Chapter 6

3PN angular momentum flux and the evolution of orbital elements for inspiralling compact binaries in quasi-elliptical orbits: instantaneous contribution

6.1 Introduction

The generation problem of gravitational waves for inspiralling compact binaries was completed at the third post-Newtonian (3PN) order both for the equation of motion of the binary and for its far-zone radiation field. The computation of the 3PN equations of motion (EOM) and the 3PN accurate mass quadrupole moment was technically more involved than the corresponding 2.5PN case due to the issues related to the ambiguities of self-field regularisation using Riesz or Hadamard regularisations [145, 70, 76, 143, 74]. A critical understanding of the origin of these ambiguities and the use of a different regularisation scheme that respects the gauge symmetries of general relativity, like dimensional regularisation, was crucial to the resolution of this problem [76, 73, 75]. The 3.5PN phasing of inspiralling compact binaries moving in quasi-circular orbits is now complete and available for use in GW data analysis [99, 75]. Further, we finally have the complete 3PN EOM for compact binaries moving in *general orbits* and an ambiguity-free fully determined 3PN accurate mass quadrupole for *general orbits* required to compute the 3PN accurate energy and angular momentum fluxes for inspiralling compact binaries moving in general *non-circular orbits*.

The prototype binary sources of GW for laser interferometer gravitational wave detectors are neutron star or black hole binaries close to their merger phase and consequently moving in quasi-circular orbits. However, astrophysical paradigms have been investigated that lead to binaries with nonzero eccentricity in the sensitive bandwidth of both terrestrial and space-based gravitational wave detectors [78, 81, 82, 93, 222]. We have already examined the possible scenarios which will produce eccentric binaries and the necessity for including the effect of eccentricity in the waveform modelling in Sec 1.8. We discuss the theoretical inputs that are required in constructing templates for the eccentric orbit case in the next section.

6.1.1 Theoretical inputs required for the construction of templates for binaries in elliptical orbits

Recently, in a series of two related papers [114, 115], we discussed the computation of the energy flux of gravitational waves (GW) from inspiralling compact binaries moving in general non-circular orbits up to third post-Newtonian (PN) order. For non-circular orbits, in addition to the conserved energy and gravitational wave energy flux, the angular momentum flux needs to be known to determine the phasing of eccentric binaries. A knowledge of the angular momentum flux of the system averaged over an orbit is mandatory to calculate the evolution of the orbital elements of non-circular, in particular, elliptic orbits under GW radiation reaction. In this chapter, we compute the angular momentum flux of inspiralling compact binaries moving in non-circular orbits up to 3PN order generalising earlier work at Newtonian order by Peters [139], at 1PN order by Junker and Schäfer [109], 1.5PN (tails and spin-orbit) by Schaféer and Rieth [110] and at 2PN order by Gopakumar and Iyer [111]. Unlike at earlier post-Newtonian orders, the 3PN contribution to angular momentum flux comes not only from *instantaneous* terms but also *hereditary* contributions. Further, the hereditary contributions comprise not only the tails-of-tails and tail-square terms as for the energy flux but also an interesting *memory* contribution at 2.5PN.

The evolution of orbital elements under gravitational radiation goes back to the classic work of Peters and Mathews [95]. This was progressively extended by Blanchet and Schäfer to 1PN in [107] and 1.5PN in [107, 110] and finally to 2PN by Gopakumar and Iyer [111]. While [109, 110] require the 1PN accurate orbital description of Damour and Deruelle [223], [111] crucially employs the generalised 2PN quasi-Keplerian parametrization of the binary's orbital motion in ADM coordinates as given in [224, 225, 226]. In this chapter we obtain the orbital average of the *instantaneous* part of the angular momentum flux at 3PN using the recently constructed 3PN generalized quasi-Keplerian parametrization of the binary's orbital motion by Memmesheimer, Gopakumar and Schäfer [227]. Combining the results for the angular momentum flux obtained in this thesis with the results for the far-zone flux of energy obtained by us earlier [115, 114], we finally evaluate the evolution of the orbital elements under the instantaneous contribution in the 3PN gravitational radiation reaction.

As far as using these inputs towards constructing templates for elliptical orbits is concerned, Damour, Gopakumar and Iyer [112] discussed an analytic method for constructing high accuracy templates for the GW signals from compact binaries in quasi-elliptical orbits in their inspiral phase. They go beyond the standard averaging over the orbital time-scale and compute the additional oscillatory contributions . Using an improved "method of variation of constants" and working up to the leading radiation reaction order of 2.5PN, they combine the three time scales involved in the elliptical orbit case - the orbital period, periastron precession and radiation reaction time scales - without making the usual approximation of treating the radiative time scale as an adiabatic process. This was extended to 3.5PN order in Ref [113].

In Sec 6.2 we start with the structure of the far-zone flux of angular momentum, use expressions relating the radiative moments to the source moments and decompose the angular momentum flux into its instantaneous and hereditary parts. Sec. 6.3 discusses the computation of the instantaneous terms in harmonic coordinates for the angular momentum flux. Sec. 6.4 recasts the flux in ADM coordinates. Secs 6.5 summarises the 3PN quasi-Keplerian representation required to average the flux expressions over an orbit. Sec. 6.6 provides the orbital average of the instantaneous part of the angular momentum flux in the ADM coordinates. The 3PN orbital averaged energy flux in ADM coordinates is summarised in Sec. 6.7 based on earlier work [190, 114]. The evolution of the orbital elements, in ADM coordinates, under instantaneous terms in the 3PN gravitational radiation reaction is presented in Sec. 6.9

6.2 The far-zone angular momentum flux

In this section, we discuss the computation of 3PN accurate angular momentum flux for binaries moving in general (non-circular) orbits. Starting from the expression for the far zone angular momentum flux in terms of the radiative multipole moments and using the relations connecting the radiative multipole moments to the source moments, we rewrite the flux of angular momentum. It consists of the instantaneous terms which are functions of the retarded time and hereditary terms which depend on the dynamics of the system in its entire past. The 3PN accurate angular momentum flux in the source's far-zone can be written in terms of the symmetric trace-free (STF) mass and current type radiative multipole moments (U_Ls and V_Ls) [61] as

$$\left(\frac{d\mathcal{J}_i}{dt}\right) = \frac{G}{c^5}\epsilon_{ipq}\Big\{\frac{2}{5}U_{pj}U^{(1)}_{qj}$$

$$+\frac{1}{c^2}\left[\frac{1}{63}U_{pjk}U^{(1)}_{qjk}+\frac{32}{45}V_{pj}V^{(1)}_{qj}\right]+\frac{1}{c^4}\left[\frac{1}{2268}U_{pjkl}U^{(1)}_{qjkl}+\frac{1}{28}V_{pjk}V^{(1)}_{qjk}\right]$$
$$+\frac{1}{c^6}\left[\frac{1}{118800}U_{pjklm}U^{(1)}_{qjklm}+\frac{16}{14175}V_{pjkl}V^{(1)}_{qjkl}\right]+\mathcal{O}(8)\Bigg\}\,. \tag{6.1}$$

In the above U_L and V_L (with $L = ijk...$ a multi-index composed of l indices) are the mass and current type radiative multipole moments respectively and $U^{(l)}_L$ and $V^{(l)}_L$ denote their l^{th} time derivatives. The moments are functions of retarded time $T_R \equiv T - \frac{R}{c}$ in radiative coordinates. ε_{ipq} is the usual Levi-Civita symbol such that $\varepsilon_{123} = +1$. The shorthand $\mathcal{O}(n)$ indicates that the post-Newtonian remainder is of order of $\mathcal{O}(c^{-n})$.

Using the MPM formalism, the radiative moments in Eq. (6.1) can be re-expressed in terms of the source moments to an accuracy sufficient for the computation of the angular momentum flux up to 3PN. For the angular momentum flux to be complete up to 3PN approximation, one must compute the mass type radiative quadrupole U_{ij} to 3PN accuracy, mass octupole U_{ijk} and current quadrupole V_{ij} to 2PN accuracy, mass hexadecupole U_{ijkm} and current octupole V_{ijk} to 1PN accuracy and finally U_{ijkmn} and V_{ijkm} to Newtonian accuracy.

6.2.1 Radiative moments in terms of source moments

The relations connecting the different radiative moments U_L and V_L to the corresponding source moments I_L and J_L [57, 59, 58] are given below. For the mass type moments we have

$$\begin{aligned}
U_{ij}(T_R) &= I^{(2)}_{ij}(T_R)+\frac{2GM}{c^3}\int_{-\infty}^{T_R}dV\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{11}{2}\right]I^{(4)}_{ij}(V) \\
&+\frac{G}{c^5}\Bigg\{-\frac{2}{7}\int_{-\infty}^{T_R}dV I^{(3)}_{a<i}(V)I^{(3)}_{j>a}(V) \\
&\quad+\frac{1}{7}I^{(5)}_{a<i}I_{j>a}-\frac{5}{7}I^{(4)}_{a<i}I^{(1)}_{j>a}-\frac{2}{7}I^{(3)}_{a<i}I^{(2)}_{j>a}+\frac{1}{3}\varepsilon_{ab<i}I^{(4)}_{j>a}J_b \\
&\quad+4\left[W^{(2)}I_{ij}-W^{(1)}I^{(1)}_{ij}\right]\Bigg\} \\
&+2\left(\frac{GM}{c^3}\right)^2\int_{-\infty}^{T_R}dV I^{(5)}_{ij}(V)\left[\ln^2\left(\frac{T_R-V}{2b}\right)+\frac{57}{70}\ln\left(\frac{T_R-V}{2b}\right)+\frac{124627}{44100}\right] \\
&+\mathcal{O}(7), && \text{(6.2a)} \\
U_{ijk}(T_R) &= I^{(3)}_{ijk}(T_R)+\frac{2GM}{c^3}\int_{-\infty}^{T_R}dV\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{97}{60}\right]I^{(5)}_{ijk}(V) \\
&+\mathcal{O}(5)\,, && \text{(6.2b)} \\
U_{ijkm}(T_R) &= I^{(4)}_{ijkm}(T_R)+\frac{G}{c^3}\Bigg\{2M\int_{-\infty}^{T_R}dV\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{59}{30}\right]I^{(6)}_{ijkm}(V)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{5}\int_{-\infty}^{T_R} dV I^{(3)}_{<ij}(V) I^{(3)}_{km>}(V) \\
&-\frac{21}{5} I^{(5)}_{<ij} I_{km>} - \frac{63}{5} I^{(4)}_{<ij} I^{(1)}_{km>} - \frac{102}{5} I^{(3)}_{<ij} I^{(2)}_{km>}\Big\} \\
&+\mathcal{O}(4)\,,
\end{aligned}
\tag{6.2c}
$$

where the bracket <> denotes STF projection. In the above formulas, M is the total ADM mass of the binary system. The I_L's and J_L's are the mass and current-type source moments, and $I_L^{(p)}$, $J_L^{(p)}$ denote their p-th time derivatives. W is the monopole corresponding to the set of gauge moments W_L [65].

For the current-type moments, on the other hand, we find

$$
\begin{aligned}
V_{ij}(T_R) &= J^{(2)}_{ij}(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R} dV \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{7}{6}\right] J^{(4)}_{ij}(V) \\
&\quad +\mathcal{O}(5)\,,
\end{aligned}
\tag{6.3a}
$$

$$
\begin{aligned}
V_{ijk}(T_R) &= J^{(3)}_{ijk}(T_R) + \frac{G}{c^3}\Big\{2M\int_{-\infty}^{T_R} dV \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{5}{3}\right] J^{(5)}_{ijk}(V) \\
&\qquad +\frac{1}{10}\varepsilon_{ab<i} I^{(5)}_{j\underline{a}} I_{k>b} - \frac{1}{2}\varepsilon_{ab<i} I^{(4)}_{j\underline{a}} I^{(1)}_{k>b} - 2J_{<i} I^{(4)}_{jk>}\Big\} \\
&\quad +\mathcal{O}(4)\,.
\end{aligned}
\tag{6.3b}
$$

[The underlined index $\underline{a}$ means that it should be excluded from the STF projection]. For all the other moments required in the computation we need only the leading order accuracy, so that

$$
U_L(T_R) = I^{(l)}_L(T_R) + \mathcal{O}(3)\,, \tag{6.4a}
$$

$$
V_L(T_R) = J^{(l)}_L(T_R) + \mathcal{O}(3)\,. \tag{6.4b}
$$

The radiative moments have two distinct contributions. The first part which is a function only of the retarded time, $T_R = T - \frac{R}{c}$, and referred to as the 'instantaneous terms' and the second part that depends on the dynamics of the system in its entire past [57] and is referred to as hereditary contributions. From the expressions for U_Ls and V_Ls, one can schematically split the total contribution to the angular momentum flux as the sum of the instantaneous and hereditary terms.

6.2.2 Structure of the 3PN angular momentum flux

Starting from the expression for the angular momentum flux in terms of the radiative multipole moments, Eq. (6.1), and the expressions for the radiative moments in terms of the source multipoles, Eq. (6.2 and (6.3), the angular momentum flux can be re-written as

$$\left(\frac{d\mathcal{J}_i}{dt}\right) = \left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm inst} + \left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm hered} . \tag{6.5}$$

Since we do not discuss the hereditary terms in the angular momentum flux here, the hereditary terms in angular momentum flux are not given here though it is easy to write them down using the expressions, Eqs (6.2) and (6.3), for the radiative moments. Following the nomenclature adopted in [102], the instantaneous terms can further be classified as inst(s), which corresponds to the terms from source multipoles I_L and J_L, inst(c) terms which arise from the transformation between the source multipoles (I_L, J_L) and the canonical moments (M_L, S_L) and inst(r) from the terms present in the radiative mutipole moments. In our case the last two types of terms start appearing at 2.5PN order. The expression for the three kinds of terms read as

$$\begin{aligned}\left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm inst(s)} &= \frac{G}{c^5}\varepsilon_{ipq}\Big\{\frac{2}{5}I^{(2)}_{pj}I^{(3)}_{qj} \\ &+\frac{1}{c^2}\left[\frac{1}{63}I^{(3)}_{pjk}I^{(4)}_{qjk} + \frac{32}{45}J^{(2)}_{pj}J^{(3)}_{qj}\right] + \frac{1}{c^4}\left[\frac{1}{2268}I^{(4)}_{pjkl}I^{(5)}_{qjkl} + \frac{1}{28}J^{(3)}_{pjk}J^{(4)}_{qjk}\right] \\ &+\frac{1}{c^6}\left[\frac{1}{118800}I^{(5)}_{pjklm}I^{(6)}_{qjklm} + \frac{16}{14175}J^{(4)}_{pjkl}J^{(5)}_{qjkl}\right]\Big\},\end{aligned} \tag{6.6}$$

$$\begin{aligned}\left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm inst(c)} &= \frac{2G}{5c^5}\varepsilon_{ipq}\left\{\frac{4G}{c^5}\left[W^{(5)}I^{(2)}_{pj}I_{qj} + 2W^{(4)}I^{(2)}_{pj}I^{(1)}_{qj} - 3W^{(2)}I^{(2)}_{pj}I^{(3)}_{qj} - W^{(1)}I^{(2)}_{pj}I^{(4)}_{qj}\right.\right. \\ &\left.\left. + W^{(4)}I_{pj}I^{(3)}_{qj} + W^{(3)}I^{(1)}_{pj}I^{(3)}_{qj} - W^{(1)}I^{(3)}_{pj}I^{(3)}_{qj}\right]\right\} ,\end{aligned} \tag{6.7}$$

$$\begin{aligned}\left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm inst(r)} &= \frac{G}{c^5}\,\frac{2}{5}\frac{G}{c^5}\varepsilon_{ipq}\Big\{I^{(3)}_{qj}\Big[-\frac{5}{7}I^{(4)}_{a<p}I^{(1)}_{j>a} - \frac{2}{7}I^{(3)}_{a<p}I^{(2)}_{j>a} + \frac{1}{7}I^{(5)}_{a<p}I_{j>a} + \frac{1}{3}\varepsilon_{ab<p}\,I^{(4)}_{j>a}\,J_b\Big] \\ &+I^{(2)}_{pj}\Big[-\frac{4}{7}I^{(5)}_{a<q}I^{(1)}_{j>a} - I^{(4)}_{a<q}I^{(2)}_{j>a} - \frac{4}{7}I^{(3)}_{a<q}I^{(3)}_{j>a} + \frac{1}{7}I^{(6)}_{a<q}I_{j>a} \\ &+\frac{1}{3}\varepsilon_{ab<q}\,I^{(5)}_{j>a}J_b\Big)\Big]\Big\}.\end{aligned} \tag{6.8}$$

6.2.3 Source multipole moments for the 3PN angular momentum flux

We provide in this section the necessary multipole moments needed for the computation of the instantaneous part of the 3PN accurate energy and angular momentum fluxes. Though algebraically long, the procedure is fairly algorithmic and discussed in detail in in [143, 72].

