Chapter 3

A magnetic field generation mechanism in the pre-recombination plasma

[1]This chapter is based on the paper: Gopal & Sethi, 2005, MNRAS, 363, 521

Summary and main results of chapter 3

The question of the origin of cosmic magnetic fields is still an unsolved problem. Various approaches and mechanisms have been put forward to generate large-scale magnetic fields. In this chapter we study a mechanism for magnetic field generation within the framework of the standard cosmological model which will operate in the pre-recombination era. The main components of the pre-recombination plasma are photons, electrons and protons. The dominant interactions are Coulomb scattering between electrons and protons and Thomson scattering between photons and electrons. The photons preferiantially exert pressure on the electrons thereby causing a difference in the velocity fields of electrons and protons thereby inducing an electric current. If the induced current has a non-vanishing curl component then magnetic fields can be generated. To the zeroth order, the plasma is homogenous and in thermal equilibrium as is evidenced by the near-isotropy of the CMBR and its Planckian spectrum. However, observed anisotropies of the CMBR also suggest that there small amplitude fluctuations over and above the uniform background. Such a fluctuating multi-component plasma holds the promise of inducing a current with a non-vanishing vorticity which thus becomes the source of the magnetic field. We studied this possibility using the formalism of perturbation theory assuming adiabatic initial conditions.

The main results are summarized below:

- ***In the linear theory there is no field generation mainly because of the fact that the fluid velocities have vanishing vorticity.***
- ***In second order, a non-vanishing vorticity can be generated by treating the collision term for photon-electron interaction to second order.***
- ***The strength of the magnetic field generated in this manner is nearly*** $10^{-30}\mathrm{G}$ ***for scales from ~ 100 Mpc to a kilo-parsec at the current epoch.***

3.1 Introduction

Magnetic fields are ubiquitous in the universe and presumably play an important role in most objects in the Universe. Their origin however is an unsolved problem. (see e.g. Parker 1979; Zeldovich, Ruzmaikin & Sokolov 1983). The present day fields in galaxies have strengths of the order of a micro-Gauss. These fields could have arisen from dynamo amplification of seed fields $\simeq 10^{-20}$ G (see e.g. Ruzmaikin, Shukurov & Sokoloff 1988; Beck et al 1996; Shukurov 2004; Brandenburg & Subramanian 2005) or could have originated from primordial magnetic fields of strength $\simeq 10^{-9}$ G generated during inflationary epoch in the early universe (Turner & Widrow 1988; Ratra 1992; Ashoorioon & Mann 2005, see Grasso & Rubinstein 2001; Giovannini 2004, for reviews).

There are various astrophysical mechanisms for the creation of cosmic magnetic fields. Broadly these can be divided into pre-recombination and post-recombination mechanisms. Most of the astrophysical mechanisms proposed after recombination are based on the Biermann battery mechanism (Biermann 1950) which was first investigated to explain stellar magnetism. Application of this mechanism in the cosmological context lead to seed fields of the order of $10^{-20} - 10^{-17}$ G on scales of a few hundreds of kilo-parsecs to a few Mpc (see, e.g. Subramanian, Narasimha & Chitre 1994; Kulsrud et al. 1997; Grasso & Rubenstein 2001; Widrow 2002 for reviews).

Harrison (1970) considered a scenario in which a small seed field $\simeq 10^{-25}$ G is generated in the radiation era owing to the vorticity in the photon-baryon fluid. In particular they showed that there is a one-one correspondence between the vorticity in the fluid and the magnetic field. In this analysis however, the source of vorticity is not specified. It can be shown that in the absence of a continual source, vorticity decays with time and hence will be insignificant by the epoch of recombination (Hu & White 1997, Rees 1987). This is the main limitation of this analysis. Hogan (2000) and Berezhiani & Dolgov (2004) also considered the generation of magnetic fields from photon pressure in the pre-recombination epoch. Berezhiani & Dolgov considered the generation of fields from a Biermann mechanism which can operate in the second order of perturbation theory as a result of photon diffusion. More recently Matarrese et al. (2005) studied magnetic field generation from vector metric perturbations which are naturally generated when second order effects of fluctuations are studied. In this chapter we study the second order effect of photon-electron scattering and deduce that vorticity is induced naturally to this order from the above effect.

In the pre-recombination Universe, photons, electrons, and protons can be treated as tightly coupled fluids. Photons, however, preferentially exert pressure on electrons [the pressure on protons is suppressed by a factor $(m_e/m_p)^2$]. During the evolution of the photon-baryon plasma a difference in velocity fields of electrons and protons can therefore be generated, and this holds the promise of generating magnetic fields from this induced current. In addition, the approximation that photons and baryons are tightly coupled, and therefore can be treated as one fluid, breaks down for scales up to several Mpc by the time of recombination. (During the process of recombination this scale exceeds the horizon scale.) At smaller scales, the photons free-stream and this in principle can lead to additional contribution to induced currents that might generate magnetic fields.

In this chapter, we study the coupled electron, proton, and photon plasma in first and second order in perturbation theory to understand the generation of magnetic fields during the evolution of the plasma. In the next section we describe the relevant equations for our study. In §3.3, we discuss the evolution of magnetic fields and its sources in the first and second order in perturbation theory. In §3.4 discuss our results and give concluding remarks. All the quantitative estimates in this chapter, are given for the spatially flat FRW model with $\Omega_{\rm m} = 0.3$ and $\Omega_{\Lambda} = 0.7$ (Spergel et al.. 2003, Reiss et al. 2004, Tonry et al. 2003, Perlmutter et al. 1999, Riess et al. 1998) with $\Omega_{\rm b}h^2 = 0.02$ (Spergel et al. 2003, Tytler et al. 2000) and $h = 0.7$ (Freedman et al. 2001).

3.2 Pre-recombination plasma

The primary components of the primordial plasma prior to the recombination epoch are photons, free electrons and protons. The dominant collisional interactions are Thomson scattering of photons by electrons and Coulomb scattering between the electrons and protons.

