Chapter 4

SPHERICALLY SYMMETRIC FLOWS

4.1 INTRODUCTION

The Ricci flow is a tensor evolution equation and therefore commutes with diffeomorphisms. It follows that isometry groups are preserved under the Ricci flow. In this chapter we will study the Ricci flow for a spherically symmetric initial data set. The idea behind the study of the symmetric initial data set is to reduce the complexity of the flow. Imposition of a large number of symmetries will simplify the situation to a trivial extent of having to solve ordinary differential equations (ODE) instead of partial ones. Studies of RF of homogeneous geometry are given in detail in [1] which reduces to solving ODE's. Too little symmetry, on the other hand, will leave many important features intractable. We wish to study the situation with spherically symmetric initial data as this will provide us with an analytical as well as numerical testing ground and it paves the way for the general treatment. This special case is useful since one can develop a physical feel for the geometrical quantities of interest and easily produce physical examples and counterexamples as a guide to intuition.

- **The Ricci Flow in brief:**

  The Ricci flow (RF) has been introduced in detail in chapter(2). Here we again recall the definition of RF for the present purpose.

  Let (Σ, h_{ab}) be an asymptotically flat, three dimensional Riemannian manifold. (a, b run over 1,2,3. We restrict our discussion to three dimensional manifolds.) Given an initial metric h_{ab}, the Ricci flow describes an evolution equation, which evolves the metric according to its Ricci tensor. The evolution parameter is τ and the family of metrics on Σ, $h_{ab}(\tau)$ satisfies the Ricci flow equation

$$\frac{\partial h_{ab}}{\partial \tau} = -2R_{ab}. \tag{4.1}$$

In the neighborhood of a point $p \in \Sigma$, we can introduce a Riemann normal co-ordinate system and then the form of (4.1) becomes parabolic (∇^2 is the Laplacian in local co-ordinates)

$$\frac{\partial h_{ab}}{\partial \tau} = \nabla^2 h_{ab} \tag{4.2}$$

and looks like a heat equation for the metric coefficients. However, in a general coordinate system, the PDE (4.1) is a *degenerate* parabolic equation, because of its diffeomorphism invariance.

- **Some useful calculations in spherically symmetric initial data set:**

  While setting up the spherically symmetric initial set of data, we will work with two forms of the initial metric, each of them having its usefulness and limitation. We will call them by "a-form" and "b-form". They are connected to each other by a coordinate transformation and described below is the analysis of both.

  – **The a-form:** In this form of the initial data set, the 3-metric is taken as

  $$ds^2 = a(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{4.3}$$

  With this metric we calculate the Ricci tensors, the scalar curvature and the Laplacian operator on any function $f(r)$ given as follows: The components of the Ricci tensors are (A prime means differentiation with respect to r),

  $$R_{rr} = \frac{a'}{ra} \tag{4.4}$$

  $$R_{\theta\theta} = \frac{a'r}{2a^2} + 1 - \frac{1}{a} \tag{4.5}$$

  $$R_{\phi\phi} = sin^2\theta(R_{\theta\theta}). \tag{4.6}$$

The scalar curvature is

$$R = \frac{2}{r^2} + \frac{2a'}{ra^2} - \frac{2}{ar^2} \tag{4.7}$$

and the Laplacian of a function $f(r)$ is

$$D^2 f = \frac{1}{a}\left[f'' - \frac{a' f'}{2a}\right] + \frac{2 f'}{a}. \tag{4.8}$$

This form is useful for the study of the evolution of Hawking mass M, defined below in (4.18), by taking

$$a(r) = \left(1 - \frac{2M(r)}{r}\right)^{-1} \tag{4.9}$$

and solving the RF PDEs. However this form of the metric is not useful if there exists an apparent horizon because in that case $a(r)$ blows up at the apparent horizon ($r = 2M$) and hence this is not useful for the study of evolution of the area of the apparent horizon under RF. To study the evolution of the area of the apparent horizon under RF we use the "b-form" of the metric discussed next.

