Chapter 1

INTRODUCTION

1.1 BLACK HOLES IN ASTROPHYSICS

Quasars are by far the most luminous objects in the universe. The spectra of quasars show large red shifts (ranging from .158 for 3C273 to 6.4 for the most distant quasar known), indicating that they are at a cosmological distance. From the apparent luminosity (the energy received here on earth per unit area, per unit time) we can compute that a bright quasar puts out 10^{42} joules of energy every second. Converted from energy to mass units, this is about twice the mass of the earth per second or two hundred solar masses per year!

The luminosity of quasars varies on a time scale of a few days. From this, astronomers conclude that the source of the radiation can be no more than a few light days in size. It was considered incredible that an object not much bigger than our solar system could emit so much power. Some astronomers started looking for other explanations for quasars. It was suggested that perhaps the observed redshifts did not indicate cosmological distance and were due to other causes. But further research eventually ruled out other explanations for the red shift, leading most astronomers to conclude that quasars really are the most distant and luminous objects in the universe.

There are very few mechanisms that produce enough energy to power a quasar. The model that best fits the observed properties of quasars is a *supermassive black hole*. A black hole is a region of space from which nothing can escape, not even light. Black holes are predicted by Einstein's general theory of relativity and result from the deaths of very massive stars. The black holes at the centers of quasars have masses of a million or even a billion times the solar mass. The origin and evolution of these supermassive black holes is the subject of current research.

The mechanism for producing the energy is believed to be accretion of gas. The black hole which powers the quasar is surrounded by a rotating cloud of gas called an accretion disc. As the gas swirls around and falls into the black hole, it is heated up by friction to millions of degrees and emits thermal radiation. This thermal radiation peaks in x-rays but spans decades over the electromagnetic spectrum, making the quasar bright in radio wavelengths, visible light as well as x-rays. With the discovery of quasars in the nineteen sixties, and the realization that these sources are energetic, small in size and efficient in conversion of matter to energy, the study of black holes ceased to be theoretical speculation and entered the realm of astrophysics.

Black holes are formed from the gravitational collapse of massive stars. Stars are born when clouds of gas are pulled together by gravity. As the cloud contracts, it heats up and finally nuclear reactions are ignited. Hydrogen atoms combine to release energy by nuclear fusion, and the star heats up some more and shines. There is competition between gravity which pulls in the stellar material and the pressure of the hot gas which pushes it out. This equilibrium is maintained for millions of years as the star burns its nuclear fuel and shines. However, the nuclear fuel is a finite resource and is spent with time unlike gravity which is ever present.

When a star has exhausted its fuel supply, gravitational forces crush the star to one of three possible end points: a white dwarf, a neutron star or a black hole. Which of these endpoints is attained depends on the mass. When the nuclear fuel is spent, the star cools and the only pressure resisting gravity comes from degeneracy pressure. A solar mass star can be supported by electron degeneracy pressure and this is called a white dwarf, the expected end point of our Sun. As the mass of the star increases, the electrons need to move faster in order to oppose the gravitational pressure. Because of the relativistic limit on their speed, the equation of state softens and a star above about 1.4 solar masses (the Chandrasekhar limit) cannot find equilibrium as a white dwarf. Stars of mass higher than 1.4 solar masses implode under gravity and then explode as the inner core of the star bounces. The result is a spectacular supernova explosion resulting in a collapsed core, a neutron star, which is supported by *neutron* degeneracy pressure. Stars of still higher mass end up as black holes.

A rough measure of the importance of gravity is given by the dimensionless Newtonian potential $\phi = 2GM/c^2R$, where M is the mass and R the radius of the object. For a white dwarf, ϕ is of order 10^{-4}, for a Neutron star ϕ is around .5 and for a black hole ϕ is of order unity. Thus, the mass to radius ratio measures the extent to which general relativistic effects are important. This ratio ϕ is sometimes known as the "compactness" of an object and we will encounter this notion again later in the thesis from a more abstract and geometric point of view.

1.2 BLACK HOLES IN GENERAL RELATIVITY

The last section dealt with black holes as astrophysical objects and emphasized that black holes are real objects in the universe and not a figment of our imaginations. From here onwards we regard black holes from a more theoretical viewpoint. The existence of black holes throws up puzzles and questions of principle and in understanding these we realize that there are deep interconnections between our theories of gravitation, quantum mechanics and thermodynamics. We consider physical processes involving interaction between black holes and their environment. We do not worry about the astrophysical relevance of the physical processes but focus rather on their logical consistency. This discussion is entirely standard and based on some lectures by Jacobson [1]. Let us start with a few dimensional considerations, beginning from the fundamental constants which appear in special relativity (c), quantum mechanics (h) and gravitation (G).