Observations of the Cosmic Microwave Background Radiation (CMBR), which is the relic of the radiation existing in this era, show that the plasma is almost homogeneous and in thermal equilibrium at the time of recombination (see e.g. Peebles 1993). However, anisotropies observed in the CMBR also indicate that there are spatial fluctuations superimposed on this uniform background density. In the present analysis, we assume that the fluctuations are adiabatic, which means that the entropy per fluid particle is conserved. Recent WMAP observations favour this initial condition (Peiris et al. 2003). The electrons interact with each other and with protons through Coulomb scattering. The mean free paths for *e*-*e*, *e*-*p* and *p*-*p* collisions are the same in this thermal plasma (see e.g. Shu 1992) and are much smaller than the astrophysically relevant scales ($\simeq$ 1 Mpc). Hence a continuum description treating them as fluids can be used. The effect of scattering between different species manifests in the form of a relative drag between the fluids which is taken into account by including a momentum exchange term in the Euler equation. For photons, however, the dominant interaction is Thomson scattering off free electrons with mean free path (comoving), $l_{\gamma\rm e}$, at $z \simeq 1000$ for a fully ionized universe being, $l_{\gamma\rm e} = 1/(a\sigma_{\rm T}n_e) \simeq 3\,$Mpc. Here, $\sigma_{\rm T}$ is the Thomson cross section for e-γ scattering, $n_{\rm e}$ is the electron number density and a is the scale factor. This is comparable to the length scales in consideration and hence a Boltzmann particle description is essential. In the next two sections we set up the mathematical equations describing photons as well as the ionised component consisting of electrons and protons.

3.2.1 Description of photons

The photons are described by the phase space distribution function $f(\mathbf{x}, \eta, p, \hat{n})$ where p is the magnitude of photon momentum, $\hat{n}$ is the propagation direction and η is the conformal time. Since the distribution is blacbody to zeroth order, it can be expanded as $f = f^0 + \delta f$ where f^0 is the Planck function and δf is a small perturbation. A further simplification of analysis can be made, since we dont consider frequency dependant effects, by describing the evolution of the perturbed distribution in terms of the brightness $\Delta(\mathbf{x}, \eta, \hat{n})$ which is a frequency averaged variable defined as:

$$\Delta = \frac{\int dpp^3 \delta f}{\int dpp^3 f^0} \tag{3.1}$$

The space-time metric is specified using comoving coordinates x^i and conformal time η for the conformal-Newton gauge and is given as (see e.g. Ma & Bertschinger 1995, and discussion therein):

$$ds^2 = a^2(\eta)[-(1 + 2\Psi)d\eta^2 + (1 - 2\Phi)dx_i dx^i] \tag{3.2}$$

Here, $a(\eta)$ is the scale factor and Ψ, Φ are the two potentials characterising scalar perturbations in this gauge. The evolution of these potentials is governed by a generalized Poisson equation (Hu & White 1997, Ma & Bertschinger 1995).

The evolution of photons in such a space-time is then given by the Boltzmann equation for the brightness:

$$\dot{\Delta} + n_i \partial_i \Delta + n_i \partial_i \Psi + \dot{\Phi} = C[f] \tag{3.3}$$

Here, over-dots denote derivative with respect to conformal time η. $C[f]$ is the collision term accounting for the scattering of photons with electrons. The linearized collision term for Thomson scattering (neglecting polarization) is given as (see, e.g. Hu & White 1997):

$$C[f] = n_e \sigma_T \left(\Delta_0 - \Delta + 4\mathbf{v}_e \cdot \hat{n} + \frac{3}{2} n_i n_j \Pi_{ij} \right) \tag{3.4}$$

Here, $\Delta_0 \equiv \int \frac{d\Omega}{4\pi} \Delta$ denotes the isotropic part of Δ and Π_{ij} is the photon anisotropic stress tensor which takes into account the angular dependence of Thomson scattering . It is given by: $\Pi_{ij} = \int \frac{d\Omega}{4\pi} \left(n_i n_j - \frac{1}{3}\delta_{ij} \right) \Delta$. The collision term also contains a term including the electron velocity which arises from the transformation of the scattering rate from the electron's rest frame to the inertial frame. The electron velocity in the linear theory of scalar perturbations does not contain any vortical component and hence this term acts like a pure dipole term maintaining azimuthal symmetry about the photon propagation direction. In our analysis however, we are looking for a possible generation of vorticity and hence this simplification cannot be made. Hence we resort to a more general multipole moment expansion in terms of spherical harmonic functions rather than the Legendre polynomial expansion used in the standard case.

3.2.2 Multipole moment expansion

In this section we discuss the complete spherical harmonic moment expansion of the Boltzmann equation without assuming azimuthal symmetry of Δ. The notations used in this section are self contained.
The Fourier transformed Boltzmann equation can be written as:

$$\dot{\Delta} + ik\mu\Delta + ik\mu\Psi + \dot{\Phi} = \frac{1}{\tau_{\gamma e}}\left(\Delta_0 - \Delta + 4\mathbf{v}_{\mathrm{e}} \cdot \hat{n} + \frac{3}{2}n_{\mathrm{i}}n_{\mathrm{j}}\Pi_{\mathrm{ij}}\right) \tag{3.5}$$

In the above μ is the angle between the wave vector $\mathbf{k}$ and the direction $\hat{n}$. The photon brightness function Δ can be expanded in terms of the scalar spherical harmonics $Y_{\ell m}$ as:

$$\Delta(\hat{n}) = \sum \sqrt{\frac{4\pi}{2\ell + 1}}\Delta_{\ell m}Y_{\ell m}(\hat{n}) \tag{3.6}$$

where, the coefficients $\Delta_{\ell m}$ are given by the inverse relation,

$$\Delta_{\ell m} = \sqrt{\frac{2\ell + 1}{4\pi}} \int d\Omega Y^{*}_{\ell m}(\hat{n})\Delta(\hat{n}) \tag{3.7}$$

Both the above relations follow from the properties of completeness and orthogonality of the spherical harmonic functions. The photon fluid variables like over-density δ_γ and velocity $\mathbf{v}_\gamma$ are then given by:

$$\delta_\gamma \equiv \int \frac{d\Omega}{4\pi}\Delta = \Delta_{00} \tag{3.8}$$

$$\hat{k} \cdot \mathbf{v}_\gamma = \frac{\Delta_{10}}{4} \tag{3.9}$$

$$\Omega_\gamma = |\hat{k} \times \mathbf{v}_\gamma| = \frac{\Delta_{11}}{4} \tag{3.10}$$