– **The b-form:** In this form of the initial data set, the 3-metric is taken as

$$ds^2 = dr^2 + b(r)(d\theta^2 + sin^2\theta d\phi^2). \tag{4.10}$$

With this metric we again calculate the Ricci tensors, the scalar curvature and the Laplacian operator on any function $f(r)$ given as follows: The components of the Ricci tensors are,

$$R_{rr} = \frac{b'^2 - 2bb''}{2b^2} \tag{4.11}$$

$$R_{\theta\theta} = 1 - \frac{b''}{2} \tag{4.12}$$

$$R_{\phi\phi} = sin^2\theta(R_{\theta\theta}). \tag{4.13}$$

The scalar curvature is

$$R = \frac{b'^2 - 4b(b'' - 1)}{2b^2} \tag{4.14}$$

and the Laplacian of a function $f(r)$ is

$$D^2 f = f'' + \frac{b' f'}{b}. \tag{4.15}$$

- **Geometric Quantities of Interest:** Let S be a closed surface in Σ, γ_{ab} the pull back or induced metric on S, $\mathcal{R}$ the scalar curvature of (S, γ) and k the trace of its extrinsic curvature. We will be interested in the evolution of some geometric properties of S under the Ricci flow. Our interest in these quantities stems from their physical significance. These are :

  the area of S,
$$A(S) = \int_S dA = \int_S d^2x\,\sqrt{\gamma} \tag{4.16}$$
  the "compactness" of S,
$$C(S) = \int_S dA(2\mathcal{R} - k^2) \tag{4.17}$$
  and its Hawking mass
$$M_H(S) = \frac{\sqrt{A(S)}}{64\pi^{3/2}}C(S). \tag{4.18}$$

  The area of apparent horizons is related to the entropy of Black Holes and the Hawking Mass is related to the Energy. The Hawking mass M_H of an asymptotic round two sphere is equal to its ADM energy. More generally, the Hawking Mass of a surface S is sometimes physically interpreted as the mass contained within the surface S. While there are some problems with this interpretation (positivity is not always assured), the Hawking mass is an useful notion [2, 3] of quasilocal mass. It vanishes in the limit that S shrinks to a point and (as we mentioned before) becomes the ADM energy for a round sphere at infinity. The "compactness" is a dimensionless quantity, which in some sense measures how much mass is concentrated within the closed surface S. (We use the word compact not in the mathematical sense, but in the physical sense of the introduction as in "A neutron star is a compact object"). The quantity $C(S)$ has been used to good effect by Geroch, Jang and Wald [2, 3] in their approach to positive mass theorem and the Penrose inequality. In fact their work forms the base for recent progress [4, 5] on the Riemann Penrose inequality. $C(S)$ tends to zero as S tends to zero and also as S tends to an asymptotic round sphere.

  We also note that by the *Gauss Bonnet theorem*, if S is of spherical topology

$$2\int_S \mathcal{R} dA = 2\int_S 2k dA = 16\pi. \tag{4.19}$$

So, in this case we can write the Hawking mass as

$$M_H := \frac{\sqrt{A}}{64\pi^{3/2}}\Big[16\pi - \int_S k^2 dA\Big]. \tag{4.20}$$

- **Spherical Symmetry:** In order to get a feel for the evolution of these quantities under the Ricci flow, let us start with a general spherically symmetric situation. We can choose co-ordinates adapted to the symmetry and write the metric in "a-form" as

$$ds^2 = a(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.21}$$

where $a(r)$ is a positive function which tends to 1 as $r \to \infty$ to ensure asymptotic flatness. This form of the metric is convenient for some purposes but, as mentioned before, is unsuitable for treating apparent horizons (since $a(r)$ diverges). We will initially assume that there are no horizons and later separately treat horizons using a different form (the "b-form") of the metric.

