to 3PN. The complete expressions required to calculate the instantaneous terms were already given in the previous chapter.

The relations connecting the different radiative moments U_L and V_L to the corresponding source moments I_L and J_L [124, 135, 134] required for this chapter are given below.

For the mass type moments we have

$$\begin{aligned}
U_{ij}(T_R) &= I^{(2)}_{ij}(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R} dV\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{11}{2}\right] I^{(4)}_{ij}(V) - \frac{2\,G}{7\,c^5}\int_{-\infty}^{T_R} dV I^{(3)}_{a<i}(V) I^{(3)}_{j>a}(V) \\
&\quad +2\left(\frac{GM}{c^3}\right)^2 \int_{-\infty}^{T_R} dV I^{(5)}_{ij}(V)\left[\ln^2\left(\frac{T_R-V}{2b}\right)+\frac{57}{70}\ln\left(\frac{T_R-V}{2b}\right)+\frac{124627}{44100}\right] \\
&\quad +\mathcal{O}(7), \qquad (5.2a)\\
U_{ijk}(T_R) &= I^{(3)}_{ijk}(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R} dV\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{97}{60}\right] I^{(5)}_{ijk}(V) + \mathcal{O}(5)\,, \qquad (5.2b)
\end{aligned}$$

where the bracket <> denotes STF projection. In the above formulas, M is the total ADM mass of the binary system. The I_L's and J_L's are the mass and current-type source moments, and $I_L^{(p)}$, $J_L^{(p)}$ denote their p-th time derivatives.

For the current-type moments, on the other hand, we find

$$V_{ij}(T_R) = J^{(2)}_{ij}(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R} dV\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{7}{6}\right] J^{(4)}_{ij}(V) + \mathcal{O}(5)\,. \qquad (5.3)$$

The radiative moments have two distinct contributions. The first part which is a function only of the retarded time, $T_R = T - \frac{R}{c}$, are the 'instantaneous terms' . The second part that depends on the dynamics of the system in its entire past [124] is referred to as hereditary contributions and forms the subject matter of this chapter.

The parameter b appearing in the logarithms of Eqs. (5.2) and (5.3) is a freely specifiable constant, having the dimension of time, entering the relation between the retarded time $T_R = T - R/c$ in radiative coordinates and the corresponding time $t-\rho/c$ in harmonic coordinates (where ρ is the distance of the source in harmonic coordinates). More precisely, we have

$$T_R = t - \frac{\rho}{c} - \frac{2\,G\,M}{c^3}\ln\left(\frac{\rho}{c\,r_0}\right)\,. \qquad (5.4)$$

We choose the constant b scaling the logarithm to be $\frac{r_0}{c}$ to match with the choice made in the computation of tails-of-tails in [135].

From the expressions for U_Ls and V_Ls, one can schematically split the total contribution to the angular momentum flux as the sum of the instantaneous and hereditary terms.

$$\mathcal{F}^{\mathcal{J}}{}_i \;=\; \left(\mathcal{F}^{\mathcal{J}}{}_i\right)_{\rm inst} + \left(\mathcal{F}^{\mathcal{J}}{}_i\right)_{\rm hered}\,. \qquad (5.5)$$

Since we do not discuss the instantaneous terms in the angular momentum flux here, they are not given beyond the Newtonian order here though it is easy to write them down using the expressions, Eqs (5.2) and (5.3) for the radiative moments. The Newtonian instantaneous term in $\mathcal{F}^{\mathcal{J}}{}_i$ which will be calculated later in this chapter for illustrating the Fourier decomposition method for calculating the hereditary terms is

$$\left(\mathcal{F}^{\mathcal{J}}{}_i\right)^{\rm Newtonian}_{\rm inst}(t) = \frac{2}{5}\frac{G}{c^5}\varepsilon_{ipq} I^{(2)}_{pj}(t) I^{(3)}_{qj}(t) \qquad (5.6)$$

The hereditary part can be decomposed, as mentioned in the earlier Section, as

$$(\mathcal{F}^{\mathcal{J}}{}_i)_{\rm hered} = (\mathcal{F}^{\mathcal{J}}{}_i)_{\rm tail} + (\mathcal{F}^{\mathcal{J}}{}_i)_{\rm tail(tail)} + (\mathcal{F}^{\mathcal{J}}{}_i)_{\rm (tail)^2} + (\mathcal{F}^{\mathcal{J}}{}_i)_{\rm memory}, \tag{5.7}$$

The quadratic-order tail integrals are explicitly given by (using Eqs 5.2 and 5.3 in Eq 5.1)

$$\begin{aligned}(\mathcal{F}^{\mathcal{J}}{}_i)_{\rm tail}(t) = \frac{G}{c^5}\Bigg\{&\frac{4GM}{5c^3}\,\varepsilon_{ijk}\int_0^{+\infty} d\tau\,\Big(I^{(2)}_{ja}(t)\,I^{(5)}_{ka}(t-\tau) - I^{(3)}_{ja}(t)\,I^{(4)}_{ka}(t-\tau)\Big)\left[\ln\left(\frac{\tau}{2r_0/c}\right)+\frac{11}{12}\right]\\
&+\frac{2GM}{63c^5}\,\varepsilon_{ijk}\int_0^{+\infty} d\tau\,\Big(I^{(3)}_{jab}(t)\,I^{(6)}_{kab}(t-\tau) - I^{(4)}_{jab}(t)\,I^{(5)}_{kab}(t-\tau)\Big)\left[\ln\left(\frac{\tau}{2r_0/c}\right)+\frac{97}{60}\right]\\
&+\frac{64GM}{45c^5}\,\varepsilon_{ijk}\int_0^{+\infty} d\tau\,\Big(J^{(2)}_{ja}(t)\,J^{(5)}_{ka}(t-\tau) - J^{(3)}_{ja}(t)\,J^{(4)}_{ka}(t-\tau)\Big)\left[\ln\left(\frac{\tau}{2r_0/c}\right)+\frac{7}{6}\right]\Bigg\},\end{aligned} \tag{5.8}$$

while the cubic-order tails (proportional to M^2) are

$$\begin{aligned}(\mathcal{F}^{\mathcal{J}}{}_i)_{\rm tail(tail)}(t) &= \frac{G}{c^5}\frac{4G^2M^2}{5c^6}\,\varepsilon_{ijk}\int_0^{+\infty} d\tau\,\Big(I^{(2)}_{ja}(t)I^{(6)}_{ka}(t-\tau) - I^{(3)}_{ja}(t)I^{(5)}_{ka}(t-\tau)\Big) &\text{(5.9a)}\\
&\qquad\times\left[\ln^2\left(\frac{\tau}{2r_0/c}\right)+\frac{57}{70}\ln\left(\frac{\tau}{2r_0/c}\right)+\frac{124627}{44100}\right],\\
(\mathcal{F}^{\mathcal{J}}{}_i)_{\rm (tail)^2}(t) &= \frac{G}{c^5}\frac{8G^2M^2}{5c^6}\,\varepsilon_{ijk}\left(\int_0^{+\infty} d\tau\, I^{(4)}_{ja}(t-\tau)\left[\ln\left(\frac{\tau}{2r_0/c}\right)+\frac{11}{12}\right]\right) &\text{(5.9b)}\\
&\qquad\times\left(\int_0^{+\infty} d\tau\, I^{(5)}_{ka}(t-\tau)\left[\ln\left(\frac{\tau}{2r_0/c}\right)+\frac{11}{12}\right]\right),\end{aligned}$$

and finally, the memory integral is

$$\left(\mathcal{F}^{\mathcal{J}}{}_i\right)_{\rm memory}(t) \;=\; \frac{G}{c^5}\frac{4G}{35c^5}\varepsilon_{ijk}\,I^{(3)}_{ja}(t)\left(\int_0^{\infty}\left[I^{(3)}_{b<k}\,I^{(3)}_{a>b}\right][t-\tau]\,d\tau\right). \tag{5.10}$$

Note that in Eq. (5.10), a term of the form $-\frac{G}{c^5}\frac{4G}{35c^5}\varepsilon_{ijk}\,I^{(2)}_{ja}(t)\frac{d}{dt}\left(\int_0^{\infty}\left[I^{(3)}_{b<k}\,I^{(3)}_{a>b}\right][t-\tau]\,d\tau\right)$ has been left out. This term simply reduces to $-\frac{G}{c^5}\frac{4G}{35c^5}\varepsilon_{ijk}\,I^{(2)}_{ja}(t)I^{(3)}_{b<k}(t)I^{(3)}_{a>b}(t)$, *i.e.*, an instantaneous term. This term, therefore, has been incorporated in the instantaneous part of the angular momentum flux and has been computed in Ref. [188]. For the energy flux case, the entire memory contribution becomes instantaneous (see Ref. [185]).

In the equations Eq. (5.8), (5.9) recall that M is the *conserved* mass monopole or total ADM mass of the source. The first term in (5.8) is the dominant tail at order 1.5PN while the second and third represent the sub-dominant tails appearing both at order 2.5PN. The higher-order tails are not given since they are at least at 3.5PN order (see [135] for their expressions). The two cubic-order tails given in Eqs. (5.9) are both at 3PN order. The memory term appears at 2.5PN order. Note that the constant b scaling the logarithms in the tail integrals in the radiative moments Eqs (5.2) & (5.3) has been replaced in the above tail, tail-of-tail etc. integrals as (r_0/c). For simplicity, we have replaced the symbol for the retarded time T_R in radiative coordinates by t.

5.3 Solution of the equations of motion of compact binaries

5.3.1 Doubly-periodic structure of the solution

To compute the integrals appearing in Eqs. (5.8), (5.9) and (5.10), a knowledge of the evolution of the source is required. For this purpose, we need to construct the solution of the equation-of-motion of compact binaries. For this purpose, we review in this Section, the general "doubly-periodic" structure of the PN solution, and the quasi-Keplerian representation of the 1PN binary motion by means of different types of eccentricities. The works [31, 189, 176] are closely followed here.

If we neglect the radiation reaction term at the 2.5PN order, the equations of motion of a compact binary system up to the 3PN order admit ten first integrals of the motion. These correspond to the conserved energy, angular and linear momenta, and position of the center of mass [145, 146]. When restricted to the frame of the center of mass, the equations admit four first integrals associated with the energy E and the angular momentum vector $\mathbf{J}$, given at 3PN order by Eqs. (4.8)–(4.9) of Ref. [149].

The motion takes place in a plane orthogonal to $\mathbf{J}$. We denote by $r = |\mathbf{x}|$ the binary's orbital separation, and by $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ the relative velocity (both $\mathbf{x}$ and $\mathbf{v}$ lie in the plane of motion). The conserved E and $\mathbf{J}$ are functions of r, $\dot{r}^2$, v^2 and $\mathbf{x}\times\mathbf{v}$ (for definiteness we employ the harmonic coordinate system of [149][1]), and depend on the total mass $m = m_1+m_2$ and reduced mass $\mu = m_1 m_2/m$. Polar coordinates r, ϕ in the orbital plane are used to express E and the norm $J = |\mathbf{J}|$ as some explicit functions of r, $\dot{r}^2$ and $\dot{\phi}$ ($v^2 = \dot{r}^2 + r^2\dot{\phi}^2$). The latter functions can be inverted (by means of straightforward PN iteration) to give $\dot{r}^2$ and $\dot{\phi}$ in terms of r and the constants of motion E and J. Thus,

$$\dot{r}^2 = \mathcal{R}[r; E, J], \tag{5.11a}$$

$$\dot{\phi} = \mathcal{G}[r; E, J], \tag{5.11b}$$

where $\mathcal{R}$ and $\mathcal{G}$ are polynomials in $1/r$, the degree of which depends on the PN approximation in question. At 3PN order, it is seventh degree for both $\mathcal{R}$ and $\mathcal{G}$ [190]. The various coefficients of the powers of $1/r$ are themselves polynomials in E and J, and also, of course, depend on m and the dimensionless reduced mass ratio $\nu \equiv \mu/m$. For bound elliptic-like orbits, one can prove [189] that the function $\mathcal{R}$ admits two real roots, r_P and r_A such that $r_\mathrm{P} < r_\mathrm{A}$, which admit some non-zero finite Newtonian limits when $c \to \infty$, and represent respectively the radii of the orbit's periastron and apastron. The other roots tend to zero in the limit $c \to \infty$.

The binary's orbital period, or time of return to the periastron, is obtained by integrating the radial motion (we drop the dependence on E and J in the following, for simplicity).

$$P = 2\int_{r_\mathrm{P}}^{r_\mathrm{A}} \frac{dr}{\sqrt{\mathcal{R}[r]}}. \tag{5.12}$$

Let us introduce the mean anomaly ℓ and the mean motion n by

$$\ell = n(t - t_\mathrm{P}) \tag{5.13}$$

[1] All calculations in this chapter will be done at the relative 1PN order, and at that order there is no difference between the harmonic and ADM coordinates.

$$n = \frac{2\pi}{P}, \tag{5.14}$$

where $t_{\rm P}$ denotes the instant of passage to the periastron. For a given value of the mean anomaly ℓ, the orbital separation r is obtained by inversion of the integral equation

$$\ell = n \int_{r_{\rm P}}^{r} \frac{dr'}{\sqrt{\mathcal{R}\,[r']}}. \tag{5.15}$$

This defines the function $r(\ell)$ which is a periodic function in ℓ with period 2π. Using this in Eq. (5.11), the orbital phase ϕ can be obtained in terms of the mean anomaly ℓ by integrating the angular motion as

$$\phi(\ell) = \phi_{\rm P} + \frac{1}{n} \int_0^{\ell} d\ell' \, \mathcal{G}\,[r(\ell')]\,, \tag{5.16}$$

where $\phi_{\rm P}$ denotes the value of the phase at the instant $t_{\rm P}$. Formally, the functions $r(\ell)$ and $\phi(\ell)$ complete the solution but does not take into account the doubly-periodic nature of the problem.

For this purpose, consider the advance of periastron per period, *i.e.*, the increase in the orbital phase $\Delta\phi$ (modulo 2π) during a single return to the periastron

$$\Delta\phi + 2\pi = 2 \int_{r_{\rm P}}^{r_{\rm A}} dr \frac{\mathcal{G}[r]}{\sqrt{\mathcal{R}[r]}}, \tag{5.17}$$

Let us define the fractional angle (*i.e.* the angle divided by 2π) of the total advance of the periastron per orbital revolution,

$$2\pi K = 2\pi + \Delta\phi. \tag{5.18}$$

Thus the precession of the periastron per period is given by $\Delta\phi = 2\pi(K-1)$. As K tends to one in the limit $c \to \infty$ (as is checked from the Newtonian limit), it is often convenient to define $k \equiv K - 1$, which will then entirely constitute the *relativistic precession*.

If, like the radial motion , we introduce another mean anomaly ℓ_ϕ and a mean angular motion ω_ϕ given by

$$\ell_\phi = \omega_\phi(t - t_{\rm P}) \tag{5.19}$$

$$\omega_\phi = \frac{2\pi}{P/K}, \tag{5.20}$$

we find that the two mean motions and mean anomalies are related by

$$\omega_\phi = K\,n \tag{5.21}$$

$$\ell_\phi = K\,\ell \tag{5.22}$$

In the case of a circular orbit, where the phase evolves linearly with time, $\dot{\phi} = \mathcal{G}\,[r] = \omega$, where ω is the orbital frequency of the circular orbit given by

$$\omega = K\,n = (1+k)\,n. \tag{5.23}$$

In the general case of a non-circular orbit we use the mean angular motion $\omega = Kn$ (we drop the subscript ϕ) and to explicitly introduce the linearly growing part of the orbital phase (5.16) by decomposing it in the form

$$\begin{aligned}\phi &= \phi_P + \omega\,(t - t_P) + W(\ell) \\ &= \phi_P + K\,\ell + W(\ell).\end{aligned} \tag{5.24}$$

Here $W(\ell)$ denotes a particular function which is periodic in ℓ (hence, periodic in time with period P). From Eq (5.16), this function is given in terms of the mean anomaly ℓ by

$$W(\ell) = \frac{1}{n}\int_0^{\ell} d\ell' \Big[\mathcal{G}\,[r(\ell')] - \omega\Big]. \tag{5.25}$$

Finally, the decomposition (5.24) exhibits clearly the "doubly periodic" nature of the binary motion, in terms of the mean anomaly ℓ with period 2π, and in terms of the periastron advance $K\,\ell$ with period $2\pi\,K$. It is worth noting that in Refs. [191, 180] the notation λ is used; it corresponds to $\lambda = K\,\ell$ and will also occasionally be used here.

5.3.2 Quasi-Keplerian representation of the motion of compact binaries

In our calculations we shall also require the explicit solution of the motion at 1PN order, in the form due to Damour & Deruelle [176]. The solution is given in parametric form in terms of the eccentric anomaly u. The radius r and the mean anomaly ℓ are expressed as

$$r = a_r\,(1 - e_r \cos u), \tag{5.26a}$$

$$\ell = u - e_t \sin u. \tag{5.26b}$$

The phase angle ϕ is given by (the additive constant ϕ_P has been set equal to zero)

$$\phi = K\,V, \tag{5.27}$$

where the true anomaly V is defined by[2]

$$V = 2\arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2} \tan\frac{u}{2}\right]. \tag{5.28}$$

In the above, K is the periastron advance given in general terms by Eq. (5.18), and a_r is the semi-major axis of the orbit. Note that there are, in this parametrization at 1PN order, three kinds of eccentricities e_r, e_t and e_ϕ (labelled after the coordinates r, t and ϕ). All these eccentricities differ from one another by 1PN terms, while the advance of the periastron per orbital revolution appears also starting at the 1PN order. Due to these features, this representation is referred to as the "quasi-Keplerian" (QK) parametrization for the 1PN orbital motion of the binary. The periodic function W of Eq. (5.25) now reads

$$W = K\,(V - \ell)\ . \tag{5.29}$$

[2]We have denoted the true anomaly by V rather than by the symbol v of earlier papers to avoid confusion with the relative speed v.