(3.11)

By substituting the expansion for Δ in Eq. (3.5) and using the familiar properties of spherical harmonics we arrive at the following hierarchy of equations for the evolution of the moments $\Delta_{\ell m}$. For details of such an expansion we refer to Hu & White (1997).

$$\dot{\Delta}_{\ell m} + ik\frac{A_{\ell,m}}{(2\ell - 1)}\Delta_{\ell-1,m} + ik\frac{A_{\ell+1,m}}{(2\ell + 3)}\Delta_{\ell+1,m} + \frac{\Delta_{\ell m}}{\tau_{\gamma e}} = S_{\ell m} \tag{3.12}$$

Here, $A_{\ell m} = \sqrt{\ell^2 - m^2}$. The source $S_{\ell m}$ is given as:

$$S_{\ell m} = \left(\frac{\Delta_{\ell m}}{\tau_{\gamma e}} + 4\dot{\Phi}\right)\delta_{\ell 0}\delta_{m0} + \left(\frac{4v_{\ell m}}{\tau_{\gamma e}} - k\Psi\delta_{m0}\right)\delta_{\ell 1} + \frac{1}{10}\Delta_{\ell m}\delta_{\ell 2} \tag{3.13}$$

In the above equations, the coefficients $v_{\ell m}$ are the coefficients in the multipole expansion of the $\mathbf{v}_e \cdot \hat{n}$ term such that $\mathbf{v}_e \cdot \hat{n} = \sum v_{\ell m} Y_{\ell m}\delta_{\ell 1}$. The vorticity of the photon fluid is tracked by the evolution of the $\ell = 1, m = 1$ moment Δ_{11}. We notice from Eq (3.13), that the source $S_{11} = 4v_{11}/\tau_{\gamma e}$. This is the only source for the evolution of the $m = 1$ moment. The first two moment equations in the hierarchy give the familiar continuity and Euler equations for photons:

$$\dot{\delta}_\gamma + \frac{4i}{3}\mathbf{k} \cdot \mathbf{v}_\gamma - 4\dot{\Phi} = 0 \tag{3.14}$$

$$\dot{\mathbf{v}}_\gamma + i\mathbf{k}\frac{\delta_\gamma}{4} + \mathbf{\Pi} - i\mathbf{k}\Psi = \frac{\mathbf{v}_e - \mathbf{v}_\gamma}{\tau_{\gamma e}} \tag{3.15}$$

In the above equation, $\mathbf{\Pi}_i = \frac{3}{4} i k_j \Pi_{ij}$.

There are some interesting conclusions which can be drawn from the above hierarchy. We can see that for a given m, each l-moment $\Delta_{\ell m}$ is coupled to an $l+1$ and an $l-1$ moment in the hierarchy. On the other hand, there is no coupling between different m modes. This implies that if there is no source $S_{\ell m}$ for a given m and if initial conditions are such that $\Delta_{\ell m} = 0$ for that m, then, $\Delta_{\ell m} = 0$ at all times even if other m moments evolve. In the present analysis, our emphasis will be on studying the effect of scalar perturbations which correspond to $m = 0$. For such perturbations Π_{ij} can be greatly simplified by using azimuthal symmetry about the axis of electron velocity (see e.g. Dodelson & Jubas 1995). However, our aim here is to study the generation of magnetic field from the evolution of the coupled photon-baryon plasma. Since the generation of this field can explicitly break this symmetry, one should consider a more general expression for the anisotropic stress tensor.

3.2.3 Fluid equations for electrons and protons

As discussed earlier, since the mean free paths of electrons and protons are very small compared to astrophysical scales, we can describe their evolution accurately using continuity and Euler equations for an ideal fluid. In linear theory, the density field $\rho_{e,p}(\mathbf{x}, \eta)$ is expanded as $\rho_{e,p}(\mathbf{x}, \eta) = \bar{\rho}_{e,p}(\eta)[1 + \delta_{e,p}[\mathbf{x}, \eta)]$, where $\bar{\rho}$ is the unperturbed background density and δ is the fractional perturbation. In what follows quantities denoted by a bar on top are background unperturbed quantities. The generalised continuity equations for each of the above species is given as (e.g. Ma & Bertschinger 1995):

$$\dot{\delta}_{e,p} + \mathbf{\nabla} \cdot \mathbf{v}_{e,p} - 3\dot{\Phi} = 0 \tag{3.16}$$

Note that we have used the continuity equations including the effect of the metric perturbation as given in the conformal Newtonian gauge.

The corresponding generalised Euler equations are:

$$\dot{\mathbf{v}}_e + \frac{\dot{a}}{a}\mathbf{v}_e = -\frac{\nabla P_e}{\rho_e} - \nabla\Psi - \frac{ae\mathbf{E}}{m_e} - \frac{ae}{m_e}\mathbf{v}_e \times \mathbf{B} + \left(\frac{\mathbf{v}_\gamma - \mathbf{v}_e}{\tau_{\gamma e}}\right)R + \frac{\mathbf{v}_p - \mathbf{v}_e}{\tau_{ep}} \tag{3.17}$$

$$\dot{\mathbf{v}}_p + \frac{\dot{a}}{a}\mathbf{v}_p = -\frac{\nabla P_p}{\rho_p} - \nabla\Psi + \frac{e\mathbf{E}}{m_p} + \frac{ae}{m_p}\mathbf{v}_p \times \mathbf{B} + \frac{\mathbf{v}_e - \mathbf{v}_p}{\tau_{ep}}\left(\frac{m_e}{m_p}\right) \tag{3.18}$$

Here, τ_{ep} is the (co-moving) electron-proton collision time scale; $\tau_{\gamma e} = 1/(n_e\sigma_T a)$ is the photon-electron Thompson scattering time scale. In the above equation we have neglected the photon-proton scattering term since the effect of this term will be suppressed compared to the electron-photon scattering term and it does not produce any qualitatively new effect. We have also included the forces due to a possible presence of the electric **E** and the magnetic **B** fields. These are not any given external fields but could be generated in a self-consistent manner. As we will see later, these fields are identically zero in linear theory whereas they are generated in the second order . The relative contribution of the photons and the electrons in the momentum equation is specified by the ratio $R \equiv 4\rho_\gamma/3\rho_e$. $P_{e,p}$ are the pressures acting on the charged fluids due to each of their internal scattering i.e $e-e$ and $p-p$ collisions. For adiabatic fluid we can write $P \equiv P(\rho)$.