Let us choose the surface S consistent with the symmetry, as $r = c$. The area of S, $A(S)$ is given by

$$A(S) = 4\pi r^2. \tag{4.22}$$

We wish to ensure that the scalar curvature of the space is positive. In order to do this conveniently, we introduce [6] a function $M(r)$ by $a(r) = (1 - 2M(r)/r)^{-1}$ or

$$M(r) = \frac{(1 - a(r)^{-1})r}{2}. \tag{4.23}$$

The scalar curvature of (4.21) is given by $R = 4M'(r)/r^2$ and so the constraint of positivity of scalar curvature simply states that $M(r)$ is a non-decreasing function of r. Assuming that the form (4.21) holds all the way to the origin, and assuming that the scalar curvature R is finite, we have $M(0) = 0, M'(0) = 0, M''(0) \geq 0$. $M(r)$ increases from zero and tends to an asymptotic value $M = \lim_{r->\infty} M(r)$ which is the ADM mass of the metric. A simple calculation gives the compactness as

$$C(r) = 32\pi \frac{M(r)}{r}. \tag{4.24}$$

Applying the general definition of Hawking mass to spherical symmetry, we find

$$M_H(r) = M(r). \tag{4.25}$$

This can be shown as follows: We consider the "a-form" (4.21) of the metric where $r = constant$ describes the surface S of a sphere of area $4\pi r^2$. Now consider the Hawking mass functional

$$M_H(S) := \frac{\sqrt{A}}{64\pi^{\frac{3}{2}}}\left(16\pi - \int_S k^2 dA\right). \tag{4.26}$$

Let n_a be a unit normal to the surface S. The normalization

$$h^{ab} n_a n_b = a^{-1}(r) n_r n_r = 1 \tag{4.27}$$

fixes

$$n_r = \sqrt{a} \tag{4.28}$$

and so

$$n^a = (\frac{1}{\sqrt{a}}, 0, 0). \tag{4.29}$$

The trace of the extrinsic curvature is

$$k = D_a n^a = \frac{2}{r\sqrt{a}}. \tag{4.30}$$

So we have,

$$\int k^2 dA = \frac{16\pi}{a(r)}. \tag{4.31}$$

For flat space $a(r) = 1$ and we have $M_H(S) = 0$ for round spheres in flat space as expected. If we take the example of the Schwarzschild space

$$a = \left(1 - \frac{2M}{r}\right)^{-1} \tag{4.32}$$

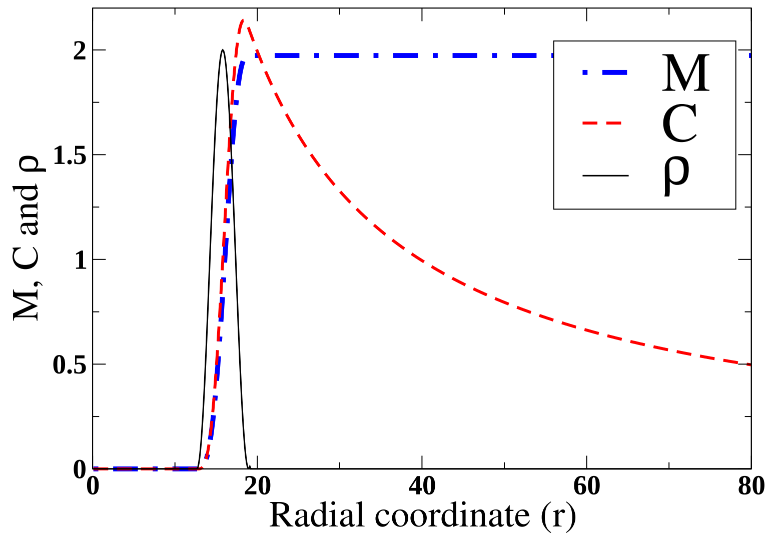

Figure 4.1: $M_H(r)$ increases with r to attain its asymptotic ADM value. But $C(r)$ increases to a maximum value and then decreases to 0 at infinity.