The above solution is closed by the explicit dependence of the orbital elements in terms of the 1PN conserved energy E and angular momentum J in the center-of-mass frame (taken as usual per unit of the reduced mass μ). This is given in Ref. [176]. Note that the semi-major axis a_r and mean motion n depend at 1PN order only on the constant of energy through

$$\begin{aligned} a_r &= -\frac{G\,m}{2E}\left\{1+\left(\frac{7}{2}-\frac{\nu}{2}\right)\frac{E}{c^2}\right\}, & (5.30a)\\ n &= \frac{(-2\,E)^{3/2}}{G\,m}\left\{1+\left(\frac{15}{4}-\frac{\nu}{4}\right)\frac{E}{c^2}\right\}. & (5.30b)\end{aligned}$$

Posing $h \equiv J/(Gm)$, the 1PN periastron precession simply reads

$$K = 1 + \frac{3}{c^2\,h^2}, \tag{5.31}$$

while the three different eccentricities are given by

$$\begin{aligned} e_r &= \left\{1+2\,E\,h^2\left[1+\left(-\frac{15}{2}+\frac{5}{2}\nu\right)\frac{E}{c^2}+\frac{-6+\nu}{c^2h^2}\right]\right\}^{1/2}, & (5.32a)\\ e_t &= \left\{1+2\,E\,h^2\left[1+\left(\frac{17}{2}-\frac{7}{2}\nu\right)\frac{E}{c^2}+\frac{2-2\nu}{c^2h^2}\right]\right\}^{1/2}, & (5.32b)\\ e_\phi &= \left\{1+2\,E\,h^2\left[1+\left(-\frac{15}{2}+\frac{\nu}{2}\right)\frac{E}{c^2}-\frac{6}{c^2h^2}\right]\right\}^{1/2}. & (5.32c)\end{aligned}$$

Notice the following simple ratios (valid at 1PN order)

$$\begin{aligned} \frac{e_t}{e_r} &= 1+(8-3\nu)\,\frac{E}{c^2}, & (5.33a)\\ \frac{e_t}{e_\phi} &= 1+(8-2\nu)\,\frac{E}{c^2}, & (5.33b)\\ \frac{e_r}{e_\phi} &= 1+\nu\frac{E}{c^2}. & (5.33c)\end{aligned}$$

The binary orbit can be characterised by either (E, J) or any two of the quasi-Keplerian orbital elements. We choose (n, e_t) and list the 1PN accurate expressions for the other orbital elements in terms of n and e_t, which we will require later in the work.

For this purpose we first invert Eq. (5.30b) to obtain the 1PN conserved energy in terms of n.

$$E = -\frac{1}{2}\,(G\,m\,n)^{2/3}\left\{1+(15-\nu)\left(\frac{G\,m\,n}{c^3}\right)^{2/3}\right\}, \tag{5.34}$$

Using this in Eq. (5.30a), we get

$$a_r = \left(\frac{G\,m}{n^2}\right)^{1/3}\left\{1+(-3+\nu)\left(\frac{G\,m\,n}{c^3}\right)^{2/3}\right\}, \tag{5.35}$$

which is a 1PN extension of Kepler's law. From Eqs (5.33) we obtain e_r and e_ϕ in terms of n

and e_t.

$$e_r = e_t \left\{ 1 + (4 - \frac{3}{2}\nu) \left(\frac{G\, m\, n}{c^3} \right)^{2/3} \right\}, \tag{5.36a}$$

$$e_\phi = e_t \left\{ 1 + (4 - \nu) \left(\frac{G\, m\, n}{c^3} \right)^{2/3} \right\}. \tag{5.36b}$$

The periastron precession, given by Eq. (5.31), also needs to be expressed as a function of n and e_t. To obtain it, we use the following expression, easily obtained from Eq. (5.32b).

$$-2E\, h^2 = (1 - e_t)^2 \left\{ 1 + \left(\frac{G\, m\, n}{c^3} \right)^{2/3} \frac{1}{4\,(1 - e_t^2)} \left(9 + \nu - (17 - 7\,\nu)\, e_t^2 \right) \right\} \tag{5.37}$$

The above relation, along with Eq. (5.34) yields

$$K = 1 + \frac{3}{(1 - e_t^2)} \left(\frac{G\, m\, n}{c^3} \right)^{2/3} \tag{5.38}$$

5.4 Fourier decomposition of the binary's multipole moments

5.4.1 Newtonian Angular Momentum flux

The method we shall use in this chapter is illustrated by the computation of the averaged angular momentum flux of compact binaries at Newtonian order using a Fourier decomposition of the Keplerian motion [171]. The GW angular momentum flux reduces at Newtonian order to (from Eq. 5.6) [3]

$$(\mathcal{F}^{\mathcal{J}}{}_i)^{(\mathrm{N})} = \frac{2}{5}\, \varepsilon_{ijk}\, \overset{(2)}{I}{}_{ja}^{(\mathrm{N})}(t)\, \overset{(3)}{I}{}_{ka}^{(\mathrm{N})}(t), \tag{5.39}$$

where (N) means the Newtonian limit, the superscript (n) refers to time differentiations. $I_{ij}^{(\mathrm{N})}$ is, by construction, the symmetric-trace-free (STF) quadrupole moment at Newtonian order given by $I_{ij}^{(\mathrm{N})} = \mu\, x^{<i} x^{j>}$. x^i is the binary's orbital separation, and the angular brackets around indices indicate the STF projection: $x^{<i} x^{j>} \equiv x^i x^j - \frac{1}{3}\delta^{ij} r^2$. However, the presence of the Levi-Civita symbol ensures that the trace part of the symmetrized quadrupole moment cancels out. Hence, unlike the energy-flux calculation in Ref. [185], in this chapter we will ignore the trace of the quadrupole moment. Thus

$$I_{ij}^{(\mathrm{N})} = \mu\, x^i x^j. \tag{5.40}$$

Peters & Mathews [171, 47] obtained the expression of the (averaged) Newtonian flux for compact binaries on eccentric orbits by two methods. The first method was to directly average in time Eq. (5.39) using the expression (5.40) computed for the Keplerian ellipse; the second method was to decompose the components of the quadrupole moment into discrete Fourier series using the known Fourier decomposition of the Kepler orbit (the two methods,

[3] From now on we set $c = 1$ and $G = 1$.

as expected, agreed on the result).

In the second method the quadrupole moment, which is a periodic function of time at Newtonian order, is thus decomposed into a Fourier series

$$
\begin{aligned}
I^{(\mathrm{N})}_{ij}(t) &= \sum_{p=-\infty}^{+\infty} \underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{ij}\, e^{\mathrm{i}\,p\,\ell}, && (5.41\mathrm{a})\\
\text{with}\quad \underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{ij} &= \int_0^{2\pi} \frac{d\ell}{2\pi}\, I^{(\mathrm{N})}_{ij}\, e^{-\mathrm{i}\,p\,\ell}, && (5.41\mathrm{b})
\end{aligned}
$$

where ℓ is the mean anomaly of the binary motion, Eq. (5.13). Since $I^{(\mathrm{N})}_{ij}$ is real the Fourier coefficients clearly satisfy ${}_{(p)}\mathcal{I}^{(\mathrm{N})}_{ij} = {}_{(-p)}\mathcal{I}^{(\mathrm{N})*}_{ij}$ ($*$ denotes the complex conjugate). Inserting Eqs. (5.41) into (5.39) we obtain

$$
(\mathcal{F}^{\mathcal{J}}{}_i)^{(\mathrm{N})} = \frac{2}{5} \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} (\mathrm{i}p\,n)^2 (\mathrm{i}q\,n)^3\, \varepsilon_{ijk}\, \underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{ja}\, \underset{(q)}{\mathcal{I}}{}^{(\mathrm{N})}_{ka}\, e^{\mathrm{i}(p+q)\ell}. \tag{5.42}
$$

After this, we perform an average over one period P. This means the average over $\ell = n\,(t - t_{\mathrm{P}})$ which is easily performed with the following

$$
\langle e^{\mathrm{i}p\ell} \rangle \equiv \int_0^{2\pi} \frac{d\ell}{2\pi}\, e^{\mathrm{i}\,p\,\ell} = \delta_{p,0}. \tag{5.43}
$$

This immediately yields the averaged angular momentum flux in the form of the Fourier series

$$
\langle\, (\mathcal{F}^{\mathcal{J}}{}_i)^{(\mathrm{N})}\, \rangle = -\frac{4}{5}\, \mathrm{i} \sum_{p=1}^{+\infty} (p\,n)^5\, \varepsilon_{ijk}\, \underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{ja}\, \underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})*}_{ka}. \tag{5.44}
$$

Using dimensional analysis (and the known circular orbit limit) this flux is necessarily of the form

$$
\langle\, (\mathcal{F}^{\mathcal{J}}{}_i)^{(\mathrm{N})}\, \rangle = \frac{32}{5}\, \nu^2\, \frac{m^{9/2}}{a^{7/2}}\, f_J(e)\, \mathbf{z}_i, \tag{5.45}
$$

where $\nu = \mu/m$ and a is the semi-major axis of the Kepler orbit, $\mathbf{z}_i$ is an unit vector parallel to the angular momentum of the binary (and perpendicular to the orbit lying in the x-y plane) and the function $f_J(e)$ is a dimensionless function depending only on the binary's eccentricity e. The coefficient in front of (5.45) is chosen in such a way that $f_J(e)$ reduces to one in the circular orbit limit ($e \to 0$). Therefore,

$$
f_J(e) = \frac{-\mathrm{i}}{8\,\mu^2\, a^4} \sum_{p=1}^{+\infty} p^5\, \varepsilon_{ijk}\, \underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{ja}\, \underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})*}_{ka}\, \mathbf{z}^i. \tag{5.46}
$$

The Fourier coefficients of the quadrupole moment are explicitly given by Eqs. (5.128) in Section 5.10. $f_J(e)$ admits an algebraically closed-form expression which is crucial for the timing of the binary pulsar PSR 1913+16 [50], and given by

$$
f_J(e) = \frac{8 + 7\,e^2}{8\,(1 - e^2)^2}\,. \tag{5.47}
$$

The method of decomposing the Newtonian moment of compact binaries as discrete Fourier series was used in Ref. [173] to compute the tail at the dominant 1.5PN order. To extend this result we need to be more systematic about the Fourier decomposition of the (not necessarily Newtonian) source multipole moments.

5.4.2 General structure of the Fourier decomposition

The two sets of source moments of the compact binary are denoted by $I_L(t)$ and $J_L(t)$ following Ref. [126]. The multi-index notation means $L \equiv i_1 i_2 \cdots i_l$, where l is the number of indices or multipolarity (which should not be confused with the mean anomaly ℓ). In this Section we study the structure of the mass and current moments I_L and say J_{L-1} (where $L-1 \equiv i_1 i_2 \cdots i_{l-1}$ is chosen in the current moment for convenience rather than L), at any PN order and for a compact binary system moving on a general non-circular orbit[4]. Their general structure can be written as

$$\begin{aligned} I_L(t) &= \sum_{k=0}^{l} \mathcal{F}_k[r, \dot{r}, v^2]\, x^{<i_1 \cdots i_k} v^{i_{k+1} \cdots i_l>}, &\quad (5.48a)\\ J_{L-1}(t) &= \sum_{k=0}^{l-2} \mathcal{G}_k[r, \dot{r}, v^2]\, x^{<i_1 \cdots i_k} v^{i_{k+1} \cdots i_{l-2}} \varepsilon^{i_{l-1}>ab} x^a v^b, &\quad (5.48b) \end{aligned}$$

where $x^i = y_1^i - y_2^i$ and $v^i = dx^i/dt = v_1^i - v_2^i$ denote the relative position and velocity of the two bodies (in a harmonic coordinate system). In (5.48) we pose for instance $x^{i_1 \cdots i_k} \equiv x^{i_1} \cdots x^{i_k}$, and the angular brackets surrounding indices refer to the usual symmetric-trace-free (STF) projection with respect to those indices.

Using polar coordinates r, ϕ in the orbital plane (as in Sec. 5.3.1), the above introduced coefficients $\mathcal{F}_k$ and $\mathcal{G}_k$ depend on the masses and on r, $\dot{r}$ and $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$. For quasi-elliptic motion one can explicitly factorize out the dependence on the orbital phase ϕ by inserting $x = r\cos\phi$, $y = r\sin\phi$, and $v_x = \dot{r}\cos\phi - r\,\dot{\phi}\sin\phi$, $v_y = \dot{r}\sin\phi + r\,\dot{\phi}\cos\phi$. Furthermore, using the explicit solution of the motion (Sec. 5.3.2) r, $\dot{r}$ and v^2, and hence the $\mathcal{F}_k$'s and

$\mathcal{G}_k$'s can be expressed as periodic functions of the mean anomaly $\ell = n\,(t - t_{\mathrm{P}})$, where $n = 2\pi/P$. We then find that the above general structure of the multipole moments can be expressed in terms of the phase angle ϕ, as the following finite sum over some "magnetic-type" index m ranging from $-l$ to $+l$,

$$\begin{aligned} I_L(t) &= \sum_{m=-l}^{l} \underset{(m)}{\mathcal{A}_L}(\ell)\, e^{\mathrm{i}\, m\phi}, &\quad (5.49a)\\ J_{L-1}(t) &= \sum_{m=-l}^{l} \underset{(m)}{\mathcal{B}_{L-1}}(\ell)\, e^{\mathrm{i}\, m\phi}, &\quad (5.49b) \end{aligned}$$

involving some coefficients ${}_{(m)}\mathcal{A}_L$ and ${}_{(m)}\mathcal{B}_{L-1}$ depending on the mean anomaly ℓ and which are complex ($\in \mathbb{C}$). (Some of these coefficients could be vanishing in particular cases.) The point for our purpose is that these coefficients are *periodic* functions of ℓ with period 2π. As

[4]However the intrinsic spins of the compact objects are neglected, so the motion takes place in a fixed orbital plane.

we can see, the structure of the mass and current moments I_L and J_{L-1} is basically the same, but their coefficients ${}_{(m)}\mathcal{A}_L$ and ${}_{(m)}\mathcal{B}_{L-1}$ will have a different parity, because of the Levi-Civita symbol entering the current moment J_{L-1}.

To proceed further, let us exploit the doubly periodic nature of the dynamics in the two variables $\lambda \equiv K\ell$ and ℓ (as reviewed in Sec. 5.3.1). The phase is given in full generality by Eq. (5.24) where $W(\ell)$ is periodic in ℓ. In the following it will be more convenient to single out in the expression of the phase the purely relativistic precession of the periastron, namely $\lambda - \ell = k\ell$ where $k = K - 1$. We insert the expression of the phase variable into Eqs. (5.49) which yields many factors modifying the coefficients of (5.49), but in such a way that they retain the periodicity in ℓ. Hence

$$I_L(t) = \sum_{m=-l}^{l} \underset{(m)}{\mathcal{I}}_L(\ell)\, e^{\mathrm{i}\, m k \ell}, \tag{5.50a}$$

$$J_{L-1}(t) = \sum_{m=-l}^{l} \underset{(m)}{\mathcal{J}}_{L-1}(\ell)\, e^{\mathrm{i}\, m k \ell}, \tag{5.50b}$$

where the coefficients ${}_{(m)}\mathcal{I}_L(\ell)$ and ${}_{(m)}\mathcal{J}_{L-1}(\ell)$ are 2π-periodic. This allows us to use a discrete Fourier series expansion in the interval $\ell \in [0, 2\pi]$ for each of these coefficients, *i.e.*,

$$\underset{(m)}{\mathcal{I}}_L(\ell) = \sum_{p=-\infty}^{+\infty} \underset{(p,m)}{\mathcal{I}}_L\, e^{\mathrm{i}\, p \ell}, \tag{5.51a}$$

$$\underset{(m)}{\mathcal{J}}_{L-1}(\ell) = \sum_{p=-\infty}^{+\infty} \underset{(p,m)}{\mathcal{J}}_{L-1}\, e^{\mathrm{i}\, p \ell}, \tag{5.51b}$$

and the inverse relations are

$$\underset{(p,m)}{\mathcal{I}}_L = \int_0^{2\pi} \frac{d\ell}{2\pi}\, \underset{(m)}{\mathcal{I}}_L(\ell)\, e^{-\mathrm{i}\, p \ell}, \tag{5.52a}$$

$$\underset{(p,m)}{\mathcal{J}}_{L-1} = \int_0^{2\pi} \frac{d\ell}{2\pi}\, \underset{(m)}{\mathcal{J}}_{L-1}(\ell)\, e^{-\mathrm{i}\, p \ell}. \tag{5.52b}$$

We now have the following final decompositions of the multipole moments,

$$I_L(t) = \sum_{p=-\infty}^{+\infty} \sum_{m=-l}^{l} \underset{(p,m)}{\mathcal{I}}_L\, e^{\mathrm{i}\,(p+mk)\,\ell}, \tag{5.53a}$$

$$J_{L-1}(t) = \sum_{p=-\infty}^{+\infty} \sum_{m=-l}^{l} \underset{(p,m)}{\mathcal{J}}_{L-1}\, e^{\mathrm{i}\,(p+mk)\,\ell}. \tag{5.53b}$$

The moments I_L and J_{L-1} being real, their Fourier coefficients satisfy ${}_{(p,m)}\mathcal{I}_L = {}_{(-p,-m)}\mathcal{I}^*_L$ and ${}_{(p,m)}\mathcal{J}_{L-1} = {}_{(-p,-m)}\mathcal{J}^*_{L-1}$.

The previous decompositions were general, but it is still useful to introduce a special notation for the particular case of the Newtonian (N) order, for which the relativistic precession $k \to 0$. In this limit, the usual periodic Fourier decomposition of the moments is recovered [generalizing Eqs. (5.41)], with only one Fourier summation over the index p, so that

$$I_L^{(\mathrm{N})}(t) = \sum_{p=-\infty}^{+\infty} \underset{(p)}{\mathcal{I}}{}_L^{(\mathrm{N})}\, e^{\mathrm{i}\, p\, \ell}, \tag{5.54a}$$

$$J_{L-1}^{(\mathrm{N})}(t) = \sum_{p=-\infty}^{+\infty} \underset{(p)}{\mathcal{J}}{}_{L-1}^{(\mathrm{N})}\, e^{\mathrm{i}\, p\, \ell}. \tag{5.54b}$$

The Newtonian Fourier coefficients are equal to the sums over m of the doubly-periodic Fourier coefficients in Eqs. (5.53) when taken in the Newtonian limit, namely

$$\underset{(p)}{\mathcal{I}}{}_L^{(\mathrm{N})} = \sum_{m=-l}^{l} \underset{(p,m)}{\mathcal{I}}{}_L^{(\mathrm{N})}, \tag{5.55a}$$

$$\underset{(p)}{\mathcal{J}}{}_{L-1}^{(\mathrm{N})} = \sum_{m=-l}^{l} \underset{(p,m)}{\mathcal{J}}{}_{L-1}^{(\mathrm{N})}. \tag{5.55b}$$

5.5 Hereditary contributions in the angular momentum flux

The technique of the previous Section is applied to the computation of the tail integrals in the angular momentum flux of compact binaries. Although the computations are effectively done up to the 3PN level, the method we propose could in principle be implemented at any PN order.

We shall compute all the tail and tail-of-tail terms (5.8)–(5.9) [*i.e.* up to the 3PN order] averaged over the mean anomaly ℓ. Together with the instantaneous terms reported in Ref. [188] we shall obtain the complete expression of the 3PN angular momentum flux. It is clear from Eqs. (5.8)–(5.9) that all the terms necessitate an evaluation at the relative Newtonian order *except* the mass-type quadrupolar tail term – first term in (5.8) – which must crucially include the 1PN corrections. We start with all the terms required at relative Newtonian order and then tackle the more difficult 1PN quadrupolar tail term.

5.5.1 Tails at relative Newtonian order

In this section we consider the mass-type quadrupolar tail term in the angular momentum flux, the first term in Eq. (5.8). However, we will not compute this term at the PN order required for this chapter, but at the relative Newtonian order [5]. This will serve as a simple illustration of the method we will use for computing the higher-order tails.