Thus the above set of equations give a complete description of the electron and proton fluids. It is also useful to know the evolution of the vorticity of each of the above fluids as it is the presence of this component which can be directly related to the magnetic field evolution. By taking curl of the Euler equations and using Maxwell's equations, we can get the evolution equation for the vorticities ($\mathbf{\Omega}_{e,p} \equiv \nabla \times \mathbf{v}_{e,p}$) of the fluids as:

$$\begin{aligned}
\dot{\mathbf{\Omega}}_e + \frac{\dot{a}}{a}\mathbf{\Omega}_e &= \frac{e}{m_e a}\frac{\mathrm{d}}{\mathrm{d}\eta}(a^2\mathbf{B}) - \frac{ae}{m_e}\nabla \times (\mathbf{v}_e \times \mathbf{B}) + \left(\frac{\mathbf{\Omega}_\gamma - \mathbf{\Omega}_e}{\tau_{\gamma e}}\right)R - \frac{\nabla^2\mathbf{B}}{4\pi n_e e\tau_{ep}} \\
\dot{\mathbf{\Omega}}_p + \frac{\dot{a}}{a}\mathbf{\Omega}_p &= -\frac{e}{m_p a}\frac{\mathrm{d}}{\mathrm{d}\eta}(a^2\mathbf{B}) + \frac{ae}{m_p}\nabla \times (\mathbf{v}_p \times \mathbf{B}) + \frac{\nabla^2\mathbf{B}}{4\pi n_e e\tau_{ep}}\left(\frac{m_e}{m_p}\right)
\end{aligned} \tag{3.19}$$

3.3 Evolution equation for the magnetic field

To arrive at the equation governing the evolution of magnetic field, we use the Euler equations for the charged fluids and Maxwell's equations appropriate to the LFRW background metric (Appendix B). Subtracting Eq. (3.17) from Eq. (3.18) and using Maxwell's equations we first obtain the evolution of the current **J** :

$$\frac{m_e}{e^2}\frac{\partial}{\partial\eta}\left(\frac{\mathbf{J}}{n_e}\right) + \frac{\dot{a}}{a}\frac{m_e}{e^2 n_e}\mathbf{J} = \frac{1}{n_e e}\nabla P_e + a\mathbf{E} + a(\mathbf{v}_e \times \mathbf{B}) - \left(\frac{\mathbf{v}_\gamma - \mathbf{v}_e}{\tau_{\gamma e}}\right)R\frac{m_e}{e} - \frac{m_e\mathbf{J}}{n_e e^2\tau_{ep}} \tag{3.20}$$

In the above equation we have neglected forces on the proton fluid due to pressure gradient and electric field since they are smaller than that for the electron fluid by the factor m_e/m_p.

Taking curl of equation (1.20) and using Maxwell's equations, we get the equation for the generation of magnetic fields:

$$\begin{aligned}\frac{1}{a}\frac{\partial}{\partial\eta}(a^2\mathbf{B}) &= \frac{m_e}{e^2}\nabla\times\frac{\partial}{\partial\eta}\left(\frac{\mathbf{J}}{n_e}\right)+\frac{m_e}{e}\nabla\times\left(\frac{\nabla P_e}{\rho_e}\right)-\nabla\times(\mathbf{v}_e\times\mathbf{B}) \\ &+\frac{m_e}{e^2}\nabla\times\left(\frac{\mathbf{J}}{n_e\tau_{ep}}\right)+\frac{m_e}{e}\nabla\times\left(\frac{R(\mathbf{v}_\gamma-\mathbf{v}_e)}{\tau_{\gamma e}}\right)-\frac{\dot{a}}{a}\frac{m_e}{e^2}\nabla\times\left(\frac{\mathbf{J}}{n_e}\right)\end{aligned} \tag{3.21}$$

Since, we are looking for possible sources of magnetic field beginning from initial no-magnetic field conditions, we can neglect the third, fourth and sixth terms on the right hand side of the above equation. Each of these terms may only act to amplify or reduce the field strength once the field has been generated and we will see later that for the strength of the generated field, these terms will have a negligible effect anyway. The first term on the right hand side can be estimated to be negligible for the scales of interest compared to the the term on the left hand side whatever be the value of the generated field. (see e.g Widrow 2002). Making the above simplifictions we get:

$$\frac{1}{a}\frac{\partial}{\partial\eta}(a^2\mathbf{B}) = \mathbf{S}(\mathbf{x},\eta) \tag{3.22}$$

with

$$\mathbf{S}(\mathbf{x},\eta) = \frac{m_e}{e}\nabla\times\left(\frac{\nabla P_e}{\rho_e}\right)+\frac{m_e}{e}\nabla\times\left(\frac{R(\mathbf{v}_\gamma-\mathbf{v}_e)}{\tau_{\gamma e}}\right) \tag{3.23}$$

Thus we have identified two possible source terms for the generation of the magnetic field. The first term is the familiar Biermann battery term which can be rewritten as $\frac{\nabla\rho_e\times\nabla P_e}{\rho_e^2}$. This term contributes whenever the gradient of pressure and the gradient of density are not collinear. The second term represents the vortical component of the drag force on the electrons as a result of its interaction with photons. Each of these terms in principle can lead to the generation of magnetic field. We now discuss the nature of these source terms of magnetic field generation in first and second order in perturbation theory.