and then for any r

$$M_H(S) = \frac{\sqrt{4\pi r^2}}{64\pi^{\frac{3}{2}}}\left(16\pi - 16\pi\left(1 - \frac{2M}{r}\right)\right) = M. \tag{4.33}$$

For a general $a(r)$ we can always write it as $\left(1 - \frac{2M(r)}{r}\right)^{-1}$ provided $\left(1 - \frac{2M(r)}{r}\right) \neq 0$ or $r \neq 2M(r)$. This is possible only in the absence of apparent horizons. Once this is done we find that

$$M_H(S) = M(r). \tag{4.34}$$

In some sense [6], $M_H(r)$ measures the total mass contained within a sphere of radius r. The functions $\rho(r) = (1/16\pi)R(r)$, $C(r)$ and $M_H(r)$ are plotted in arbitrary units in figure (4.1) for a particular spherically symmetric distribution of matter.

Note that $M_H(r)$ increases with r to attain its asymptotic ADM value. But $C(r)$ increases to a maximum value and then decreases to 0 at infinity. For a star $C(r)$ attains its maximum near the surface of the star. For a star surrounded by infalling shells of matter, the behaviour is more complex (Fig. (4.2)). From (4.24) in the Newtonian limit $C(r)$ is a constant times the dimensionless Newtonian potential, or the mass to radius ratio. Hence the name compactness is justified.

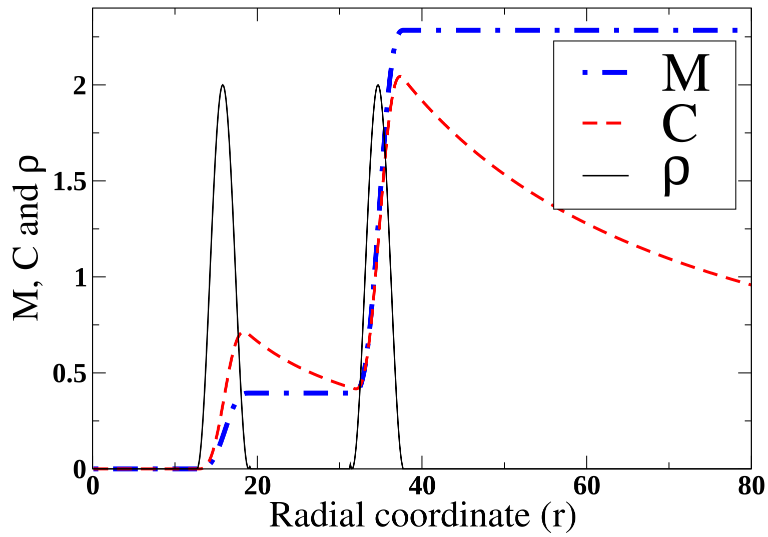

Figure 4.2: Two shells of matter.

4.2 AREA OF APPARENT HORIZON UNDER RICCI FLOW

Let (Σ, h) be a three dimensional manifold and $\mathcal{H}$ be a minimal surface in Σ. $\mathcal{H}$ is a closed two manifold embedded in Σ with the property that the trace of the extrinsic curvature vanishes. We want to see how the area of $\mathcal{H}$ varies under the RF. We start with the spherically symmetric "b-form" of the metric

$$ds^2 = dr^2 + b(r)(d\theta^2 + sin^2\theta d\phi^2). \tag{4.35}$$

Let the location of the apparent horizon be at $r = r_0$. Next we evaluate the condition that the surface $r = r_0$ be an apparent horizon condition. For this, let n_i be the radial, unit normal to the surface of the horizon. The trace of the extrinsic curvature, k, then is