The 1.5PN mass-quadrupole tail contribution is, from Eq. (5.8)

$$\begin{aligned}\langle(\mathcal{F}^{\mathcal{J}}{}_i)^{(\mathrm{N})}_{\text{mass quad}}\rangle_{\text{tail}} &= \langle\frac{4M}{5}\,\varepsilon_{ijk}\int_0^{+\infty} d\tau\left(\overset{(2)}{I}{}_{ja}^{(\mathrm{N})}(t)\,\overset{(5)}{I}{}_{ka}^{(\mathrm{N})}(t-\tau) - \overset{(3)}{I}{}_{ja}^{(\mathrm{N})}(t)\,\overset{(4)}{I}{}_{ka}^{(\mathrm{N})}(t-\tau)\right)\\ &\qquad\times\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{11}{12}\right]\rangle,\end{aligned} \tag{5.56}$$

[5]We shall compute this term at 1PN relative order in Sec. 5.5.3.

where the brackets $\langle\rangle$ refer to the average over the mean anomaly ℓ as defined by Eq. (5.43). The term (5.56) was already computed using a Fourier series at Newtonian order in Ref. [173]; note that the method of [173] is valid only for periodic motion and thus is applicable only at the Newtonian level. In this Section we recover the Newtonian result of [173].

The Fourier decomposition of the Newtonian quadrupole moment was already given in general form by Eqs. (5.41). Inserting that into the flux (5.56), we evaluate the tail integral by using the fact that if $\ell(t) = n\,(t-t_{\rm P})$ corresponds to the current time t, then $\ell(t-\tau) = \ell(t) - n\,\tau$ corresponds to the retarded time $t-\tau$. Next we perform the average over the current value $\ell(t)$ with the help of the formula (5.43). We get

$$\langle(\mathcal{F}^{\mathcal{J}}{}_i)^{\rm (N)}_{\rm mass\ quad}\rangle_{\rm tail} = \frac{8M}{5}\,\mathrm{i}\sum_{p=-\infty}^{+\infty}(p\,n)^7\,\varepsilon_{ijk}\,\underset{(p)}{\mathcal{I}}{}^{\rm (N)}_{ja}\,\underset{(-p)}{\mathcal{I}}{}^{\rm (N)}_{ka}\int_0^{+\infty}d\tau\,e^{\mathrm{i}p\,n\,\tau}\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{11}{12}\right]. \tag{5.57}$$

Note the crucial replacement of the Fourier decomposition of the quadrupole moment $I_{ka}{}^{\rm (N)}(t-\tau)$ at the *retarded time* $t-\tau$ in the tail integral in Eq. (5.56) by the Fourier coefficients at the *current time* t, defined by Eq. (5.41). This permits us to take the Fourier coefficients of $I_{ka}{}^{\rm (N)}$ outside the tail integral. This replacement makes the result derived below "exact" only in a PN sense, as we have neglected the effect of the binary's adiabatic evolution by radiation reaction in the past. Consequently, this replacement introduces a remainder term in Eq. (5.57) given by the order of magnitude of the adiabatic parameter $\xi_{\rm rad} \equiv \dot{\omega}/\omega^2$. From Refs. [124, 158], we know that the above replacement of the current motion in the tail integral is valid only modulo some remainder $\mathcal{O}(\xi_{\rm rad})$ or, more precisely, $\mathcal{O}(\xi_{\rm rad}\ln\xi_{\rm rad})$. This remainder corresponds to a correction term of relative 2.5PN order which is always negligible for our purposes (the 1.5PN order of the tails makes the correction terms due to the influence of the binary's past at 4PN order).

To tackle the last factor in (5.57) which is the tail integral in the Fourier domain, we use the closed-form formula

$$\int_0^{+\infty}d\tau\,e^{\mathrm{i}\sigma\tau}\ln\left(\frac{\tau}{2r_0}\right) = -\frac{1}{\sigma}\left[\frac{\pi}{2}\mathrm{sign}(\sigma)+\mathrm{i}\Big(\ln(2|\sigma|r_0)+C\Big)\right], \tag{5.58}$$

where $\sigma \equiv p\,n$, $\mathrm{sign}(\sigma) = \pm 1$ and $C = 0.577\cdots$ denotes the Euler constant. Inserting Eq. (5.58) into (5.57) we have

$$\langle(\mathcal{F}^{\mathcal{J}}{}_i)^{\rm (N)}_{\rm mass\ quad}\rangle_{\rm tail} = -\frac{8\pi\,M}{5}\,\mathrm{i}\sum_{p=1}^{+\infty}(p\,n)^6\,\varepsilon_{ijk}\,\underset{(p)}{\mathcal{I}}{}^{\rm (N)}_{ja}\,\underset{(p)}{\mathcal{I}}{}^{\rm (N)*}_{ka}. \tag{5.59}$$

Note that the range of p's corresponds to positive frequencies only.

The remaining tail integrals, given by the second and third terms in Eq. (5.8), are evaluated in exactly the same way. With the PN accuracy of the present calculation these integrals are truly Newtonian so the mass octupole moment I_{ijk} and current quadrupole moment J_{ij} are required at Newtonian order only. For simplicity, we drop the supercript (N) because there can be no confusion with other results. We thus need to evaluate the time-averaged fluxes

$$\langle(\mathcal{F}^{\mathcal{J}}{}_i)_{\rm mass\ oct}\rangle_{\rm tail} \;=\; \langle\frac{2M}{63}\,\varepsilon_{ijk}\int_0^{+\infty}d\tau\,\left(I^{(3)}_{jab}(t)\,I^{(6)}_{kab}(t-\tau) - I^{(4)}_{jab}(t)\,I^{(5)}_{kab}(t-\tau)\right)$$

$$\times\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{97}{60}\right]\rangle, \tag{5.60}$$

$$\langle\,(\mathcal{F}^{\mathcal{J}}{}_i)_{\text{curr quad}}\rangle_{\text{tail}} \;=\; \langle\frac{64M}{45}\,\varepsilon_{ijk}\int_0^{+\infty}d\tau\,\left(J^{(2)}_{ja}(t)\,J^{(5)}_{ka}(t-\tau)-J^{(3)}_{ja}(t)\,J^{(4)}_{ka}(t-\tau)\right)$$
$$\times\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{7}{6}\right]\rangle. \tag{5.61}$$

Note that, as in the case of the mass-quadrupole moment, the trace of J_{ij} also does not contribute to the angular momentum flux (see the argument after Eq. (5.39)). However, for I_{ijk}, we do have to take into account its trace. Inserting the Fourier decomposition of the moments, performing the average using Eq. (5.43) and using the integration formula (5.58) gives us

$$\langle(\mathcal{F}^{\mathcal{J}}{}_i)_{\text{mass oct}}\rangle_{\text{tail}} \;=\; -\frac{4\pi M}{63}\,\mathrm{i}\sum_{p=1}^{+\infty}(p\,n)^8\,\varepsilon_{ijk}\,\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{jab}\,\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})*}_{kab} \tag{5.62a}$$

$$\langle\,(\mathcal{F}^{\mathcal{J}}{}_i)_{\text{curr quad}}\rangle_{\text{tail}} \;=\; -\frac{128\pi M}{45}\,\mathrm{i}\sum_{p=1}^{+\infty}(p\,n)^6\,\varepsilon_{ijk}\,\underset{(p)}{\mathcal{J}}{}^{(\mathrm{N})}_{ja}\,\underset{(p)}{\mathcal{J}}{}^{(\mathrm{N})*}_{ka}. \tag{5.62b}$$

In Sec. 5.6 we shall provide some numerical plots for the enhancement eccentricity-dependent factors associated with Eqs. (5.62), since they do not have a closed-form expression.

5.5.2 Tails-of-tails and tails squared

At the 3PN order (*i.e.* 1.5PN beyond the dominant tail) the first cubic non-linear interaction, between the quadrupole moment I_{ij} and two mass monopole factors M, appears. From Eqs. (5.9) we have to compute the "tail-of-tail" contribution and the so-called "tail squared" one,

$$\langle(\mathcal{F}^{\mathcal{J}}{}_i)_{\text{tail(tail)}}\rangle = \langle\frac{4M^2}{5}\,\varepsilon_{ijk}\int_0^{+\infty}d\tau\,\left(I^{(2)}_{ja}(t)I^{(6)}_{ka}(t-\tau)-I^{(3)}_{ja}(t)I^{(5)}_{ka}(t-\tau)\right)$$
$$\times\left[\ln^2\left(\frac{\tau}{2r_0}\right)+\frac{57}{70}\ln\left(\frac{\tau}{2r_0}\right)+\frac{124627}{44100}\right]\rangle, \tag{5.63a}$$

$$\langle(\mathcal{F}^{\mathcal{J}}{}_i)_{(\text{tail})^2}\rangle = \langle\frac{8M^2}{5}\,\varepsilon_{ijk}\left(\int_0^{+\infty}d\tau\,I^{(4)}_{ja}(t-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{11}{12}\right]\right)$$
$$\times\left(\int_0^{+\infty}d\tau\,I^{(5)}_{ka}(t-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{11}{12}\right]\right)\rangle. \tag{5.63b}$$

(5.63c)

Both terms are evaluated at relative Newtonian order. We insert the Fourier decomposition of the Newtonian quadrupole moment (5.41) [again suppressing the superscript (N) for simplicity]. The new feature with respect to the tails is the appearance of a logarithm *squared* in the tail-of-tail integral (5.63). We have again replaced the motion in the infinite past of the binary by the motion in the current time (see the argument following Eq. (5.57)). The closed-form formula required to deal with this term is [compare with Eq. (5.58)]

$$\int_0^{+\infty} d\tau\, e^{\mathrm{i}\sigma\tau} \ln^2\left(\frac{\tau}{2r_0}\right) = \frac{\mathrm{i}}{\sigma}\left\{\frac{\pi^2}{6} - \left[\frac{\pi}{2}\mathrm{sign}(\sigma) + \mathrm{i}\Big(\ln(2|\sigma|r_0) + C\Big)\right]^2\right\}, \tag{5.64}$$

and with this formula, together with (5.58), we obtain the result

$$\langle (\mathcal{F}^{\mathcal{J}}{}_i)_{\mathrm{tail(tail)}}\rangle = -\frac{8M^2}{5}\,\mathrm{i}\sum_{p=1}^{+\infty}(p\,n)^7\,\varepsilon_{ijk}\,\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{ja}\,\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})*}_{ka} \tag{5.65a}$$
$$\left\{\frac{\pi^2}{6} - 2\Big(\ln(2p\,n\,r_0) + C\Big)^2 + \frac{57}{35}\Big(\ln(2p\,n\,r_0) + C\Big) - \frac{124627}{22050}\right\}.$$

The tail squared term is readily computed with (5.58) and is

$$\langle (\mathcal{F}^{\mathcal{J}}{}_i)_{(\mathrm{tail})^2}\rangle = -\frac{8M^2}{5}\,\mathrm{i}\sum_{p=1}^{+\infty}(p\,n)^7\,\varepsilon_{ijk}\,\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{ja}\,\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})*}_{ka}\left\{\frac{\pi^2}{2} + 2\left(\ln(2p\,n\,r_0) + C - \frac{11}{12}\right)^2\right\}. \tag{5.66}$$

Adding the two results (5.65a) and (5.66) we finally get

$$\langle (\mathcal{F}^{\mathcal{J}}{}_i)_{\mathrm{tail(tail)}+(\mathrm{tail})^2}\rangle = -\frac{8M^2}{5}\,\mathrm{i}\sum_{p=1}^{+\infty}(p\,n)^7\,\varepsilon_{ijk}\,\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{ja}\,\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})*}_{ka} \tag{5.67a}$$
$$\left\{\frac{2\pi^2}{3} - \frac{214}{105}\ln(2p\,n\,r_0) - \frac{214}{105}C - \frac{116761}{29400}\right\}.$$

Note that the contribution from logarithms *squared* has cancelled out between the two terms (5.65a)–(5.66). Such cancellation is known to occur for general sources [135]. Note also that the result (5.67a) still depends on the arbitrary length scale r_0. It is important to trace out the fate of this constant and check that the complete angular momentum flux we obtain at the end (including all the instantaneous contributions computed in [188]) is independent of r_0.

5.5.3 The mass quadrupole tail at 1PN order

In this subsection, we calculate the mass quadrupole tail at the relative 1PN order, namely

$$\langle (\mathcal{F}^{\mathcal{J}}{}_i)_{\text{mass quad}}\rangle_{\mathrm{tail}} = \langle\frac{4M}{5}\,\varepsilon_{ijk}\int_0^{+\infty} d\tau\,\left(I^{(2)}_{ja}(t)\,I^{(5)}_{ka}(t-\tau) - I^{(3)}_{ja}(t)\,I^{(4)}_{ka}(t-\tau)\right)$$
$$\times\left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{11}{12}\right]\rangle, \tag{5.68}$$

At the 1PN order (and similarly at any higher PN orders), we must take care of the doubly-periodic structure of the solution of the motion [Sec. 5.3.1], and decompose the multipole moments according to the general formulas (5.53). Hence the 1PN mass quadrupole moment I_{ij} entering Eq. (5.68) is decomposed as

$$I_{ij}(t) = \sum_{p=-\infty}^{+\infty} \sum_{m=-2}^{2} \underset{(p,m)}{\mathcal{I}}{}_{ij}\, e^{\mathrm{i}\,(p+m\,k)\,\ell}, \tag{5.69}$$

with doubly-indexed Fourier coefficients ${}_{(p,m)}\mathcal{I}_{ij}$ which are valid through order 1PN. The harmonics for which $m = \pm 1$ are zero at the 1PN order, so Eq. (5.69) reduces to

$$I_{ij}(t) = \sum_{p=-\infty}^{+\infty} \left\{ \underset{(p,-2)}{\mathcal{I}}{}_{ij}\, e^{\mathrm{i}\,(p-2k)\,\ell} + \underset{(p,0)}{\mathcal{I}}{}_{ij}\, e^{\mathrm{i}\,p\,\ell} + \underset{(p,2)}{\mathcal{I}}{}_{ij}\, e^{\mathrm{i}\,(p+2k)\,\ell} \right\}, \tag{5.70}$$

but for our purposes, Eq. (5.69) is more convenient, keeping in mind that the terms with $m = \pm 1$ are absent. As before we insert Eq. (5.69) into Eq. (5.68) to obtain [after neglecting 2.5PN radiation reaction terms $\mathcal{O}(\xi_{\mathrm{rad}})$ mentioned before]

$$\begin{aligned}\langle (\mathcal{F}^{\mathcal{J}}{}_i)_{\text{mass quad}} \rangle_{\text{tail}} \;=\; & -\frac{4M}{5}\,\mathrm{i} \sum_{p,p';m,m'} n^7 \Big((p+mk)^2 (p'+m'k)^5 - (p+mk)^3 (p'+m'k)^4 \Big) \\ & \varepsilon_{ijk}\, \underset{(p,m)}{\mathcal{I}}{}_{ja}\, \underset{(p',m')}{\mathcal{I}}{}_{ka} \langle e^{\mathrm{i}(p+p'+(m+m')k)\ell} \rangle \int_0^{+\infty} d\tau\, e^{-\mathrm{i}\,(p'+m'k)\,n\,\tau} \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right],\end{aligned} \tag{5.71}$$

where the summations range from $-\infty$ to $+\infty$ for p and p', and from -2 to 2 for m and m'. The factors $(p + mk)^2$, $(p' + m'k)^5$ etc. come from the time-derivatives of the quadrupole moment. We leave the last two factors in Eq. (5.71) as they are, namely the average over ℓ of an elementary "doubly-periodic" complex exponential, and the Fourier transform of the tail integral.

The expression in Eq. (5.71) is to be calculated at the 1PN order. Since the relativistic advance of the periastron k is a small 1PN quantity, we first evaluate Eq. (5.71) at *linear* order in k [*i.e.*, neglecting $\mathcal{O}(k^2)$ which is at least 2PN]. Afterwards we shall insert the explicit PN expressions for the 1PN quadrupole moment and ADM mass. The necessary formulas for performing the linear-order expansion in k of the last two factors in Eq. (5.71) are provided below. The average we perform is over the orbital period (time to return to the periastron) and so is defined by

$$\langle e^{\mathrm{i}\,(p+m\,k)\,\ell} \rangle \equiv \int_0^{2\pi} \frac{d\ell}{2\pi}\, e^{\mathrm{i}\,(p+m\,k)\,\ell}. \tag{5.72}$$

Using the fact that $m\,k \ll 1$ since we are in the limit where $k \to 0$ (hence $p + m\,k$ is never an integer unless $k = 0$), we readily find

$$\langle e^{\mathrm{i}\,(p+m\,k)\,\ell} \rangle = \left\{ \begin{array}{ll} \dfrac{m}{p}\,k & \text{if } p \neq 0 \\ & \\ 1 + \mathrm{i}\,\pi\,m\,k & \text{if } p = 0 \end{array} \right\} + \mathcal{O}(k^2). \tag{5.73}$$

The above result depends only on whether p is zero or not, and is true for any integer m, except that when $m = 0$ the result (5.73) becomes "exact" as there is no remainder term $\mathcal{O}(k^2)$ in this case.

To compute the tail integral given by the last factor in Eq. (5.71), we expand it at first order in k, obtaining

$$\int_0^{+\infty} d\tau\, e^{\mathrm{i}\,(p+m\,k)\,n\,\tau} \ln\left(\frac{\tau}{2r_0}\right) = \left(1 - \frac{m\,k}{p}\right) \int_0^{+\infty} d\tau\, e^{\mathrm{i}p\,n\,\tau} \ln\left(\frac{\tau}{2r_0}\right) - \mathrm{i}\frac{m\,k}{p^2 n} + O(k^2), \qquad (5.74)$$

and we apply for the remaining integral in Eq (5.74) the formula Eq. (5.58).

Using Eqs. (5.73) and (5.74) we can explicitly compute the tail expression Eq. (5.71) at first order in k (the extension to higher order in k would in principle be straightforward). The result is left in the form of the multiple Fourier series Eq. (5.71), into which the results (5.73)–(5.74) have been inserted (we do not give a more explicit form for this result which is given by a complicated Mathematica expression). In the next Section we shall re-express this series in terms of some elementary enhancement functions which we evaluate numerically.