Dimensional Considerations: In Newtonian physics, the escape velocity of a spherical mass M of radius R is defined by

$$\frac{1}{2}v_{esc}^2 = GM/R => v_{esc} = \sqrt{2GM/R} \tag{1.1}$$

which is independent of the mass of the escaping object. v_{esc} exceeds the speed of light if $R < R_s := 2GM/c^2$. The radius R_s is called the "Schwarzschild radius" or "geometrical radius" for the mass M. In classical GR mass and length have the same dimensions. This is the opposite of what happens in quantum physics, where the Compton wavelength of a particle of mass M goes *inversely* as its mass and is equal to $\lambda_c = \hbar/Mc$. If we want to find the mass for which these two length scales are equal then we have to equate the Schwarzschild radius, R_s to the Compton wavelength, ($\lambda_c = \hbar/Mc$) to get the mass $M = M_p$ called the Planck mass. R_s is then the Planck length L_p and E_p is the Planck energy. Their values are

$$M_p = (\hbar c/G)^{1/2} \sim 10^{-5} gm \tag{1.2}$$

$$E_p = (\hbar c^5/G)^{1/2} \sim 10^{19} GeV \tag{1.3}$$

$$L_p = (\hbar G/c^3)^{1/2} \sim 10^{-33} cm. \tag{1.4}$$

The Planck time T_p is

$$T_p = L_p/c \sim 10^{-43} sec. \tag{1.5}$$

These are the scales at which the quantum effects of gravity become important. In this thesis we are dealing with classical gravity and so it is assumed that we are dealing with length and time scales much greater than the Planck scales. However as we will see quantum effects are necessary for a consistent thermodynamic interpretation of black holes.

Schwarzschild Metric: In general relativity (GR), the unique spherically symmetric vacuum solution of Einstein's equation is the *Schwarzschild metric* which, in Schwarzschild coordinates, is given by

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{1.6}$$

From the form of the metric, it is clear that something goes wrong at $r = R_s$, the Schwarzschild radius. However, this is only a co-ordinate singularity and can be removed by choice of coordinates. The Eddington-Finkelstein (EF) coordinate system is an example of such a *good* coordinate system in which the line element is given by

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{1.7}$$

Here $R_s = 2GM/c^2$ and M is the mass. We will suppose that R_s is positive. If $R_s = 0$ then we get flat spacetime.

The surface $r = R_s$ is the event horizon. It is a regular part of the spacetime. A local observer will not see anything unusual here. A local neighborhood of a point on this surface is identical to a local neighborhood of any other point. The event horizon is a global notion. At $r = 0$ the curvature diverges so there is a *true* singularity there. However this singularity cannot be "seen" from infinity since light rays do not escape to infinity from points within the event horizon. In this case the spacetime is called a black hole.

Energy in GR: Energy in special relativity (SR) is naturally defined and it is associated with time translational symmetry of Minkowski space. In GR no such symmetry is available in general and therefore the total energy is not a particularly natural concept in GR. To define energy in GR one needs to work harder. One can only define energy in certain restricted contexts. For instance the energy density seen by a local observer *is* well defined even in GR. If we compare the role of energy in GR with that in Newtonian gravitation[2] then we find that the energy density ρ of a source in Newtonian gravitation has a counterpart T_{ab}, the energy momentum tensor, in GR. However, there are no analogous quantities in GR for the total energy of the source, the energy density of the gravitational field and the total energy density of the gravitational field which, in Newtonian gravitation are given as $\int dV\rho$, $-(1/8\pi G)(\nabla\phi)(\nabla\phi)$ and $-(1/8\pi G \int dV(\nabla\phi)(\nabla\phi)$ respectively, where ϕ is the gravitational potential. It turns out, however, that the total energy including the contributions from both the source and the field, given as $\int dV\rho - (1/8\pi G \int dV(\nabla\phi)(\nabla\phi)$ in Newtonian gravitation, can be defined in GR provided the spacetime is asymptotically flat in a suitable sense.