$$k := D_a n^a = \frac{b'}{b} \tag{4.36}$$

(a prime indicates a differentiation with respect to r) where D_a is the covariant derivative operator with respect to the metric form (4.35) The condition that the surface $r = r_0$ be an apparent horizon is

$$k = 0. \tag{4.37}$$

We therefore have, since $b \neq 0$, the condition that the surface $r = r_0$ be an apparent horizon is

$$b'|_{r=r_0} = 0. \tag{4.38}$$

During the RF the location of the horizon will change and so $r_0 = r_0(\tau)$ where τ is the parameter of the RF. Also the geometry of $\mathcal{H}$ will change. In principle both these effects could lead to change of area. The area is given by

$$A(r) = \int_S \sqrt{\gamma}d\theta d\phi = 4\pi b(r) \tag{4.39}$$

where $\gamma = b^2 sin^2\theta$ is the determinant of the induced metric γ_{ij} on S. The rate of change of area, therefore, is

$$\frac{dA}{d\tau} = 4\pi\frac{db}{d\tau} = 4\pi\left[\frac{\partial b}{\partial r}\bigg|_{r=r_0}\frac{dr_0}{d\tau} + \frac{\partial b}{\partial \tau}\bigg|_{r=r_0}\right]. \tag{4.40}$$

The first term vanishes because of the apparent horizon condition (4.38). The second term is obtained from the RF

$$\frac{\partial h_{\theta\theta}}{\partial \tau} = -2R_{\theta\theta}. \tag{4.41}$$

Note here that we do not add a diffeomorphism term in the RF as even if we add a diffeomorphism, it will only displace the horizon $\mathcal{H}$ but will not change its area. We also do not add a term responsible for an overall scaling of the metric as this will modify the asymptotic structure of the manifold while we want a fixed metric at spatial infinity. The scalar curvature is

$$R = \frac{b'^2 - 4b(b'' - 1)}{2b^2}. \tag{4.42}$$

Since $b' = 0$ for an apparent horizon the scalar curvature becomes

$$R = \frac{2(1 - b'')}{b}. \tag{4.43}$$

Assuming that $R > 0$ (weak energy condition), since $b > 0$, we have $b'' < 1$. Then

$$\frac{\partial b}{\partial \tau} = \frac{\partial h_{\theta\theta}}{\partial \tau} = -2R_{\theta\theta} = -2\left[1 - \frac{b''}{2}\right] = -1 + (b'' - 1). \tag{4.44}$$

We then have, since $b'' < 1$,

$$\frac{\partial b}{\partial \tau} < -1. \tag{4.45}$$

So the area $A = 4\pi b$ satisfies

$$\frac{\partial A}{\partial \tau} < -4\pi. \tag{4.46}$$

This implies that the area of the horizon is linearly decreasing with τ. Since the area was finite to begin with, we find that b evaluated at the horizon goes to zero in a finite τ. Next we see that as $b \to 0$ we approach a singularity. To show this we will suppose that R is finite and will arrive at a contradiction.

On the surface of $\mathcal{H}$ the scalar curvature

$$R = \frac{2(1 - b'')}{b} \tag{4.47}$$

is finite only if $b'' = 1$. We Taylor expand in powers of $r - r_0$ about r_0, the location of the apparent horizon

$$b(r) = b(r_0) + b'(r_0)(r - r_0) + \frac{1}{2}b''(r_0)(r - r_0)^2 + ... \tag{4.48}$$

As $b(r_0) \to 0$ and $b''(r_0) = 1$, since $b'(r_0) = 0$ due to the apparent horizon condition (4.38), we have

$$b(r) = \frac{1}{2}(r - r_0)^2. \tag{4.49}$$

So the metric is

$$ds^2 = dr^2 + \frac{1}{2}(r - r_0)^2(d\theta^2 + sin^2\theta d\phi^2). \tag{4.50}$$