We skip these details and list the final expression for these relevant source multipoles.

The mass quadrupole I_{ij} is already available in [74] and the procedure used for its computation is outlined there. We list it here for the sake of completeness and choice of notation.

$$
\begin{aligned}
I_{ij} &= \nu\, m \left\{ \left[\mathcal{A} - \frac{24}{7}\,\frac{\nu}{c^5}\,\frac{G^2 m^2}{r^2}\,\dot{r} \right] x_{\langle i}x_{j\rangle} + \mathcal{B}\,\frac{r^2}{c^2}\,v_{\langle i}v_{j\rangle} \right. \\
&\qquad \left. + 2\left[\mathcal{C}\,\frac{r\,\dot{r}}{c^2} + \frac{24}{7}\,\frac{\nu}{c^5}\,\frac{G^2 m^2}{r} \right] x_{\langle i}v_{j\rangle} \right\},
\end{aligned}
\tag{6.9a}
$$

where

$$
\begin{aligned}
\mathcal{A} &= 1 + \frac{1}{c^2}\left[v^2 \left(\frac{29}{42} - \frac{29\,\nu}{14} \right) + \frac{G\,m}{r}\left(-\frac{5}{7} + \frac{8}{7}\,\nu \right) \right] \\
&\quad + \frac{1}{c^4}\left[\frac{G\,m}{r}\,v^2 \left(\frac{2021}{756} - \frac{5947}{756}\,\nu - \frac{4883}{756}\,\nu^2 \right) \right. \\
&\qquad + \frac{G^2\,m^2}{r^2}\left(-\frac{355}{252} - \frac{953}{126}\,\nu + \frac{337}{252}\,\nu^2 \right) \\
&\qquad + v^4 \left(\frac{253}{504} - \frac{1835}{504}\,\nu + \frac{3545}{504}\,\nu^2 \right) \\
&\qquad \left. + \frac{G\,m}{r}\,\dot{r}^2 \left(-\frac{131}{756} + \frac{907}{756}\,\nu - \frac{1273}{756}\,\nu^2 \right) \right] \\
&\quad + \frac{1}{c^6}\left[v^6 \left(\frac{4561}{11088} - \frac{7993}{1584}\,\nu + \frac{117067}{5544}\,\nu^2 - \frac{328663}{11088}\,\nu^3 \right) \right. \\
&\qquad + v^4\,\frac{G\,m}{r}\left(\frac{307}{77} - \frac{94475}{4158}\,\nu + \frac{218411}{8316}\,\nu^2 + \frac{299857}{8316}\,\nu^3 \right) \\
&\qquad + \frac{G^3\,m^3}{r^3}\left(\frac{6285233}{207900} + \frac{15502}{385}\,\nu - \frac{3632}{693}\,\nu^2 + \frac{13289}{8316}\,\nu^3 \right. \\
&\qquad\qquad \left. - \frac{428}{105}\,\ln\left(\frac{r}{r_0}\right) - \frac{44}{3}\,\nu\,\ln\left(\frac{r}{r'_0}\right) \right) \\
&\qquad + \frac{G^2\,m^2}{r^2}\,\dot{r}^2 \left(-\frac{8539}{20790} + \frac{52153}{4158}\,\nu - \frac{4652}{231}\,\nu^2 - \frac{54121}{5544}\,\nu^3 \right) \\
&\qquad + \frac{G\,m}{r}\,\dot{r}^4 \left(\frac{2}{99} - \frac{1745}{2772}\,\nu + \frac{16319}{5544}\,\nu^2 - \frac{311}{99}\,\nu^3 \right) \\
&\qquad + \frac{G^2\,m^2}{r^2}\,v^2 \left(\frac{187183}{83160} - \frac{605419}{16632}\,\nu + \frac{434909}{16632}\,\nu^2 - \frac{37369}{2772}\,\nu^3 \right) \\
&\qquad \left. + \frac{G\,m}{r}\,v^2\,\dot{r}^2 \left(-\frac{757}{5544} + \frac{5545}{8316}\,\nu - \frac{98311}{16632}\,\nu^2 + \frac{153407}{8316}\,\nu^3 \right) \right],
\end{aligned}
\tag{6.9b}
$$

$$
\begin{aligned}
\mathcal{B} &= \frac{11}{21} - \frac{11}{7}\,\nu \\
&\quad + \frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{106}{27} - \frac{335}{189}\,\nu - \frac{985}{189}\,\nu^2 \right) \right. \\
&\qquad \left. + v^2 \left(\frac{41}{126} - \frac{337}{126}\,\nu + \frac{733}{126}\,\nu^2 \right) + \dot{r}^2 \left(\frac{5}{63} - \frac{25}{63}\,\nu + \frac{25}{63}\,\nu^2 \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{c^4}\left[v^4\left(\frac{1369}{5544}-\frac{19351}{5544}\,\nu+\frac{45421}{2772}\,\nu^2-\frac{139999}{5544}\,\nu^3\right)\right.\\
&\qquad+\frac{G^2\,m^2}{r^2}\left(-\frac{40716}{1925}-\frac{10762}{2079}\,\nu+\frac{62576}{2079}\,\nu^2-\frac{24314}{2079}\,\nu^3\right.\\
&\qquad\qquad\left.+\frac{428}{105}\,\ln\left(\frac{r}{r_0}\right)\right)\\
&\qquad+\frac{Gm}{r}\,\dot{r}^2\left(\frac{79}{77}-\frac{5807}{1386}\,\nu+\frac{515}{1386}\,\nu^2+\frac{8245}{693}\,\nu^3\right)\\
&\qquad+\frac{Gm}{r}\,v^2\left(\frac{587}{154}-\frac{67933}{4158}\,\nu+\frac{25660}{2079}\,\nu^2+\frac{129781}{4158}\,\nu^3\right)\\
&\qquad\left.+\,v^2\,\dot{r}^2\left(\frac{115}{1386}-\frac{1135}{1386}\,\nu+\frac{1795}{693}\,\nu^2-\frac{3445}{1386}\,\nu^3\right)\right], \qquad (6.9c)
\end{aligned}
$$

$$
\begin{aligned}
C \;=\;& -\frac{2}{7}+\frac{6}{7}\,\nu\\
&+\frac{1}{c^2}\left[v^2\left(-\frac{13}{63}+\frac{101}{63}\,\nu-\frac{209}{63}\,\nu^2\right)\right.\\
&\qquad\left.+\frac{G\,m}{r}\left(-\frac{155}{108}+\frac{4057}{756}\,\nu+\frac{209}{108}\,\nu^2\right)\right]\\
&+\frac{1}{c^4}\left[\frac{G\,m}{r}\,v^2\left(-\frac{2839}{1386}+\frac{237893}{16632}\,\nu-\frac{188063}{8316}\,\nu^2-\frac{58565}{4158}\,\nu^3\right)\right.\\
&\qquad+\frac{G^2\,m^2}{r^2}\left(-\frac{12587}{41580}+\frac{406333}{16632}\,\nu-\frac{2713}{396}\,\nu^2+\frac{4441}{2772}\,\nu^3\right)\\
&\qquad+v^4\left(-\frac{457}{2772}+\frac{6103}{2772}\,\nu-\frac{13693}{1386}\,\nu^2+\frac{40687}{2772}\,\nu^3\right)\\
&\qquad\left.+\frac{Gm}{r}\,\dot{r}^2\left(\frac{305}{5544}+\frac{3233}{5544}\,\nu-\frac{8611}{5544}\,\nu^2-\frac{895}{154}\,\nu^3\right)\right]. \qquad (6.9d)
\end{aligned}
$$

In the above equation r_0 is an arbitrary scale that is introduced in the general MPM formalism and which then appears in the definition of the source multipole moments. r'_0 is related to other scales r'_1 and r'_2 by $m\log r'_0 = m_1\log r'_1 + m_2\log r'_2$ specific to application of the formalism to point particle systems and come from regularizing self-field effects. By definition of the ambiguity parameters these scales are taken to be the *same* as the two scales that appear in the final expression of the 3PN equations of motion in harmonic coordinates computed in [148, 70]. r'_1, r'_2 and hence r'_0 are 'unphysical' in the sense that they can be arbitrarily moved by a coordinate transformation of the 'bulk' metric outside the particles or more appropriately when considering the renormalisation which follows the regularization by relevant shifts of the particles' world lines [75].

The 2PN mass octupole for general orbits is the next of the non-trivial moments required for what follows. It is given by:

$$
\begin{aligned}
I_{ijk} \;=\;& \nu m \sqrt{1-4\nu}\;\mathrm{STF}_{ijk}\Bigg\{ x_{ijk}\Bigg[-1+\frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{5}{6}-\frac{13\,\nu}{6}\right)+v^2\left(-\frac{5}{6}+\frac{19}{6}\,\nu\right)\right] \\
&+\frac{1}{c^4}\Bigg[v^4\Big(-\frac{257}{440}+\frac{7319}{1320}\,\nu-\frac{5501}{440}\,\nu^2\Big)+\frac{G^2m^2}{r^2}\Big(\frac{47}{33}+\frac{1591}{132}\,\nu-\frac{235}{66}\,\nu^2\Big) \\
&+\frac{Gm}{r}\dot{r}^2\Big(\frac{247}{1320}-\frac{531}{440}\,\nu+\frac{1347}{440}\,\nu^2\Big)+\frac{Gm}{r}v^2\Big(-\frac{3853}{1320}+\frac{14257}{1320}\,\nu+\frac{17371}{1320}\,\nu^2\Big)\Bigg]\Bigg] \\
&+x_{ij}v_k\frac{r\dot{r}}{c^2}\Bigg[1-2\,\nu+\frac{1}{c^2}\Big[\frac{Gm}{r}\Big(\frac{2461}{660}-\frac{8689}{660}\,\nu-\frac{1389}{220}\,\nu^2\Big) \\
&+v^2\Big(\frac{13}{22}-\frac{107}{22}\,\nu+\frac{102}{11}\,\nu^2\Big)\Big]\Bigg]+\;x_iv_{jk}\frac{r^2}{c^2}\Bigg[-1+2\,\nu+\frac{1}{c^2}\Big[v^2\Big(-\frac{61}{110}+\frac{519}{110}\,\nu-\frac{504}{55}\,\nu^2\Big) \\
&+\dot{r}^2\Big(\frac{1}{11}-\frac{4}{11}\,\nu+\frac{3}{11}\,\nu^2\Big)+\frac{Gm}{r}\Big(-\frac{1949}{330}-\frac{62}{165}\,\nu+\frac{483}{55}\,\nu^2\Big)\Big]\Bigg] \\
&+v_{ijk}\frac{\dot{r}\,r^3}{c^4}\Big(-\frac{13}{55}+\frac{52}{55}\,\nu-\frac{39}{55}\,\nu^2\Big)\Bigg\}\;+\mathcal{O}(5)\,.
\end{aligned}
\tag{6.10}
$$

The other mass-type moments needed in this work read as,

$$
\begin{aligned}
I_{ijkl} \;=\;& \nu\,m\,\mathrm{STF}_{ijkl}\Bigg\{x_{ijkl}\Bigg[1-3\nu+\frac{1}{c^2}\Bigg[\left(\frac{103}{110}-\frac{147}{22}\nu+\frac{279}{22}\nu^2\right)v^2 \\
&-\left(\frac{10}{11}-\frac{61}{11}\nu+\frac{105}{11}\nu^2\right)\frac{Gm}{r}\Bigg]\Bigg]-\frac{72}{55}v_ix_{jkl}\frac{r\dot{r}}{c^2}(1-5\nu+5\nu^2) \\
&+\frac{78}{55}v_{ij}x_{kl}\frac{r^2}{c^2}(1-5\nu+5\nu^2)\Bigg\}\,,
\end{aligned}
\tag{6.11}
$$

$$
I_{ijklm} \;=\; -\nu\,m\,\sqrt{1-4\nu}\,(1-2\nu)\mathrm{STF}_{ijklm}\left\{x_{ijklm}\right\}, \tag{6.12}
$$

$$
I_{ijklmn} \;=\; \nu\,m(1-5\nu+5\nu^2)\mathrm{STF}_{ijklmn}\left\{x_{ijklmn}\right\}. \tag{6.13}
$$

In the above and what follows, $x_{ijk...}=x_ix_jx_k...$ and $v_{ijk...}=v_iv_jv_k...$, $\nu=m_1m_2/m^2$ and STF_L denotes that the terms inside the bracket are symmetric and trace-free in the indices listed. W, the monopole corresponding to the set of gauge moments W_L is given by:

$$
W=\frac{1}{3}\,\nu\,m\,\mathbf{x}.\mathbf{v}+\mathcal{O}(2). \tag{6.14}
$$

The other new input needed is the current quadrupole to 2PN accuracy which reads as:

$$
\begin{aligned}
J_{ij} \;=\;& m\,\nu\,\sqrt{1-4\,\nu}\;\mathrm{STF}_{ij}\Bigg\{\varepsilon_{abi}x_{ja}v_b\Bigg[-1+\frac{1}{c^2}\left[\frac{G\,m}{r}\left(-\frac{27}{14}-\frac{15}{7}\,\nu\right)+v^2\left(-\frac{13}{28}+\frac{17}{7}\,\nu\right)\right] \\
&+\frac{1}{c^4}\Bigg[v^4\left(-\frac{29}{84}+\frac{11}{3}\,\nu-\frac{505}{56}\,\nu^2\right)+\frac{G^2\,m^2}{r^2}\left(\frac{43}{252}+\frac{1543}{126}\,\nu-\frac{293}{84}\,\nu^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{Gm}{r}\dot{r}^2\left(\frac{5}{252}+\frac{241}{252}\nu+\frac{335}{84}\nu^2\right)+\frac{Gm}{r}v^2\left(-\frac{671}{252}+\frac{1297}{126}\nu+\frac{121}{12}\nu^2\right)\Bigg]\Bigg]\\
&+\varepsilon_{abi}x^a v^{jb}\,\frac{r\dot{r}}{c^2}\Bigg[-\frac{5}{28}+\frac{5}{14}\nu+\frac{1}{c^2}\Bigg[v^2\left(-\frac{25}{168}+\frac{25}{24}\nu-\frac{25}{14}\nu^2\right)\\
&+\frac{G\,m}{r}\left(-\frac{103}{63}-\frac{337}{126}\nu+\frac{173}{84}\nu^2\right)\Bigg]\Bigg]+\mathcal{O}(5)\,.
\end{aligned}
\tag{6.15}
$$