There are, in fact two distinct regimes in which a total energy can be defined in GR. The first is spatial infinity, i.e., in the limit as one approaches the asymptotic region along spacelike directions (Fig. 1.1). This expression for energy is called the Arnowitt-Deser-Misner (ADM) energy or mass. More precisely the ADM mass is defined in terms of the behaviour of initial data on a spacelike three dimensional surface Σ, which asymptotically, approaches the behaviour of a $t = const.$ plane in Minkowski spacetime. The ADM mass

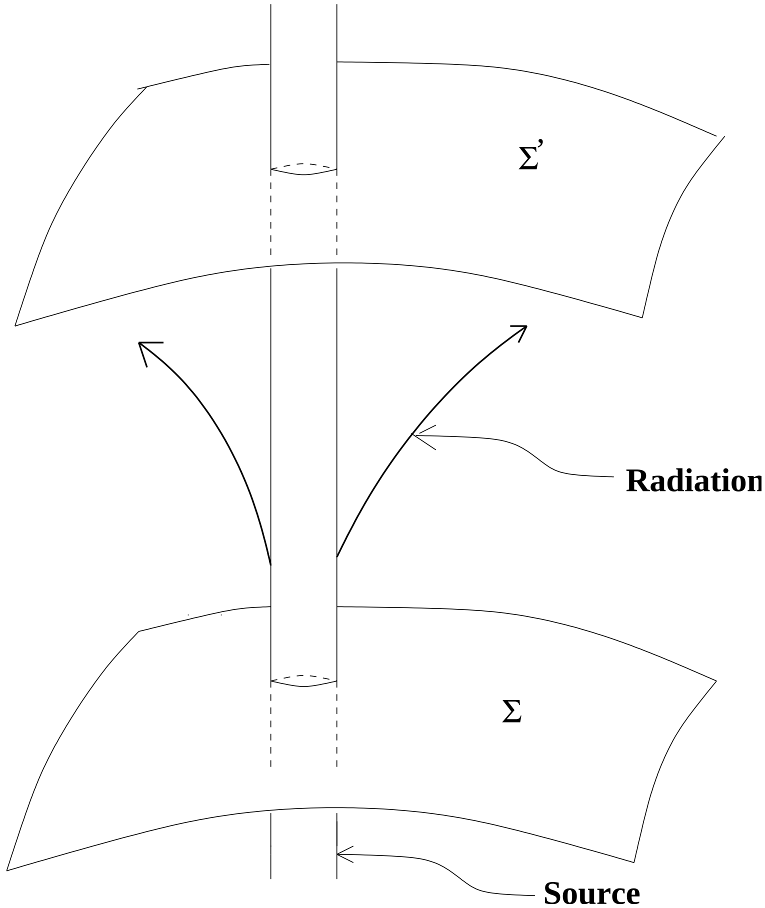

Figure 1.1: The Arnowitt-Deser-Misner energy. The energy is defined in terms of asymptotic behaviour on a spacelike, three dimensional surface Σ. Since radiation emitted between two such surfaces is incident on the second, the energy is invariant under translation of Σ to any Σ'in future.

is invariant under translation of the surface Σ to any other surface Σ' in the future as the radiation emitted between two such surfaces is incident on the second (Fig. 1.1)

The second regime in which total energy can be defined is at null infinity and this energy is called the Bondi energy. The Bondi energy is defined in terms of the rate of approach of the metric to flatness along a null surface that, asymptotically looks like a null cone. Since gravitational radiation can escape between successive surfaces, the energy decreases under translation of the surface into the future (Fig. 1.2). In this thesis we will deal only with the ADM energy, which is defined on a spatial slice of spacetime.

Let $(\mathcal{M}, g_{\mu\nu})$ $(\mu, \nu = 0, 1, 2, 3)$ be an asymptotically flat, globally hyperbolic spacetime (with signature $(-, +, +, +)$) and (Σ, h_{ab}, K_{ab}) be a spatial slice in $\mathcal{M}$. The induced metric of Σ