3.3.1 Evaluation of the source term: Linear theory

The source term for any Fourier mode $\mathbf{S}(\mathbf{k},\eta)$ can be simplified for the linear case. In this case, $R = 4\bar{\rho}_\gamma/(3\bar{\rho}_b)$, and $\tau_{\gamma e} = 1/(\bar{n}_e\sigma_T)$, are unperturbed quantities and hence don't carry any spatial dependence. The first term of the right hand side of Eq. (3.23) identically vanishes in this case. The source term can then be written as:

$$\mathbf{S}(\mathbf{k},\eta) = \frac{m_e R}{e\tau_{\gamma e}}(\mathbf{\Omega}_\gamma-\mathbf{\Omega}_e) \tag{3.24}$$

Thus, we see that the source for the magnetic field in the plasma is the differential vorticity between electrons and photons. The evolution of photon vorticity is tracked by the Boltzmann moment equation for $l = 1, m = 1$ (Eq. (3.12)). We note that the source of photon vorticity is $4v_{\ell 1}/\tau_{\gamma e} \propto \Omega_e$. This implies that the only source which can excite any l-moment for $m = 1$ is the electron fluid vorticity. The evolution equation for the electron fluid vorticity (Eq. (3.19)) shows that the only sources of vorticity are the magnetic field and the photon vorticity. This means that if the vorticities were zero in the initial condition, as is the case with initial zero-vorticity conditions we consider here, none of these quantities can be generated for any scale in the linear regime. In particular we can conclude that no magnetic field is generated in linear order for scalar perturbations. It should be noted that using Eq. (3.12) and Eq. (3.13) allows us to follow modes at which photons are free-streaming at any given epoch. Therefore the above conclusion holds for all scales larger than the scales at which electrons and protons can be treated as fluids.

3.3.2 Source term in the second order

In the previous section we argued that to the first order in perturbation theory, no vorticity and consequently no magnetic field is generated on any scale. Hence we should study the perturbation to the next order. There are various terms which have to be included in going to second order in perturbation theory. Second order terms can arise from treating metric perturbations to second order (Martinez-Gonzalez, Sanz & Silk 1992) or by including the second order terms in the electron-photon scattering (Vishniac 1987, Dodelson & Jubas 1995, Hu, Scott, & Silk 1994). Vishniac (1987) adopted the simple procedure of including the spatial dependence of densities to include the second order effects. Detailed analyses (Dodelson & Jubas 1995, Hu, Scott, & Silk 1994) showed that Vishniac's procedure gives the most important second order effect in the electron-proton scattering for sub-horizon scales. This allows us to study scales smaller than the horizon at the last scattering surface, $H^{-1} \simeq 100\,\mathrm{Mpc}$. At larger scales other second order effects from electron-photon scattering and the second order metric perturbations might be comparable or dominate.

We adopt Vishiniac's procedure here and obtain the second order term from treating the spatial dependence of densities i.e. in R, $\tau_{\gamma e}$ and ρ_e in the source term for magnetic field generation (Eq. (3.23)). To get estimates of the generated magnetic field we solve for the difference in photon and baryonic bulk velocity in the tight-coupling approximation. We argue below that the main contribution to the source $S(\mathbf{k}, \eta)$ for any scale comes from epochs at which the tight-coupling approximation is valid.

The source term is evaluated in the tight-coupling approximation in Appendix A and given by Eq. (3.29). We argue there that for adiabatic evolution, the only term that can source the magnetic field generation is given by Eq. (3.30).

This allows us to solve the evolution of the generated magnetic field at any scale:

$$a^2\mathbf{B}(\mathbf{k},\eta) \simeq \frac{\bar{R}am_e}{3e}\int_0^{\eta} d\eta' \frac{\dot{a}}{a}\int d^3k'\delta_e(\mathbf{k}'),\eta')\mathbf{k}'\times\mathbf{v}_e((\mathbf{k}-\mathbf{k}'),\eta') \tag{3.25}$$

Note that $a\bar{R}$ is independent of time. Eq. (3.30) can be solved using linear theory evolution of density and velocity perturbations for each scale from initial conditions at the time at which the scale is super-horizon to the epoch of recombination, $\eta_{\rm rec}$ (Eq. (3.36)). At the epoch of recombination the photons decouple from the baryons and therefore the source for magnetic field generation vanishes. We do not attempt an explicit solution here but seek an approximate understanding of the generated magnetic field. We first justify our use of the tight-coupling approximation. As discussed in Appendix A, for each scale the tight-coupling approximation is valid for epochs before the Silk damping regime (Eq. (3.34)). From Eq. (3.25), the magnetic field at a given scale $\mathbf{k}$ gets contribution from density and velocity perturbations at all scales. It should however be noted that if $\mathbf{k}$ corresponds to a scale at which the density and velocity perturbations are in damping regime the source of magnetic field is also in the damping regime. (More precisely one is interested in the power spectrum of the magnetic field, which, from Eq. (3.25), is a four-point function containing density fields. For magnetic field at scale $\mathbf{k}$, the integrand of the source is $\propto P(k')P(|\mathbf{k}-\mathbf{k}'|)$, here $P(k)$ is the power spectrum of the density field; $|\mathbf{k}-\mathbf{k}'| \simeq k$ if both k and k' are not in the damping or free-streaming regime.) This means that most of the contribution to the magnetic field comes from epochs at which the tight-coupling approximation is valid. More precisely much of the contribution to magnetic field at a scale k comes from epochs $\eta \lesssim \eta_d \simeq \omega_d^{-1}$ (Eq. (3.35)). For scales that are not in the damping regime at $\eta_{\rm rec}$ the upper limit of the integral in Eq. (3.25) is $\eta_{\rm rec}$. For smaller scales, the upper limit is $\simeq \eta_d$. Having identified some generic features of the magnetic field source terms we can give an order-of-magnitude estimate of the generated field from Eq. (3.30). For all scales a reasonable upper limit on the generated magnetic field for the currently-favoured ΛCDM model, at a scale $L \simeq k^{-1}$, is:

$$(a^2B)(L,\eta_0) = (a^2B)(L,\eta_1) \simeq 10^{-30}\,\mathrm{G}\left(\frac{\delta_e(\eta_1)}{10^{-3}}\right)\left(\frac{v_e(\eta_1,L)}{10\,\mathrm{km\,sec^{-1}}}\right)\left(\frac{10\,\mathrm{Mpc}}{L}\right) \tag{3.26}$$