The remaining current moments required are given by

$$
\begin{aligned}
J_{ijk} &= \nu m\,\mathrm{STF}_{ijk}\epsilon_{kab}\Bigg\{x_{aij}v_b\Bigg[1-3\nu\\
&\quad+\frac{1}{c^2}\Bigg[\left(\frac{41}{90}-\frac{77}{18}\nu+\frac{185}{18}\nu^2\right)v^2+\left(\frac{14}{9}-\frac{16}{9}\nu-\frac{86}{9}\nu^2\right)\frac{Gm}{r}\Bigg]\Bigg]\\
&\quad+\frac{7}{45}x^a v^{ijb}\frac{r^2}{c^2}(1-5\nu+5\nu^2)+\frac{2}{9}x_{ai}v_{bj}\frac{r\dot{r}}{c^2}(1-5\nu+5\nu^2)\Bigg\}\,,
\end{aligned}
\tag{6.16}
$$

$$
J_{ijkl} = -\nu\, m\,\sqrt{1-4\nu}(1-2\nu)\mathrm{STF}_{ijkl}\left\{\epsilon_{lab}\,x_{aijk}v_b\right\}, \tag{6.17}
$$

$$
J_{ijklm} = \nu\, m\,(1-5\nu+5\nu^2)\,\mathrm{STF}_{ijklm}\left\{\epsilon_{mab}\,x_{aijkl}v_b\right\}. \tag{6.18}
$$

Starting from the input multipole moments listed above, one can calculate the angular momentum flux. The next step involved in the computation is to obtain the time derivatives of the multipole moments. This requires the equation of motion of the binary up to the relevant order, maximum being 3PN for the mass quadrupole. The equation of motion (EOM) of the inspiralling compact binary is complete up to 3PN in ADM and harmonic coordinates [144, 228, 148, 76, 73]. In our computation we use the 3PN EOM in the centre of mass (CM) coordinates in the harmonic gauge obtained in [151]. The ambiguity parameter λ in the expression is no longer arbitrary but now uniquely determined and given by $\lambda = -3080/1987$. The full 3PN EOM is given below for completeness.

$$
\frac{dv^i}{dt} = -\frac{m}{r^2}\left[(1+\mathcal{P})\,n^i+\mathcal{Q}\,v^i\right]+\mathcal{O}\left(\frac{1}{c^7}\right)\,, \tag{6.19}
$$

where the coefficients $\mathcal{P}$ and $\mathcal{Q}$ are

$$
\begin{aligned}
\mathcal{P} &= \frac{1}{c^2}\left\{-\frac{3\,\dot{r}^2\,\nu}{2}+v^2+3\,\nu\,v^2-\frac{m}{r}\,(4+2\,\nu)\right\}\\
&+\frac{1}{c^4}\Bigg\{\frac{15\,\dot{r}^4\,\nu}{8}-\frac{45\,\dot{r}^4\,\nu^2}{8}-\frac{9\,\dot{r}^2\,\nu\,v^2}{2}+6\,\dot{r}^2\,\nu^2\,v^2+3\,\nu\,v^4-4\,\nu^2\,v^4\\
&\qquad+\frac{m}{r}\left(-2\,\dot{r}^2-25\,\dot{r}^2\,\nu-2\,\dot{r}^2\,\nu^2-\frac{13\,\nu\,v^2}{2}+2\,\nu^2\,v^2\right)
\end{aligned}
$$

$$
+ \frac{m^2}{r^2}\left(9 + \frac{87\,\nu}{4}\right)\Bigg\}
$$

$$
+ \frac{1}{c^5}\left\{-\frac{24\,\dot{r}\,\nu\,v^2}{5}\frac{m}{r} - \frac{136\,\dot{r}\,\nu}{15}\frac{m^2}{r^2}\right\}
$$

$$
\begin{aligned}
+ \frac{1}{c^6}\Bigg\{&-\frac{35\,\dot{r}^6\,\nu}{16} + \frac{175\,\dot{r}^6\,\nu^2}{16} - \frac{175\,\dot{r}^6\,\nu^3}{16} + \frac{15\,\dot{r}^4\,\nu\,v^2}{2} \\
&- \frac{135\,\dot{r}^4\,\nu^2\,v^2}{4} + \frac{255\,\dot{r}^4\,\nu^3\,v^2}{8} - \frac{15\,\dot{r}^2\,\nu\,v^4}{2} + \frac{237\,\dot{r}^2\,\nu^2\,v^4}{8} \\
&- \frac{45\,\dot{r}^2\,\nu^3\,v^4}{2} + \frac{11\,\nu\,v^6}{4} - \frac{49\,\nu^2\,v^6}{4} + 13\,\nu^3\,v^6 \\
&+ \frac{m}{r}\Bigg(79\,\dot{r}^4\,\nu - \frac{69\,\dot{r}^4\,\nu^2}{2} - 30\,\dot{r}^4\,\nu^3 - 121\,\dot{r}^2\,\nu\,v^2 + 16\,\dot{r}^2\,\nu^2\,v^2 \\
&\qquad + 20\,\dot{r}^2\,\nu^3\,v^2 + \frac{75\,\nu\,v^4}{4} + 8\,\nu^2\,v^4 - 10\,\nu^3\,v^4\Bigg) \\
&+ \frac{m^2}{r^2}\Bigg(\dot{r}^2 + \frac{32573\,\dot{r}^2\,\nu}{168} + \frac{11\,\dot{r}^2\,\nu^2}{8} - 7\,\dot{r}^2\,\nu^3 + \frac{615\,\dot{r}^2\,\nu\,\pi^2}{64} - \frac{26987\,\nu\,v^2}{840} \\
&\qquad + \nu^3\,v^2 - \frac{123\,\nu\,\pi^2\,v^2}{64} - 110\,\dot{r}^2\,\nu\,\ln\left(\frac{r}{r'_0}\right) + 22\,\nu\,v^2\,\ln\left(\frac{r}{r'_0}\right)\Bigg) \\
&+ \frac{m^3}{r^3}\left(-16 - \frac{41911\,\nu}{420} + \frac{44\,\lambda\,\nu}{3} - \frac{71\,\nu^2}{2} + \frac{41\,\nu\,\pi^2}{16}\right)\Bigg\}\,,
\end{aligned}
\tag{6.20a}
$$

$$
\begin{aligned}
\mathcal{Q} = {}& \frac{1}{c^2}\left\{-4\,\dot{r} + 2\,\dot{r}\,\nu\right\} \\
&+ \frac{1}{c^4}\Bigg\{\frac{9\,\dot{r}^3\,\nu}{2} + 3\,\dot{r}^3\,\nu^2 - \frac{15\,\dot{r}\,\nu\,v^2}{2} - 2\,\dot{r}\,\nu^2\,v^2 \\
&\qquad + \frac{m}{r}\left(2\,\dot{r} + \frac{41\,\dot{r}\,\nu}{2} + 4\,\dot{r}\,\nu^2\right)\Bigg\} \\
&+ \frac{1}{c^5}\left\{\frac{8\,\nu\,v^2}{5}\frac{m}{r} + \frac{24\,\nu}{5}\frac{m^2}{r^2}\right\} \\
&+ \frac{1}{c^6}\Bigg\{-\frac{45\,\dot{r}^5\,\nu}{8} + 15\,\dot{r}^5\,\nu^2 + \frac{15\,\dot{r}^5\,\nu^3}{4} + 12\,\dot{r}^3\,\nu\,v^2 \\
&\qquad - \frac{111\,\dot{r}^3\,\nu^2\,v^2}{4} - 12\,\dot{r}^3\,\nu^3\,v^2 - \frac{65\,\dot{r}\,\nu\,v^4}{8} + 19\,\dot{r}\,\nu^2\,v^4 + 6\,\dot{r}\,\nu^3\,v^4 \\
&\qquad + \frac{m}{r}\left(\frac{329\,\dot{r}^3\,\nu}{6} + \frac{59\,\dot{r}^3\,\nu^2}{2} + 18\,\dot{r}^3\,\nu^3 - 15\,\dot{r}\,\nu\,v^2 - 27\,\dot{r}\,\nu^2\,v^2 - 10\,\dot{r}\,\nu^3\,v^2\right) \\
&\qquad + \frac{m^2}{r^2}\Bigg(-4\,\dot{r} - \frac{18169\,\dot{r}\,\nu}{840} + 25\,\dot{r}\,\nu^2 + 8\,\dot{r}\,\nu^3 - \frac{123\,\dot{r}\,\nu\,\pi^2}{32} \\
&\qquad\qquad + 44\,\dot{r}\,\nu\,\ln\left(\frac{r}{r'_0}\right)\Bigg)\Bigg\}\,.
\end{aligned}
\tag{6.20b}
$$

6.3 Angular momentum flux up to 3PN in standard harmonic coordinates: Instantaneous terms

Using the multipole moments given in the previous section and the EOM in the CM frame, one computes the required time derivatives as required in Eq. (6.6). It is then straightforward to compute the different instantaneous contributions to the angular momentum flux at 3PN order. This has to be supplemented by the hereditary contributions to the 2.5PN and 3PN orders, an issue *not* addressed in this thesis. We thus have,

$$\begin{aligned}\left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm SHar} &= \Bigg[\left(\frac{d\mathcal{J}_i}{dt}\right)^{N} + \left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 1PN} + \left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 2PN} \\ &\quad + \left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 2.5PN} + \left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 3PN}\Bigg]_{\rm inst} + \left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm her} + \mathcal{O}(7). \end{aligned} \tag{6.21}$$

We next discuss the various contributions to the angular momentum flux.

6.3.1 The angular momentum flux up to 2PN order

The first important check on our algebra is the reproduction of the terms computed earlier in the angular momentum flux up to 2PN order [139, 109, 111] [139, 109, 111]. For completeness we list the the angular momentum flux up to 2PN in this section.

$$\left(\frac{d\mathcal{J}_i}{dt}\right)^{N} = \frac{G^2m^3\nu^2}{c^5r^3}\,\tilde{\mathbf{L}}_i\left\{\frac{16}{5}v^2 - \frac{24}{5}\dot{r}^2 - \frac{16}{5}\frac{G\,m}{r}\right\}, \tag{6.22a}$$

$$\begin{aligned}\left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 1PN} &= \frac{G^2m^3\nu^2}{c^7r^3}\mathbf{L}_i\Bigg\{v^4\left(\frac{614}{105} - \frac{1096}{105}\nu\right) + v^2\,\dot{r}^2\left(-\frac{296}{35} + \frac{1108}{35}\nu\right) \\ &\quad + \frac{G\,m}{r}\,v^2\left(-\frac{464}{105} + \frac{152}{21}\nu\right) + \dot{r}^4\left(\frac{38}{7} - \frac{144}{7}\nu\right) \\ &\quad + \frac{G\,m}{r}\,\dot{r}^2\left(\frac{496}{35} + \frac{788}{105}\nu\right) + \frac{G^2\,m^2}{r^2}\left(-\frac{596}{21} + \frac{8}{105}\nu\right)\Bigg\}, \end{aligned} \tag{6.22b}$$

$$\begin{aligned}\left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 2PN} &= \frac{G^2m^3\nu^2}{c^9r^3}\mathbf{L}_i\Bigg\{v^6\left(\frac{53}{63} - \frac{353}{9}\nu + \frac{614}{15}\nu^2\right) + v^4\,\dot{r}^2\left(-\frac{2246}{105} + \frac{12653}{105}\nu - \frac{15637}{105}\nu^2\right) \\ &\quad + \frac{G\,m}{r}\,v^4\left(\frac{11}{21} - \frac{491}{315}\nu + \frac{4022}{315}\nu^2\right) + v^2\,\dot{r}^4\left(\frac{715}{21} - \frac{3361}{21}\nu + \frac{448}{3}\nu^2\right) \\ &\quad + \frac{G\,m}{r}\,v^2\,\dot{r}^2\left(\frac{21853}{315} - \frac{7201}{105}\nu + \frac{2551}{315}\nu^2\right) \\ &\quad + \frac{G^2\,m^2}{r^2}\,v^2\left(-\frac{21302}{315} + \frac{2262}{35}\nu - \frac{6856}{315}\nu^2\right) + \dot{r}^6\left(-\frac{52}{3} + \frac{652}{9}\nu - \frac{388}{9}\nu^2\right) \\ &\quad + \frac{G\,m}{r}\,\dot{r}^4\left(-\frac{22312}{315} + \frac{5914}{45}\nu - \frac{277}{9}\nu^2\right) + \frac{G^2\,m^2}{r^2}\,\dot{r}^2\left(\frac{5624}{105} - \frac{7172}{45}\nu + \frac{3058}{105}\nu^2\right) \end{aligned}$$

$$+\frac{G^3\,m^3}{r^3}\left(\frac{340724}{2835}+\frac{15658}{315}\nu+\frac{44}{45}\nu^2\right)\Bigg\}. \tag{6.22c}$$

In the above expressions $\tilde{\mathbf{L}}_{\mathbf{i}} = \varepsilon_{ijk} x_j\, v_k$.

6.3.2 Instantaneous terms at 2.5PN order

At 2.5PN order, as Eqs (6.7) and (6.8) would reveal, there are two kinds of terms apart from the inst(s) type ones. We present below the respective expressions obtained for the three kinds of contributions. Its worth noticing that all the three terms have an overall $\dot{r}$ dependence.

$$\begin{aligned}\left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm inst(s)} &= \frac{G}{c^5}\Bigg\{-\frac{32G^2m^4\nu^3\,\dot{r}}{175c^5r^4}\mathbf{L}_i\Bigg[822v^4-2460\,\dot{r}^2v^2+1154\frac{Gm}{r}v^2+1575\,\dot{r}^4\\ &\quad +324\frac{Gm^2}{r}-1460\frac{Gm}{r}\,\dot{r}^2\Bigg]\Bigg\}. \end{aligned} \tag{6.23}$$

$$\begin{aligned}\left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm inst(r)} &= \frac{G}{c^5}\Bigg\{-\frac{8G^2m^4\nu^3\,\dot{r}}{105c^5r^4}\mathbf{L}_i\Bigg[66v^4-15\,\dot{r}^2v^2+154\frac{Gm}{r}v^2-105\,\dot{r}^4+88\frac{Gm^2}{r}\\ &\quad -30\frac{Gm}{r}\,\dot{r}^2\Bigg]\Bigg\}. \end{aligned} \tag{6.24}$$

$$\begin{aligned}\left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm inst(c)} &= \frac{G}{c^5}\Bigg\{-\frac{8G^2m^4\nu^3\,\dot{r}}{105c^5r^4}42\,\mathbf{L}_i\,\Bigg[v^4-30\,\dot{r}^2v^2+35\,\dot{r}^4\\ &\quad -9\left(\frac{Gm}{r}\right)^2+2v^2\frac{G\,m}{r}-2\frac{Gm}{r}\,\dot{r}^2\Bigg]\Bigg\}. \end{aligned} \tag{6.25}$$

The total 2.5PN instantaneous contribution is the sum of these three contributions which finally reads as

$$\begin{aligned}\left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 2.5PN} &= \frac{G^2m^3\nu^2}{5\,c^{10}r^3}\mathbf{L}_i\Bigg\{\dot{r}\nu\Bigg[-\frac{27744}{35}\frac{G\,m}{r}\,v^4+\frac{19144}{7}\frac{G\,m}{r}\,v^2\,\dot{r}^2-\frac{116944}{105}\frac{G^2\,m^2}{r^2}v^2\\ &\quad +\frac{8976}{7}\left(\frac{Gm}{r}\right)^2\,\dot{r}^2-1960\frac{G\,m}{r}\,\dot{r}^4-\frac{22864}{105}\left(\frac{Gm}{r}\right)^3\Bigg]\Bigg\} \end{aligned} \tag{6.26}$$