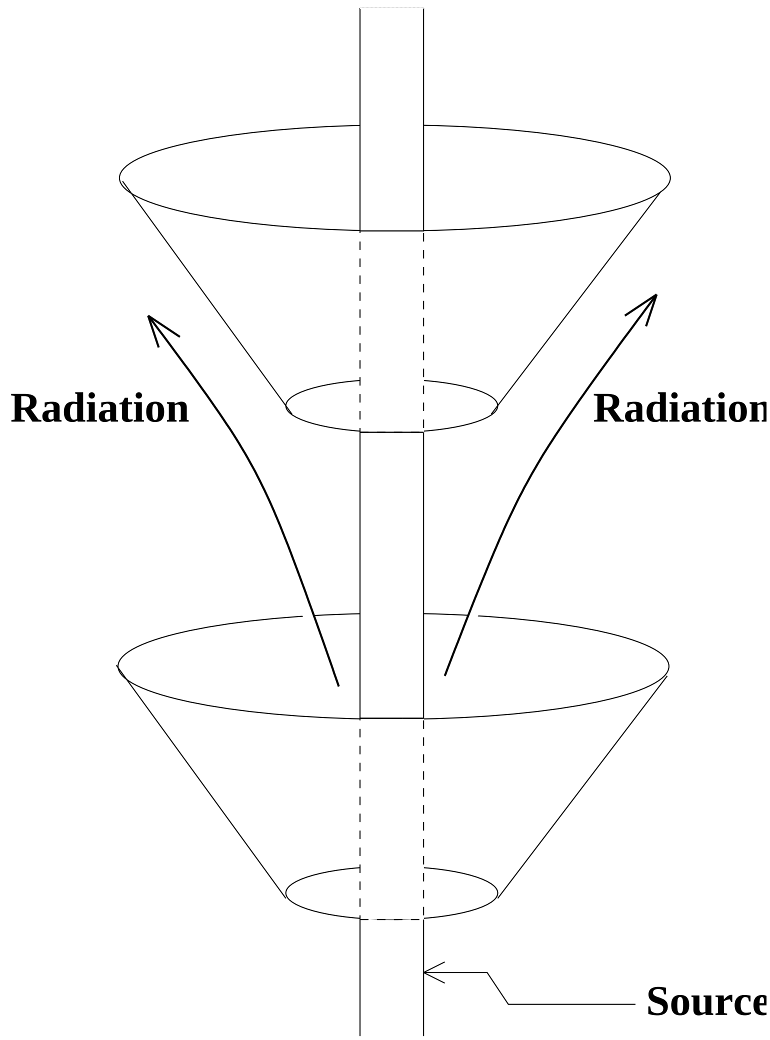

Figure 1.2: The Bondi energy. The energy is defined in terms of asymptotic behaviour on a three-dimensional null surface. Since radiation can escape between successive surfaces, the energy decreases under translation of the surface into the future.

is h_{ab} $(a, b = 1, 2, 3)$ and its extrinsic curvature is K_{ab}. The pair (h_{ab}, K_{ab}) is an initial data set for the spacetime. We also suppose that the slicing is asymptotically flat so that the spatial metric h_{ab} has the form :

$$ds^2 = h_{ab}dx^a dx^b \approx \left(1 + \frac{2M}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.8}$$

as $r \to \infty$. Note that we require that h_{ab} tends to a *fixed* flat metric δ_{ab} at infinity. The constant M is the ADM mass of the spacetime. It is expressed in terms of the initial data as a surface integral over a surface $\mathcal{S}$ at spatial infinity.

$$M := -\left(\frac{1}{8\pi}\right)\lim_{r\to\infty}\oint_{\mathcal{S}}(k - k_o)\sqrt{\sigma}dS \tag{1.9}$$

where σ_{ij} is the induced metric on $\mathcal{S}$, $k = \sigma^{ij}k_{ij}$ is the trace of the extrinsic curvature of $\mathcal{S}$ embedded in Σ, and k_o is the trace of the extrinsic curvature of $\mathcal{S}$ embedded in flat space. We suppose that the space time $(\mathcal{M}, g_{\mu\nu})$ satisfies the dominant energy condition [3]. If $G_{\mu\nu}$ is the Einstein tensor, we require

$$G_{\mu\nu}v^\mu w^\nu \geq 0 \tag{1.10}$$

for v, w future pointing timelike vectors.

Positive Mass Theorem: Even if we assume that an energy condition is satisfied, it is logically possible that the total ADM energy of an isolated gravitating system is negative! The positive rest energy of the matter forming the system may be cancelled by the negative potential energy of gravity. This does not happen and this is the content of the positive mass theorem. If one assumes that (1) the spacetime can be spanned by a nonsingular Cauchy surface whose only boundary is the one at infinity, and (2) the stress energy tensor T_{ab} satisfies the *dominant energy condition (DEC)* then it can be proved that the total energy of the spacetime is necessarily positive. This was first proved as a theorem in differential geometry by Schoen and Yau. Later Edward Witten provided a physically motivated proof using spinors. Yet another proof using focussing of light has been given by Penrose, Sorkin and Woolgar [4]. Positivity of the total energy at infinity does not necessarily mean that the system cannot radiate an infinite amount of energy while collapsing, since both the energy of the radiation and the energy of the leftover system are included in the total energy. The Bondi energy allows one to evaluate just the leftover energy. The Bondi energy is the gravitating mass as seen by the light rays propagating out to infinity in the light like direction, rather than spacelike direction. Like the ADM energy it can also be shown that the Bondi energy is also necessarily non negative. Thus *only a finite energy can be radiated away*.

