GEOMETRIC FLOWS
AND
BLACK HOLE ENTROPY

by

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Certificate:

This is to certify that the thesis entitled “GEOMETRIC FLOWS AND BLACK HOLE ENTROPY” submitted by SUTIRTHA ROY CHOWDHURY for the award of the degree of Doctor of Philosophy of Jawaharlal Nehru University is his original work. This has not been published or submitted to any other University for any other Degree or Diploma.

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Declaration:

I hereby declare that the work reported in this thesis is entirely original. This thesis is composed independently by me at Raman Research Institute under the supervision of Prof. Joseph Samuel. I further declare that the subject matter presented in this thesis has not previously formed the basis for the award of any degree, diploma, membership, associateship, fellowship or any other similar title of any university or institution.

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Dedicated to the memory of my uncle

Rathin Roy Chowdhury

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Contents

PREFACE

This thesis is a study of geometric flows and the thermodynamic features of black holes. Geometric flows are differential equations which arise in a geometric context. Such equations have been used in the past to understand aspects of general relativity. We start with a general introduction to black holes and summarize some important results, which are theorems in differential geometry. We see that, in the physics of black holes, there is a delicate interplay between thermodynamics and differential geometry. Differential geometric equations and inequalities from black hole physics show strong thermodynamic analogies. An example of this is the area theorem which represents the second law of thermodynamics in geometric form. Similarly, the ADM energy in general relativity is expressed as an integral of geometric quantities over a sphere at infinity.

Our interest here is in understanding the physics from a geometric point of view. While the physics acts as a guide for keeping the study firmly in touch with reality, differential geometry provides a mathematical language for expressing these physically motivated intuitive ideas. A good part of this thesis is devoted to applications of the Ricci flow in general relativity. The Ricci flow is a heat equation for metrics and we apply this flow to Riemannian metrics on three dimensional manifolds. These metrics are viewed as time symmetric initial data for Einstein's general theory of relativity. Specifically we wish to pursue analogies between the Ricci flow which evolves metrics and the approach of a thermodynamic system to equilibrium.

In the second chapter we give an introduction to the the Ricci flow. The Ricci flow first appeared in physics in Friedan's work on the renormalization of σ models [Friedan 1980 *Phys. Rev. Lett.* **45**, 1057]. In an independent mathematical development it was used by Hamilton in order to study the topology of three-manifolds by geometric methods [R. S. Hamilton 1982 Three-manifolds with positive Ricci curvature *J. Differential Geom.* 17]. The problem of classifying three-manifolds has been at the forefront of modern topology since Riemann gave a classification theorem for two-dimensional manifolds via uniformization theorem. The work of W.Thurston in the late 1970's led to the hope of a similar classification of three-manifolds. The search for homogeneous metrics led Hamilton to use the Ricci flow as a way of making spaces homogeneous. With important recent contributions by Grigory Perelman, the problem has been almost completely solved and Thurston's program has been implemented to a large extent. Perelman gave a gradient formulation of Ricci flow [Grisha Perelman 2002 The entropy formula for the Ricci flow and its geometric applications *Preprint* math.DG/0211159] and this led to a catalogue of the three dimensional spaces with the exception of some still elusive hyperbolic spaces. In the process Pereleman also solved the century old Poincare conjecture. This thesis can be described as an attempt to apply these mathematical techniques to general relativity.

In the third chapter we describe the thermodynamic motivation for our work and give some simple models from physics where geometric differential equations lead a system to its maximum entropy final state. By analogy we try to attack the Riemannian Penrose inequality from this point of view. The Penrose inequality is a statement about initial data sets in general relativity (GR). It states that the ADM mass of an initial data set in GR is greater than a fixed constant ($1/4\sqrt{\pi}$) times the square root of the area of any apparent horizon it contains. This inequality in its general form remains unproved. In the special case of time symmetric initial data, Penrose's inequality reduces to a statement about Riemannian geometry and has been proved by mathematicians (Huisken-Ilmanen and Bray) using geometric flows. Our work tries to develop a new approach to the Riemannian Penrose inequality using the Ricci flow. Our motivating hope is that this new approach will have a clearer physical and thermodynamic interpretation and possibly lead to new ways of thinking about the general Penrose inequality. As we proceed, we see that this hope is not realized in its initial form, but along the way we derive a few geometric results which can be physically interpreted in terms of energy and entropy.

In the fourth chapter, we start with the simple situation of spherically symmetric asymptotically flat three manifolds. In this case the equations are quite tractable and the physical interpretation quite transparent. We write down the Ricci flow adapted to spherical symmetry and develop our intuition for studying the general case. We introduce the geometric quantities of interest, the area of a closed two surface, its “compactness” and its Hawking mass. We study the behaviour of area, compactness and Hawking mass under Ricci flow. We note that if a three manifold contains an apparent horizon (minimal surface), the flow ends in a singularity at a finite time. We also note that there is a maximum principle for compactness. If the maximum value of the compactness is less than 16π, it always decreases under a Ricci flow. This leads to the result that the flow exists for all time. We thus have a clear understanding of the existence of the Ricci flow in the spherically symmetric asymptotically flat case. Chapter five treats the same physical situation but using numerical methods to evolve the Ricci flow. We are able to evolve the flow numerically and recover our analytical results from an independent method.

In Chapter six, we try to generalize some of our results to a general (not spherically symmetric) asymptotically flat three manifold. In this we are partially successful. We derive an inequality relating the evolution of area of a surface and its Hawking mass. We study the evolution of compactness of a surface and its Hawking mass. These quantities have a clear physical significance in terms of entropy and energy. While these results are interesting, they do not lead to a new proof of the Riemannian Penrose inequality.

The problem seems to be that while the entropy introduced by Perelman in his gradient formulation of the Ricci flow has some features in common with black hole entropy, it is not quite the same. In Chapter seven, we first show that it is not possible to identify Perelman entropy with black hole entropy. The proof involves showing that the Schwarzschild metric (which maximizes black hole entropy for a fixed energy) is not a fixed point of the Perelman flow. We then propose a new flow, which is very similar to the Perelman flow, differing only in that the diffusion constant is space dependent. The new flow does have the virtue of having the Schwarzschild space as a fixed point. Chapter eight is a concluding discussion.

Appendix A describes earlier approaches to the positive mass theorem and the Riemannian Penrose inequality using the inverse mean curvature flow. This can be reinterpreted as a geometric flow on the space of metrics: changing the metric by a diffeomorphism is equivalent to flowing a surface in a fixed manifold. Appendices B, C and D are mathematical appendices which contain some of the calculations.

SOME OF THE WORK DESCRIBED IN THIS THESIS APPEARS IN THE FOLLOWING PAPERS:

[1] Joseph Samuel and Sutirtha Roy Chowdhury 2007 Geometric Flows and Black Hole Entropy *Class. Quantum Grav.* **24** F47.

[2] Joseph Samuel and Sutirtha Roy Chowdhury 2008 Energy, Entropy and Ricci Flow *Class. Quantum Grav.* **25** 035012.