A shift r coordinate $r \to (r - r_0)$ gives the form

$$ds^2 = dr^2 + \frac{1}{2}r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{4.51}$$

So the volume of a ball of radius r is

$$\int d\theta d\phi dr \frac{\sqrt{r^4 sin^2\theta}}{4} = \frac{1}{4}\left(\frac{4\pi r^3}{3}\right). \tag{4.52}$$

We then find from the expression for the volume of a ball of radius r (as $r \to 0$) centered at point p that

$$volume\ \ \mathcal{B}(p, r) = \left(\frac{4\pi}{3}\right)\left(r^3 - \frac{1}{30}R(p)r^5\right) \tag{4.53}$$

and so the scalar curvature, $R(p) \sim r^{-2}$, blows up which is a contradiction to the assumption of finite R that we started with. So an apparent horizon results in a singularity.

4.3 AREA UNDER RICCI FLOW

We now consider a general round sphere $S \subset \Sigma$ given by $r = constant$ consistent with the spherically symmetric "b-form" of the metric

$$ds^2 = dr^2 + b(r)(d\theta^2 + sin^2\theta d\phi^2). \tag{4.54}$$

The trace of the extrinsic curvature k of S is

$$k := D_a n^a = \frac{b'}{b} \tag{4.55}$$

where n_a is the radial, unit normal to the surface S (a prime indicates a differentiation with respect to r).

The area A of S is given as

$$A(r) = \int_S \sqrt{\gamma} d\theta d\phi = 4\pi b(r) \tag{4.56}$$

where $\gamma = b^2 sin^2\theta$ is the determinant of the induced metric γ_{ij} on S.

We consider a pure Ricci flow without a diffeomorphism term, we view S as a fixed surface of Σ and so the location of the surface S does not change and hence $\frac{dr}{d\tau} = 0$.

We then have

$$\frac{dA}{d\tau} = 4\pi\frac{\partial b}{\partial\tau} = -4\pi(2 - b'') \tag{4.57}$$

which is obtained from the Ricci flow

$$\frac{\partial h_{\theta\theta}}{\partial\tau} = -2R_{\theta\theta}. \tag{4.58}$$

The scalar curvature R for the "b-form" of the metric considered above is

$$R = \frac{b'^2 - 4b(b'' - 1)}{2b^2}. \tag{4.59}$$

With this we see that

$$\int_S \sqrt{\gamma} d\theta d\phi R = \frac{2\pi b'^2}{b} - 8\pi(b'' - 1) \tag{4.60}$$

and also that the compactness,

$$C = 16\pi - \int_S \sqrt{\gamma} d\theta d\phi k^2 = 16\pi - \frac{4\pi b'^2}{b}. \tag{4.61}$$

From the above calculations we see that, in spherical symmetry,

$$\frac{dA}{d\tau} = -\frac{1}{2}\int_S \sqrt{\gamma} d\theta d\phi R - \frac{1}{4}C \tag{4.62}$$

and so we arrive at the inequality (since $R \geq 0$)

$$\frac{dA}{d\tau} \leq -\frac{1}{4}C. \tag{4.63}$$

In the case of the Schwarzschild space, $R = 0$ and so the first integral in (4.62) vanishes and we have

$$\frac{dA}{d\tau} = -\frac{1}{4}C(S). \tag{4.64}$$

Thus our inequality (4.63) is saturated by the Schwarzschild space.