Singularity theorems: Unlike the co-ordinate singularity at $r = R_s$, the origin of the Schwarzschild metric $r = 0$ has a true curvature singularity. It was first believed that this singularity was an artifact of spherical symmetry and that a generic collapse would evade the singularity. However, work by Hawking and Penrose showed that this was not so and that singularities were generic rather than special. The ubiquity of singularities is guaranteed by the *singularity theorems* by Hawking and Penrose. The idea of the proof rests on the concept of a trapped surface. If one considers a closed spacelike 2-sphere in ordinary Minkowski space and emits a flash of light from it, the outward moving wave front expands at all points and the inward moving wavefront contracts. However, in a strong gravitational field, it can happen that there are 2-surfaces for which both these wave fronts are everywhere contracting. Such a surface is called a trapped surface. Light from such a surface is doomed to stay within the finite part of space-time and will never make it to infinity. The boundary of the future of the 2-sphere then contains inextendible geodesics, indicating a singularity. However, the singularity is not visible from infinity.

Negative mass and cosmic censorship: If $R_s < 0$ then there is no event horizon and the singularity is *naked*. Light rays can escape from the singularity to infinity and an observer at infinity can see the singularity. There is no telling what will come out of the singularity and the observer will be unable to predict the future. This situation is generally believed not to arise and this is the content of the *cosmic censorship conjecture*. The conjecture states that given non-singular initial data and energy conditions, evolution using Einstein's equations does not produce naked singularities. We would like to believe this conjecture is true and that the naked singularities do not occur in nature. While there has been much work in this area there is not much progress as there is no precise formulation of the cosmic censorship conjecture. One of the approaches is to assume some intuitive formulation of the cosmic censorship conjecture and explore its consequences. One example of this approach is Penrose's observation that the cosmic censorship conjecture implies constraints on the initial data of GR. These constraints are known as *Penrose's Inequalities*. Since these inequalities are formulated on the space of initial data, they are easier to test than carrying out the difficult task of evolving Einstein's equations to check for cosmic censorship.

We saw in the last section that black holes are good sources of energy. It is also true that they are good sinks of information. One can hide information in black holes and then it becomes impossible to retrieve. It is as good as lost! For instance, a star which collapses to form a black hole can be characterized by its multipole moments and its chemical composition. The black hole which forms has no memory of the initial star. It is characterized by just a few parameters, its mass, its charge and its angular momentum. This is formalized in the

- **No hair theorem:**: *Stationary asymptotically flat black hole solutions to GR coupled to electromagnetism that are nonsingular outside the event horizon are fully characterized by the parameters of mass, electric and magnetic charge and angular momentum*.

The family of stationary black hole solutions is the Kerr-Newman family which contains three parameters corresponding to mass, charge and angular momentum.

1.3 ENERGY EXTRACTION FROM BLACK HOLES

Energy extraction: A black hole can be used to extract the rest energy of a particle as useful work. An example of this is the mechanism by which astrophysical quasars are believed to convert up to about 5% of the rest mass of the infalling gas into energy. This is considerably more efficient than say nuclear reactions in a star, which only convert 1% of the rest mass into energy. In fact, theoretically we can use black holes to convert mass into energy with 100% efficiency (barring quantum effects). Let us consider some of these energy extraction processes in idealized theoretical form. Suppose a mass m is slowly lowered by means of a

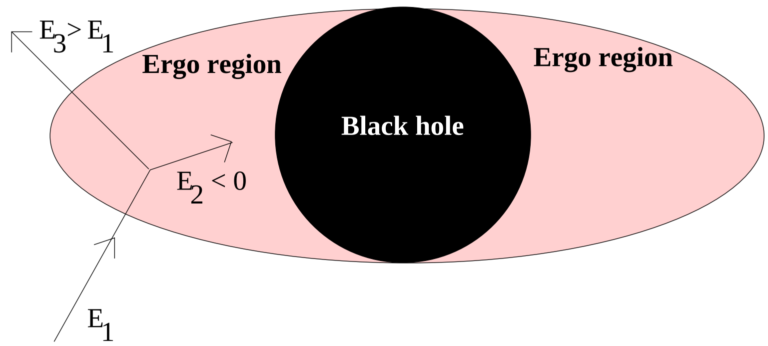

Figure 1.3: Penrose process to extract rotational energy by exploiting the ergo region of a rotating black hole: The incoming particle entering into the ergoregion with a positive energy E_1 breaks into two particles inside the ergoregion. This process can be so arranged that the energy of one particle is negative ($E_2 < 0$). This particle falls into the black hole while the other particle escapes to infinity with an energy E_3 which is more than E_1, the energy of the original particle. Thus $(E_3 - E_1)$ amount of energy is extracted from the black hole at the expense of its rotational energy.