5.5.4 Memory Integral at 2.5PN order

The memory contribution is, from Eq. (5.10),

$$\left(\mathcal{F}^J{}_i\right)_{mem} = \frac{4}{35}\varepsilon_{ijk}\, I^{(3)}_{ja}(t)\left(\int_0^{\infty} \left[I^{(3)}_{b<k}\, I^{(3)}_{a>b}\right][t-\tau]\, d\tau\right) \qquad (5.75)$$

in which the symmetrisation over k & a in the integrand can be removed because it is manifestly symmetric and the tracelessness condition can also be removed because of the presence of ε_{ijk} and the symmetry of I_{ij}. Fourier decomposing I_{ij} we get

$$\left(\mathcal{F}^J{}_i\right)_{mem} = \frac{4}{35}\varepsilon_{ijk}\left(\sum_{p=-\infty}^{+\infty}(i\,p\,n)^3 \underset{(p)}{\mathcal{I}}_{ja}\, e^{ipl}\right)\left(\sum_{q=-\infty}^{+\infty}(i\,q\,n)^3 \underset{(p)}{\mathcal{I}}_{bk}\, e^{iql}\right)\left(\sum_{r=-\infty}^{+\infty}(i\,r\,n)^3 \underset{(p)}{\mathcal{I}}_{ab}\, e^{irl}\right) \left(\int_0 e^{-i(q+r)n\,\tau}\, d\tau\right) \qquad (5.76)$$

where, like in the tail integrals we have neglected the adiabatic orbital evolution of the binary and replaced it by the motion at the current time. The integrand for the memory does not contain the log kernel, but it being highly oscillatory the crests and troughs cancel out and the only contribution comes from the boundaries. However the infinite past contribution (corresponding to $\tau \to \infty$) is zero if we assume stationarity in the past.

On performing an average over an orbit we get using

$$\langle e^{i(p+q+r)l}\rangle = \delta_{p+q+r,0}. \qquad (5.77)$$

$$\langle\left(\mathcal{F}^J{}_i\right)_{mem}\rangle = \frac{4}{35}\, n^8\, \varepsilon_{ijk} \sum_{p=-\infty}^{+\infty}\sum_{q=-\infty}^{+\infty} p^2\, q^3\, (p+q)^3\, \underset{(p)}{\mathcal{I}}_{ja}\, \underset{(p)}{\mathcal{I}}_{bk}\, \underset{-(p+q)}{\mathcal{I}}_{ab} \qquad (5.78)$$

This, on simplification, reduces to

$$\langle \left(\mathcal{F}^J{}_i\right)_{mem} \rangle = \frac{8}{35}\, n^8\, \varepsilon_{ijk} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} p^2\, q^3 \left((p+q)^3\, \Re[\underset{(p)}{\mathcal{I}}_{ja}\, \underset{(q)}{\mathcal{I}}_{bk}\, \underset{(p+q)}{\mathcal{I}}{}^*_{ab}] \right.$$
$$\left. -(p-q)^3\, \Re[\underset{(p)}{\mathcal{I}}_{ja}\, \underset{(q)}{\mathcal{I}}{}^*_{bk}\, \underset{(p-q)}{\mathcal{I}}{}^*_{ab}] \right) \tag{5.79}$$

where $\Re[x]$ stands for real part of x. The angular momentum flux contribution from the memory, in terms of the PN parameter $x = (m\,\omega)^{2/3}$, becomes

$$\langle \left(\mathcal{F}^J{}_i\right)_{mem} \rangle = \frac{32}{5} \nu^2\, m\, x^{7/2} \left(\frac{1}{28}\, x^{5/2}\, \rho_J(e) \right) \mathbf{z}_i \tag{5.80}$$

where $\rho_J(e)$ is the enhancement function corresponding to the memory and goes to zero as $e \to 0$. It is given by

$$\rho_J(e) = \varepsilon_{ijk} \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} p^2\, q^3 \left((p+q)^3\, \Re[\underset{(p)}{\mathcal{I}}_{ja}\, \underset{(q)}{\mathcal{I}}_{bk}\, \underset{(p+q)}{\mathcal{I}}{}^*_{ab}] - (p-q)^3\, \Re[\underset{(p)}{\mathcal{I}}_{ja}\, \underset{(q)}{\mathcal{I}}{}^*_{bk}\, \underset{(p-q)}{\mathcal{I}}{}^*_{ab}] \right) \mathbf{z}^i \tag{5.81}$$

We find that the functions of the quadrupole moment Fourier coefficients appearing in each of the two terms in Eq. (5.81) are pure imaginary and therefore, we have, like the circular orbit case,

$$\rho_J(e) = 0. \tag{5.82}$$

In the future, we would like to look at this result in more detail, specially a proof of the vanishing of this term in a time-domain calculation. Also, the validity of the assumption of the replacement of the past motion by the current motion inside the memory integral needs to be treated more rigorously, perhaps by the use of Fourier transforms rather than Fourier series.

5.6 Numerical calculation of the tail integrals

5.6.1 Definition of the eccentricity enhancement factors

We define here some functions of the eccentricity by certain Fourier series of the components of the *Newtonian* multipole moments $I_L^{(\mathrm{N})}$ and $J_{L-1}^{(\mathrm{N})}$ for a Keplerian ellipse with eccentricity e, semi-major axis a, frequency $n = 2\pi/P$ (such that Kepler's law $n^2 a^3 = m$ holds at Newtonian order). In the center of mass frame $I_L^{(\mathrm{N})} = \mu s_l(\nu) x^{<L>}$ and $J_{L-1}^{(\mathrm{N})} = \mu s_l(\nu) x^{<L-2} \varepsilon^{i_{l-1}>ab} x^a v^b$ where $\mu = m_1 m_2/m = \nu m$. We pose $s_l(\nu) \equiv X_2^{l-1} + (-)^l X_1^{l-1}$, where, $X_1 \equiv \frac{m_1}{m} = \frac{1}{2}\left(1 + \sqrt{1-4\nu}\right)$, and, $X_2 \equiv \frac{m_2}{m} = \frac{1}{2}\left(1 - \sqrt{1-4\nu}\right)$. Let us rescale the latter Newtonian moments in order to make them dimensionless

$$I_L^{(\mathrm{N})} \equiv \mu\, a^l\, s_l(\nu)\, \hat{I}_L\,, \tag{5.83a}$$
$$J_{L-1}^{(\mathrm{N})} \equiv \mu\, a^l\, n\, s_l(\nu)\, \hat{J}_{L-1}\,. \tag{5.83b}$$

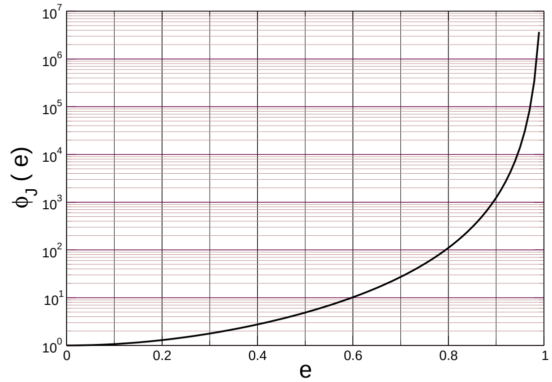

Figure 5.1: Variation of $\varphi_J(e)$ with the eccentricity e. In the circular orbit limit we have $\varphi_J(0) = 1$.

Our first "enhancement" function is the Peters & Mathews [171, 47] function which we have already expressed in Eq. (5.46) as a Fourier series [and which turns out to admit the analytically closed form (5.47)]. This series becomes, in terms of the Fourier components of the rescaled quadrupole moment $\hat{I}_{ij}$

$$f_J(e) = -\frac{\mathrm{i}}{8} \sum_{p=1}^{+\infty} p^5\, \varepsilon_{ijk}\, \underset{(p)}{\hat{\mathcal{I}}}_{jb}\, \underset{(p)}{\hat{\mathcal{I}}}^{*}_{kb}\, \mathbf{z}^i, \tag{5.84}$$

and is such that the averaged Newtonian angular momentum flux of compact binaries reads

$$\langle (\mathcal{F}^{\mathcal{J}}{}_i)^{(\mathrm{N})} \rangle = \frac{32}{5}\, \nu^2\, m\, x^{7/2}\, f_J(e)\, \mathbf{z}_i. \tag{5.85}$$

In the above we have defined, for future convenience, the frequency-related PN parameter $x = (m\,\omega)^{2/3}$ where ω is the binary's orbital frequency defined for general orbits by Eq. (5.23). Note that in Eq. (5.85) (which is Newtonian) we can replace ω by n (hence x reduces to m/a).

Next, we define several other "enhancement" functions of the eccentricity which will permit to usefully parametrize the tail terms at Newtonian order. First we pose

$$\varphi_J(e) = -\frac{\mathrm{i}}{16} \sum_{p=1}^{+\infty} p^6\, \varepsilon_{ijk}\, \underset{(p)}{\hat{\mathcal{I}}}_{jb}\, \underset{(p)}{\hat{\mathcal{I}}}^{*}_{kb}\, \mathbf{z}^i. \tag{5.86}$$

Like for $f_J(e)$ this function is defined in such a way that it tends to one in the circular orbit limit, when $e \to 0$. However, unlike $f_J(e)$, it does not admit a closed-form expression, and we leave it in the form of a Fourier series. The function $\varphi_J(e)$ parametrizes the mass quadrupole tail at Newtonian order, in the sense that we have, from Eq. (5.59),

$$\langle (\mathcal{F}^{\mathcal{J}}{}_i)^{(\mathrm{N})}_{\text{mass quad}} \rangle = \frac{32}{5}\, \nu^2\, m\, x^{7/2} \left[4\pi\, x^{3/2}\, \varphi_J(e)\right] \mathbf{z}_i. \tag{5.87}$$

For circular orbits, $\varphi_J(0) = 1$ and we recognize the coefficient 4π of the 1.5PN tail term ($\propto x^{3/2}$) as computed analytically in Refs. [173]. The function $\varphi_J(e)$ has already been computed

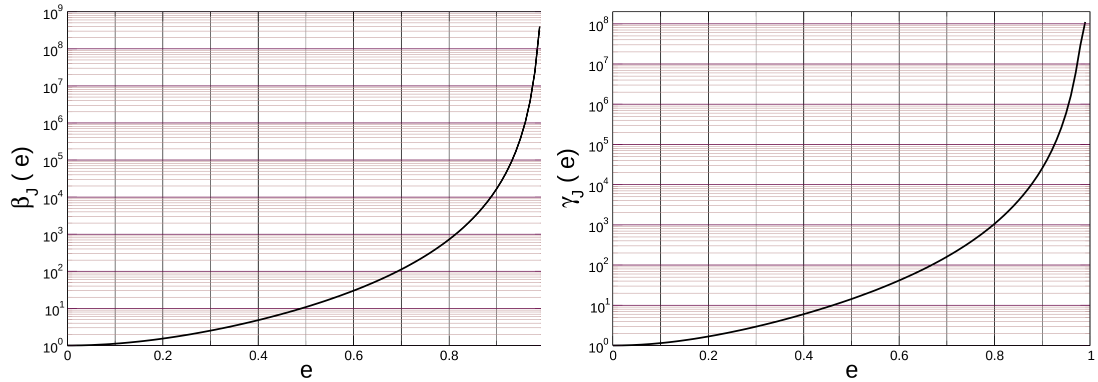

Figure 5.2: Variation of $\beta_J(e)$ (left panel) and $\gamma_J(e)$ (right panel) with the eccentricity e. In the circular orbit limit we have $\beta_J(0) = \gamma_J(0) = 1$.

numerically from its Fourier series (5.86) in Ref. [173]. Here we show the plot of $\varphi_J(e)$ in Fig. 5.1 (see Sec. 5.6.2 for details on the numerical computation) [6].

We next provide similar expressions for the 2.5PN mass octupole and current quadrupole tails by posing

$$\beta_J(e) = -\frac{20\,\mathrm{i}}{16403}\sum_{p=1}^{+\infty} p^8\,\varepsilon_{ijk}\,\underset{(p)}{\hat{\mathcal{I}}}_{jab}\,\underset{(p)}{\hat{\mathcal{I}}}{}^{*}_{kab}\,\mathbf{z}^i, \tag{5.88a}$$

$$\gamma_J(e) = -8\,\mathrm{i}\sum_{p=1}^{+\infty} p^6\,\varepsilon_{ijk}\,\underset{(p)}{\hat{\mathcal{J}}}_{ja}\,\underset{(p)}{\hat{\mathcal{J}}}{}^{*}_{ka}\,\mathbf{z}^i. \tag{5.88b}$$

These functions also tend to one when $e \to 0$ (as will be checked later) and most probably do not admit any closed-form expressions. The tail terms ($\propto x^{5/2}$) of Eqs. (5.60) reduce to

$$\langle(\mathcal{F}^{\mathcal{J}}{}_i)_{\text{mass oct}}\rangle_{\text{tail}} = \frac{32}{5}\nu^2\,m\,x^{7/2}\left[\frac{16403}{2016}\,\pi\,(1-4\,\nu)\,x^{5/2}\,\beta_J(e)\right]\mathbf{z}_i, \tag{5.89}$$

$$\langle(\mathcal{F}^{\mathcal{J}}{}_i)_{\text{curr quad}}\rangle_{\text{tail}} = \frac{32}{5}\nu^2\,m\,x^{7/2}\left[\frac{\pi}{18}\,(1-4\,\nu)\,x^{5/2}\,\gamma_J(e)\right]\mathbf{z}_i. \tag{5.90}$$

The numerical graphs of the functions $\beta_J(e)$ and $\gamma_J(e)$ are shown in Fig. 5.2.

Two further enhancement factors are needed for the tail-of-tail and tail squared integrals (which are Newtonian). The first of these functions looks very much like $f_J(e)$, Eq. (5.84), in the sense that its Fourier series involves *odd* powers of the modes p. Namely we define

$$F_J(e) = -\frac{\mathrm{i}}{32}\sum_{p=1}^{+\infty} p^7\,\varepsilon_{ijk}\,\underset{(p)}{\hat{\mathcal{I}}}_{ja}\,\underset{(p)}{\hat{\mathcal{I}}}{}^{*}_{ka}\,\mathbf{z}^i. \tag{5.91}$$

Thanks to this odd power $\propto p^7$ we find that $F_J(e)$ admits like for $f_J(e)$ an analytic closed

[6] Our notation is different from the one in Rieth & Schäfer [173]; the function $\varphi_{\mathrm{RS}}(e)$ there is related to our definition by $\varphi_{\mathrm{RS}}(e) = \varphi_J(e)/f_J(e)$. In the present work it is better not to divide the various functions by the Peters & Mathews function $f_J(e)$.

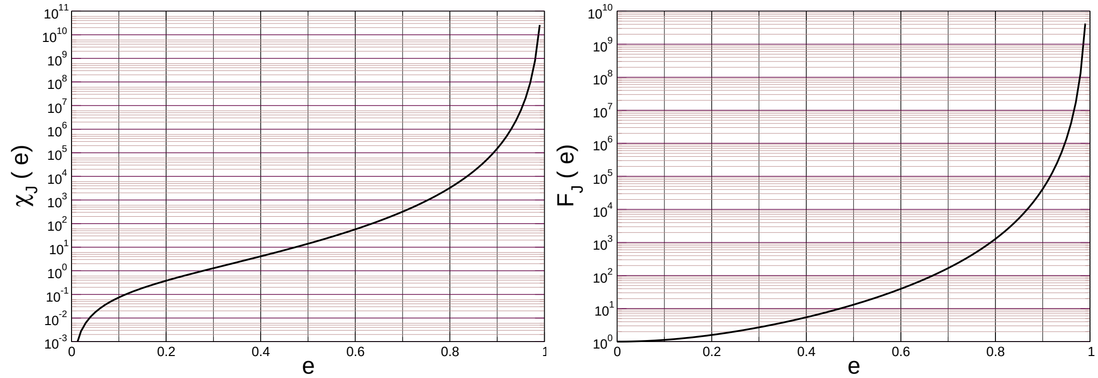

Figure 5.3: Variation of $\chi_J(e)$ (left panel) and $F_J(e)$ (right panel) with the eccentricity e. In the right panel, the exact expression of $F_J(e)$ given by Eq. (5.92) is used. In the circular orbit limit we have $\chi_J(0) = 0$ and $F_J(0) = 1$.

form which is

$$F_J(e) = \frac{1 + \frac{229}{32}\, e^2 + \frac{327}{64}\, e^4 + \frac{69}{256}\, e^6}{(1 - e^2)^5}. \tag{5.92}$$

We need another function whose Fourier transform differs from the one of $F_J(e)$ by the presence of the *logarithm* of modes, namely

$$\chi_J(e) = -\frac{\mathrm{i}}{32} \sum_{p=1}^{+\infty} p^7 \ln\left(\frac{p}{2}\right) \varepsilon_{ijk} \underset{(p)}{\hat{\mathcal{I}}}_{ja} \underset{(p)}{\hat{\mathcal{I}}}{}^{*}_{ka} \mathbf{z}^i. \tag{5.93}$$

Most probably $\chi_J(e)$ does not admit any analytic form [hence we name it using the Greek alphabet – in contrast to $f_J(e)$ and $F_J(e)$]. Note that $\chi_J(e)$ has been exceptionally defined in such a way that it *vanishes* when $e \to 0$. This is easily checked since in the circular orbit limit (and at Newtonian order) the quadrupole moment $I^{(\mathrm{N})}_{ij}$ possesses only one harmonic, which is the one for which $p = 2$, and consequently the log-term in $\chi_J(e)$ becomes zero. In Fig. 5.3 we show the numerical plot of the function $\chi_J(e)$ [and also the one for $F_J(e)$]. In Fig. 5.3 we show the numerical plot of the function $\chi_J(e)$ [and also the one for $F_J(e)$].

With the above definitions the sum of tail-of-tail and tail squared contributions obtained in Eq. (5.67a) becomes

$$\begin{aligned} \langle (\mathcal{F}^{\mathcal{J}}{}_i)_{\mathrm{tail(tail)+(tail)^2}} \rangle &= \frac{32}{5}\, \nu^2\, m\, x^{13/2} \Bigg\{ \Bigg[-\frac{116761}{3675} + \frac{16}{3}\pi^2 - \frac{1712}{105} C - \frac{1712}{105} \ln(4\,\omega\, r_0) \Bigg] F_J(e) \\ &\quad - \frac{1712}{105}\, \chi_J(e) \Bigg\}\, \mathbf{z}_i. \end{aligned} \tag{5.94}$$

(5.95)

The circular-orbit limit is read off and seen to agree with Eq. (5.9) in Ref. [135] or Eq. (12.7) in Ref. [116].