4.4 COMPACTNESS UNDER RICCI FLOW

The compactness C is

$$C(\tau) = \int_S \sqrt{\gamma} d^2x(2\mathcal{R} - k^2). \tag{4.65}$$

We consider, again, a fixed, closed surface $S \subset \Sigma$ and take the "a-form" of the metric

$$ds^2 = a(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{4.66}$$

We calculate the trace of the extrinsic curvature k of S

$$k := D_a n^a = \frac{2}{r\sqrt{a}} \tag{4.67}$$

and we have

$$\int_S \sqrt{\gamma} d\theta d\phi k^2 = 4\pi\frac{8\pi}{a} \tag{4.68}$$

where n_a is the radial, unit normal to the surface S (a prime indicates a differentiation with respect to r) and γ_{ij} is the induced metric on S. A calculation shows that under Ricci flow

$$\frac{dC}{d\tau} = 16\pi(-2M'(r)/r^2 + 2M''(r)/r) \tag{4.69}$$

which for the Schwarzschild exterior space ($M = constant$ and ($r > 2M$)) gives

$$\frac{dC}{d\tau} = 0. \tag{4.70}$$

- **A maximum principle for the compactness:**

  We notice an important fact about the rate of change of the maximum value of compactness. We consider the the metric in the "a-form" as in the equation (4.3) and we take the RF equation with a diffeomorphism term as

$$\frac{dh_{ab}}{d\tau} = -2R_{ab} + 2D_a D_b f. \tag{4.71}$$

  Then the evolution of $a(r)$ under this flow is

$$\frac{\partial a(r)}{\partial \tau} = \frac{a''(r)}{a(r)} - \frac{3(a'(r))^2}{2a(r)^2} - \frac{2(a(r)-1) + ra'(r)(1-a(r))/a(r)}{r^2} \tag{4.72}$$

  where a prime denotes differentiation with respect to r.

  If we take $a(r) = ((1 - 2M(r))/r)^{-1}$ and assume that there will be no apparent horizon formation i.e., $r > 2M(r)$ always, then $a(r) > 1$ and at the maximum value of $a(r)$, we have $a'(r)|_{max} = 0$ and $a''(r)|_{max} < 0$. So from equation (4.72) we see that the maximum value of $a(r)$ decreases as the flow parameter τ increases i.e.,

$$\frac{\partial a(r)_{max}}{\partial \tau} \leq \frac{a''(r)}{a(r)} < 0. \tag{4.73}$$

  Now as $M(r) = (r/2)(1-1/a(r))$ we have, from equation (4.24), $C(r) = 16\pi(1-1/a(r))$. Clearly the maximum value of $a(r)$ will imply the maximum value of $C(r)$ and as the maximum value of $a(r)$ decreases as τ increases, the maximum value of $C(r)$ also decreases and we have the inequality

$$\frac{dC(r)_{max}}{d\tau} \leq 0. \tag{4.74}$$

This ensures the existence of Ricci flow when we do not start with an apparent horizon. This can be understood by the following argument: We know that if we start with an apparent horizon then a Ricci flow will make it approach the singularity in a finite value of τ. Since apparent horizons shrink under Ricci flow, they cannot form if they were initially absent. C_{max} then decreases with the flow and finally approaches zero. The only space with $C_{max} = 0$ is flat space as can be seen by invoking the positive mass theorem.

4.5 HAWKING MASS UNDER RICCI FLOW

The Hawking mass is given by

$$M_H = \frac{\sqrt{A}}{64\pi^{3/2}} C(S). \tag{4.75}$$

So we have

$$\frac{d}{d\tau} M_H = \left(\frac{1}{64\pi^{3/2}}\right)\left(\frac{1}{2\sqrt{A}} \frac{dA}{d\tau} C + \sqrt{A} \frac{dC}{d\tau}\right). \tag{4.76}$$

As we have seen already that for the Schwarzschild metric $\frac{dC}{d\tau} = 0$ and $\frac{dA}{d\tau} = -\frac{1}{4}C$ (equations (4.70) and (4.64) respectively), we have

$$\frac{d}{d\tau} M_H = -\left(\frac{1}{512\sqrt{A}\pi^{3/2}}\right) C^2 \leq 0. \tag{4.77}$$

In general, Hawking mass is not monotonic under Ricci flow. By choosing the mass distribution, one can get either sign for $\frac{d}{d\tau} M_H$ under Ricci flow.

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