rope into a black hole, and its potential energy stored at infinity. When the mass reaches the event horizon, all the rest energy has been converted into energy at infinity. Dropping the particle into the black hole does not increase the mass of the black hole since the energy has all been extracted.

Penrose process: Penrose suggested that energy can be extracted from a black hole if it is spinning or charged. This process is not of interest astrophysically, but from the theoretical point of view it tells us a lot about the interaction of a black hole with its environment. One can exploit the existence of the ergoregion to extract the rotational energy of a rotating black hole. A particle is sent into the ergoregion, where it breaks up into two particles, arranged so that one particle enters a negative energy orbit and falls across the horizon, while the other escapes to infinity with energy greater than the initial energy. This follows from the conservation of energy. The extracted energy comes at the expense of the rotational energy of the black hole, so the negative energy particle must have an angular momentum opposite to that of the black hole. The most efficient energy extraction process would be the one for which the ratio of energy to angular momentum extracted is maximized. This limiting efficiency is reached [1] when the horizon area is unchanged by the process. For a black hole with a positive electrical charge, one can extract energy from it by dropping in negative charges. These charges can go into negative energy orbits and give us a net positive energy at infinity. The net effect of this process is to neutralize the charged black hole.

Limits on energy extraction: It was shown by Hawking that the area of an event horizon can never decrease under quite general assumptions. Hawking's area theorem puts limit on the efficiency of energy extraction. We saw that the most efficient energy extraction occurs when the black hole area is unchanged and in less efficient processes the area always increases. This is very similar to the thermodynamic idea that reversible engines are the most efficient and that irreversible processes waste energy by converting it into entropy.

If there is no limit on energy extraction then we could extract an infinite amount of energy from a system. This is a physically absurd possibility and if it was allowed, would point to some inconsistency in the theory. For this discussion it is necessary to impose some kind of energy condition. For, if matter with negative local energy were allowed, one could decrease the energy further and extract energy making the local energy even more negative. In this case, there would certainly be no limit on the maximum amount of energy that can be extracted from a system. No matter with negative local energy is known and it is generally believed that no such matter exists. In fact the area theorem and the singularity theorem quoted above explicitly assume an energy condition. There are various forms for the energy condition e.g. strong, weak and dominant energy condition. All these forms imply that the local energy density is always positive.

Even given a local energy condition, it could have been that total ADM energy is negative. If such a situation were possible, this too would point to an absurdity in the theory. For, if the total energy were allowed to be negative, by scaling the system we can make it arbitrarily negative. There is then no lower bound to the energy and there again arises the possibility of infinite energy extraction. The positive mass theorem saves GR from this absurd possibility.

Another possibility arises when naked singularities are permitted. The ADM energy, which is the numerical value of the Hamiltonian generating the time translation symmetry at infinity can be negative e.g. if we simply put $R_s < 0$ in the EF metric (1.7), this yields a spacetime which is vacuum everywhere and has a naked singularity. $R_s < 0 \Rightarrow M < 0$ will then imply that by a suitable scaling of the metric we may make $M \to -\infty$ and can extract an infinite amount of energy from a black hole which is absurd in physics. Thus the cosmic censorship hypothesis is closely related to the positive mass theorem, a fact first noticed by Jang and Wald [5]. These are both needed for the physical consistency of GR.

1.4 BLACK HOLE THERMODYNAMICS

In the last section, we saw that that black holes can exchange energy with their surroundings. We can also think of dumping entropy into a black hole, thereby reducing the entropy of the universe. This appears to violate the second law of thermodynamics. In the nineteen seventies, it was proposed by Bekenstein that black holes have entropy proportional to their area. It would then seem that one can include black holes within the framework of thermodynamics and talk of a total entropy which still increases. This viewpoint is supported by the Hawking's area theorem which states that (given an energy condition) the total area of black holes never decreases. Hawking's area theorem is a result in differential geometry, but it seems to express the second law of thermodynamics! While this analogy was initially thought to be formal, it was subsequently realized that this analogy becomes real and physical once quantum effects are included.