6.3.3 Instantaneous terms at 3PN order

The 3PN instantaneous contribution to the angular momentum expression, which is one of the most important results of this chapter is presented below. It forms the starting point for the computation of evolution of orbital elements, our final goal.

$$
\begin{aligned}
\left(\frac{d\mathcal{J}_i}{dt}\right)^{3\mathrm{PN}} = \; & \frac{G^2 m^3\, \nu^2 \mathbf{L}_i}{r^3\, c^{11}} \left\{ v^8 \left[\frac{145919}{13860} - \frac{110423\,\nu}{1260} + \frac{1079083\,\nu^2}{4620} - \frac{30229\,\nu^3}{165} \right] \right. \\
& + \dot{r}^2 v^6 \left[-\frac{2473}{70} + \frac{763409\,\nu}{2310} - \frac{2155249\,\nu^2}{2310} + \frac{543171\,\nu^3}{770} \right] \\
& + \frac{G\,m}{r} v^6 \left[\frac{483097}{13860} - \frac{60913\,\nu}{1540} + \frac{28711\,\nu^2}{4620} + \frac{91\,\nu^3}{165} \right] \\
& + \dot{r}^4 v^4 \left[\frac{18695}{231} - \frac{632111\,\nu}{924} + \frac{1552525\,\nu^2}{924} - \frac{61970\,\nu^3}{77} \right] \\
& + \dot{r}^2 \frac{G\,m}{r} v^4 \left[\frac{205817}{13860} - \frac{74689\,\nu}{140} + \frac{6423539\,\nu^2}{13860} - \frac{9979\,\nu^3}{66} \right] \\
& + \frac{G^2\,m^2}{r^2} v^4 \left[\frac{5112059}{28875} - \frac{6848\log(\frac{r}{r_0})}{175} + \frac{3152431\,\nu}{6930} \right. \\
& \left. - \frac{369\pi^2\,\nu}{40} - \frac{407999\,\nu^2}{3465} + \frac{154421\,\nu^3}{1155} \right] \\
& + \dot{r}^6 v^2 \left(-\frac{451}{6} + \frac{59870\,\nu}{99} - \frac{14841\,\nu^2}{11} + \frac{32342\,\nu^3}{99} \right] \\
& + \dot{r}^4 \frac{G\,m}{r} v^2 \left[-\frac{100999}{1980} + \frac{3716239\,\nu}{2772} - \frac{22275889\,\nu^2}{13860} + \frac{738973\,\nu^3}{3465} \right] \\
& + \dot{r}^2 \frac{G^2\,m^2}{r^2} v^2 \left[-\frac{14402404}{5775} + \frac{1712\log(\frac{r}{r_0})}{5} - \frac{1394339\,\nu}{495} \right. \\
& \left. + \frac{369\pi^2\,\nu}{4} + \frac{317813\,\nu^2}{495} - \frac{869048\,\nu^3}{3465} \right] \\
& + \frac{G^3\,m^3}{r^3} v^2 \left[\frac{1229915081}{1559250} - \frac{37664\log(\frac{r}{r_0})}{525} - \frac{8689013\,\nu}{28350} - \frac{352\ \log(\frac{r}{r_0'})\ \nu}{15} \right. \\
& \left. - \frac{41\pi^2\,\nu}{10} + \frac{184003\,\nu^2}{990} - \frac{155339\,\nu^3}{3465} \right] \\
& + \dot{r}^8 \left[\frac{93}{4} - \frac{4035\,\nu}{22} + \frac{4294\,\nu^2}{11} - \frac{410\,\nu^3}{11} \right] \\
& + \dot{r}^6 \frac{G\,m}{r} \left[\frac{1932907}{69300} - \frac{106499\,\nu}{140} + \frac{13641581\,\nu^2}{13860} - \frac{50525\,\nu^3}{1386} \right] \\
& + \dot{r}^4 \frac{G^2\,m^2}{r^2} \left[\frac{46903957}{17325} - \frac{1712\log(\frac{r}{r_0})}{5} + \frac{3002737\,\nu}{1155} - \frac{861\pi^2\,\nu}{8} \right. \\
& \left. - \frac{28913\,\nu^2}{330} + \frac{85543\,\nu^3}{3465} \right] \\
& + \dot{r}^2 \frac{G^3\,m^3}{r^3} \left[-\frac{152347309}{103950} + \frac{5136\log(\frac{r}{r_0})}{35} + \frac{102197341\,\nu}{103950} - \frac{176\ \log(\frac{r}{r_0'})\ \nu}{5} \right.
\end{aligned}
$$

$$+\frac{123\pi^2\,\nu}{20}-\frac{3265967\,\nu^2}{6930}+\frac{267188\,\nu^3}{3465}\Bigg]$$
$$+\frac{G^4\,m^4}{r^4}\Bigg[-\frac{47779894}{111375}+\frac{3424\log(\frac{r}{r_0})}{525}-\frac{49735624\,\nu}{51975}+\frac{704\ \log(\frac{r}{r_0'})\ \nu}{15}$$
$$+\frac{779\pi^2\,\nu}{40}-\frac{818\,\nu^2}{63}+\frac{1780\,\nu^3}{693}\Bigg]\Bigg\}. \tag{6.27}$$

In harmonic coordinates, as can be seen from above, the 3PN term contains two kinds of log terms, $\log r_0$ and $\log r_0'$. It was shown in the circular orbit case [143] that a gauge independent representation of angular momentum flux does not contain the $\log r_0'$ terms and hence it is a gauge dependent quantity. We discuss the cancellation of log terms in the general orbit case in Sec. 6.8.2

6.4 Angular momentum flux in ADM coordinates

The first prominent application of the present computation will be discussed in 6.9 where the evolution of the orbital elements under gravitational wave radiation reaction will be studied to order 3PN beyond the leading quadrupolar radiation reaction. This is based on the generalized quasi-Keplerian representation of the orbit which can be written down in both harmonic and ADM coordinates. However, since many related numerical relativity studies are in ADM-type coordinates we present the applications in the later sections of this chapter in ADM coordinates. This would require us to re-express the instantaneous expressions for the fluxes in harmonic coordinates in terms of the corresponding expressions in the ADM coordinates for later convenience of averaging over an orbit. The standard harmonic coordinates also contain gauge-dependent logarithm terms which are not the most convenient for numerical calculations.

The transformation from the standard harmonic to ADM coordinates require the contact transformations between the standard harmonic and ADM coordinates. The transformation of $\mathbf{r}$ in the standard harmonic coordinates to that in the ADM is obtained in Ref. [151] and reads as

$$\begin{aligned}
\mathbf{r}_{\rm SHar} \;=\; & \mathbf{r}_{\rm ADM}+\Bigg\{\frac{Gm}{c^4}\left[v^2\left(\frac{5}{8}\nu\right)+\dot{r}^2\left(-\frac{1}{8}\nu\right)+\frac{Gm}{r}\left(3\nu+\frac{1}{4}\right)\right] \\
& +\frac{Gm\nu}{c^6}\Bigg[v^4\left(\frac{1}{2}-\frac{11}{8}\nu\right)+\dot{r}^2v^2\left(-\frac{5}{16}+\frac{15}{16}\nu\right)+\dot{r}^4\left(\frac{1}{16}-\frac{5}{16}\nu\right) \\
& +\frac{Gm}{r}v^2\left(\frac{451}{48}+\frac{3}{8}\nu\right)+\frac{Gm}{r}\dot{r}^2\left(-\frac{161}{48}+\frac{5}{2}\nu\right) \\
& +\frac{G^2m^2}{r^2}\left(-\frac{2773}{280}+\frac{22}{3}\ln\left(\frac{r}{r_0'}\right)-\frac{21}{32}\pi^2\right)\Bigg]\Bigg\}\,\mathbf{n}
\end{aligned}$$

$$+\left\{\frac{Gm}{c^4}\dot{r}\left(-\frac{9}{4}\nu\right)\right.$$
$$\left.+\frac{Gm\nu}{c^6}\dot{r}\left[v^2\left(-\frac{17}{8}+\frac{21}{4}\nu\right)+\dot{r}^2\left(\frac{5}{12}-\frac{29}{24}\nu\right)+\frac{Gm}{r}\left(-\frac{43}{3}-5\nu\right)\right]\right\}\mathbf{v}\,. \quad (6.28)$$

Using this expression and equations of motion up to 3PN given earlier in Eq. (6.20), we calculate the expressions for transforming $\mathbf{v}$, r, v and $\dot{r}$ in standard harmonic to ADM coordinates which are,

$$\begin{aligned}
\mathbf{v}_{\mathrm{SHar}} &= \mathbf{v}_{\mathrm{ADM}}+\left\{\frac{Gm}{c^4 r}\left[v^2\left(-\frac{13}{8}\nu\right)+\dot{r}^2\left(\frac{17}{8}\nu\right)+\frac{Gm}{r}\left(\frac{1}{4}+\frac{21}{4}\nu\right)\right]\right.\\
&+\frac{Gm\nu}{c^6 r}\left[v^4\left(-\frac{13}{8}+\frac{31}{8}\nu\right)+\dot{r}^2 v^2\left(\frac{49}{16}-\frac{127}{16}\nu\right)\right.\\
&+\dot{r}^4\left(-\frac{19}{16}+\frac{53}{16}\nu\right)+\frac{Gm}{r}v^2\left(-\frac{9}{16}-\frac{25}{8}\nu\right)+\frac{Gm}{r}\dot{r}^2\left(\frac{165}{16}+\frac{45}{4}\nu\right)\\
&\left.\left.+\frac{G^2m^2}{r^2}\left(-\frac{3839}{840}+\frac{22}{3}\ln\left(\frac{r}{r'_0}\right)-\frac{21}{32}\pi^2+\frac{1}{2}\nu\right)\right]\right\}\mathbf{v}\\
&+\left\{\frac{Gm}{c^4 r}\dot{r}\left[v^2\left(-\frac{7}{8}\nu\right)+\dot{r}^2\left(\frac{3}{8}\nu\right)+\frac{Gm}{r}\left(-\frac{1}{2}-\frac{19}{4}\nu\right)\right]\right.\\
&+\frac{Gm\nu}{c^6 r}\dot{r}\left[v^4\left(-\frac{9}{8}+\frac{13}{4}\nu\right)+\dot{r}^2 v^2\left(\frac{19}{16}-\frac{65}{16}\nu\right)+\dot{r}^4\left(-\frac{5}{16}+\frac{25}{16}\nu\right)\right.\\
&+\frac{Gm}{r}v^2\left(-\frac{37}{2}+\frac{31}{8}\nu\right)+\frac{Gm}{r}\dot{r}^2\left(\frac{99}{8}-\frac{259}{24}\nu\right)\\
&\left.\left.+\frac{G^2m^2}{r^2}\left(\frac{28807}{840}-22\ln\left(\frac{r}{r'_0}\right)+\frac{63}{32}\pi^2-\frac{13}{4}\nu\right)\right]\right\}\mathbf{n}\,, \qquad (6.29a)\\
r_{\mathrm{SHar}} &= r_{\mathrm{ADM}}+\frac{Gm}{c^4}\left\{v^2\left(\frac{5}{8}\nu\right)+\dot{r}^2\left(-\frac{19}{8}\nu\right)+\frac{Gm}{r}\left(\frac{1}{4}+3\nu\right)\right\}\\
&+\frac{Gm\nu}{c^6}\left\{v^4\left(\frac{1}{2}-\frac{11}{8}\nu\right)+\dot{r}^2 v^2\left(-\frac{39}{16}+\frac{99}{16}\nu\right)\right.\\
&+\dot{r}^4\left(\frac{23}{48}-\frac{73}{48}\nu\right)+\frac{Gm}{r}v^2\left(\frac{451}{48}+\frac{3}{8}\nu\right)\\
&\left.+\frac{Gm}{r}\dot{r}^2\left(-\frac{283}{16}-\frac{5}{2}\nu\right)+\frac{G^2m^2}{r^2}\left(-\frac{2773}{280}+\frac{22}{3}\ln\left(\frac{r}{r'_0}\right)-\frac{21}{32}\pi^2\right)\right\}, \qquad (6.29b)\\
v^2{}_{\mathrm{SHar}} &= v^2_{\mathrm{ADM}}+\frac{Gm}{c^4 r}\left\{v^4\left(-\frac{13}{4}\nu\right)+\dot{r}^2 v^2\left(\frac{5}{2}\nu\right)+\dot{r}^4\left(\frac{3}{4}\nu\right)\right.\\
&\left.+\frac{Gm}{r}v^2\left(\frac{1}{2}+\frac{21}{2}\nu\right)+\frac{Gm}{r}\dot{r}^2\left(-1-\frac{19}{2}\nu\right)\right\}\\
&+\frac{Gm\nu}{c^6 r}\left\{v^6\left(-\frac{13}{4}+\frac{31}{4}\nu\right)+\dot{r}^2 v^4\left(\frac{31}{8}-\frac{75}{8}\nu\right)+\dot{r}^4 v^2\left(-\frac{3}{2}\nu\right)+\dot{r}^6\left(-\frac{5}{8}+\frac{25}{8}\nu\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{Gm}{r}v^4\left(-\frac{9}{8}-\frac{25}{4}\nu\right)+\frac{Gm}{r}\dot{r}^2v^2\left(-\frac{131}{8}+\frac{121}{4}\nu\right)+\frac{Gm}{r}\dot{r}^4\left(\frac{99}{4}-\frac{259}{12}\nu\right)\\
&+\frac{G^2m^2}{r^2}v^2\left(-\frac{3839}{420}+\frac{44}{3}\ln\left(\frac{r}{r_0'}\right)-\frac{21}{16}\pi^2+\nu\right)\\
&+\frac{G^2m^2}{r^2}\dot{r}^2\left(\frac{28807}{420}-44\ln\left(\frac{r}{r_0'}\right)+\frac{63}{16}\pi^2-\frac{13}{2}\nu\right)\Bigg\}, \qquad (6.29c)
\end{aligned}
$$

$$
\begin{aligned}
\dot{r}^2_{\rm SHar} \;=\;& \dot{r}^2_{\rm ADM}+\frac{Gm}{c^4r}\dot{r}^2\left\{v^2\left(-\frac{19}{2}\nu\right)+\dot{r}^2\left(\frac{19}{2}\nu\right)+\frac{Gm}{r}\left(-\frac{1}{2}+\nu\right)\right\}\\
&+\frac{Gm\nu}{c^6r}\dot{r}^2\left\{v^4\left(-\frac{39}{4}+\frac{99}{4}\nu\right)+\dot{r}^2v^2\left(\frac{163}{12}-\frac{443}{12}\nu\right)+\dot{r}^4\left(-\frac{23}{6}+\frac{73}{6}\nu\right)\right.\\
&+\frac{Gm}{r}v^2\left(-\frac{1603}{24}-\frac{17}{2}\nu\right)+\frac{Gm}{r}\dot{r}^2\left(\frac{1777}{24}+\frac{131}{12}\nu\right)\\
&\left.+\frac{G^2m^2}{r^2}\left(\frac{6242}{105}-\frac{88}{3}\ln\left(\frac{r}{r_0'}\right)+\frac{21}{8}\pi^2-\frac{11}{2}\nu\right)\right\}. \qquad (6.29d)
\end{aligned}
$$

It is important to note that the transformation equation contains $\log(r/r_0')$ and this, as we shall see later in this section, makes the fluxes in ADM coordinates free of $\log r_0'$. Employing the equations above, one can concisely rewrite a similar decomposition for the angular momentum flux in ADM coordinates. Since the transformations between the coordinate systems start at 2PN order, only the 2PN and 3PN terms in the flux get modified and we list these two terms below. The coordinates in the expression below, including $\tilde{\mathbf{L}}_{\mathbf{i}}=\varepsilon_{ijk}x_j\,v_k$ are all in ADM coordinates though we do not label them as such to avoid a heavy notation.