Finally we provide the mass quadrupole tail at 1PN order, whose computation is much more involved (see Sec. 5.5.3) as the Fourier series Eq. (5.71) contains several summations, and depend on the intermediate results (5.73) and (5.74). The computation must also incor-

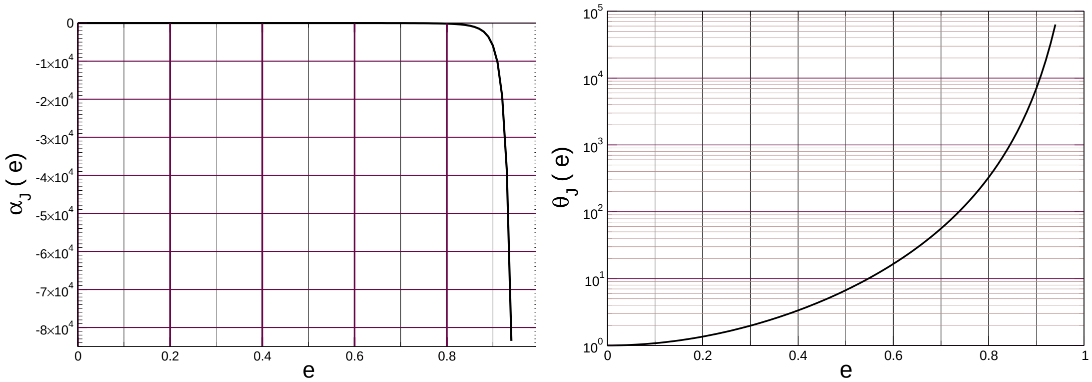

Figure 5.4: Variation of $\alpha_J(e)$ (left panel) and $\theta_J(e)$ (right panel) with the eccentricity e. In the circular orbit limit we have $\alpha_J(0) = 0$ and $\theta_J(0) = 1$.

porate the 1PN relativistic correction in the mass quadrupole moment and ADM mass; we provide them in Eqs. (5.98) and (5.99) below. Probably no simple way exists [*i.e.* no simple Fourier series like for instance (5.93)] for expressing the new enhancement functions of eccentricity which appear at the 1PN order. However it can be easily checked that the 1PN term is a linear function of the symmetric mass ratio ν, hence we must introduce two enhancement functions, denoted below α_J and θ_J. As usual, we normalize them so that $\alpha_J(0) = 1$ and $\theta_J(0) = 1$. We thus have [extending Eq. (5.87) at the 1PN order]

$$\langle(\mathcal{F}^{\mathcal{J}}{}_i)^{(\mathrm{N})}_{\text{mass quad}}\rangle = \frac{32}{5}\,\nu^2\, m\, x^5\left\{4\pi\,\varphi_J(e_t) + \pi\, x\left[-\frac{428}{21}\,\alpha_J(e_t) + \frac{178}{21}\nu\,\theta_J(e_t)\right]\right\}\mathbf{z}_i. \tag{5.96}$$

This equation defines the two enhancement functions α_J and θ_J, and we use Mathematica to compute them as complicated Fourier decompositions, which will then be directly computed numerically using the method outlined in Sec. 5.6.2. Notice that since we are at the 1PN order we must be specific about eccentricity we use. We adopted here the "time" eccentricity e_t entering the Kepler equation (5.26b) in Sec. 5.3.2. The other eccentricities are related to it by Eqs. (5.33) at the 1PN order. On the other hand, the frequency-related PN parameter, given by

$$x = (m\,\omega)^{2/3}, \tag{5.97}$$

crucially includes the 1PN relativistic correction coming from the periastron advance $K = 1 + k$, through the definition $\omega = n\,K$ of Sec. 5.3.1. All the 1PN corrections arising from the formulas (5.73) and (5.74), the multipole moments M and I_{ij}, the use of the time eccentricity e_t and the specific PN variable x, are incorporated in a Mathematica program dealing with the decomposition (5.71) and used to obtain (5.96). The plots of the enhancement functions $\alpha(e)$ and $\theta(e)$ are given in Fig. 5.4.

5.6.2 Numerical evaluation of the Fourier coefficients

Let us now describe the numerical implementation of the computation of the Fourier coefficients of the multipole moments that lead to the numerical plots of the previous Section. We focus on the computation of the crucial coefficients ${}_{(p,m)}\mathcal{I}_{ij}$ at 1PN order which are the more difficult to obtain. The mass quadrupole moment with 1PN accuracy is given by [compare with the general structure (5.48a)]

$$\begin{aligned} I_{ij} &= \mu\left\{1+\left[v^2\left(\frac{29}{42}-\frac{29}{14}\nu\right)+\frac{m}{r}\left(-\frac{5}{7}+\frac{8}{7}\nu\right)\right]x^{\langle i}x^{j\rangle}\right. \\ &\qquad \left.+\left(\frac{11}{21}-\frac{11}{7}\nu\right)r^2\,v^{\langle i}v^{j\rangle}+\left(-\frac{4}{7}+\frac{12}{7}\nu\right)r\,\dot{r}\,x^{\langle i}v^{j\rangle}\right\}, \end{aligned} \tag{5.98}$$

where x^i and $v^i = dx^i/dt$ are the relative position and velocity in harmonic coordinates, and $r = |x^i|$ (like in Sec. 5.3.2). Equation (5.98) is valid for non-spinning compact binaries on an arbitrary quasi-Keplerian orbit in the center-of-mass frame (see *e.g.* [132]). Since we calculate tails with 1PN relative accuracy we need to know how the ADM mass M relates to the total mass $m = m_1 + m_2$ at 1PN order,

$$M = m\left[1+\nu\left(\frac{v^2}{2}-\frac{m}{r}\right)\right]. \tag{5.99}$$

With the help of the quasi-Keplerian representation [Sec. 5.3.2], the dependence of I_{ij} on x^i, v^i, r, v^2 and $\dot{r}$ is parametrized by the eccentric anomaly u. However, as explained previously we require $I_{ij}(\ell)$ in the time domain to proceed. The steps of our numerical implementation can be schematically expressed as :

1. Firstly, each component of the 1PN mass quadrupole is expressed in terms of the quasi-Keplerian parameters using Eqs. (5.26)–(5.28). The components of the mass quadrupole become functions of the eccentric anomaly u, and are parametrized by the mean motion n and by one of the eccentricities (we chose to be e_t – the "time" eccentricity in Kepler's equation (5.26b) [7])

2. Now we numerically invert the equation for the mean anomaly $\ell = u - e_t \sin u$ to obtain the function $u(\ell)$. This can be done either by using the series representation in terms of Bessel functions,

   $$u = \ell + 2\sum_{s=1}^{+\infty}\frac{1}{s}J_s(s\,e_t)\sin(s\,\ell), \tag{5.100}$$

   or numerically by finding the root of $\ell = u - e_t \sin u$. We find the latter method to be more efficient and accurate method and employ it using the FindRoot routine in Mathematica. A table of 20 000 points of u and ℓ between 0 and 2π (for each value of e_t) was generated for this purpose. The above inversion allows us to re-express all functions of the eccentric anomaly u as functions of the mean anomaly ℓ. If required, one can attempt a more accurate implementation, in the future, for solving Kepler's equation along the lines of [192].

3. There is a discontinuity in the u dependence of V in Eq. (5.28). To avoid it we use

   $$V(u) = u + 2\,\arctan\left(\frac{\beta_\phi\,\sin u}{1-\beta_\phi\,\cos u}\right), \tag{5.101}$$

[7] The semi-major axis a_r and the other eccentricities e_r and e_ϕ are deduced from n and e_t using Eqs. (5.30)–(5.33).

where $\beta_\phi \equiv [1-(1-e_\phi^2)^{1/2}]/e_\phi$. We thus have the Fourier coefficients ${}_{(m)}\mathcal{I}_{ij}(\ell)$ defined in Eq. (5.50a) as explicit (numerical) functions of ℓ.

4. These functions also have a dependence on the mass ratio ν and the PN parameter x defined by $(m\,\omega)^{2/3}$ where $\omega = n\,K$. To avoid assuming numerical values for ν and x and hence to preserve the full generality of the result, we split the function ${}_{(m)}\mathcal{I}_{ij}$ into

$$\underset{(m)}{\mathcal{I}}_{ij}(\ell, e_t, \nu, x) = \underset{(m)}{\mathcal{I}}{}^{00}_{ij}(\ell, e_t) + x\left[\underset{(m)}{\mathcal{I}}{}^{10}_{ij}(\ell, e_t) + \nu\,\underset{(m)}{\mathcal{I}}{}^{11}_{ij}(\ell, e_t)\right]. \tag{5.102}$$

The various ${}_{(m)}\mathcal{I}^{ab}_{ij}$ are now functions of ℓ and e_t only. Fourier coefficients of these terms are evaluated in the next step.

5. For a fixed value of e_t, it is straightforward to obtain the plot of ${}_{(m)}\mathcal{I}^{00}_{ij}$ versus ℓ. Equivalently, we can also express the Fourier decomposition of ${}_{(m)}\mathcal{I}^{00}_{ij}(\ell)$ as

$$\underset{(m)}{\mathcal{I}}{}^{00}_{ij}(\ell) = \sum_{p=-\infty}^{+\infty} \underset{(p,m)}{\mathcal{I}}{}^{00}_{ij}\, e^{\mathrm{i}\,p\,\ell}. \tag{5.103}$$

Next we find a numerical fit to Eq. (5.102), in powers of $e^{ip\ell}$, to extract out the coefficients ${}_{(p,m)}\mathcal{I}^{00}_{ij}$. The same procedure is adopted for different values of e_t and for ${}_{(p,m)}\mathcal{I}^{10}_{ij}$ and ${}_{(p,m)}\mathcal{I}^{11}_{ij}$.

6. Substituting the Fourier coefficients into Eq. (5.71) we generate numerical values of the averaged angular momentum flux $\langle \mathcal{F}_{\mathcal{J}_{\text{mass quad}}}\rangle$ for the different values of e_t, and hence get the numerical values of the enhancement functions, and most importantly of the 1PN ones $\alpha_J(e_t)$ and $\theta_J(e_t)$ defined by (5.96). The plots of these functions reported in Sec. 5.6.1 readily follow.

The above procedure is quite general, and provides a method which could be extended to higher PN orders. However, at the Newtonian order it is in fact much more efficient to make use of the well-known Fourier decomposition of the Keplerian motion. Using this we can derive the components of the multipole moments (at Newtonian order) as series of combinations of Bessel functions. Then it is simple to compute numerically the associated "Newtonian" enhancement functions [namely the functions $\varphi_J(e)$, $\beta_J(e)$, $\gamma_J(e)$ and $\chi_J(e)$ defined in Sec. 5.6.1]. For the convenience of the reader we provide in Appendix 5.10 all the expressions of the components of the required Newtonian moments [$I^{(\mathrm{N})}_{ij}$, $I^{(\mathrm{N})}_{ijk}$ and $J^{(\mathrm{N})}_{ij}$] as series of Bessel functions, we have used to compute numerically the functions $\varphi(e)$, $\beta_J(e)$, $\gamma_J(e)$ and $\chi_J(e)$[8].

[8] On the other hand, for the Newtonian tail terms, we could proceed exactly in the same way as for the 1PN term, following the steps 1 - 8. We have verified that both methods agree well.

5.7 The hereditary contribution to the 3PN angular momentum flux

5.7.1 Final expression of the tail terms

Based on the treatment outlined above of a numerical scheme for the computation of the orbital average of the hereditary part of the angular momentum flux up to 3PN, we finally provide the complete results for the numerical plots of the dimensionless enhancement factors. It is convenient for the final presentation to redefine in a minor way the "elementary" enhancement functions of Sec. 5.6.1, which were directly given by simple Fourier decompositions. Let us choose

$$\psi_J(e) \equiv \frac{13\,696}{8191}\,\alpha_J(e) - \frac{16\,403}{24\,573}\beta_J(e) - \frac{112}{24\,573}\,\gamma_J(e), \tag{5.104a}$$

$$\zeta_J(e) \equiv -\frac{1424}{4081}\,\theta_J(e) + \frac{16\,403}{12\,243}\beta_J(e) + \frac{16}{1749}\,\gamma_J(e), \tag{5.104b}$$

$$\kappa_J(e) \equiv F_J(e) + \frac{59\,920}{11\,6761}\chi_J(e). \tag{5.104c}$$

Considering thus the 1.5PN and 2.5PN terms, composed of tails, and the 3PN terms, composed of the tail-of-tail and the tail-squared terms, the total hereditary contribution to the angular momentum flux (5.7) when averaged over ℓ (and normalized to the Newtonian value for circular orbits) finally reads

$$\begin{aligned}\langle(\mathcal{F}^{\mathcal{J}}{}_i)_{\text{hered}}\rangle = \frac{32}{5}\,\nu^2\,m\,x^{7/2}\Bigg\{&4\pi\,x^{3/2}\,\varphi_J(e_t) + \pi\,x^{5/2}\left[-\frac{8191}{672}\,\psi_J(e_t) - \frac{583}{24}\nu\,\zeta_J(e_t)\right] \\ &+x^3\left[-\frac{116\,761}{3675}\,\kappa_J(e_t) + \left[\frac{16}{3}\,\pi^2 - \frac{1712}{105}\,C - \frac{1712}{105}\ln(4\omega\,r_0)\right]F_J(e_t)\right]\Bigg\}\mathbf{z}_i.\end{aligned} \tag{5.105}$$

All enhancement functions above reduce to one in the circular case, when $e_t = 0$, so the circular-limit is easily obtained from Eq. (5.105). It is seen to be in complete agreement with Refs. [135, 116] where expressions for the hereditary contributions to the energy flux for circular orbits are given. The comparison is readily made when one notes that, in the circular-limit, $\langle\mathcal{F}\rangle = \omega\,\langle\mathcal{F}^{\mathcal{J}}{}_i\rangle\,\mathbf{z}^i$. The function $F_J(e_t)$ was computed in Section. 5.6, and we recall here its expression,

$$F_J(e_t) = \frac{1 + \frac{229}{32}\,e_t^2 + \frac{327}{64}\,e_t^4 + \frac{69}{256}\,e_t^6}{(1 - e_t^2)^5}. \tag{5.106}$$

However the other enhancement functions $\varphi_J(e_t)$, $\psi(e_t)_J$, $\zeta(e_t)_J$ and $\kappa(e_t)_J$ in Eq. (5.105) (very likely) do not admit any analytic closed-form expressions. We have given the details of the numerical calculation of these functions in Sec. 5.6.2. Here we provide the numerical plots of the final functions $\psi_J(e_t)$, $\zeta_J(e_t)$ and $\kappa_J(e_t)$ in Figs. 5.5–5.6 as functions of the eccentricity e_t [recall that the function $\varphi_J(e_t)$ has already been given in Fig. 5.1] [9].

As seen from Eq. (5.105) the final result depends on the constant r_0 at the 3PN order. The

[9] The numerical results used for the figures 1-6 are available in the form of Tables on request

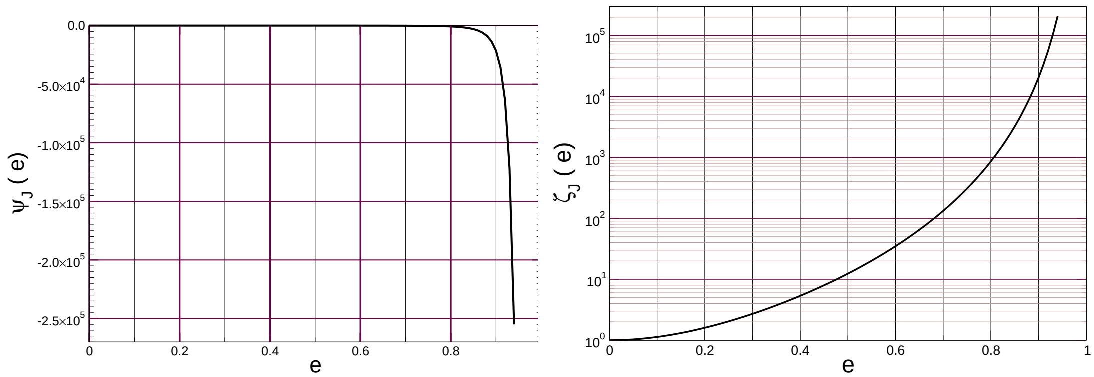

Figure 5.5: Variation of $\psi_J(e)$ and $\zeta_J(e)$ with eccentricity e. In the circular orbit limit, $\psi_J(0) = 1$ & $\zeta_J(0) = 1$.

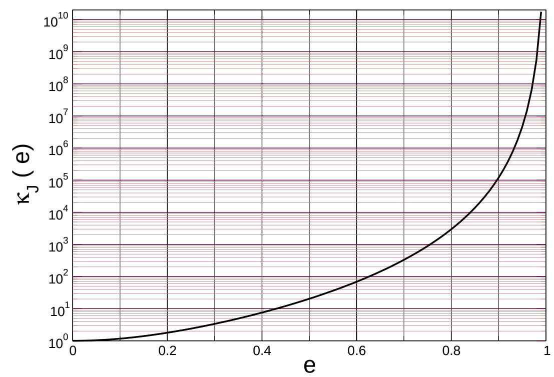

Figure 5.6: Variation of $\kappa_J(e)$ with the eccentricity e. In the circular orbit limit we have $\kappa_J(0) = 1$.

dependence on the constant r_0 comes exclusively from the contribution of tails-of-tails (*i.e.* the cubic multipole interaction $M^2 \times I_{ij}$) as can be seen from the expression of the radiative mass-quadrupole in Eq. (5.2). We refer the reader to Section VI of Ref. [185] for a rigorous confirmation of this fact. Now, we know from the study of the circular-orbit case (*cf.* [116]) that the dependence on r_0 is cancelled out with a similar term contained in the expression of the source-type quadrupole moment I_{ij} at 3PN order. This cancellation must be true for general sources, and has been proved on general grounds in Ref. [135]. In fact, the expression for $F_J(e_t)$ was *guessed* demanding the cancellation of the $\log r_0$ term in the total angular momentum flux in Ref. [188] (see Eq. (6.55) of [188]). Our derivation confirms this *guess* and thereby provides an interesting check of our calculations in this chapter. Expansions of our final enhancement functions upto the first order in e_t^2 in the small eccentricity regime ($e_t \to 0$) are useful for compare the perturbative limit of the complete angular momentum flux at 3PN order (including all instantaneous terms) with the result of black-hole perturbations. These expansions are obtained analytically as follows. For the functions which are Newtonian we can either use the Fourier coefficients in the Appendix 5.10 and expand them at first order in e_t^2 or follow the general procedure explained in Sec. 5.6.2 for the relevant moments but expanding Eq. (5.100) to first order in e_t^2, namely,

$$u = \ell + e_t \sin \ell + \frac{e_t^2}{2} \sin 2\ell + O\left(e_t^3\right). \tag{5.107}$$

The two 1PN functions [$\psi_J(e_t)$ and $\zeta_J(e_t)$], on the other hand, are obtained directly using the latter procedure. We find

$$\begin{aligned}
\varphi_J(e) &= 1 + \frac{209}{32} e^2 + O\left(e^4\right), && (5.108a)\\
\psi_J(e) &= 1 - \frac{17\,416}{8191} e^2 + O\left(e^4\right), && (5.108b)\\
\zeta_J(e) &= 1 + \frac{102\,371}{8162} e^2 + O\left(e^4\right), && (5.108c)\\
\kappa_J(e) &= 1 + \left(\frac{389}{32} - \frac{549}{32} \ln 2 + \frac{2187}{128} \ln 3\right) e^2 + O\left(e^4\right), && (5.108d)
\end{aligned}$$

and of course [since this is immediately deduced from Eq. (5.106)]

$$F_J(e) = 1 + \frac{389}{32} e^2 + O\left(e^4\right). \tag{5.109}$$

We have checked that the numerical results of Figs. 5.1, 5.5 and 5.6 agree well with Eqs. (5.108) in the limit of small eccentricities.