If an object of energy E has entropy $S(E)$, it follows that it has a temperature given by

$$1/T = \frac{\partial S}{\partial E}.$$

Applying this to black holes one expects that black holes have a temperature inversely proportional to their mass (since their area goes as E^2). Amazingly, this turned out to be true. Hawking showed that quantum mechanically a black hole radiates a thermal spectrum in accord with this expectation. This calculation fixed the constant of proportionality between entropy and area $S = A/4$ in Planck units. The analogue of the temperature is the surface gravity κ of the black hole also in Planck units.

If one considers rotating black holes and energy extraction one finds that there is an analogue of the first law of thermodynamics: the energy change in the black hole can be split up into "heat" and "work" terms. Finally, the analogue of the zeroth law also holds: The surface gravity is constant over the horizon. For Schwarzschild black holes, this is trivial from spherical symmetry. But the zeroth law also holds for rotating black holes, where it is far from obvious. Thus the laws of thermodynamics hold for black holes. To summarize:

- First Law: The energy change of a black hole is the sum of heat and work terms. For a charged rotating black hole, the first law takes the form

$$dM = \kappa dA/8\pi G + \Omega dJ + \phi dQ \tag{1.11}$$

  where the first term on the RHS is the heat and the last two are the work terms.

- Second Law: The total area of black holes never decreases.

- zeroth Law: the surface gravity of a black hole is constant over the event horizon.

In thermodynamics, one sometimes hears of a "third law", the Nernst heat theorem. This law is not obeyed by black holes. In fact the third law of thermodynamics is not as fundamental as the other two. Even in ordinary thermodynamics, one can conceive of systems violating the third law without encountering any contradiction. In this thesis we will restrict ourselves to classical thermodynamics of black holes. There are slight corrections when one includes quantum effects. For example, the area of black holes may decrease due to emission of Hawking radiation.

1.5 THE PENROSE INEQUALITY

Let (Σ, h_{ab}, K_{ab}) be an initial data set with mass M. Let A be the area of the outermost apparent horizon [3] in Σ. The Penrose inequality states that if the energy condition holds,

$$M \geq \sqrt{\frac{A}{16\pi}}.$$

The Penrose inequality clearly implies PMT, which only states that $M \geq 0$.

The Penrose inequality was proposed as a test of the cosmic censorship conjecture. As mentioned before, we start with the assumption that the cosmic censorship conjecture is right and thereby derive an inequality which relates quantities that are determined directly by the initial data set. Next we look for an initial data set which violates the inequality and thereby derive a contradiction to the initial assumptions about the validity of the cosmic censorship conjecture. A generalized version of Penrose's argument goes as follows: We consider a spacetime, $(\mathcal{M}, g_{\mu\nu})$ $(\mu, \nu = 0, 1, 2, 3)$ which is an asymptotically flat, globally hyperbolic spacetime (with signature $(-, +, +, +)$) and let (Σ, h_{ab}, K_{ab}) be a spatial slice in $\mathcal{M}$. The induced metric of Σ is h_{ab} $(a, b = 1, 2, 3)$ and its extrinsic curvature is K_{ab}. The pair (h_{ab}, K_{ab}) is an initial data set for the spacetime. We consider a gravitational collapse taking place in this spacetime. The singularity theorems assure us that a singularity will form. The assumption that the cosmic censorship conjecture is valid will then imply that the collapse must produce a black hole which will settle down to a stationary final state. The only stationary vacuum black holes are the Kerr solutions (from the theorems of Israel, Hawking, and Robinson). The area of a Kerr black hole is

$$A_f = 8\pi[M_f^2 + (M_f^4 - J^2)^{1/2}] \leq 16\pi M_f^2 \tag{1.12}$$

where M_f is the mass (the final mass at the end of collapse) and J is the angular momentum of the black hole. Now let us consider the initial data for this spacetime on the earlier spacelike slice Σ. By the Hawking area theorem (which is also based on cosmic censorship conjecture) the initial surface area A_i of the horizon cannot be larger than the final area, so that

$$A_i \leq A_f. \tag{1.13}$$

Furthermore, since the final mass M_f at the end of the collapse cannot exceed the initial total mass (ADM energy of the initial data set) M, or in other words, since the energy carried off to infinity by radiation cannot be negative, assuming that no further extra matter or radiation falls in from the infinity during the collapse, we have