$$
\begin{aligned}
\left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 2PN}_{\rm ADM} \;=\;& \frac{G^2m^3\;\nu^2\,\mathbf{L}_i}{r^3\,c^9}\left\{v^6\left[\frac{533}{63}-\frac{353\nu}{9}+\frac{614\nu^2}{15}\right]\right.\\
&+\dot{r}^2v^4\left[-\frac{2246}{105}+\frac{12653\,\nu}{105}-\frac{15637\nu^2}{105}\right]\\
&+\left(\frac{G\,m}{r}\right)v^4\left[\frac{11}{21}-\frac{1333\,\nu}{63}+\frac{4022\nu^2}{315}\right]\\
&+\dot{r}^4v^2\left[\frac{715}{21}-\frac{3361\,\nu}{21}+\frac{448\nu^2}{3}\right]\\
&+\left(\frac{G\,m}{r}\right)\dot{r}^2v^2\left[\frac{21853}{315}+\frac{2942\,\nu}{105}+\frac{2551\nu^2}{315}\right]\\
&+\left(\frac{G\,m}{r}\right)^2v^2\left[-\frac{4210}{63}+\frac{2962\,\nu}{35}-\frac{6856\nu^2}{315}\right)\\
&+\dot{r}^6\left[-\frac{52}{3}+\frac{652\,\nu}{9}-\frac{388\nu^2}{9}\right]\\
&+\left(\frac{G\,m}{r}\right)\dot{r}^4\left[-\frac{22312}{315}+\frac{1999\,\nu}{45}-\frac{277\nu^2}{9}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left(\frac{G\,m}{r}\right)^2 \dot{r}^2 \bigg[\frac{5666}{105} - \frac{6938\,\nu}{45} + \frac{3058\nu^2}{105}\bigg] \\
&\quad + \left(\frac{G\,m}{r}\right)^3 \bigg[\frac{336188}{2835} + \frac{11878\,\nu}{315} + \frac{44\nu^2}{45}\bigg]\bigg\}, \qquad (6.30) \\
\left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 3PN}_{\rm ADM} &= \frac{G^2 m^3\,\nu^2 \mathbf{L}_i}{r^3\,c^{11}} \bigg\{ v^8 \bigg[\frac{145919}{13860} - \frac{110423\,\nu}{1260} + \frac{1079083\nu^2}{4620} - \frac{30229\nu^3}{165}\bigg] \\
&\quad + \dot{r}^2 v^6 \bigg[-\frac{2473}{70} + \frac{763409\,\nu}{2310} - \frac{2155249\nu^2}{2310} + \frac{543171\nu^3}{770}\bigg] \\
&\quad + \left(\frac{G\,m}{r}\right) v^6 \bigg[\frac{483097}{13860} - \frac{17429\,\nu}{154} + \frac{693331\nu^2}{4620} + \frac{91\nu^3}{165}\bigg] \\
&\quad + \dot{r}^4 v^4 \bigg[\frac{18695}{231} - \frac{632111\,\nu}{924} + \frac{1552525\nu^2}{924} - \frac{61970\nu^3}{77}\bigg] \\
&\quad + \left(\frac{G\,m}{r}\right) \dot{r}^2 v^4 \bigg[\frac{205817}{13860} - \frac{22549\,\nu}{105} - \frac{6112567\nu^2}{13860} - \frac{9979\nu^3}{66}\bigg] \\
&\quad + \left(\frac{G\,m}{r}\right)^2 v^4 \bigg[\frac{10477393}{57750} + \frac{1801028\,\nu}{3465} - \frac{369\pi^2\nu}{40} - \frac{994673\nu^2}{3465} \\
&\quad + \frac{154421\,\nu^3}{1155} - \frac{6848}{175}\log\left(\frac{r}{r_0}\right)\bigg] \\
&\quad + \dot{r}^6 v^2 \bigg[-\frac{451}{6} + \frac{59870\,\nu}{99} - \frac{14841\nu^2}{11} + \frac{32342\nu^3}{99}\bigg] \\
&\quad + \left(\frac{G\,m}{r}\right) \dot{r}^4 v^2 \bigg[-\frac{100999}{1980} + \frac{6446971\,\nu}{6930} - \frac{512285\nu^2}{2772} + \frac{738973\nu^3}{3465}\bigg] \\
&\quad + \left(\frac{G\,m}{r}\right)^2 \dot{r}^2 v^2 \bigg[-\frac{14457734}{5775} - \frac{18425707\,\nu}{6930} + \frac{369\pi^2\nu}{4} + \frac{3826709\nu^2}{3465} \\
&\quad - \frac{869048\,\nu^3}{3465} + \frac{1712}{5}\log\left(\frac{r}{r_0}\right)\bigg] \\
&\quad + \left(\frac{G\,m}{r}\right)^3 v^2 \bigg[\frac{1229915081}{1559250} - \frac{827081\,\nu}{2835} - \frac{31\pi^2\nu}{5} + \frac{886279\nu^2}{6930} \\
&\quad - \frac{155339\,\nu^3}{3465} - \frac{37664}{525}\log\left(\frac{r}{r_0}\right)\bigg] \\
&\quad + \dot{r}^8 \bigg[\frac{93}{4} - \frac{4035\,\nu}{22} + \frac{4294\nu^2}{11} - \frac{410\nu^3}{11}\bigg] \\
&\quad + \left(\frac{G\,m}{r}\right) \dot{r}^6 \bigg[\frac{1932907}{69300} - \frac{40997\,\nu}{70} + \frac{955543\nu^2}{2772} - \frac{50525\nu^3}{1386}\bigg] \\
&\quad + \left(\frac{G\,m}{r}\right)^2 \dot{r}^4 \bigg[\frac{93865829}{34650} + \frac{924466\,\nu}{385} - \frac{861\pi^2\nu}{8} - \frac{210811\nu^2}{462} \\
&\quad + \frac{85543\,\nu^3}{3465} - \frac{1712}{5}\log\left(\frac{r}{r_0}\right)\bigg] \\
&\quad + \left(\frac{G\,m}{r}\right)^3 \dot{r}^2 \bigg[-\frac{153361069}{103950} + \frac{7294789\,\nu}{10395} + 3\pi^2\nu
\end{aligned}
$$

$$
-\frac{2376287\,\nu^2}{6930}+\frac{267188\nu^3}{3465}+\frac{5136}{35}\log\left(\frac{r}{r_0}\right)\Bigg]
$$
$$
+\left(\frac{G\,m}{r}\right)^4\Bigg[-\frac{317864383}{779625}-\frac{7120061\,\nu}{10395}+\frac{947\pi^2\nu}{40}-\frac{3748\nu^2}{315}
$$
$$
+\frac{1780\,\nu^3}{693}+\frac{3424}{525}\log\left(\frac{r}{r_0}\right)\Bigg]\Bigg\}. \tag{6.31}
$$

Before we discuss the calculation of the orbital average of the angular momentum flux in Sec. 6.6, we summarize the generalized quasi-Keplerian representation at 3PN order in the ADM coordinates recently obtained in [227] which, as mentioned earlier, is another essential input for the computations to follow.

6.5 3PN quasi-Keplerian representation: Summary

The quasi-Keplerian representation at 1PN was introduced by Damour and Deruelle[223] to discuss the problem of binary pulsar timing. This elegant formulation has recently played a crucial role in the our computation of the hereditary terms in the energy flux in [115] and been concisely summarised there. The 2PN generalized quasi-Keplerian parametrization in the ADM coordinates was given by Damour, Schäfer and Wex [224, 225, 226] (see Fig 6.1 for pictorial representation of the orbit). In a more recent work, the 3PN parametrization of the orbital motion of the binary was constructed by Memmeshiemer, Gopakumar and Schäfer [227]. In ADM-type coordinates Eq. (19) of Ref. [227] provides the 3PN parametrization which reads as

$$
r = a_r\,(1-e_r\,\cos u)\,, \tag{6.32a}
$$
$$
l \equiv n\,(t-t_0) = u - e_t\,\sin u + \left(\frac{g_{4t}}{c^4}+\frac{g_{6t}}{c^6}\right)(v-u) + \left(\frac{f_{4t}}{c^4}+\frac{f_{6t}}{c^6}\right)\sin v + \frac{i_{6t}}{c^6}\,\sin 2\,v + \frac{h_{6t}}{c^6}\,\sin 3\,v\,, \tag{6.32b}
$$
$$
\frac{2\,\pi}{\Phi}\,(\phi-\phi_0) = v + \left(\frac{f_{4\phi}}{c^4}+\frac{f_{6\phi}}{c^6}\right)\sin 2v + \left(\frac{g_{4\phi}}{c^4}+\frac{g_{6\phi}}{c^6}\right)\sin 3v + \frac{i_{6\phi}}{c^6}\,\sin 4v + \frac{h_{6\phi}}{c^6}\,\sin 5v\,, \tag{6.32c}
$$

where

$$
v = 2\,\arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2}\,\tan\frac{u}{2}\right]. \tag{6.33}
$$

In the above equations a_r, e_r, l, u, n, e_t, e_ϕ, and $2\pi/\Phi$ are the 3PN accurate semi-major axis, radial eccentricity, mean anomaly, eccentric anomaly, mean motion, 'time' eccentricity, angular eccentricity and the angle of advance of periastron per orbital revolution, respec-

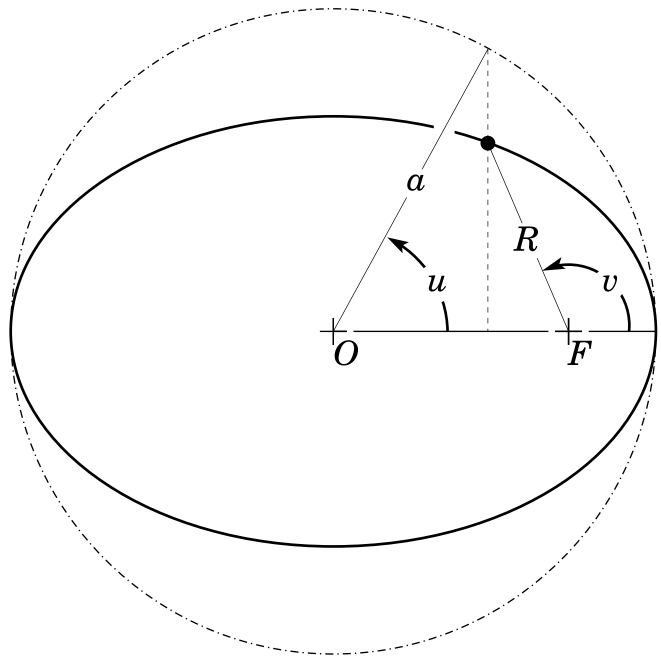

Figure 6.1: The geometrical interpretation of the eccentric and true anomalies, u and v, appearing in the Keplerian parametrization. The auxiliary circle of radius a circumscribes the orbital ellipse with semi-major axis a. The points O and F stand for the origin and the focus of the ellipse. Figure taken from Ref. [227].

tively. Notice that the equations contain three types of eccentricities $e_r\,, e_t\,, e_\phi$ named after the three coordinates $r\,, t\,,$ and ϕ and are different from each other starting at 1PN order. $\Phi/2\pi \equiv K \equiv 1 + k$ is the notation followed in other related works. The explicit dependence of the orbital elements and all the coefficients above in ADM coordinates in terms of the 3PN conserved orbital energy and 3PN conserved angular momentum are given in Eq. (25) of [227] as a PN series and they form the basis for our computation of the average angular momentum flux at 3PN order. We list these equations which forms the basis of our later computations. The 3PN accurate expressions for the orbital elements $a_r, e_r^2, n, e_t^2, \Phi$, and e_ϕ^2 and the post-Newtonian orbital functions $g_{4t}, g_{6t}, f_{4t}, f_{6t}, i_{6t}, h_{6t}, f_{4\phi}, f_{6\phi}, g_{4\phi}, g_{6\phi}, i_{6\phi}$, and $h_{6\phi}$, in terms of $E, h = \frac{|\mathbf{J}|}{G\,m}$ and ν read

$$
\begin{aligned}
a_r \quad = \quad & \frac{1}{(-2\,E)}\Big\{1 + \frac{(-2\,E)}{4\,c^2}\,(-7+\nu) + \frac{(-2\,E)^2}{16c^4}\Big[(1+10\,\nu+\nu^2) \\
& + \frac{1}{(-2\,E\,h^2)}(-68+44\,\nu)\Big] + \frac{(-2\,E)^3}{192\,c^6}\Big[3 - 9\,\nu - 6\,\nu^2 \\
& + 3\,\nu^3 + \frac{1}{(-2\,E\,h^2)}\Big(864 + \left(-3\,\pi^2 - 2212\right)\nu + 432\,\nu^2\Big)
\end{aligned}
$$

$$+\frac{1}{(-2\,E\,h^2)^2}\Big(-6432+\Big(13488-240\,\pi^2\Big)\,\nu-768\,\nu^2\Big)\Big]\Big\}\,, \tag{6.34a}$$

$$\begin{aligned}
{e_r}^2 \;=\;& 1+2\,E\,h^2+\frac{(-2\,E)}{4\,c^2}\Big\{24-4\,\nu+5\;(-3+\nu)\,(-2\,E\,h^2)\Big\}\\
&+\frac{(-2\,E)^2}{8\,c^4}\Big\{52+2\,\nu+2\,\nu^2-\Big(80-55\,\nu+4\,\nu^2\Big)\,(-2\,E\,h^2)\\
&-\frac{8}{(-2\,E\,h^2)}\,(-17+11\,\nu)\Big\}+\frac{(-2\,E)^3}{192\,c^6}\Big\{-768-6\,\nu\,\pi^2\\
&-344\,\nu-216\,\nu^2+3(-2\,E\,h^2)\Big(-1488+1556\,\nu-319\,\nu^2\\
&+4\,\nu^3\Big)-\frac{4}{(-2\,E\,h^2)}\Big(588-8212\,\nu+177\,\nu\,\pi^2+480\,\nu^2\Big)\\
&+\frac{192}{(-2\,E\,h^2)^2}\Big(134-281\,\nu+5\,\nu\,\pi^2+16\,\nu^2\Big)\Big\}\,,
\end{aligned} \tag{6.34b}$$

$$\begin{aligned}
n \;=\;& (-2\,E)^{3/2}\Big\{1+\frac{(-2\,E)}{8\,c^2}\;(-15+\nu)+\frac{(-2\,E)^2}{128c^4}\Big[555+30\,\nu\\
&+11\,\nu^2+\frac{192}{\sqrt{(-2\,E\,h^2)}}\,(-5+2\,\nu)\Big]+\frac{(-2\,E)^3}{3072\,c^6}\Big[-29385\\
&-4995\,\nu-315\,\nu^2+135\,\nu^3-\frac{16}{(-2\,E\,h^2)^{3/2}}\Big(10080+123\,\nu\,\pi^2\\
&-13952\,\nu+1440\,\nu^2\Big)+\frac{5760}{\sqrt{(-2\,E\,h^2)}}\Big(17-9\,\nu+2\,\nu^2\Big)\Big]\Big\}\,,
\end{aligned} \tag{6.34c}$$