5.8 Evolution of orbital elements under radiation reaction: hereditary contributions

The most important application of the hereditary part of the 3PN angular momentum flux computed in this chapter and the hereditary terms in the energy flux obtained in Ref [185] is to obtain the time evolution of the orbital elements of the binary under gravitational radiation reaction. Note that, by 3PN evolution of orbital elements under gravitational radiation reaction we mean its evolution under 5.5PN terms beyond leading Newtonian order in the equation of motion. In this Section, we compute the evolution of n, e_t, a_r and k (averaged over an orbit) due to the hereditary terms upto 3PN accuracy. This extends the earlier works at Newtonian order by Peters [47], 1PN computation of Refs [172, 175] and at 2PN order by Ref [174, 180]. The 1.5PN hereditary effects also have been incorporated in the orbital element evolution in Refs [158, 173].

To this end, we start with expressions for the orbital elements in terms of the conserved energy (E) and angular momentum (J) (taken, as usual, per unit reduced mass). The expressions for a_r ,n, e_t, and k in terms of E & J were already given in Section 5.3.2. We recall them here for convenience.

$$\begin{aligned}
a_r &= -\frac{m}{2E}\left\{1 + \left(\frac{7}{2} - \frac{\nu}{2}\right) E\right\}, && (5.110a)\\
n &= \frac{(-2\,E)^{3/2}}{m}\left\{1 + \left(\frac{15}{4} - \frac{\nu}{4}\right) E\right\}, && (5.110b)
\end{aligned}$$

$$e_t = \left\{1 + \frac{2\,E\,J^2}{m^2}\left[1 + \left(\frac{17}{2} - \frac{7}{2}\nu\right)E + \frac{(2-2\nu)m^2}{J^2}\right]\right\}^{1/2}, \tag{5.110c}$$

$$k = \frac{3\,m^2}{J^2}. \tag{5.110d}$$

Note that we have taken all the above relations to be 1.5PN accurate, (see Section 5.3.2, where they are written with G & c) a decision which we justify soon. The 3PN accurate evolution of these orbital elements due to instantaneous terms are computed in Ref. [188]. Taking time derivatives of Eq. (5.110) we get expressions for rate-of-changes of the orbital elements linear in $\frac{dE}{dt}$ and $\frac{dJ}{dt}$. For example,

$$\frac{da_r}{dt} = \frac{m}{2E^2}\frac{dE}{dt}, \tag{5.111a}$$

$$\frac{dk}{dt} = -\frac{6\,m^2}{J^3}\frac{dJ}{dt}. \tag{5.111b}$$

Let us now use the heuristic balance equations for energy and angular momentum flux, *i.e.*

$$\frac{dE}{dt} = -\frac{\mathcal{F}}{\nu\,m}, \tag{5.112a}$$

$$\frac{dJ}{dt} = -\frac{\mathcal{F}^{\mathcal{J}}}{\nu\,m}. \tag{5.112b}$$

where $\mathcal{F}$ & $\mathcal{F}^{\mathcal{J}}$ are the far-zone energy and angular momentum fluxes respectively and $\mathcal{F}^{\mathcal{J}}$ being simply ${\mathcal{F}^{\mathcal{J}}}_i.\mathbf{z}^i$. Recall that E & J are defined per-unit-reduced-mass.

On replacing $\mathcal{F}$ & $\mathcal{F}^{\mathcal{J}}$ with $\langle(\mathcal{F})_{\rm hered}\rangle$ & $\langle(\mathcal{F}^{\mathcal{J}})_{\rm hered}\rangle$ respectively, we finally obtain the evolution of the orbital elements due to hereditary terms averaged over the binary orbit. $\langle(\mathcal{F})_{\rm hered}\rangle$ has been calculated upto 3PN order in Ref. [185]. Here we simply reproduce the final result (taken from Eq.(6.2) in Ref. [185]), expressed in terms of enhancement functions of eccentricity all of which reduce to 1 in the circular-orbit case.

$$\begin{aligned}\langle(\mathcal{F})_{\rm hered}\rangle = \frac{32}{5}\,\nu^2\,x^5\Bigg\{&4\pi\,x^{3/2}\,\varphi(e_t) + \pi\,x^{5/2}\left[-\frac{8191}{672}\,\psi(e_t) - \frac{583}{24}\nu\,\zeta(e_t)\right]\\&+x^3\left[-\frac{116\,761}{3675}\,\kappa(e_t) + \left[\frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{1712}{105}\ln(4\omega\,r_0)\right]F(e_t)\right]\Bigg\}.\end{aligned} \tag{5.113}$$

Like the angular momentum flux, closed form expressions exist only for $F(e_t)$. All the other enhancement functions have been numerically computed at different values of e_t in Ref. [185]. $F(e_t)$ comes from the contribution of tails-of-tails and is given by (compare with Eq. (5.106)),

$$F(e_t) = \frac{1 + \frac{85}{6}e_t^2 + \frac{5171}{192}e_t^4 + \frac{1751}{192}e_t^6 + \frac{297}{1024}e_t^8}{(1-e_t^2)^{13/2}}. \tag{5.114}$$

Inspection of Eqs. (5.113) & (5.105) show that both the fluxes start at relative 1.5PN order beyond the leading order. As the time derivatives of the orbital elements are linear in the fluxes of E & J (which in turn start from 1.5PN order), 1.5PN accurate relations for the orbital elements are sufficient to calculate their 3PN evolution.

We also need 1.5PN accurate expressions for E & J, which we express below in terms of the PN parameter x.

$$E = -\frac{1}{2}x\left[1 + x\left(\frac{5}{4} - \frac{\nu}{12} - \frac{2}{(1-e^2)}\right)\right], \tag{5.115a}$$

$$J = m\sqrt{1-e^2}\,x^{-1/2}\left[1 + x\left(\frac{3}{2} + \frac{1+5e^2}{6(1-e^2)}\,\nu\right)\right]. \tag{5.115b}$$

5.8.1 Evolution of a_r & n

From Eqs. (5.110), we see that the orbital elements a_r & n depend only on E upto the accuracy we need. Therefore hereditary terms in the angular momentum flux do not contribute in the time evolution of a_r & n upto 3PN order. Using Eqs. (5.112), (5.113), & (5.115) in the expressions of the derivatives of the orbital elements (like Eqs. (5.111)) , we get the averaged evolution of the orbital elements a_r & n due to *hereditary* terms accurate upto 3PN order.

$$\begin{aligned}\left\langle\frac{da_r}{dt}\right\rangle = & -\frac{64}{5}\nu\,x^3\Bigg\{4\pi\,x^{3/2}\,\varphi(e_t) + \pi\,x^{5/2}\Big[-\frac{4159}{672}\,\psi^{(a_r)}(e_t) - \frac{189}{8}\,\nu\,\zeta^{(a_r)}(e_t)\Big] \\ & + x^3\left[-\frac{116\,761}{3675}\,\kappa(e_t) + \left[\frac{16}{3}\,\pi^2 - \frac{1712}{105}\,C - \frac{1712}{105}\ln\left(4\omega\,r_0\right)\right]F(e_t)\right]\Bigg\},\end{aligned} \tag{5.116a}$$

$$\begin{aligned}\left\langle\frac{dn}{dt}\right\rangle = & \frac{96}{5}\frac{\nu}{m^2}\,x^{11/2}\Bigg\{4\pi\,x^{3/2}\,\varphi(e_t) + \pi\,x^{5/2}\Big[-\frac{17599}{672}\,\psi^{(n)}(e_t) - \frac{189}{8}\,\nu\,\zeta^{(n)}(e_t)\Big] \\ & + x^3\left[-\frac{116\,761}{3675}\,\kappa(e_t) + \left[\frac{16}{3}\,\pi^2 - \frac{1712}{105}\,C - \frac{1712}{105}\ln\left(4\omega\,r_0\right)\right]F(e_t)\right]\Bigg\}.\end{aligned} \tag{5.116b}$$

The functions of eccentricity appearing above are either present in Eq. (5.113), or are linear combinations of them. All the new functions of eccentricity introduced above reduce to one in the circular orbit limit ($e_t \to 0$). The plots for the enhancement functions in the expression for $\langle\frac{dn}{dt}\rangle$ are provided in Fig. 5.7. Below, we provide the explicit expressions of the new enhancement functions in Eq. (5.116). These are

$$\psi^{(a_r)}(e_t) = -\frac{1344}{4159}\,\frac{3+5e_t^2}{1-e_t^2}\,\varphi(e_t) + \frac{8191}{4159}\,\psi(e_t), \tag{5.117a}$$

$$\zeta^{(a_r)}(e_t) = \frac{583}{567}\,\zeta(e_t) - \frac{16}{567}\,\varphi(e_t), \tag{5.117b}$$

$$\psi^{(n)}(e_t) = \frac{1344}{17\,599}\,\frac{7-5e_t^2}{1-e_t^2}\,\varphi(e_t) + \frac{8191}{17\,599}\,\psi(e_t) \tag{5.117c}$$

$$\zeta^{(n)}(e_t) = \frac{583}{567}\,\zeta(e_t) - \frac{16}{567}\,\varphi(e_t). \tag{5.117d}$$

The expansions of the new enhancement functions for $\langle\frac{dn}{dt}\rangle$ in the limit of small eccentricities are also easily obtained. We use the small e_t expansions already obtained in Eq. (5.108) and

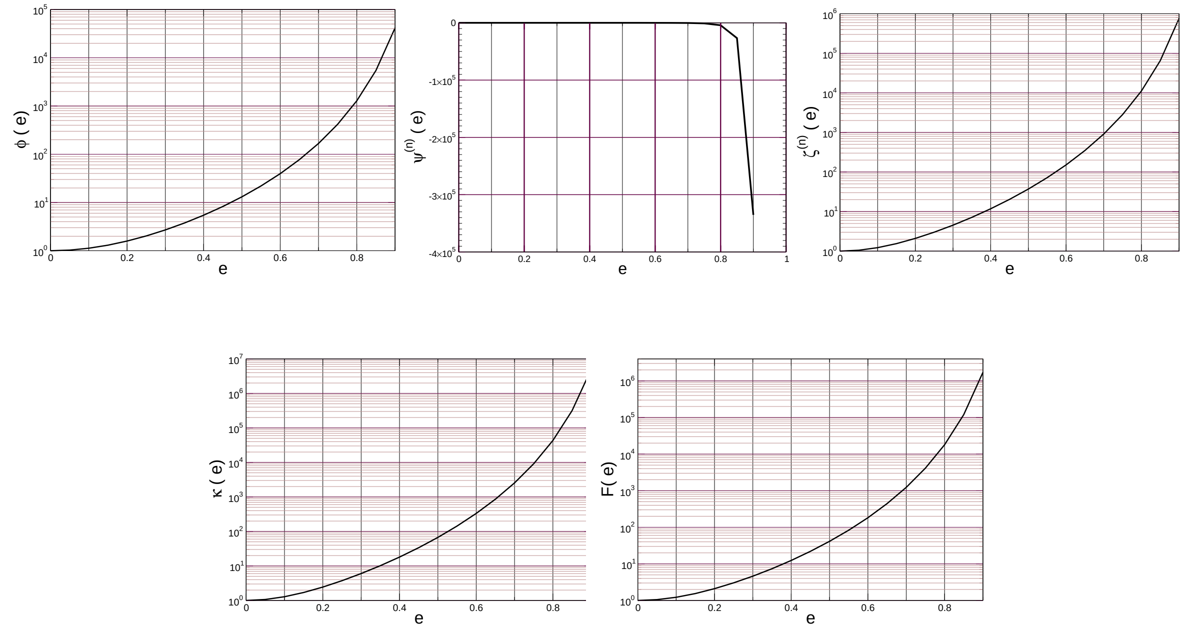

Figure 5.7: Variation of $\varphi(e)$, $\psi^{(n)}(e)$, $\zeta^{(n)}(e)$, $\kappa(e)$ and $F(e)$ with eccentricity e. In the circular orbit limit, *i.e.*, $e \to 0$, all these functions approach unity. Note the similarity of $\psi^{(n)}(e)$ with $\psi_J(e)$ in Fig. 5.5.

also similar expansions in Eqs. (6.9 & 6.10) and insert them in Ref. [185] in Eq. (5.117).

$$\psi^{(n)}(e) \;=\; 1 + \frac{94\,115}{17\,599}\, e^2 + O\left(e^4\right), \tag{5.118a}$$

$$\zeta^{(n)}(e) \;=\; 1 + \frac{9215}{441}\, e^2 + O\left(e^4\right). \tag{5.118b}$$

5.8.2 Evolution of e_t

From Eq. (5.110), we see that e_t is the only orbital element which depends on *both* the conserved energy and angular momentum upto 1.5PN order. Therefore, the hereditary effects upto 3PN in the fluxes of both E & J affect the evolution of e_t. We therefore require both Eqs. (5.113), & (5.105) for the respective fluxes. Plugging these expressions into the time derivative of e_t together with Eq. (5.115), we obtain the averaged evolution of the time-eccentricity e_t due to hereditary contributions. Recall that the other eccentricities e_r and e_ϕ appearing in the quasi-Keplerian representation Eq. (5.26) are related by simple 1PN relations (see Eq. (5.33)) and are the same in the Newtonian limit. The averaged time-evolution of e_t is

$$\begin{aligned}\langle\frac{de_t}{dt}\rangle \;=\; & -\frac{32}{5}\, e_t\, \frac{\nu}{m}\, x^4 \Big\{ 4\pi\, x^{3/2}\, \varphi^{(e_t)}(e_t) + \pi\, x^{5/2} \Big[-\psi^{(e_t)}(e_t) - \nu\, \zeta^{(e_t)}(e_t) \Big] \\ & + x^3 \left[-\frac{116\,761}{3675}\, \kappa^{(e_t)}(e_t) + \left[\frac{16}{3}\pi^2 - \frac{1712}{105} C - \frac{1712}{105} \ln\left(4\omega\, r_0\right) \right] F^{(e_t)}(e_t) \right] \Big\}.\end{aligned} \tag{5.119}$$

The functions of eccentricity appearing Eq. (5.119) all reduce to zero in the circular orbit limit. This is expected, because the circular orbit is always of eccentricity zero. This behaviour obviously holds true for the instantaneous terms also. We provide the plots of the enhancement functions in Eq. (5.119) in Fig. 5.8. In terms of the enhancement functions appearing in Eqs. (5.113) &(5.105) they are

$$\varphi^{(e_t)}(e_t) = \left(\frac{1-e_t^2}{e_t}\right)\varphi(e_t) - \left(\frac{\sqrt{1-e_t^2}}{e_t}\right)\varphi_J(e_t), \tag{5.120a}$$

$$\begin{aligned}\psi^{(e_t)}(e_t) &= \frac{14}{e_t}\left[\left(1-\frac{11}{7}e_t^2\right)\varphi(e_t) - \frac{1-\frac{3}{7}e_t^2}{\sqrt{1-e_t^2}}\varphi_J(e_t)\right] \\ &+\frac{8191}{672}\left[\left(\frac{1-e_t^2}{e_t}\right)\psi(e_t) - \left(\frac{\sqrt{1-e_t^2}}{e_t}\right)\psi_J(e_t)\right],\end{aligned} \tag{5.120b}$$

$$\begin{aligned}\zeta^{(e_t)}(e_t) &= -\frac{22}{3\,e_t}\left[(1-e_t^2)\,\varphi(e_t) - \frac{1-\frac{5}{11}e_t^2}{\sqrt{1-e_t^2}}\varphi_J(e_t)\right] \\ &+\frac{583}{24}\left[\left(\frac{1-e_t^2}{e_t}\right)\zeta(e_t) - \left(\frac{\sqrt{1-e_t^2}}{e_t}\right)\zeta_J(e_t)\right],\end{aligned} \tag{5.120c}$$

$$\kappa^{(e_t)}(e_t) = \left(\frac{1-e_t^2}{e_t}\right)\kappa(e_t) - \left(\frac{\sqrt{1-e_t^2}}{e_t}\right)\kappa_J(e_t), \tag{5.120d}$$

$$F^{(e_t)}(e_t) = \left(\frac{1-e_t^2}{e_t}\right)F(e_t) - \left(\frac{\sqrt{1-e_t^2}}{e_t}\right)F_J(e_t). \tag{5.120e}$$

The function $F^{(e_t)}(e_t)$ is known in closed form and is given by (using Eqs. (5.114) & (5.106))

$$F^{(e_t)}(e_t) = \frac{769\,e_t}{96\,(1-e_t^2)^{11/2}}\left(1+\frac{2782}{769}e_t^2+\frac{10\,721}{6152}e_t^4+\frac{1719}{24\,608}e_t^6\right). \tag{5.121}$$

The small eccentricity limits of the new enhancement functions are obtained below. Like in the previous subsection, we use the known small e_t limits of the enhancement functions in the RHS of Eq. (5.120).

$$\varphi^{(e_t)}(e) = \frac{985}{192}\,e + O\left(e^3\right), \tag{5.122a}$$

$$\psi^{(e_t)}(e) = \frac{55\,691}{1344}\,e + O\left(e^3\right), \tag{5.122b}$$

$$\zeta^{(e_t)}(e) = \frac{19\,067}{126}\,e + O\left(e^3\right), \tag{5.122c}$$

$$\kappa^{(e_t)}(e) = \left(\frac{769}{96}+\frac{44\,662\,487}{11\,209\,056}\ln 2 - \frac{58\,788\,747}{14\,945\,408}\ln 3\right)e + O\left(e^3\right). \tag{5.122d}$$

The last of the functions in Eq. (5.120) is easily reduced to its small eccentricity limit with the help of its closed form expression in Eq. (5.121). It is

$$F^{(e_t)}(e) = \frac{769}{96}\,e + O\left(e^3\right). \tag{5.123}$$

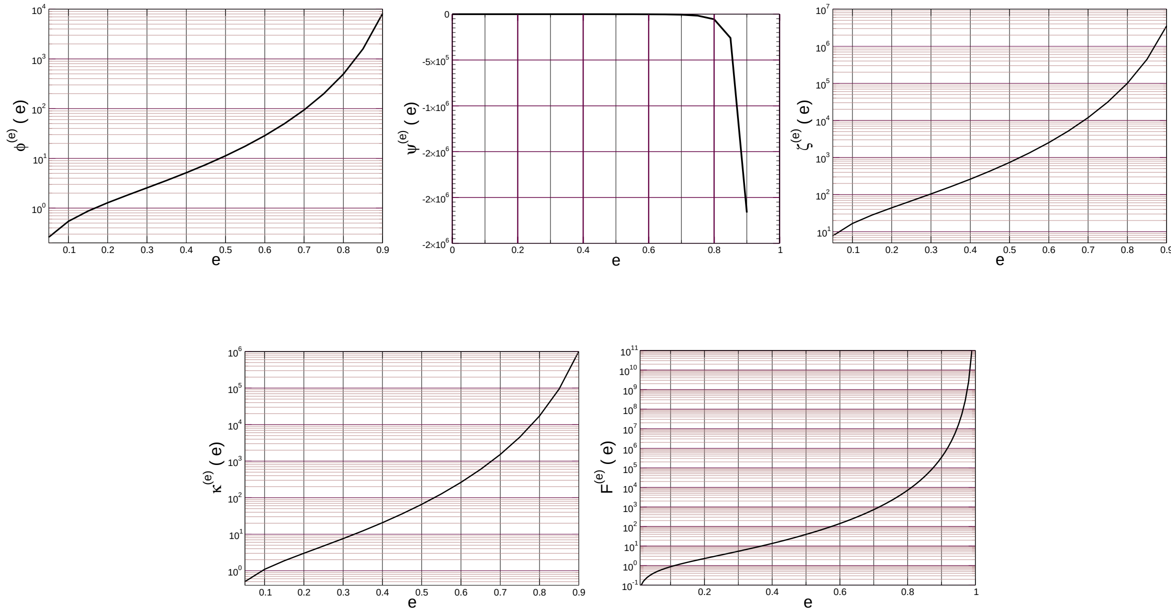

Figure 5.8: Variation of $\varphi^{(e)}(e)$, $\psi^{(e)}(e)$, $\zeta^{(e)}(e)$, $\kappa^{(e)}(e)$ and $F^{(e)}(e)$ with eccentricity e. In the circular orbit limit, *i.e.*, $e \to 0$, all these functions approach zero. Again note the similarity in behaviour of $\psi^{(e)}(e)$ with $\psi_J(e)$ in Fig. 5.5.