Here $\eta_1 = \eta_{\rm rec}$ for scales that are not in Silk damping regime at the epoch of last scattering and $\eta_1 \simeq \eta_d$ for smaller scales. In Eq. (3.26) we have used the fact that once the sources of generating magnetic fields vanish, the magnetic field evolves such that a^2B remains constant (see e.g. Wasserman 1978). In Eq. (3.26), $v_e(L) \simeq (kP(k))^{1/2}$ (the matter power spectrum is given e.g. by Bardeen et al. 1986) and $\delta_e(\eta_1)$ is the RMS of the density field. One can compare the magnetic fields at different scales by evaluating the sources at the epoch of recombination $\eta_{\rm rec}$. As seen from Eq. (3.36) the density field is either non-evolving or in the oscillatory phase for much of the period prior to recombination (except for modes $\eta^{-1} \lesssim k \lesssim \left(\eta/\sqrt{3}\right)^{-1}$ in the matter-dominated epoch). The velocity field however grows $\propto \eta$ for super-horizon scales in the radiation dominated era and for $k \lesssim (\eta/\sqrt{3})^{-1}$ in the matter dominated era. Therefore if Eq. (3.26) is evaluated at $\eta_{\rm rec}$, the small scale magnetic field is approximately smaller by a factor $(\eta_{\rm ent}/\eta_{\rm rec})$; here $\eta_{\rm ent} \simeq k^{-1}$ is the epoch of horizon entry of the mode k. Using this to write the sources of the magnetic field at $\eta_{\rm rec}$ in Eq. (3.26), it is seen that, for all scales, $B(\eta_0) \lesssim 10^{-30}$ G.

3.4 Conclusion and discussion

We have studied the possibility of generating magnetic fields during the evolution of the photon-baryon plasma in the pre-recombination universe. For scalar perturbation in linear theory magnetic field is not generated at any scale; this includes scales at which the photon-baryon coupling approximation breaks down. We show that in the second order in perturbation theory a small magnetic field is generated. The strength of the generated magnetic field is $\lesssim 10^{-30}$ for scales from $\simeq 100$ Mpc to sub-kpc at the present epoch.

In Eq. (3.22), we have neglected several terms which could back-react on the generated magnetic field. It can be readily seen that, for the strength of the generated field, these terms are always much smaller than the source term of the magnetic field. And therefore we were justified in neglecting those terms for studying the generation of magnetic field. As discussed above magnetic fields at small scales are frozen in the plasma from epochs $\simeq \omega_d^{-1}$. It could be asked whether the radiative viscosity prior to the recombination can damp these fields. The maximum length scale damped by pre-recombination radiative viscosity is $\propto B$ (Jedamzik, Katalinić, & Olinto 1998, Subramanian & Barrow 1998). For the small magnetic fields we obtain, the maximum scale of dissipation can be shown to be much smaller than any relevant length scales.

Appendix A

In this section we discuss the initial conditions and the tight coupling approximation relevant to the study of photon-electron fluctuations.

The electron-proton plasma recombines at a redshift $z_{\rm rec} \simeq 10^3$ (see e.g. Peebles 1993). At any epoch in the universe before recombination, $\eta \lesssim \eta_{\rm rec} \simeq 2H_0^{-1}(1+z_{\rm rec})^{-1/2}/\Omega_m^{1/2}$, there are roughly five physically relevant length scales: (a) Super-horizon scale, $k \lesssim \eta^{-1}$ (b) scales that are sub-horizon but larger than the sound Horizon, $\eta/\sqrt{3} \gtrsim k \gtrsim \eta^{-1}$. At these scales the evolution of velocity fields is determined by gravitational potentials. (c) scales smaller than the sound horizon scale but larger than the Silk damping scale, $\eta/\sqrt{3} \lesssim k \lesssim k_{\rm silk}$. At these scales the baryon velocity evolution is determined by both gravitational potentials and the photon pressure, (d) scales that are in the damping regime but larger than the scales at which photon free-stream, $k_{\rm silk} \lesssim k \lesssim k_{\rm fs}$, $k_{\rm fs} \simeq (2\,{\rm Mpc})^{-1}(10^3/(1+{\rm z}))^{-2}$. The densities and velocities of baryons decay exponentially in this regime (see e.g. Peebles 1980) and (e) $k \gtrsim k_{\rm fs}$, at these scales photons are free-streaming and therefore photons and baryons cannot be treated as coupled fluids. During the evolution in the expanding universe before recombination, the electron velocity and density perturbations at most scales first pass through the Silk damping regime before reaching this phase. Therefore during this phase $\delta_e, v_e \simeq 0$. The only exception to this occurs around the epoch of recombination when the free-streaming length increases very rapidly. As the sources of magnetic field generation are nearly zero in this regime, the dynamics of plasma at these scales play an unimportant role for our study. In the evolution in linear theory all scales undergo either some or all of these phases of evolution.

Initial condition for each mode is set outside the horizon. Up to phase (c) discussed above, $k \ll k_{\rm fs}$. During this phase the photons are tightly coupled to the baryons and this greatly simplifies the problem (Peebles & Yu 1970, Peebles 1980, Hu & Sugiyama 1995). In this approximation, to zeroth order in $\tau_{\gamma e}$: $\mathbf{v}_\gamma = \mathbf{v}_e$; and $\Pi^{ij} = 0$. Also to this order in $\tau_{\gamma e}$: $\delta_e = 3/4\delta_\gamma$; this can be readily obtained by subtracting the electron continuity equation from the photon continuity equation (Eqs. (3.14) and (3.16)). To solve for the difference between electron and photon bulk velocity we need to expand to the first order in $\tau_{\gamma e}$. To this order, from Eq. (3.15) (Peebles & Yu 1970, Peebles 1980):

$$\mathbf{v}_\gamma - \mathbf{v}_e = \tau_{\gamma e}\left(\frac{\partial \mathbf{v}^i_\gamma}{\partial \eta} + \frac{1}{4}\boldsymbol{\nabla}\delta_\gamma + \boldsymbol{\nabla}\boldsymbol{\Pi} + \boldsymbol{\nabla}\Psi\right) \tag{3.27}$$

Here the quantities in the bracket on the right hand side are to be evaluated to the zeroth order in $\tau_{\gamma e}$. Eq. (3.27) along with the evolution of electron velocity (Eq. (3.18)) and the difference of electron and proton velocities (Eq. (3.20)) can be used to give the following expression for the electric field in the tight coupling approximation:

$$a\mathbf{E}(\mathbf{x}, t) = \frac{m_e}{e}\left(\frac{\dot{a}}{a}R\mathbf{v}_e - \frac{\boldsymbol{\nabla} p_e}{\rho_e} - \frac{1}{4}R\boldsymbol{\nabla}\delta_\gamma\right) \tag{3.28}$$