$$\begin{aligned}
{e_t}^2 \;=\;& 1+2\,E\,h^2+\frac{(-2\,E)}{4\,c^2}\Big\{-8+8\,\nu-(-17+7\,\nu)\,(-2\,E\,h^2)\Big\}\\
&+\frac{(-2\,E)^2}{8\,c^4}\Big\{8+4\,\nu+20\,\nu^2-(-2\,E\,h^2)(112-47\,\nu+16\,\nu^2)\\
&-24\,\sqrt{(-2\,E\,h^2)}\,(-5+2\,\nu)+\frac{4}{(-2\,E\,h^2)}\,(17-11\,\nu)\\
&-\frac{24}{\sqrt{(-2\,E\,h^2)}}\;(5-2\,\nu)\Big\}\\
&+\frac{(-2\,E)^3}{192\,c^6}\Big\{24\;(-2+5\,\nu)\Big(-23+10\,\nu+4\,\nu^2\Big)-15\Big(-528\\
&+200\,\nu-77\,\nu^2+24\,\nu^3\Big)(-2\,E\,h^2)-72\,(265-193\,\nu\\
&+46\,\nu^2)\,\sqrt{(-2\,E\,h^2)}-\frac{2}{(-2\,E\,h^2)}\Big(6732+117\,\nu\,\pi^2-12508\,\nu\\
&+2004\,\nu^2\Big)+\frac{2}{\sqrt{(-2\,E\,h^2)}}\Big(16380-19964\,\nu+123\,\nu\,\pi^2\\
&+3240\,\nu^2\Big)-\frac{2}{(-2\,E\,h^2)^{3/2}}\Big(10080+123\,\nu\,\pi^2-13952\,\nu\\
&+1440\,\nu^2\Big)+\frac{96}{(-2\,E\,h^2)^2}\Big(134-281\,\nu+5\,\nu\,\pi^2+16\,\nu^2\Big)\Big\}\,,
\end{aligned} \tag{6.34d}$$

$$g_{4t} = \frac{3\,(-2\,E)^2}{2}\left\{\frac{5-2\,\nu}{\sqrt{(-2\,E\,h^2)}}\right\}, \tag{6.34e}$$

$$\begin{aligned} g_{6t} &= \frac{(-2\,E)^3}{192}\left\{\frac{1}{(-2\,E\,h^2)^{3/2}}\Big(10080 + 123\,\nu\pi^2 - 13952\,\nu \\ &\quad +1440\,\nu^2\Big) + \frac{1}{\sqrt{(-2\,E\,h^2)}}\left(-3420 + 1980\,\nu - 648\,\nu^2\right)\right\}, \end{aligned} \tag{6.34f}$$

$$f_{4t} = -\frac{1}{8}\,\frac{(-2\,E)^2}{\sqrt{(-2\,E\,h^2)}}\left\{(4+\nu)\,\nu\,\sqrt{(1+2\,E\,h^2)}\right\}, \tag{6.34g}$$

$$\begin{aligned} f_{6t} &= \frac{(-2\,E)^3}{192}\left\{\frac{1}{(-2\,E\,h^2)^{3/2}}\,\frac{1}{\sqrt{1+2\,E\,h^2}}\Big(1728 - 4148\,\nu + 3\,\nu\pi^2 \right. \\ &\quad +600\,\nu^2 + 33\,\nu^3\Big) + 3\,\frac{\sqrt{(-2\,E\,h^2)}}{\sqrt{(1+2\,E\,h^2)}}\nu\left(-64 - 4\,\nu + 23\,\nu^2\right) \\ &\quad + \frac{1}{\sqrt{(-2\,E\,h^2)\,(1+2\,E\,h^2)}}\Big(-1728 + 4232\,\nu - 3\,\nu\pi^2 \\ &\quad \left. -627\,\nu^2 - 105\,\nu^3\Big)\right\}, \end{aligned} \tag{6.34h}$$

$$i_{6t} = \frac{(-2\,E)^3}{32}\,\nu\left\{\frac{(1+2\,E\,h^2)}{(-2\,E\,h^2)^{3/2}}\left(23 + 12\,\nu + 6\,\nu^2\right)\right\}, \tag{6.34i}$$

$$h_{6t} = \frac{13\,(-2\,E)^3}{192}\nu^3\left(\frac{1+2\,E\,h^2}{-2\,E\,h^2}\right)^{3/2}, \tag{6.34j}$$

$$\begin{aligned} \Phi &= 2\,\pi\left\{1 + \frac{3}{c^2h^2} + \frac{(-2\,E)^2}{4\,c^4}\left[\frac{3}{(-2\,E\,h^2)}\,(-5+2\,\nu) \right.\right. \\ &\quad \left. + \frac{15}{(-2\,E\,h^2)^2}\,(7-2\,\nu)\right] + \frac{(-2\,E)^3}{128\,c^6}\left[\frac{24}{(-2\,E\,h^2)}(5-5\nu \right. \\ &\quad +4\nu^2) - \frac{1}{(-2\,E\,h^2)^2}\left(10080 - 13952\,\nu + 123\,\nu\pi^2 + 1440\,\nu^2\right) \\ &\quad \left.\left. + \frac{5}{(-2\,E\,h^2)^3}\left(7392 - 8000\,\nu + 123\,\nu\pi^2 + 336\,\nu^2\right)\right]\right\}, \end{aligned} \tag{6.34k}$$

$$f_{4\phi} = \frac{(-2\,E)^2}{8}\,\frac{(1+2\,E\,h^2)}{(-2\,E\,h^2)^2}\,\nu\,(1-3\,\nu), \tag{6.34l}$$

$$\begin{aligned} f_{6\phi} &= \frac{(-2\,E)^3}{256}\left\{\frac{4\,\nu}{(-2\,E\,h^2)}\left(-11 - 40\,\nu + 24\,\nu^2\right)\right. \\ &\quad + \frac{1}{(-2\,E\,h^2)^2}\left(-256 + 1192\,\nu - 49\,\nu\pi^2 + 336\,\nu^2 - 80\,\nu^3\right) \\ &\quad \left. + \frac{1}{(-2\,E\,h^2)^3}\left(256 + 49\,\nu\pi^2 - 1076\,\nu - 384\,\nu^2 - 40\,\nu^3\right)\right\}, \end{aligned} \tag{6.34m}$$

$$g_{4\phi} = -\frac{3(-2\,E)^2}{32}\,\frac{\nu^2}{(-2\,E\,h^2)^2}(1+2\,E\,h^2)^{3/2}, \tag{6.34n}$$

$$g_{6\phi} = \frac{(-2\,E)^3}{768}\,\sqrt{(1+2\,E\,h^2)}\left\{-\frac{3}{(-2\,E\,h^2)}\,\nu^2\,(9-26\,\nu)\right.$$

$$
\begin{aligned}
& -\frac{1}{(-2\,E\,h^2)^2}\,\nu\Big(220+3\,\pi^2+312\,\nu+150\,\nu^2\Big) \\
& +\frac{1}{(-2\,E\,h^2)^3}\,\nu\Big(220+3\,\pi^2+96\,\nu+45\,\nu^2\Big)\Big\}\,, \qquad (6.34o)
\end{aligned}
$$

$$
i_{6\phi} = \frac{(-2\,E)^3}{128}\,\frac{(1+2\,E\,h^2)^2}{(-2\,E\,h^2)^3}\,\nu\Big(5+28\,\nu+10\,\nu^2\Big)\,, \qquad (6.34p)
$$

$$
h_{6\phi} = \frac{5\,(-2\,E)^3}{256}\,\frac{\nu^3}{(-2\,E\,h^2)^3}\,(1+2\,E\,h^2)^{5/2}\,, \qquad (6.34q)
$$

$$
\begin{aligned}
{e_\phi}^2 = {} & 1+2\,E\,h^2+\frac{(-2\,E)}{4\,c^2}\Big\{24+(-15+\nu)\,(-2\,E\,h^2)\Big\} \\
& +\frac{(-2\,E)^2}{16\,c^4}\Big\{-32+176\,\nu+18\,\nu^2-(-2\,E\,h^2)(160-30\,\nu \\
& +3\,\nu^2)+\frac{1}{(-2\,E\,h^2)}\Big(408-232\,\nu-15\,\nu^2\Big) \\
& +\frac{(-2\,E)^3}{384\,c^6}\Big\{-16032+2764\,\nu+3\,\nu\,\pi^2+4536\,\nu^2+234\,\nu^3 \\
& -36\Big(248-80\,\nu+13\,\nu^2+\nu^3\Big)(-2\,E\,h^2)-\frac{6}{(-2\,E\,h^2)}\Big(2456 \\
& -26860\,\nu+581\,\nu\,\pi^2+2689\,\nu^2+10\,\nu^3\Big)+\frac{3}{(-2\,E\,h^2)^2}\Big(27776 \\
& -65436\,\nu+1325\,\nu\,\pi^2+3440\,\nu^2-70\,\nu^3\Big)\Big\}\,. \qquad (6.34r)
\end{aligned}
$$

The three eccentricities e_r, e_t and e_ϕ, which differ from each other at PN orders, are related by

$$
\begin{aligned}
e_t = {} & e_r\Big\{1+\frac{(-2E)}{2c^2}(-8+3\nu)+\frac{(-2E)^2}{8c^4}\,\frac{1}{(-2Eh^2)}\Big[-34+22\nu \\
& +(-60+24\nu)\,\sqrt{(-2Eh^2)}+\Big(72-33\nu+12\nu^2\Big)(-2Eh^2)\Big] \\
& +\frac{(-2E)^3}{192c^6}\,\frac{1}{(-2Eh^2)^2}\Big[-6432+13488\nu-240\nu\pi^2 \\
& -768\nu^2+\Big(-10080+13952\nu-123\nu\pi^2 \\
& -1440\nu^2\Big)\sqrt{(-2Eh^2)}+(2700-4420\nu-3\nu\pi^2 \\
& +1092\nu^2)(-2Eh^2)+(9180-6444\nu+1512\nu^2)(-2Eh^2)^{3/2} \\
& +\Big(-3840+1284\nu-672\nu^2+240\nu^3\Big)(-2Eh^2)^2\Big]\Big\}\,, \qquad (6.35a)
\end{aligned}
$$

$$
\begin{aligned}
e_\phi = {} & e_r\Big\{1+\frac{(-2E)}{2c^2}\nu+\frac{(-2E)^2}{32c^4}\,\frac{1}{(-2Eh^2)}\Big[136-56\nu-15\nu^2 \\
& +\nu(20+11\nu)(-2Eh^2)\Big]+\frac{(-2E)^3}{768c^6}\,\frac{1}{(-2Eh^2)^2}\Big[31872
\end{aligned}
$$

$$-88404\nu + 2055\nu\pi^2 + 4176\nu^2 - 210\nu^3 + \Big(2256$$
$$+10228\nu - 15\nu\pi^2 - 2406\nu^2 - 450\nu^3\Big)(-2Eh^2) + 6\nu\,(136$$
$$+34\nu + 31\nu^2)(-2Eh^2)^2\Big]\Big\}\,. \tag{6.35b}$$

These relations allow one to choose a specific eccentricity, while describing a PN accurate non-circular orbit.

We find the following expressions useful in obtaining the orbital average of different quantities.

$$\begin{aligned}
-E &= \frac{1}{2}c^2\zeta^{2/3}\Bigg\{1 + \zeta^{2/3}\left[\frac{5}{4} - \frac{\nu}{12}\right] + \zeta^{4/3}\left[\frac{5}{8} + \frac{5-2\nu}{\sqrt{1-e_t^2}} - \frac{5\nu}{8} - \frac{\nu^2}{24}\right] \\
&\quad + \zeta^2\Bigg[-\frac{185}{192} - \frac{75\nu}{64} - \frac{25\nu^2}{288} - \frac{35\nu^3}{5184} + \frac{\frac{5}{2} + \frac{23\nu}{6} - \frac{10\nu^2}{3}}{\sqrt{1-e_t^2}} \\
&\quad + \frac{40 - \frac{499\nu}{9} + \frac{41\pi^2\nu}{96} + 7\nu^2}{\left(1-e_t^2\right)^{3/2}}\Bigg]\Bigg\}\,, \\
-E\,h^2 &= \frac{(1-e_t^2)}{2} + \zeta^{2/3}\left[-1 + \nu + (1-e_t^2)\left[\frac{17}{8} - \frac{7\nu}{8}\right]\right] \\
&\quad + \zeta^{4/3}\Bigg[-5 + \frac{91\nu}{12} - \frac{7\nu^2}{12} + (1-e_t^2)\left[\frac{75}{16} - \frac{277\nu}{48} + \frac{29\nu^2}{48}\right] \\
&\quad + \sqrt{(1-e_t^2)}\left[\frac{15}{2} - 3\nu\right] + \frac{-\frac{15}{2} + 3\nu}{\sqrt{(1-e_t^2)}} + \frac{\frac{17}{4} - \frac{11\nu}{4}}{(1-e_t^2)}\Bigg] \\
&\quad + \zeta^2\Bigg[-\frac{145}{16} + \frac{357\nu}{16} - \frac{87\nu^2}{16} + \frac{\nu^3}{6} \\
&\quad + (1-e_t^2)\left[\frac{757}{128} - \frac{7381\nu}{384} + \frac{1207\nu^2}{192} - \frac{65\nu^3}{384}\right] \\
&\quad + \sqrt{(1-e_t^2)}\left[\frac{55}{2} - 20\nu + \frac{3\nu^2}{2}\right] \\
&\quad + \frac{\frac{305}{8} + \left[-\frac{1555}{24} + \frac{41\pi^2}{64}\right]\nu + \frac{35\nu^2}{4}}{\sqrt{(1-e_t^2)}} \\
&\quad + \frac{-\frac{391}{16} + \left[\frac{921}{16} - \frac{39\pi^2}{64}\right]\nu - \frac{479\nu^2}{48}}{(1-e_t^2)} \\
&\quad + \frac{-60 + \left[\frac{499}{6} - \frac{41\pi^2}{64}\right]\nu - \frac{21\nu^2}{2}}{(1-e_t^2)^{3/2}} \\
&\quad + \frac{42 + \left[-\frac{337}{4} + \frac{5\pi^2}{4}\right]\nu + \frac{19\nu^2}{2}}{(1-e_t^2)^2}\Bigg]\,,
\end{aligned} \tag{6.36}$$

where $\zeta = \frac{G m n}{c^3}$.