5.8.3 Evolution of k

The final orbital element whose evolution we discuss in this chapter is k. Recall that k represents the relativistic precession of periastron. The advance of the angle of periastron during a single return to periastron is given by (modulo 2π) $2\pi k$. It first appears in the 1PN description of the orbit. We also note that, below 2PN, it is only a function of J (see Eq. (5.110)) and thus, hereditary terms solely from the angular momentum flux control its evolution upto 3P order due to hereditary effects. The orbital time-scale averaged evolution equation for k is

$$\begin{aligned}\langle\frac{dk}{dt}\rangle &= \frac{192}{5}\,\frac{\nu}{m}\,x^5\Big\{4\pi\,x^{3/2}\,\varphi^{(k)}(e_t) + \pi\,x^{5/2}\Big[-\frac{20287}{672}\,\psi^{(k)}(e_t) - \frac{631}{24}\,\zeta^{(k)}(e_t)\Big] \\ &\quad + x^3\left[-\frac{116761}{3675}\,\kappa^{(k)}(e_t) + \left[\frac{16}{3}\,\pi^2 - \frac{1712}{105}\,C - \frac{1712}{105}\ln(4\omega\,r_0)\right]F^{(k)}(e_t)\right]\Big\},\end{aligned} \tag{5.124}$$

where we note that the PN order of the leading term in $\frac{dk}{dt}$ is 1PN higher than the same for $\frac{de_t}{dt}$ even though they are dimensionally same (compare Eqs. (5.119) & (5.124) . This is obviously due to the fact that k itself is zero for the Kepler ellipse and first appears at 1PN order. All the new functions of eccentricity appearing in Eq. (5.124) reduce to one in the circular orbit limit ($e_t \to 0$). These are

$$\varphi^{(k)}(e_t) = \frac{\varphi_J(e_t)}{(1-e_t^2)^{3/2}}, \tag{5.125a}$$

$$\psi^{(k)}(e_t) = \frac{12096}{20287}\,\frac{\varphi_J(e_t)}{(1-e_t^2)^{3/2}} + \frac{8191}{20287}\,\frac{\psi_J(e_t)}{(1-e_t^2)^{3/2}}, \tag{5.125b}$$

$$\zeta^{(k)}(e_t) = \frac{583}{631}\frac{\zeta_J(e_t)}{(1-e_t^2)^{3/2}} + \frac{48}{631}\frac{(1+5e_t^2)}{(1-e_t^2)^{5/2}}\varphi_J(e_t), \tag{5.125c}$$

$$\kappa^{(k)}(e_t) = \frac{\kappa_J(e_t)}{(1-e_t^2)^{3/2}}, \tag{5.125d}$$

$$F^{(k)}(e_t) = \frac{F_J(e_t)}{(1-e_t^2)^{3/2}}. \tag{5.125e}$$

5.9 Conclusion and future directions

The far-zone angular momentum flux comprises hereditary contributions that depend on the entire past history of the source. Using the GW generation formalism consisting of a multipolar post-Minkowskian expansion and supplemented by matching to a PN source, we have computed the hereditary contributions from the inspiral phase of a binary system of compact objects moving on quasi-elliptical orbits up to 3PN order using a semi-analytical method. The method requires the 1PN quasi-Keplerian representation of elliptical orbits and exploits the doubly periodic nature of the motion to average the flux over the binary's orbit. Together with the instantaneous contributions evaluated in Ref [188] and the complete 3PN energy flux obtained in Refs. [186, 185], it provides crucial inputs for the construction of ready-to-use templates for binaries moving on eccentric orbits, an interesting class of sources for the ground based gravitational wave detectors LIGO/Virgo and especially space based detectors like LISA. The extension to compute the 3.5PN terms for elliptical orbits is currently not possible due to some as yet uncalculated terms in the generation formalism at this order for general orbits. It would also require the use of the 2PN generalised quasi-Keplerian representation for some of the leading multipole moments.

Recent advances in the field of numerical relativity (NR) [37, 38, 39, 40] has led to high-accuracy comparisons between the PN predictions and the numerically-generated waveforms. Such comparisons and matching to the PN results for quasi-circular orbits have proved currently to be very successful [41, 42, 43, 44]. However, evolution of quasi-circular orbit binaries are relatively simple in the sense that the waveforms exhibit a steady "chirp", that is, a monotonic increase in amplitude and frequency. Eccentric binaries have the feature that the waveform amplitude and frequency have oscillations. Therefore, a comparison with NR for eccentric binaries provide a more stringent test of the PN formalism. The first NR studies of binaries in eccentric orbits have been completed [193, 194] and have naturally been followed up by comparison exercises [195]. In Ref. [195], an equal-mass non-spinning binary with initial eccentricity $e \simeq 0.1$ has been evolved using NR. The evolution carried out over 8 cycles show an agreement of within 0.1 radians when the phase is compared with that of an eccentric PN model with 2PN radiation reaction. The NR & PN phase difference grows to about 0.8 radians about 5 cycles before merger. The agreement is better when one uses the gauge-invariant PN expansion parameter $x = (m\,\omega)^{2/3}$ (which is also our choice in this chapter) instead of the mean-motion related parameter $\xi = m\,n$.

5.10 Appendix: Fourier coefficients of the multipole moments

In this Appendix we provide the expressions of the Fourier coefficients of the *Newtonian* multipole moments in terms of combinations of Bessel functions. We decompose the components of the moments as Fourier series,

$$I_L^{(\mathrm{N})}(t) = \sum_{p=-\infty}^{+\infty} \underset{(p)}{\mathcal{I}}{}_L^{(\mathrm{N})}\, e^{\mathrm{i}p\ell}, \tag{5.126a}$$

$$J_{L-1}^{(\mathrm{N})}(t) = \sum_{p=-\infty}^{+\infty} \underset{(p)}{\mathcal{J}}{}_{L-1}^{(\mathrm{N})}\, e^{\mathrm{i}p\ell}, \tag{5.126b}$$

where the Fourier coefficients can be obtained by evaluating the following integrals

$$\underset{(p)}{\mathcal{I}}{}_L^{(\mathrm{N})} = \frac{1}{2\pi}\int_0^{2\pi} d\ell\, I_L^{(\mathrm{N})}(t)\, e^{-\mathrm{i}p\ell}, \tag{5.127a}$$

$$\underset{(p)}{\mathcal{J}}{}_{L-1}^{(\mathrm{N})} = \frac{1}{2\pi}\int_0^{2\pi} d\ell\, J_{L-1}^{(\mathrm{N})}(t)\, e^{-\mathrm{i}p\ell}. \tag{5.127b}$$

The evaluation of these integrals is standard. The reader is referred to the book by Plummer [196] for details. For the mass quadrupole moment at Newtonian order we have[10]

$$\underset{(p)}{\mathcal{I}}{}_{xx}^{(\mathrm{N})} = -\frac{2}{e_t^2}\, J_p\,(p\, e_t) + \left(\frac{2}{p\, e_t} - \frac{2\, e_t}{p}\right) J_p'(p\, e_t)\,, \tag{5.128a}$$

$$\underset{(p)}{\mathcal{I}}{}_{xy}^{(\mathrm{N})} = \mathrm{i}\sqrt{1-e_t^2}\left[\left(\frac{2}{p} - \frac{2}{e_t^2\, p}\right) J_p\,(p\, e_t) + \frac{2}{e_t\, p^2}\, J_p'(p\, e_t)\right], \tag{5.128b}$$

$$\underset{(p)}{\mathcal{I}}{}_{yy}^{(\mathrm{N})} = \left(-\frac{2}{p^2} + \frac{2}{e_t^2\, p^2}\right) J_p\,(p\, e_t) + \left(-\frac{2}{p\, e_t} + \frac{2\, e_t}{p}\right) J_p'(p\, e_t)\,, \tag{5.128c}$$

where a $'$ denotes derivative with respect to the argument. For the mass octopole moment we find

$$\underset{(p)}{\mathcal{I}}{}_{xxx}^{(\mathrm{N})} = \left(-\frac{9}{e_t^3\, p^2} + \frac{12}{e_t\, p^2} - \frac{3\, e_t}{p^2}\right) J_p\,(p\, e_t) + \left(-\frac{18}{5\, p^3} + \frac{6}{e^2\, p^3} - \frac{6}{p} + \frac{3}{e_t^2\, p} + \frac{3\, e_t^2}{p}\right) J_p'(p\, e_t)\,, \tag{5.129a}$$

$$\underset{(p)}{\mathcal{I}}{}_{xxy}^{(\mathrm{N})} = \mathrm{i}\sqrt{1-e_t^2}\left[\left(-\frac{6}{e_t^3 p^3} + \frac{6}{5 e_t p^3} - \frac{3}{e_t^3 p} + \frac{6}{e_t p} - \frac{3e_t}{p}\right) J_p\,(pe_t) + \left(-\frac{5}{p^2} + \frac{9}{e_t^2 p^2}\right) J_p'(pe_t)\right], \tag{5.129b}$$

$$\underset{(p)}{\mathcal{I}}{}_{xyy}^{(\mathrm{N})} = \left(\frac{9}{e_t^3\, p^2} - \frac{13}{e_t\, p^2} + \frac{4\, e_t}{p^2}\right) J_p\,(pe_t) + \left(\frac{24}{5\, p^3} - \frac{6}{e_t^2\, p^3} + \frac{6}{p} - \frac{3}{e_t^2\, p} - \frac{3\, e_t^2}{p}\right) J_p'(p\, e_t)\,, \tag{5.129c}$$

[10] Note that the Fourier coefficients we provide are for *normalized* multipole moments as defined in Eqs (5.83a)–(5.83b).

$$\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{yyy} = \mathrm{i}\sqrt{1-e_t^2}\left[\left(\frac{6}{e_t^3 p^3} - \frac{12}{5 e_t p^3} + \frac{3}{e_t^3 p} - \frac{6}{e_t p} + \frac{3 e_t}{p}\right) J_p\,(p\,e_t) + \left(\frac{6}{p^2} - \frac{9}{e_t^2\,p^2}\right) J_p'(p\,e_t)\right], \tag{5.129d}$$

$$\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{zzx} = \left(\frac{1}{e_t\,p^2} - \frac{e_t}{p^2}\right) J_p\,(p\,e_t) - \frac{6}{5\,p^3}\,J_p'(p\,e_t)\,, \tag{5.129e}$$

$$\underset{(p)}{\mathcal{I}}{}^{(\mathrm{N})}_{zzy} = \mathrm{i}\sqrt{1-e_t^2}\left(\frac{6}{5\,e_t\,p^3} J_p\,(p\,e_t) - \frac{1}{p^2}\,J_p'(p\,e_t)\right). \tag{5.129f}$$

Finally, for the current quadrupole moment,

$$\underset{(p)}{\mathcal{J}}{}^{(\mathrm{N})}_{xz} = \frac{\sqrt{1-e_t^2}}{2\,p}\,J_p'\,(p\,e_t)\,, \tag{5.130a}$$

$$\underset{(p)}{\mathcal{J}}{}^{(\mathrm{N})}_{yz} = -\mathrm{i}\frac{1-e_t^2}{2\,e_t\,p}\,J_p\,(p\,e_t)\,. \tag{5.130b}$$

Bibliography

[1] D. N. Spergel *et al.* (WMAP) (2007). astro-ph/0603449.

[2] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner. Royal Society of London Proceedings Series A **269**, 21 (1962).

[3] R. A. Hulse and J. H. Taylor. Astrophys. J. **195**, L51 (1975).

[4] J. M. Weisberg and J. H. Taylor. The relativistic binary pulsar 1913+16: Thirty years of observations and analysis. In F. A. Rasio and I. H. Stairs, eds., *ASP Conf. Ser. 328: Binary Radio Pulsars*, p. 25 (2005).

[5] J. Taylor and J. Weisberg. Astrophys. J. **345**, 434 (1982).

[6] V. Fock. *Theory of space, time and gravitation* (Pergamon, London, 1959).

[7] L. Landau and E. Lifshitz. *The classical theory of fields* (Pergamon, Oxford, 1971).

[8] B. F. Schutz (2001). gr-qc/0111095.

[9] A. Abramovici *et al.* Science **256**, 325 (1992).

[10] K. S. Thorne. Gravitational radiation. In S. Hawking and W. Israel, eds., *Three hundred years of gravitation*, pp. 330–458 (Cambridge University Press, 1987).

[11] J. Weber. Phys. Rev. Lett. **22**, 1320 (1969).

[12] http://www.ligo.caltech.edu.

[13] http://www.virgo.infn.it.

[14] http://www.geo600.uni-hannover.de.

[15] tamago.mtk.nao.ac.jp.

[16] S. E. Whitcomb. Class. Quant. Grav. **25**, 114013 (2008).

[17] B. F. Schutz. Sources of radiation from neutron stars. Tech. Rep. AEI 57 (1998). Report-no: AEI 57, gr-qc/9802020.

[18] B. Abbott *et al.* (LIGO Scientific). Phys. Rev. Lett. **94**, 181103 (2005). gr-qc/0410007.

[19] A. Buonanno. Gravitational waves from the early universe. In *TASI lectures* (2003). `gr-qc/0303085`.

[20] E. S. Phinney (2003).

[21] N. Seto, S. Kawamura, and T. Nakamura. Phys. Rev. Lett. **87**, 221103 (2001). `astro-ph/0108011`.

[22] T. Damour and G. Esposito-Farèse. Class. Quantum Grav. **9**, 2093 (1992).

[23] C. M. Will. *Theory and experiments in gravitational physics* (Cambridge University Press, New York, USA, 1981).

[24] D. M. Eardley, D. L. Lee, and A. P. Lightman. Phys. Rev. D **8**, 3308 (1973).

[25] T. Damour and G. Esposito-Farèse. Phys. Rev. D **58**, 042001 (1998). `gr-qc/9803031`.

[26] B. F. Schutz. Gravitational radiation. Notes to lectures at Hanover University.

[27] L. P. Grishchuk, V. M. Lipunov, K. A. Postnov, M. E. Prokhorov, and B. S. Sathyaprakash. Physics Uspekhi **44**, 1 (2001). `astro-ph/0008481`.

[28] L. Finn. Phys. Rev. D **46**, 5236 (1992).

[29] A. Buonanno and T. Damour. Phys. Rev. D **62**, 064015 (2000). `gr-qc/0001013`.

[30] L. Blanchet. Living Rev. Rel. **5**, 3 (2002). `gr-qc/0202016`.

[31] T. Damour. Gravitational radiation and the motion of compact bodies. In N. Deruelle and T. Piran, eds., *Gravitational Radiation*, pp. 59–144 (North-Holland Company, Amsterdam, 1983).

[32] L. Blanchet, G. Faye, B. R. Iyer, and B. Joguet. Phys. Rev. D **65**, 061501(R) (2002). Erratum-ibid **71**, 129902(E) (2005), `gr-qc/0105099`.

[33] T. Damour, B. R. Iyer, and B. S. Sathyaprakash. Phys. Rev. D **57**, 885 (1998). `gr-qc/9708034`.

[34] T. Damour, B. R. Iyer, and B. S. Sathyaprakash. Phys. Rev. D **62**, 084036 (2000). `gr-qc/0001023`.

[35] T. Damour, B. R. Iyer, and B. S. Sathyaprakash. Phys. Rev. D **63**, 044023 (2001). Erratum-ibid. **D** 72 (2005) 029902, `gr-qc/0010009`.

[36] A. Buonanno and T. Damour. Phys. Rev. D **59**, 084006 (1999). `gr-qc/9811091`.

[37] F. Pretorius. Phys. Rev. Lett. **95**, 121101 (2005). `gr-qc/0507014`.

[38] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter. Phys. Rev. Lett. **96**, 111102 (2006). `gr-qc/0511103`.

[39] M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower. Phys. Rev. Lett. **96**, 111101 (2006). `gr-qc/0511048`.

[40] J. Baker, M. Campanelli, F. Pretorius, and Y. Zlochower. Class. Quant. Grav **24**, S25 (2007).

[41] A. Buonanno, G. B. Cook, and F. Pretorius. Phys. Rev. D **75**, 124018 (2007). `gr-qc/0610122`.

[42] E. Berti, V. Cardoso, J. A. Gonzalez, U. Sperhake, M. Hannam, S. Husa, and B. Bruegmann. Phys. Rev. D **76**, 064034 (2007). `gr-qc/0703053`.

[43] M. Hannam, S. Husa, U. Sperhake, B. Brugmann, and J. A. Gonzalez. Phys. Rev. D **77**, 044020 (2008). `arXiv:0706.1305`.

[44] M. Boyle *et al.* Phys. Rev. D **76**, 124038 (2007). `arXiv:0710.0158`.

[45] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter. Phys. Rev. D **73**, 104002 (2006). `gr-qc/0602026`.

[46] K. D. Kokkotas and B. G. Schmidt. Living Rev. Rel. **2**, 2 (1999). `gr-qc/9909058`.

[47] P. Peters. Phys. Rev. **136**, B1224 (1964).

[48] Y. Kozai. Astron. J **67**, 591 (1962).

[49] M. C. Miller and D. P. Hamilton. Astrophys. J. **576**, 894 (2002). `astro-ph/0202298`.

[50] J. Taylor and J. Weisberg. Astrophys. J. **253**, 908 (1982).

[51] `http://lisa.jpl.nasa.gov`.

[52] P. L. Bender. Lisa: Laser interferometer space antenna for the detection and observation of gravitational waves: Pre-phase: A report (1995). Unpublished.

[53] C. Cutler. Phys. Rev. D **57**, 7089 (1998). `gr-qc/9703068`.

[54] N. J. Cornish and S. L. Larson. Phys. Rev. D **67**(10), 103001 (pages 15) (2003).

[55] J. W. Armstrong, F. B. Estabrook, and M. Tinto. Astrophys. J. **527**, 814 (1999).

[56] F. B. Estabrook, M. Tinto, and J. W. Armstrong. Phys. Rev. D **62**(4), 042002 (2000).