In deriving Eq. (3.28) all terms proportional to the magnetic field were dropped as they cannot act as sources for generating magnetic field. Taking the curl of this equation and using Maxwell's equation (Eq. (3.39)) one obtains the equation for magnetic field generation:

$$\frac{1}{a}\frac{\partial}{\partial \eta}(a^2\mathbf{B}) = \frac{m_e}{e}\boldsymbol{\nabla}\times\left(-\frac{\dot{a}}{a}R\mathbf{v}_e + \frac{\boldsymbol{\nabla} p_e}{\rho_e} + \frac{1}{4}R\boldsymbol{\nabla}\delta_\gamma\right) \tag{3.29}$$

This equation verifies the discussion above that the source of magnetic field generation is electron vorticity in the linear theory. With non-vortical initial conditions, Eq. (3.29) shows that all the sources of magnetic field generation are zero in the linear perturbation theory. We wish to consider the second order effect by considering the spatial dependence of densities. This gives: $R = \bar{R}(\delta_e - \delta_\gamma) \simeq -1/3\bar{R}\delta_e$ in the tight coupling approximation, as $\delta_e = 3/4\delta_\gamma$ during adiabatic expansion (see e.g. Peebles 1980). In second order, the second term on the right hand side is the Biermann battery term. In the adiabatic initial condition we consider here, $p = \rho^\gamma$ with $\gamma = 5/3$. And the source term $\propto \nabla\delta_e \times \nabla\delta_e = 0$ and therefore in this limit the Biermann battery term doesn't contribute. It should be noted that the plasma evolves adiabatically only for scales that are not affected by Silk damping (see below). However as the densities and velocities damp in this regime one doesn't expect much contribution from these scales. Biermann battery term can also contribute for initial conditions different from the adiabatic initial conditions. The third term on the right hand side also vanishes even in the second order in the tightly-coupled regime. Therefore the only source of magnetic field generation is the first term on the right hand side of Eq. (3.29). Eq. (3.29) can therefore be simplified to:

$$\frac{1}{a}\frac{\partial}{\partial\eta}(a^2\mathbf{B}) = \frac{\bar{R}m_e}{3e}\boldsymbol{\nabla}\times\left(\frac{\dot{a}}{a}\delta_e\mathbf{v}_e\right) \tag{3.30}$$

This equation can be used to get an order-of-magnitude estimate of the generated magnetic field.

Eq. (3.27) can be used to calculate, to the zeroth order, the evolution equation of electron velocity field (Peebles & Yu 1970). This equation along with the continuity equation (Eq. (3.16)) and $\nabla.\mathbf{E} = 0$, Eq. (3.42), and dropping all terms proportional to $\mathbf{B}$, gives:

$$\ddot{\delta}_e = -\frac{\dot{a}}{a}\frac{\dot{\delta}_e}{(1+R)} - \frac{k^2 p_e}{\rho_e(1+R)} - k^2\Psi - \frac{R}{4(1+R)}k^2\delta_\gamma + \frac{3\dot{a}}{a}\frac{\dot{\Phi}}{(1+R)} + 3\ddot{\Phi} \tag{3.31}$$

This equation can be solved along with the evolution equation of δ_γ by WKB approximation and these solutions can be matched to large scale solutions (Hu & Sugiyama 1995). We discuss here approximate solutions at different epochs. First we discuss solutions during phase (c) of the evolution. We note that all the terms on the right hand side except for the δ_γ term are smaller as compared to this term for scales smaller than the sound horizon scale $\simeq 1/\sqrt{3}\eta$. The electron pressure is always negligible as compared to the photon pressure in the pre-recombination universe. With these simplification and bearing in mind that $1/R \ll 1$ during the evolution, Eq. (3.31) is solved to give:

$$\delta_e(\mathbf{k},\eta) = A(\mathbf{k})\cos\left(\int_0^\eta \omega_o d\eta'\right) \tag{3.32}$$

Here we have only retained the solution compatible with adiabatic initial conditions (see e.g. Hu & Sugiyama 1995) and

$$\omega_o = \frac{k}{\sqrt{3(1+1/R)}} \tag{3.33}$$

In phase (d) of the evolution of the plasma, the tight-coupling approximation breaks down and the photon-slip which damps perturbations (Silk damping) must be taken into account. The solution including the Silk damping is (see e.g. Peebles 1980):

$$\delta_e(\mathbf{k},\eta) = A(\mathbf{k})\cos(\omega_o\eta)\exp(-\omega_d\eta) \tag{3.34}$$

Here,

$$\omega_d \simeq \frac{2k^2\tau_{\gamma e}}{15} \tag{3.35}$$

The silk damping scale, at any epoch, can be obtained from this expression: $k_{\rm silk} \simeq (15/(2\tau_{\gamma e}\eta))^{1/2} \simeq (4\,{\rm Mpc})^{-1}((1+z)/10^3)^{-5/4}$ in the matter-dominated era. The velocity field in the linear evolution remains non-vortical, and hence can be found from the continuity equation (Eq. (3.16)). It should be noted that solutions for the baryon density and velocity fields differ from the corresponding quantities for the electrons only by replacing R defined here as $R' = 4\rho_\gamma/(3\rho_b)$ (see e.g. Hu & Sugiyama 1995). As $R' \gg 1$ for the evolution of the plasma in the pre-recombination universe, for baryonic densities compatible with primordial nucleosynthesis, the baryon and electron quantities can be used interchangeably in evaluating the second order expression above.