6.6 Orbital average of angular momentum flux: instantaneous terms in the ADM coordinates up to 3PN

Using the QK representation of the orbit in ADM coordinates summarised in the earlier section and the instantaneous angular momentum flux in ADM coordinates obtained in section 6.4, one transforms the expression for the magnitude of the angular momentum flux $d\mathcal{J}/dt\,(r, \dot{r}^2, v^2) \equiv |d\mathcal{J}_i/dt|$ to $d\mathcal{J}/dt\,(E, h, e_r, u)$ where E is the conserved orbital energy and h is related the conserved angular momentum $\mathbf{J}$ as $h = |\mathbf{J}|/Gm$. This expression up to 3PN order is schematically given as

$$\frac{d\mathcal{J}}{dt} = \frac{du}{ndt} \sum_{N=2}^{10} \left[\frac{\alpha_N(e_t)}{(1 - e_t \cos u)^N} + \beta_N(e_t) \frac{\sin u}{(1 - e_t \, \cos u)^N} + \gamma_N(e_t) \frac{\ln(1 - e_t \cos u}{(1 - e_t \cos u)^N} \right], \quad (6.37)$$

where,

$$\alpha_{\rm N}(E, h) = \frac{\nu^2}{G\, c^5} (-E)^5 \beta_{\rm N}(E, h)\,. \quad (6.38)$$

The coefficients $\beta_N(E, h)$ appearing in Eq. (6.38) can be written down as a PN series (See Eq. (4.12) of [111] for instance) but too long to be listed here. The computation of the orbital average involves the evaluation of the integral,

$$\langle \frac{d\mathcal{J}}{dt} \rangle = \frac{1}{P} \int_0^P \frac{d\mathcal{J}}{dt}(t)\, dt = \frac{1}{2\,\pi} \int_0^{2\,\pi} \left(\frac{ndt}{du} \right) \frac{d\mathcal{J}}{dt}(u) du\,. \quad (6.39)$$

Rewriting the angular momentum flux using the generalized QK representation, the flux can be averaged over an orbit to order 3PN extending the results of [111] at 2PN. Recall here the useful formulae

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{du}{(1 - e \cos u)^{N+1}} = \frac{1}{(1 - e^2)^{(N+1)/2}} P_N \left(\frac{1}{\sqrt{1 - e^2}} \right), \quad (6.40a)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\sin u}{(1 - e \cos u)^N} du = 0\,, \quad (6.40b)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{du}{(1 - e \cos u)^N} = \frac{1}{(1 - e^2)^{N/2}} P_{(N-1)} \left(\frac{1}{\sqrt{1 - e^2}} \right), \quad (6.40c)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\ln(1 - e \cos u)}{(1 - e \cos u)^N} du = \frac{(-1)^N}{N!} \frac{d^N \mathcal{Y}(e, y)}{dy^N} \bigg|_{y=1}, \quad (6.40d)$$

where,

$$\mathcal{Y}(et,y) = \frac{1}{\sqrt{y^2-e^2}}\left\{\ln\left[\frac{1}{2}\left(1+\sqrt{1-e^2}\right)\right]-2\ln\left[y+\sqrt{y^2-e^2}\right]\right.$$
$$\left.+2\ln\left[y-1+\sqrt{1-e^2}+\sqrt{y^2-e^2}\right]\right\}. \tag{6.41}$$

The expression for the orbital averaged angular momentum flux can be schematically written as:

$$\langle\frac{d\mathcal{J}}{dt}\rangle^{\mathrm{ADM}}_{\mathrm{inst}} = \frac{4}{5}c^2\,m\,\zeta^{7/3}\,\nu^2\,\frac{1}{(1-e_t{}^2)^{7/2}}\left[\langle\frac{d\mathcal{J}}{dt}\rangle^{\mathrm{Newt}}+\langle\frac{d\mathcal{J}}{dt}\rangle^{\mathrm{1PN}}+\langle\frac{d\mathcal{J}}{dt}\rangle^{\mathrm{2PN}}+\langle\frac{d\mathcal{J}}{dt}\rangle^{\mathrm{2.5PN}}\right.$$
$$\left.+\langle\frac{d\mathcal{J}}{dt}\rangle^{\mathrm{3PN}}\right], \tag{6.42}$$

where $\zeta = \frac{G\,m\,n}{c^3}$ and the individual terms read as:

$$\langle\frac{d\mathcal{J}}{dt}\rangle^{\mathrm{Newt}} = \frac{8+7e_t^2}{\left(1-e_t^2\right)^2}, \tag{6.43a}$$

$$\langle\frac{d\mathcal{J}}{dt}\rangle^{\mathrm{1PN}} = \zeta^{2/3}\frac{1}{(1-e_t)^3}\left\{\frac{1105}{42}-\frac{70\nu}{3}+e_t^2\left[\frac{5077}{42}-\frac{335\nu}{3}\right]\right.$$
$$\left.+e_t^4\left[\frac{8399}{336}-\frac{275\nu}{12}\right]\right\}, \tag{6.43b}$$

$$\langle\frac{d\mathcal{J}}{dt}\rangle^{\mathrm{2PN}} = \zeta^{4/3}\frac{1}{\left(1-e_t^2\right)^4}\left\{\left[\frac{7238}{81}-\frac{10175\,\nu}{63}+\frac{260\,\nu^2}{9}\right]\right.$$
$$+e_t^2\left[\frac{376751}{756}-\frac{37047\,\nu}{28}+\frac{1546\,\nu^2}{3}\right]$$
$$+e_t^4\left[\frac{377845}{756}-\frac{168863\,\nu}{168}+569\,\nu^2\right]$$
$$+e_t^6\left[\frac{30505}{2016}-\frac{2201\,\nu}{56}+\frac{1519\,\nu^2}{36}\right]$$
$$\left.+\sqrt{1-e_t^2}\left[80-32\,\nu+e_t^2(335-134\,\nu)+e_t^4(35-14\,\nu)\right]\right\}, \tag{6.43c}$$

$$\langle\frac{d\mathcal{J}}{dt}\rangle^{\mathrm{3PN}} = \zeta^2\frac{1}{\left(1-e_t^2\right)^5}\left\{\left[\frac{265845199}{138600}-\frac{20318135\,\nu}{6804}+\frac{287\pi^2\,\nu}{4}+\frac{187249\,\nu^2}{378}-\frac{1550\,\nu^3}{81}\right]\right.$$
$$+e_t^2\left[\frac{1476919051}{178200}-\frac{82215823\,\nu}{6804}+\frac{5171\pi^2\,\nu}{32}+\frac{387467\,\nu^2}{54}-\frac{96973\,\nu^3}{81}\right]$$
$$+e_t^4\left[\frac{669008149}{103950}-\frac{206700631\,\nu}{18144}-\frac{2799\pi^2\,\nu}{256}+\frac{13341787\,\nu^2}{1008}-\frac{438907\,\nu^3}{108}\right]$$
$$+e_t^6\left[\frac{114553217}{123200}-\frac{39280525\,\nu}{18144}-\frac{615\pi^2\,\nu}{128}+\frac{1092025\,\nu^2}{336}-\frac{283205\,\nu^3}{162}\right]$$

$$
\begin{aligned}
&+e_t^8\left[-\frac{10305073}{709632}+\frac{417923\,\nu}{12096}+\frac{95413\,\nu^2}{8064}-\frac{146671\,\nu^3}{2592}\right]\\
&+\left[-\frac{13696}{105}-\frac{98012e_t^2}{105}-\frac{23326e_t^4}{35}-\frac{2461e_t^6}{70}\right]\log\left[\frac{2\left(1-e_t^2\right)G\,m}{c^2\left(\sqrt{1-e_t^2}+1\right)r_0\,\zeta^{2/3}}\right]\\
&+\sqrt{1-e_t^2}\left[\frac{351577}{630}-\frac{78139\,\nu}{63}+\frac{41\pi^2\,\nu}{6}+\frac{580\,\nu^2}{3}\right.\\
&+e_t^2\left[\frac{1723433}{315}-\frac{569398\,\nu}{63}+\frac{2747\pi^2\,\nu}{96}+1902\,\nu^2\right]\\
&+e_t^4\left[\frac{17557661}{5040}-\frac{2444195\,\nu}{504}+\frac{287\pi^2\,\nu}{96}+\frac{2703\,\nu^2}{2}\right]\\
&\left.\left.+e_t^6\left[70-\frac{203\,\nu}{3}+\frac{77\,\nu^2}{3}\right]\right]\right\}. \qquad (6.43\text{d})
\end{aligned}
$$

6.7 Orbital average of energy flux: instantaneous terms in ADM coordinates up to 3PN

The other crucial input required for the computation of orbital element evolution is the energy flux up to 3PN order in ADM coordinates. The energy flux in the standard harmonic coordinates is already obtained in Refs [190, 114]. The results are reproduced by our independent calculation. The *instantaneous contribution* to the energy flux in the standard harmonic coordinates up to 3PN reads

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)^{\text{SHar}}_{\text{inst}} &= \left[\left(\frac{d\mathcal{E}}{dt}\right)^{\text{N}}+\left(\frac{d\mathcal{E}}{dt}\right)^{\text{1PN}}+\left(\frac{d\mathcal{E}}{dt}\right)^{\text{2PN}}\right.\\
&\left.+\left(\frac{d\mathcal{E}}{dt}\right)^{\text{2.5PN}}+\left(\frac{d\mathcal{E}}{dt}\right)^{\text{3PN}}\right]+\mathcal{O}(7)\,, \qquad (6.44)
\end{aligned}
$$

$$
\left(\frac{d\mathcal{E}}{dt}\right)^{\text{N}} = \frac{32}{5}\frac{G^3\,m^4\,\nu^2}{c^5\,r^4}\left\{v^2-\frac{11}{12}\dot{r}^2\right\}, \qquad (6.45\text{a})
$$

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)^{\text{1PN}} &= \frac{32}{5}\frac{G^3\,m^4\,\nu^2}{c^7\,r^4}\left\{v^4\left(\frac{785}{336}-\frac{71}{28}\nu\right)+\dot{r}^2\,v^2\left(-\frac{1487}{168}+\frac{58}{7}\nu\right)\right.\\
&+\frac{G\,m}{r}v^2\left(-\frac{170}{21}+\frac{10}{21}\nu\right)+\dot{r}^4\left(\frac{687}{112}-\frac{155}{28}\nu\right)\\
&\left.+\frac{G\,m}{r}\dot{r}^2\left(\frac{367}{42}-\frac{5}{14}\nu\right)+\frac{G^2\,m^2}{r^2}\left(\frac{1}{21}-\frac{4}{21}\nu\right)\right\}, \qquad (6.45\text{b})
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)^{2\mathrm{PN}} = \; & \frac{32}{5}\frac{G^3\, m^4\, \nu^2}{c^9\, r^4}\left\{ v^6\left(\frac{47}{14} - \frac{5497}{504}\nu + \frac{2215}{252}\nu^2\right)\right. \\
& + \dot{r}^2 v^4\left(-\frac{573}{56} + \frac{1713}{28}\nu - \frac{1573}{42}\nu^2\right) \\
& + \frac{G\, m}{r} v^4\left(-\frac{247}{14} + \frac{5237}{252}\nu - \frac{199}{36}\nu^2\right) \\
& + \dot{r}^4 v^2\left(\frac{1009}{84} - \frac{5069}{56}\nu + \frac{631}{14}\nu^2\right) \\
& + \frac{G\, m}{r}\dot{r}^2 v^2\left(\frac{4987}{84} - \frac{8513}{84}\nu + \frac{2165}{84}\nu^2\right) \\
& + \frac{G^2\, m^2}{r^2} v^2\left(\frac{281473}{9072} + \frac{2273}{252}\nu + \frac{13}{27}\nu^2\right) \\
& + \dot{r}^6\left(-\frac{2501}{504} + \frac{10117}{252}\nu - \frac{2101}{126}\nu^2\right) \\
& + \frac{G\, m}{r}\dot{r}^4\left(-\frac{5585}{126} + \frac{60971}{756}\nu - \frac{7145}{378}\nu^2\right) \\
& + \frac{G^2\, m^2}{r^2}\dot{r}^2\left(-\frac{106319}{3024} - \frac{1633}{504}\nu - \frac{16}{9}\nu^2\right) \\
& \left. + \frac{G^3\, m^3}{r^3}\left(-\frac{253}{378} + \frac{19}{7}\nu - \frac{4}{27}\nu^2\right)\right\}, \qquad (6.45\mathrm{c})
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)^{2.5\mathrm{PN}} = \; & \frac{32}{5}\,\frac{G^3\, m^4\, \nu^2}{c^{10}\, r^4}\left\{\dot{r}\,\nu\left(-\frac{12349}{210}\,\frac{G\, m}{r}v^4 + \frac{4524}{35}\frac{G\, m}{r}v^2\,\dot{r}^2 - \frac{2753}{126}\frac{G^2\, m^2}{r^2}v^2\right.\right. \\
& \left.\left. - \frac{985}{14}\,\frac{G\, m}{r}\,\dot{r}^4 + \frac{13981}{630}\,\frac{G^2\, m^2}{r^2}\,\dot{r}^2 - \frac{1}{315}\frac{G^3\, m^3}{r^3}\right)\right\}, \qquad (6.45\mathrm{d})
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)^{3\mathrm{PN}} = \; & \frac{32}{5}\frac{G^3\, m^4\, \nu^2}{c^{11}\, r^4}\left\{ v^8\left(\frac{80315}{14784} - \frac{694427}{22176}\nu + \frac{604085}{11088}\nu^2 - \frac{16985}{462}\nu^3\right)\right. \\
& + \dot{r}^2 v^6\left(-\frac{31499}{1008} + \frac{1119913}{5544}\nu - \frac{44701}{132}\nu^2 + \frac{38725}{231}\nu^3\right) \\
& + \frac{G\, m}{r} v^6\left(-\frac{61669}{3696} + \frac{95321}{1008}\nu - \frac{955013}{11088}\nu^2 + \frac{47255}{1386}\nu^3\right) \\
& + \dot{r}^4 v^4\left(\frac{204349}{2464} - \frac{3522149}{7392}\nu + \frac{2354753}{3696}\nu^2 - \frac{109447}{462}\nu^3\right) \\
& + \frac{G\, m}{r}\dot{r}^2 v^4\left(\frac{136695}{1232} - \frac{202693}{336}\nu + \frac{744377}{1232}\nu^2 - \frac{931099}{5544}\nu^3\right) \\
& + \frac{G^2\, m^2}{r^2} v^4\left(\frac{598614941}{2494800} - \frac{856}{35}\ln\left(\frac{r}{r_0}\right)\right. \\
& \left. + \left[\frac{39896}{2079} - \frac{369}{64}\pi^2\right]\nu + \frac{1300907}{33264}\nu^2 - \frac{161783}{24948}\nu^3\right) \\
& + \dot{r}^6 v^2\left(-\frac{1005979}{11088} + \frac{2589599}{5544}\nu - \frac{1322141}{2772}\nu^2 + \frac{90455}{693}\nu^3\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G\,m}{r}\,\dot{r}^4 v^2\left(-\frac{715157}{3696}+\frac{35158037}{33264}\nu-\frac{3672143}{3696}\nu^2+\frac{871025}{4158}\nu^3\right)\\
&+\frac{G^2m^2}{r^2}\,\dot{r}^2 v^2\left(-\frac{35629009}{37800}+\frac{3424}{35}\ln\left(\frac{r}{r_0}\right)\right.\\
&\left.+\left[-\frac{150739}{1232}+\frac{861}{32}\pi^2\right]\nu-\frac{453247}{1848}\nu^2+\frac{496081}{8316}\nu^3\right)\\
&+\frac{G^3\,m^3}{r^3}v^2\left(-\frac{24608492}{155925}+\frac{856}{105}\ln\left(\frac{r}{r_0}\right)\right.\\
&\left.+\left[-\frac{6356291}{22680}+\frac{44}{3}\ln\left(\frac{r}{r'_0}\right)+\frac{451}{64}\pi^2\right]\nu+\frac{3725}{462}\nu^2-\frac{841}{2268}\nu^3\right)\\
&+\dot{r}^8\left(\frac{1507925}{44352}-\frac{20365}{126}\nu+\frac{687305}{5544}\nu^2-\frac{32755}{1386}\nu^3\right)\\
&+\frac{G\,m}{r}\,\dot{r}^6\left(\frac{5476951}{55440}-\frac{671765}{1232}\nu+\frac{5205019}{11088}\nu^2-\frac{860477}{11088}\nu^3\right)\\
&+\frac{G^2\,m^2}{r^2}\,\dot{r}^4\left(\frac{115627817}{166320}-\frac{214}{3}\ln\left(\frac{r}{r_0}\right)\right.\\
&\left.+\left[\frac{42671}{792}-\frac{697}{32}\pi^2\right]\nu+\frac{1099355}{4752}\nu^2-\frac{825331}{16632}\nu^3\right)\\
&+\frac{G^3\,m^3}{r^3}\,\dot{r}^2\left(\frac{3202601}{23100}-\frac{1712}{315}\ln\left(\frac{r}{r_0}\right)\right.\\
&\left.+\left[\frac{6220199}{22680}-\frac{88}{9}\ln\left(\frac{r}{r'_0}\right)-\frac{1763}{192}\pi^2\right]\nu+\frac{57577}{1848}\nu^2-\frac{43018}{6237}\nu^3\right)\\
&\left.+\frac{G^4\,m^4}{r^4}\left(\frac{37571}{8316}-\frac{14962}{891}\nu-\frac{3019}{594}\nu^2-\frac{866}{6237}\nu^3\right)\right\}. \qquad (6.45e)
\end{aligned}
$$