[57] S. V. Dhurandhar, K. R. Nayak, and J.-Y. Vinet. Phys. Rev. D **65**(10), 102002 (2002).

[58] M. Tinto and S. Dhurandhar. Living Rev. Relativity **8**, 4 (2005).

[59] L. Ferrarese and D. Merritt. Astrophys. J. **539**, L9 (2000). `astro-ph/0006053`.

[60] D. Richstone, E. A. Ajhar, R. Bender, G. Bower, A. Dressler, S. M. Faber, A. V. Filippenko, K. Gebhardt, R. Green, L. C. Ho, J. Kormendy, T. Lauer, J. Magorrian, and S. Tremaine. Nature **395**, A14 (1998). `astro-ph/9810378`.

[61] S. A. Hughes. Mon. Not. R. Astron Soc. **331**, 805 (2002). astro-ph/0108483.

[62] E. Berti, A. Buonanno, and C. M. Will. Phys. Rev. D **71**(8), 084025 (2005). gr-qc/0411129.

[63] E. Berti, A. Buonanno, and C. M. Will. Class. Quantum Grav. **22**, S943 (2005). gr-qc/0504017.

[64] K. G. Arun, B. R. Iyer, M. S. S. Qusailah, and B. S. Sathyaprakash. Class. Quantum Grav. **23**, L37 (2006). gr-qc/0604018.

[65] K. G. Arun, B. R. Iyer, M. S. S. Qusailah, and B. S. Sathyaprakash. Phys. Rev. D **74**, 024025 (2006). gr-qc/0604067.

[66] S. A. Hughes and K. Menou. Astrophys. J. **623**, 689 (2005). astro-ph/0410148.

[67] B. F. Schutz. Nature (London) **323**, 310 (1986).

[68] P. J. E. Peebles and B. Ratra. Rev. Mod. Phys. **75**, 559 (2003). astro-ph/0207347.

[69] D. E. Holz and S. A. Hughes. Astrophys. J **629**, 15 (2005). astro-ph/0504616.

[70] N. Dalal, D. E. Holz, S. A. Hughes, and B. Jain. Phys. Rev. D **74**, 063006 (2006). astro-ph/0601275.

[71] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer. Phys. Rev. Lett. **93**, 091101 (2004). gr-qc/0406012.

[72] C. Cutler *et al.* Phys. Rev. Lett. **70**, 2984 (1993). astro-ph/9208005.

[73] C. Cutler, T. Apostolatos, L. Bildsten, L. Finn, E. Flanagan, D. Kennefick, D. Markovic, A. Ori, E. Poisson, G. Sussman, and K. Thorne. Phys. Rev. Lett. **70**, 2984 (1993).

[74] C. Cutler and E. Flanagan. Phys. Rev. D **49**, 2658 (1994).

[75] L. Blanchet, T. Damour, B. R. Iyer, C. M. Will, and A. G. Wiseman. Phys. Rev. Lett. **74**, 3515 (1995). gr-qc/9501027.

[76] L. Blanchet. Phys. Rev. D **54**, 1417 (1996). Erratum-ibid.**71**, 129904(E) (2005), gr-qc/9603048.

[77] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah. Class. Quantum Grav. **21**, 3771 (2004). Erratum-ibid. **22**, 3115 (2005), gr-qc/0404185.

[78] A. M. Sintes and A. Vecchio. Detection of gravitational waves from inspiraling compact binaries using non-restricted post-newtonian approximations. In J. Dumarchez, ed., *Rencontres de Moriond: Gravitational waves and experimental gravity* (Frontières, Paris, 2000). gr-qc/0005058.

[79] A. M. Sintes and A. Vecchio. Lisa observations of massive black hole binaries using post-newtonian waveforms. In S. Meshkov, ed., *Third Amaldi conference on Gravitational Waves*, p. 403 (American Institute of Physics Conference Series, New York, 2000). `gr-qc/0005059`.

[80] R. W. Hellings and T. A. Moore. Class. Quant. Grav. **20**, S181 (2003). `gr-qc/0207102`.

[81] C. Van Den Broeck. Class. Quantum Grav. **23**, L51 (2006). `gr-qc/0604032`.

[82] C. Van Den Broeck and A. Sengupta. Class. Quantum Grav. **24**, 155 (2007). `gr-qc/0607092`.

[83] C. Van Den Broeck and A. S. Sengupta. Class. Quantum Grav. **24**, 1089 (2007). `gr-qc/0610126`.

[84] L. Blanchet, B. R. Iyer, C. M. Will, and A. G. Wiseman. Class. Quantum Grav. **13**, 575 (1996). `gr-qc/9602024`.

[85] S. Komossa *et al.* Astrophys. J. **582**, L15 (2003). `astro-ph/0212099`.

[86] `http://srl.caltech.edu/shane/sensitivity/MakeCurve.html`.

[87] J. G. Baker and J. Centrella. Class. Quantum Grav. **22**, S355 (2005). `astro-ph/0411616`.

[88] K. G. Arun. Phys. Rev. D **74**, 024025 (2006). `gr-qc/0605021`.

[89] M. Trias and A. M. Sintes. Phys. Rev. D **77**, 024030 (2008). `arXiv:0707.4434v1 [gr-qc]`.

[90] L. Kidder, C. Will, and A. Wiseman. Phys. Rev. D **47**, R4183 (1993).

[91] L. Blanchet, A. Buonanno, and G. Faye. Phys. Rev. D **74**, 104034 (2006). Erratum-ibid. **75**, 049903 (E) (2007), `gr-qc/0605140`.

[92] E. Poisson and C. Will. Phys. Rev. D **52**, 848 (1995).

[93] B. Bruegmann, W. Tichy, and N. Jansen. Phys. Rev. Lett. **92**, 211101 (2004). `gr-qc/0312112`.

[94] T. A. Moore and R. W. Hellings. Phys. Rev. D **65**, 062001 (2002). `gr-qc/9910116`.

[95] T. Damour, B. R. Iyer, and B. S. Sathyaprakash. Phys. Rev. D **66**, 027502 (2002). Erratum-ibid **66**, 027502 (2002), `gr-qc/0207021`.

[96] L. E. Kidder, L. Blanchet, and B. R. Iyer. Class. Quant. Grav. **24**, 5307 (2007). `arXiv:0706.0726`.

[97] K. G. Arun, B. R. Iyer, B. S. Sathyaprakash, and S. Sinha. Phys. Rev. **D 75**, 124002 (2007). `arXiv:0704.1086 [gr-qc]`.

[98] L. Barack and C. Cutler. Phys. Rev. D **69**, 082005 (2004). gr-qc/0310125.

[99] L. Finn and D. Chernoff. Phys. Rev. D **47**, 2198 (1993).

[100] B. Kocsis, Z. Haiman, K. Menou, and Z. Frei. Phys. Rev. D **76**, 022003 (2007). astro-ph/0701629.

[101] C. Cutler and M. Vallisneri. Phys. Rev. D **76**, 104018 (2007). arXiv:0707.2982 [gr-qc].

[102] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes in Fortran, 2nd Edition* (Cambridge University Press, Cambridge, England, 1992).

[103] M. Vallisneri. Phys. Rev. D **77**, 042001 (2008). gr-qc/0703086.

[104] N. J. Cornish and E. K. Porter. Class. Quant. Grav. **23**, S761 (2006). gr-qc/0605085.

[105] See Chapter 3 of Baker et al., LISA Science Document LISA-LIST-RP-436 (2007).

[106] S. Babak, M. Hannam, S. Husa, and B. Schutz (2008). gr-qc/0806.1591.

[107] N. Bahcall *et al.* Astrophys. J. Suppl. **148**, 243 (2003). astro-ph/0305202.

[108] M. Milosavljevic and E. S. Phinney. Ap. J. **622**, L93 (2005). astro-ph/0410343.

[109] P. J. Armitage and P. Natarajan. Ap. J **567**, L9 (2002). arXiv:astro-ph/0201318.

[110] B. Kocsis, Z. Frei, Z. Haiman, and K. Menou. Astrophys. J. **637**, 27 (2006). astro-ph/0505394.

[111] A. Vecchio. Phys. Rev. D **70**, 042001 (2004). astro-ph/0304051.

[112] R. N. Lang and S. A. Hughes. Phys. Rev. D **74**, 122001 (2006). Erratum-ibid. **D** 75, 089902 (2007), gr-qc/0608062.

[113] R. Takahashi and T. Nakamura. Astrophys. J. **595**, 1039 (2003). astro-ph/0305055.

[114] N. Seto. Phys. Rev. D **69**, 022002 (2004). astro-ph/0305605.

[115] E. K. Porter and N. J. Cornish gr-qc/0804.0332.

[116] L. Blanchet, B. R. Iyer, and B. Joguet. Phys. Rev. D **65**, 064005 (2002). Erratum-ibid **71**, 129903(E) (2005), gr-qc/0105098.

[117] L. E. Kidder. Phys. Rev. D **D77**, 044016 (2008). arXiv:0710.0614.

[118] B. S. Sathyaprakash. Phys. Rev. D **50**(12), R7111 (1994).

[119] R. Balasubramanian, B. S. Sathyaprakash, and S. V. Dhurandhar. Phys. Rev. D **53**, 3033 (1996). Erratum-ibid. D **54**, 1860 (1996), gr-qc/9508011.

[120] A. Buonanno, Y. Chen, and M. Vallisneri. Phys. Rev. D **67**, 024016 (2003). Erratum-ibid. **D** 74, 029903(E) (2006), gr-qc/0205122.

[121] K. G. Arun, B. R. Iyer, B. S. Sathyaprakash, S. Sinha, and C. V. D. Broeck. Phys. Rev. **D76**, 104016 (2007). Erratum-ibid. **76**, 129903 (2007), `arXiv:0707.3920 [astro-ph]`.

[122] L. Blanchet and T. Damour. Phil. Trans. Roy. Soc. Lond. A **320**, 379 (1986).

[123] L. Blanchet. Proc. Roy. Soc. Lond. A **409**, 383 (1987).

[124] L. Blanchet and T. Damour. Phys. Rev. D **46**, 4304 (1992).

[125] L. Blanchet. Phys. Rev. D **51**, 2559 (1995). `gr-qc/9501030`.

[126] L. Blanchet. Class. Quantum Grav. **15**, 1971 (1998). `gr-qc/9801101`.

[127] C. Will and A. Wiseman. Phys. Rev. D **54**, 4813 (1996).

[128] M. E. Pati and C. M. Will. Phys. Rev. D **62**, 124015 (2000). `gr-qc/0007087`.

[129] M. E. Pati and C. M. Will. Phys. Rev. D **65**, 104008 (2002). `gr-qc/0201001`.

[130] K. Thorne. Rev. Mod. Phys. **52**, 299 (1980).

[131] L. Blanchet, T. Damour, and B. R. Iyer. Phys. Rev. D **51**, 5360 (1995). `gr-qc/9501029`.

[132] L. Blanchet and B. R. Iyer. Phys. Rev. D **71**, 024004 (2004). `gr-qc/0409094`.

[133] L. Blanchet and T. Damour. Phys. Rev. D **37**, 1410 (1988).

[134] L. Blanchet. Class. Quantum Grav. **15**, 89 (1998). `gr-qc/9710037`.

[135] L. Blanchet. Class. Quantum Grav. **15**, 113 (1998). Erratum-ibid **22**, 3381 (2005), `gr-qc/9710038`.

[136] W. Bonnor. Philos. Trans. R. Soc. London, Ser. A **251**, 233 (1959).

[137] L. Blanchet and T. Damour. Annales Inst. H. Poincaré Phys. Théor. **50**, 377 (1989).

[138] T. Damour and B. R. Iyer. Annales Inst. H. Poincaré, Phys. Théor. **54**, 115 (1991).

[139] T. Damour and B. R. Iyer. Phys. Rev. D **43**, 3259 (1991).

[140] R. Epstein and R. Wagoner. Astrophys. J. **197**, 717 (1975).

[141] P. Jaranowski and G. Schäfer. Phys. Rev. D **57**, 7274 (1998). `gr-qc/9712075`.

[142] P. Jaranowski and G. Schäfer. Phys. Rev. D **60**, 124003 (1999). `gr-qc/9906092`.

[143] L. Blanchet and G. Faye. Phys. Lett. A **271**, 58 (2000). `gr-qc/0004009`.

[144] L. Blanchet and G. Faye. Phys. Rev. D **63**, 062005 (2001). `gr-qc/0007051`.

[145] T. Damour, P. Jaranowski, and G. Schäfer. Phys. Rev. D **63**, 044021 (2001). Erratum-ibid **66**, 029901(E) (2002).

[146] V. de Andrade, L. Blanchet, and G. Faye. Class. Quantum Grav. **18**, 753 (2001).

[147] T. Damour, P. Jaranowski, and G. Schäfer. Phys. Lett. B **513**, 147 (2001). `gr-qc/0105038`.

[148] L. Blanchet, T. Damour, and G. Esposito-Farèse. Phys. Rev. D **69**, 124007 (2004). `gr-qc/0311052`.

[149] L. Blanchet and B. R. Iyer. Class. Quantum Grav. **20**, 755 (2003). `gr-qc/0209089`.

[150] L. Blanchet, G. Faye, B. R. Iyer, and S. Sinha. Class. Quant. Grav **25**, 165003 (2008). `arXiv:0802.1249`.

[151] Y. Itoh, T. Futamase, and H. Asada. Phys. Rev. D **63**, 064038 (2001). `gr-qc/0101114`.

[152] Y. Itoh and T. Futamase. Phys. Rev. D **68**, 121501(R) (2003). `gr-qc/0310028`.

[153] Y. Itoh. Phys. Rev. D **69**, 064018 (2004). `gr-qc/0310029`.

[154] D. Christodoulou. Phys. Rev. Lett. **67**, 1486 (1991).

[155] A. Wiseman and C. Will. Phys. Rev. D **44**, R2945 (1991).

[156] K. Thorne. Phys. Rev. D **45**, 520 (1992).

[157] I. Gradshteyn and I. Ryzhik. *Table of Integrals, Series and Products* (Academic Press, 1980).

[158] L. Blanchet and G. Schäfer. Class. Quantum Grav. **10**, 2699 (1993).

[159] M. Sasaki. Prog. Theor. Phys. **92**, 17 (1994).

[160] H. Tagoshi and M. Sasaki. Prog. Theor. Phys. **92**, 745 (1994). `gr-qc/9405062`.

[161] T. Tanaka, H. Tagoshi, and M. Sasaki. Prog. Theor. Phys. **96**, 1087 (1996). `gr-qc/9701050`.

[162] K. G. Arun, C. Mishra, C. V. D. Broeck, B. R. Iyer, B. S. Sathyaprakash, and S. Sinha. *LISA as a dark energy probe* (2008). To appear in Class. Quant. Grav. Conf. Ser.

[163] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer. Phys. Rev. D **71**, 124004 (2005). `gr-qc/0503044`.

[164] M. J. Benacquista. Liv. Rev. Rel. **5**, 2 (2002). `astro-ph/0202056`.

[165] K. Gültekin, M. C. Miller, and H. D. P. Astrophys. J. **616**, 221 (2004). `astro-ph/0402532`.

[166] O. Blaes, M. H. Lee, and A. Socrates. Astrophys. J **578**, 775 (2002). `astro-ph/0203370`.

[167] M. B. Davies, A. J. Levan, and A. R. King. Mon. Not. R. Astron. Soc **356**, 54 (2005). astro-ph/0409681.

[168] K. L. Page *et al.* Astrophys. J **637**, L13 (2006). astro-ph/0512358.

[169] J. Grindlay, S. Portegies Zwart, and S. McMillan. Nature Physics **2**, 116 (2006). astro-ph/0512654.

[170] H. K. Chaurasia and M. Bailes. Astrophys. J. **632**, 1054 (2005). astro-ph/0504021.

[171] P. Peters and J. Mathews. Phys. Rev. **131**, 435 (1963).

[172] L. Blanchet and G. Schäfer. Mon. Not. Roy. Astron. Soc. **239**, 845 (1989).

[173] R. Rieth and G. Schäfer. Class. Quantum Grav. **14**, 2357 (1997).

[174] A. Gopakumar and B. R. Iyer. Phys. Rev. D **56**, 7708 (1997). gr-qc/9710075.

[175] W. Junker and G. Schäfer. Mon. Not. R. Astron. Soc **254**, 146 (1992).

[176] T. Damour and N. Deruelle. Annales Inst. H. Poincaré Phys. Théor. **43**, 107 (1985).

[177] T. Damour and G. Schäfer. Nuovo Cim. **B101**, 127 (1988).

[178] G. Schäfer and N. Wex. Phys. Lett. A **196**, 177 (1993). Erratum, *ibid* 177, 461(E), 1993.

[179] N. Wex. Class. Quant. Grav. **12**, 983 (1995).

[180] T. Damour, A. Gopakumar, and B. R. Iyer. Phys. Rev. D **70**, 064028 (2004). gr-qc/0404128.

[181] C. Konigsdorffer and A. Gopakumar. Phys. Rev. D **73**, 124012 (2006). gr-qc/0603056.

[182] B. J. Owen. Phys. Rev. D **53**, 6749 (1996). gr-qc/9511032.

[183] K. Martel and E. Poisson. Phys. Rev. D **60**, 124008 (1999). gr-qc/9907006.

[184] M. Tessmer and A. Gopakumar (2007). arXiv:0712.3199.

[185] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah. Phys. Rev. D **77**, 064034 (2008). arXiv:0711.0250.

[186] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah. Phys. Rev. D **77**, 064035 (2008). arXiv:0711.0302.

[187] M. S. S. Qusailah. *Generation of gravitational waves from inspiralling compact binaries: 3PN luminosity, 2PN linear momentum flux, and applications*. Phd thesis, Jawaharlal Nehru University, New Delhi (2006).

[188] K. G. Arun. *Gravitational waves from inspiralling compact binaries: 2.5PN waveform, 3PN angular momentum flux and 3.5PN parameter estimation*. Phd thesis, Jawaharlal Nehru University, New Delhi (2006).

[189] T. Damour. Phys. Rev. Lett. **51**, 1019 (1983).

[190] R. Memmesheimer, A. Gopakumar, and G. Schäfer. Phys. Rev. D **70**, 104011 (2004). `gr-qc/0407049`.

[191] A. Gopakumar and B. R. Iyer. Phys. Rev. D **65**, 084011 (2002). `gr-qc/0110100`.

[192] M. Tessmer and A. Gopakumar. Mon. Not. Roy. Astron. Soc. **374**, 721 (2007). `gr-qc/0610139`.

[193] U. Sperhake *et al.* (2007). `arXiv:0710.3823`.

[194] I. Hinder, B. Vaishnav, F. Herrmann, D. Shoemaker, and P. Laguna. Phys. Rev. D **77**, 081502 (2008). `arXiv:0710.5167`.

[195] I. Hinder, F. Herrmann, P. Laguna, and D. Shoemaker (2008). `arXiv:0806.1037`.

[196] H. C. Plummer. *An Introductory Treatise on Dynamical Astronomy* (Dover Publications, New York, 1918).