$$M_f \leq M. \tag{1.14}$$

Thus we obtain

$$A_i \leq 16\pi M^2. \tag{1.15}$$

We cannot make much use of this inequality since we must know the entire evolution before we can determine the location of the event horizon on Σ. However, there is a way to obtain a more useful inequality starting form the earlier one as follows: The apparent horizon $\mathcal{H}$ is defined as the outer boundary of the region of Σ which contains trapped or marginally trapped surfaces. $\mathcal{H}$ itself must be a marginally trapped surface, and thus it satisfies [5]

$$k + K^{ab}(h_{ab} - n_a n_b) = 0 \tag{1.16}$$

where k is the trace of the extrinsic curvature of $\mathcal{H}$ as a submanifold of Σ and n^a is the unit outward normal to $\mathcal{H}$ on Σ. One can show that $\mathcal{H}$ must be a topological 2-sphere or a disjoint union of spheres and must necessarily be inside of (or coincide with) the event horizon. Let A denote the greatest lower bound of the area of surfaces which enclose $\mathcal{H}$. Then clearly $A \leq A_i$ and, thus from (1.15),

$$A \leq 16\pi M^2 \tag{1.17}$$

which is Penrose's Inequality. It is clear that the inequality (1.17) relates quantities which are determined directly by the initial data on Σ and Penrose's idea was to find an initial data set which violates inequality (1.17). If such a data set exists, something must be seriously wrong with the assumptions which went into the above derivation of the inequality. The cosmic censorship conjecture is by far the weakest link in the chain of logic, so if a data set is found which violates the inequality (1.17), it will provide us with a counterexample to the cosmic censorship conjecture. It is also to be noted that the validity of the inequality (1.17) for all initial data sets would not imply that the cosmic censorship conjecture must be true.

Penrose originally proposed the above test of the cosmic censorship conjecture in the context of the collapse of a shell composed of null fluid, with the spacetime inside the shell being flat and with the shell coinciding with the apparent horizon at some instant of time. He and Gibbons ruled out the existence of violations of the above inequality in a wide class of cases, though they did not succeed in ruling out all the possible counterexamples of this type. Later, extending Geroch's argument to prove PMT [2], Jang and Wald gave a proof of nonexistence of a class of counterexamples proposed by Penrose [5]. This proof used a geometric flow, the inverse mean curvature flow to show that Penrose's inequality was valid for time symmetric initial data if the flow existed. The proof of Jang and Wald was only complete when the existence of this flow (slightly modified to allow for jumps) was established in 1997 by Huisken and Ilmanen[6]. Bray[7] considered a flow on the space of three metrics and generalised the earlier work by extending the result to higher dimensions and trapped surfaces with disconnected components.

1.6 GEOMETRIC FLOWS AND GENERAL RELATIVITY

This thesis deals with another flow called the Ricci flow which has recently attracted some attention due to its use by Perelman in proving the Poincare conjecture. The Ricci flow first appeared in physics in the renormalization of σ models [8]. In an independent mathematical development it was used by Hamilton [9] in order to study the topology of three-manifolds by geometric methods. This is now an active area of mathematical research [10]. Some papers on the Ricci flow have also started appearing in the physics literature [11, 12, 13]. The time seems ripe for applications of this new and powerful mathematical technique to physics. Our larger motivation is to better understand physical quantities like energy and entropy in GR. This thesis in an effort to use the Ricci flow to better understand geometric physical quantities in GR. Gravitational energy is a subtle and elusive concept since it cannot be localized. However, there *is* a notion of total mass of an isolated gravitating system, the ADM mass [3]. The PMT, guarantees (assuming energy conditions [3]) that the ADM mass of a gravitating system is positive. This is a result in differential geometry which has been proved by a variety of methods [4, 14, 15]. At one level this result is a purely differential geometric one. No reference to GR or the real world is needed in order to formulate and prove the theorem. However, the theorem is extremely important from the physical point of view. It is needed for the physical consistency of general relativity. If the PMT were false, it would be possible to extract an infinite amount of energy from an isolated gravitating system. Such a possibility is clearly absurd and the PMT saves general relativity from this inconsistency.

The PMT is implied by Penrose's Inequality which is a more general and as yet unproved conjecture [16] due to Penrose. Like the PMT, the Penrose inequality can be expressed in purely differential geometric terms. If a counterexample to the Penrose inequality were found, it would imply again that general relativity is inconsistent: the theory can produce naked singularities starting form nonsingular initial data. The equations and inequalities relating the physically significant quantities in GR can be converted into purely differential geometric statements and it seems logical and natural to apply the techniques of geometric flows to better understand their behaviour.

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