The evolution of electron density and velocity in the oscillatory regime and the super-horizon solutions, prior to the epoch of recombination, can be summarized as (for solutions at super-horizon scales in this conformal-Newton gauge see e.g. Ma & Bertschinger 1995):

$$
\begin{aligned}
\delta_e &\propto \Psi = \text{constant} \quad \text{for} \quad k \lesssim \eta^{-1} \quad \text{(RD and MD)} \\
\delta_e &= \text{oscillatory} \quad \text{for} \quad k \gtrsim \left(\frac{\eta}{\sqrt{3}}\right)^{-1} \quad \text{(RD and MD)} \quad \text{and for} \quad k \gtrsim \eta^{-1} \quad \text{(RD)} \\
\delta_e &\propto \eta^2 \quad \text{for} \quad \eta^{-1} \lesssim k \lesssim \left(\frac{\eta}{\sqrt{3}}\right)^{-1} \quad \text{(MD)} \\
v_e &\propto k\Psi\eta \quad \text{for} \quad k \lesssim \eta^{-1} \quad \text{(RD and MD)} \quad \text{and for} \quad \left(\frac{\eta}{\sqrt{3}}\right)^{-1} \gtrsim k \gtrsim \eta^{-1} \quad \text{(MD)} \\
v_e &= \text{oscillatory} \quad \text{for} \quad k \gtrsim \left(\frac{\eta}{\sqrt{3}}\right)^{-1} \quad \text{(RD and MD)}
\end{aligned} \tag{3.36}
$$

Here RD and MD correspond to radiation and matter dominated epochs, respectively.

Appendix B

The Maxwell's equations for the LFRW metric in terms of physical fields, $\mathbf{E}, \mathbf{B}, \mathbf{J}$ are as follows:

$$\nabla \times (a^2\mathbf{B}) = 4\pi a^3 \mathbf{J} + \frac{\partial(a^2\mathbf{E})}{\partial\eta} \tag{3.37}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.38}$$

$$\nabla \times (a^2\mathbf{E}) = -\frac{\partial(a^2\mathbf{B})}{\partial\tau} \tag{3.39}$$

$$\nabla \cdot (a^2\mathbf{E}) = 4\pi a^3 e(n_p - n_e) \tag{3.40}$$

The current $\mathbf{J}$ is written in terms of fluid quantities as:

$$\mathbf{J} = e(n_p\mathbf{v}_p - n_e\mathbf{v}_e) \tag{3.41}$$

Here, $n_{e,p}$ are the electronic and protonic number densities which are assumed to be equal to the lowest order i.e $\bar{n}_e = \bar{n}_p = n$. From Eq. (3.37) it follows that $\nabla.\mathbf{J} = 0$ if the second term can be neglected, which is the case here (see e.g. Parker 1979). Eq. (3.16) along with Eq. (3.41) then shows that:

$$\boldsymbol{\nabla}.\mathbf{E} = 0 \tag{3.42}$$

in the linear theory.

Bibliography

Bardeen J.M., Bond J.R., Kaiser N. & Szalay A.S., 1986, ApJ, 304, 15

Beck R., Brandenburg A., Moss D., Shukurov A. M., Sokoloff D. D., 1996, Ann. Rev. Astron. Astrophys., 3 4, 155

Berezhiani Z. & Dolgov A. D., 2004, Astroparticle Physics, 21, 59

Biermann, L. 1950, Zs. Naturforsh., 5 , 65

Brandenburg, A. & S ubramanian, K., 2005, Phys. Rep., 417, 1

Dodelson S. & Jubas J.M. 19 95, ApJ, 439, 503

Freedman W.L. et al. , 2001, ApJ, 553, 47

Giovannini M., 2004, Int.J.Mod.Phys. , D13, 391

Grasso D., & Rubinstein H., 2001, Phys. Rep., 348, 161

Harrison E.R., 1970, MNRAS, 147, 279

Hogan C., 2000, astro-ph/0005380

Hu W. & Sugiyama N., 1995, ApJ, 444, 489

Hu W. & White M., 1997, Phys. Rev D, 56, 596

Hu W., Scott D. & Silk J., 1994, Phys. Rev. D, 49, 648

Jedamzik K., Katalinić V. & Olinto A.V., 1998, Phys. Rev. D., 57, 3264

Kulsrud R.M., Cen R., Ostriker J.P. & Ryu D., 1997, ApJ, 480, 481

Ma C. & Bertschinger E., 1995, ApJ, 455, 7

Matarrese S., Mollerach S., Notari A., Riotto A., 2005, Phys. Rev. D., 71, 043502

Parker E.N., 1979, *Cosmic Magnetic Fields*, Oxford, Clarendon Press; New York, Oxford University Press, 858

Peebles P.J.E., 1993, *Principles of Physical Cosmology*, Princeton University Press

Peebles P.J.E., 1980, *The Large Scale Structure of the Universe*, Princeton University Press

Peebles P.J.E. & Yu J.T., 19 70, ApJ, 162, 815

Peiris H.V. et al. , 2003, ApJS, 14 8, 213

Perlmutter S. et al. , 1999, ApJ, 517, 565

Ratra B., 1992, ApJ Lett., 391, L1

Rees M.J., 1987, QJRAS, 28, 197

Riess A.G. et al. , 2004, ApJ, 607, 665

Riess A.G. et al. , 1998, Astron. Journal , 116, 1009

Ruzmaikin A. A., Shukurov A. M., Sokoloff D. D., 1988, *Magnetic Fields of Galaxies*, Kluwer, Dordrecht (1988)

Shu F., 1992, *The Physics of Astrophysics, Vol.2: Gas Dynamics.*, Univ. Sci. Books, Mill Valley

Spergel D.N. et al. , 2003, ApJS, 148, 175

Subramanian K. & Barrow J.D., 1998, Phys. Rev. D., 58, 83502

Subramanian K., Narasimha D., & Chitre S.M., 1994, MNRAS, 271, L15

Shukurov A., 2004, Introduction to galactic dynamos, In Mathematical aspects of natural dynamos, Ed. E. Dormy, Kluwer Acad. Publ., Dordrecht.

Tonry J.L. et al. , 2003, ApJ, 594, 1

Turner M.S. & Widrow L.M., 198 8, Phys. Rev. D., 37, 2743

Tytler D., O'Meara J.M., Suzuki N., & Lubin D., 2000, Physics Reports, 333, 409

Vishniac E.T., 1987, ApJ, 322, 597

Wasserman I., 1978, ApJ, 224, 337

Zeldovich I.B. , Ruzmaikin A.A., & Sokolov D.D., 1983, New York, Gordon and Breach Science Publishers (The Fluid Mechanics of Astrophysics and Geophysics. Volume 3), 1983, 381