Following a similar procedure as for the angular momentum flux, employing the contact transformations, one can transform the energy flux in the standard harmonic coordinates to an expression for energy flux in the ADM coordinates

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)_{\mathrm{ADM}} &= \left(\frac{d\mathcal{E}}{dt}\right)_{\mathrm{SHar}\to\mathrm{ADM}}-\frac{G^4m^5\nu^2}{c^9r^5}\left\{v^4\left(\frac{184}{5}\nu\right)-\dot{r}^2v^2\left(\frac{736}{5}\,\nu\right)\right.\\
&\left.+\frac{Gm}{r}v^2\left(\frac{16}{5}+\frac{48}{5}\nu\right)+\dot{r}^4\left(\frac{320}{3}\,\nu\right)+\frac{Gm}{r}\,\dot{r}^2\left(-\frac{12}{5}-\frac{56}{15}\nu\right)\right\}\\
&+\frac{G^4m^5\nu^2}{15\,c^{11}r^5}\left\{v^6\left(-\frac{17658\,\nu}{7}+\frac{24240\,\nu^2}{7}\right)\right.\\
&+\dot{r}^2v^4\left(\frac{129866\,\nu}{7}-21598\,\nu^2\right)+\left(\frac{Gm}{r}\right)v^4\left(\frac{22798\,\nu}{7}-\frac{22584\,\nu^2}{7}\right)\\
&+\dot{r}^4v^2\left(-\frac{689434\,\nu}{21}+\frac{714608\,\nu^2}{21}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\dot{r}^2\left(\frac{Gm}{r}\right)v^2\left(\frac{2808}{7}-\frac{80515\,\nu}{7}+\frac{70430\,\nu^2}{7}\right)\\
&+\left(\frac{Gm}{r}\right)^2 v^2\left(\frac{4080}{7}+\frac{31856\,\nu}{5}+126\pi^2\,\nu-\frac{288\,\nu^2}{7}-1408\,\nu\log\left(\frac{r}{r_0'}\right)\right)\\
&+\dot{r}^6\left(\frac{116138\,\nu}{7}-\frac{110986\,\nu^2}{7}\right)\\
&+\dot{r}^4\left(\frac{Gm}{r}\right)\left(-328+\frac{198097\,\nu}{21}-\frac{143924\,\nu^2}{21}\right)\\
&+\left(\frac{Gm}{r}\right)^3\left(-\frac{48}{7}-\frac{384\,\nu}{7}+\frac{2304\,\nu^2}{7}\right)\\
&+\dot{r}^2\left(\frac{Gm}{r}\right)^2\left(-\frac{4836}{7}-\frac{673544\,\nu}{105}-84\pi^2\,\nu\right.\\
&\left.\left.-\frac{2484\,\nu^2}{7}+\frac{2816}{3}\,\nu\log\left(\frac{r}{r_0'}\right)\right)\right\}\,. \qquad (6.46)
\end{aligned}
$$

The notation $\left(\frac{d\mathcal{E}}{dt}\right)_{\mathrm{SHar}\to\mathrm{ADM}}$ refers to the expression where all the variables in harmonic coordinates (r_{SHar}, v_{SHar}, and $\dot{r}_{\mathrm{SHar}}$) are replaced with the corresponding in ADM (r_{ADM}, v_{ADM}, and $\dot{r}_{\mathrm{ADM}}$). Using the energy flux in ADM coordinates thus obtained, the orbital average of the energy flux is given by

$$
\langle\frac{d\mathcal{E}}{dt}\rangle^{\mathrm{ADM}}_{\mathrm{inst}}=\frac{32c^5\nu^2\,\zeta^{10/3}}{5\left(1-e_t^2\right)^{7/2}G}\left[\langle\frac{d\mathcal{E}}{dt}\rangle^{\mathrm{Newt}}+\langle\frac{d\mathcal{E}}{dt}\rangle^{1\mathrm{PN}}+\langle\frac{d\mathcal{E}}{dt}\rangle^{2\mathrm{PN}}+\langle\frac{d\mathcal{E}}{dt}\rangle^{3\mathrm{PN}}\right], \qquad (6.47)
$$

where

$$
\langle\frac{d\mathcal{E}}{dt}\rangle^{\mathrm{Newt}} = \left\{1+\frac{73e_t^2}{24}+\frac{37e_t^4}{96}\right\}, \qquad (6.48)
$$

$$
\begin{aligned}
\langle\frac{d\mathcal{E}}{dt}\rangle^{1\mathrm{PN}} &= \frac{\zeta^{2/3}}{1-e_t^2}\left\{\frac{2113}{336}-\frac{35\nu}{12}+e_t^2\left[\frac{10305}{224}-\frac{1081\nu}{36}\right]+e_t^4\left[\frac{3841}{128}-\frac{311\nu}{12}\right]\right.\\
&\left.+e_t^6\left[\frac{2179}{1792}-\frac{851\nu}{576}\right]\right\}, \qquad (6.49)
\end{aligned}
$$

$$
\begin{aligned}
\langle\frac{d\mathcal{E}}{dt}\rangle^{2\mathrm{PN}} &= \frac{\zeta^{4/3}}{\left(1-e_t^2\right)^2}\left\{\frac{299701}{9072}-\frac{16601\,\nu}{504}+\frac{65\nu^2}{18}\right.\\
&+e_t^2\left[\frac{5817277}{18144}-\frac{908501\,\nu}{2016}+\frac{5935\nu^2}{54}\right]\\
&+e_t^4\left[\frac{11282477}{24192}-\frac{6150947\,\nu}{8064}+\frac{247805\nu^2}{864}\right]\\
&+e_t^6\left[\frac{1801955}{16128}-\frac{750385\,\nu}{4032}+\frac{185305\nu^2}{1728}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+e_t^8\left[\frac{86567}{64512}-\frac{9769\,\nu}{4608}+\frac{21275\nu^2}{6912}\right]\\
&+\sqrt{1-e_t^2}\left[\frac{35}{2}-7\,\nu+e_t^2\left[\frac{6425}{48}-\frac{1285\nu}{24}\right]\right.\\
&\left.\left.+e_t^4\left[\frac{5065}{64}-\frac{1013\,\nu}{32}\right]+e_t^6\left[\frac{185}{96}-\frac{37\nu}{48}\right]\right]\right\}\,, \qquad (6.50)
\end{aligned}
$$

$$
\begin{aligned}
\langle\frac{d\mathcal{E}}{dt}\rangle^{3\mathrm{PN}} = & \frac{\zeta^2}{\left(1-e_t^2\right)^3}\left\{\frac{62181833}{158400}-\frac{32799587\,\nu}{54432}+\frac{779\pi^2\nu}{64}\right.\\
&+\frac{261337\,\nu^2}{3024}-\frac{775\nu^3}{324}\\
&+e_t^2\left[\frac{412054561}{105600}-\frac{609644827\,\nu}{108864}+\frac{96035\pi^2\nu}{1536}\right.\\
&\left.+\frac{12559915\,\nu^2}{6048}-\frac{53696\nu^3}{243}\right]\\
&+e_t^4\left[\frac{1531759583}{211200}-\frac{1121623015\,\nu}{108864}+\frac{22723\pi^2\nu}{3072}\right.\\
&\left.+\frac{12398711\,\nu^2}{1728}-\frac{10816087\nu^3}{7776}\right]\\
&+e_t^6\left[\frac{89256796753}{26611200}-\frac{731813647\,\nu}{145152}-\frac{219685\pi^2\nu}{12288}\right.\\
&\left.+\frac{2206527\,\nu^2}{448}-\frac{983251\nu^3}{648}\right]\\
&+e_t^8\left[\frac{12105629567}{47308800}-\frac{56556509\,\nu}{129024}-\frac{4059\pi^2\nu}{4096}\right.\\
&\left.+\frac{103625201\,\nu^2}{193536}-\frac{4586539\nu^3}{15552}\right]\\
&+e_t^{10}\left[-\frac{8977637}{11354112}+\frac{9287\,\nu}{48384}+\frac{8977\nu^2}{55296}-\frac{567617\nu^3}{124416}\right]\\
&+\sqrt{1-e_t^2}\left[\frac{30556517}{151200}-\frac{284705\,\nu}{1008}+\frac{287\pi^2\nu}{192}+\frac{455\nu^2}{12}\right.\\
&+e_t^2\left[\frac{251168231}{100800}-\frac{20078741\,\nu}{6048}+\frac{52685\pi^2\nu}{4608}+\frac{43559\nu^2}{72}\right]\\
&+e_t^4\left[\frac{1336667951}{403200}-\frac{35699627\,\nu}{8064}+\frac{41533\pi^2\nu}{6144}+\frac{303985\nu^2}{288}\right]\\
&+e_t^6\left[\frac{1558169203}{2419200}-\frac{42190997\,\nu}{48384}+\frac{1517\pi^2\nu}{9216}+\frac{73357\nu^2}{288}\right]\\
&\left.+e_t^8\left[\frac{185}{48}-\frac{1073\,\nu}{288}+\frac{407\nu^2}{288}\right]\right]
\end{aligned}
$$

$$+\frac{107\left[3072+43520e_t^2+82736e_t^4+28016e_t^6+891e_t^8\right]\mathrm{Log}[X]}{20160}\Bigg\}. \tag{6.51}$$

In the above expression

$$X=\left[\frac{c^2\left(\sqrt{1-e_t^2}+1\right)r_0\,\zeta^{2/3}}{2\left(1-e_t^2\right)G\,m}\right]. \tag{6.52}$$

Up to 2PN our results reproduce those in [107] and [111][1].

6.8 Checks on the calculation of the angular momentum flux

6.8.1 Circular orbit limit ($e_t = 0$)

As an algebraic check, we take the circular orbit limit of the orbital average of angular momentum flux and the energy flux in ADM coordinates expressed in terms of ζ and e_t. For circular orbit binaries the angular momentum flux and the energy flux must be simply related as

$$\frac{d\mathcal{E}}{dt}=\omega\,\frac{d\mathcal{J}}{dt} \tag{6.53}$$

in any coordinate system. Here $\frac{d\mathcal{J}}{dt}$ is the magnitude of the angular momentum flux. Th circular orbit limit of our calculation agrees with the above expression with ω and is given by

$$\begin{aligned}\omega &= \left(\frac{c^3\,\zeta}{G\,m}\right)\Bigg\{1+3\,\zeta^{2/3}+\zeta^{4/3}\left[\frac{39}{2}-7\nu\right]\\ &\quad+\zeta^2\left[\frac{315}{2}+\frac{1}{32}\left(-6536+123\pi^2\right)\nu+7\nu^2\right]\Bigg\},\end{aligned} \tag{6.54}$$

where $\zeta=\frac{G\,m\,n}{c^3}$. The ω above is consistent with the ω_{ADM} of [151] at 3PN providing the required check.

6.8.2 Comments on the logarithm terms in the angular momentum flux expression

The final expression in ADM coordinates, similar to the energy flux case, does not contain $\log r_0'$. This is consistent with the argument made earlier that $\log r_0'$ is a gauge dependent

[1]The results of [111] was corrected in Ref. [112] and our results match with those with corrected transformation. See [112]for details

quantity specific to the standard harmonic coordinates. However, even in the final expression for the averaged ADM angular momentum flux, the $\log r_0$ term still persists. In the circular orbit case, this term gets *exactly* cancelled by the contribution from the tail of tail terms at 3PN. In Refs [190, 114], this cancellation was proved in the case of the 3PN energy flux for general orbits. Recall that r_0 is an arbitrary length scale introduced in the general MPM formalism (to regularise divergences at infinity), which then appears in the definition of the multipole moments explicitly at 3PN. This is what leads to the $\log r_0$ dependence of the instantaneous terms in the angular momentum flux. However, the r_0 dependence of the 3PN radiative type mass quadrupole moment at infinity arises exclusively from the tail-of-tails. This should lead to a specific dependence of the hereditary terms on r_0 just appropriate to cancel the $\log r_0$ dependence of the instantaneous terms and produce a total angular momentum flux independent of $\log r_0$. In the present work the hereditary terms are not addressed and such an explicit verification not possible. However, the general argument for the cancellation allows us to conjecture the following expression for one of the 'enhancement' functions in the angular momentum flux:

$$G(e_t) = \frac{1 + \frac{229e_t^2}{32} + \frac{327e_t^4}{64} + \frac{69e_t^6}{256}}{(1 - e_t^2)^5}. \tag{6.55}$$

Reproducing the above proposed enhancement function will be a good check on the overall algebra when eventually the hereditary contribution to the angular momentum flux is computed.

6.9 Evolution of orbital elements under gravitational radiation reaction

The most important application of the 3PN angular momentum flux obtained here and the energy flux obtained in Ref [114] is to calculate how the orbital elements of the binary evolve with time under gravitational radiation reaction. Let us emphasize that by 3PN evolution of orbital elements under gravitational radiation reaction we mean its evolution under 5.5PN terms beyond leading newtonian order in the EOM. In this section, we compute the rate of change of n, e_t and a_r averaged over an orbit, due to gravitational radiation reaction.

The way to proceed towards the computation of the evolution of orbital elements is the following. We start with the 3PN accurate expressions for n and e_t in terms of the 3PN conserved energy (E) and angular momentum (J) of [227]. Differentiating them w.r.t time and using heuristic balance equations for energy and angular momentum up to 3PN order, we compute the rate of change of the orbital elements. This extends the earlier analyses at

Newtonian order by Peters [139], 1PN computation of Refs [107, 109] and at 2PN order by Ref [111, 112]. The 1.5PN hereditary effects also have been accounted in the orbital element evolution in Refs [108, 110].

The 3PN accurate expressions for the mean motion n, eccentricity e_t and semi-major axis a_r read are listed in Eqs (6.34). Let us use the example of n to outline the procedure adopted for the computation of orbital elements in more detail. Starting from Eq. (6.34c), the expression for n is symbolically written as

$$n = n(E, J). \tag{6.56}$$

Differentiating with respect to t one obtains

$$\frac{dn}{dt} = \gamma_1(e_t, \zeta, \nu)\,\frac{dE}{dt} + \gamma_2(e_t, \zeta, \nu)\,\frac{d|\mathbf{J}|}{dt}, \tag{6.57}$$

where γ_1 and γ_2 are PN expansions in powers of ζ. Now we use the balance equations,

$$\frac{dE}{dt} = -\frac{d\mathcal{E}}{dt}, \tag{6.58a}$$

$$\frac{d|\mathbf{J}|}{dt} = -\frac{d\mathcal{J}}{dt}. \tag{6.58b}$$

and replace the time derivatives of the conserved energy and angular momentum (on the right side of the expression for $\frac{dn}{dt}$) with the energy and angular momentum fluxes and compute the final expression for the orbital average by using the orbital averages of the energy and angular momentum fluxes up to 3PN. It may be noted from Eq. 6.34c that, the angular momentum flux is needed only up to 1PN accuracy for the computation of $\langle\frac{dn}{dt}\rangle$ where as the energy flux is needed up to 3PN. The structure of the evolution equations is similar for the other orbital elements also and the same procedure can be employed. The final expression for the 3PN evolution of n reads

$$\langle\frac{dn}{dt}\rangle^{\rm ADM}_{\rm inst} = \frac{c^6}{G^2\,m^2}\,\zeta^{11/3}\left[\langle\frac{dn}{dt}\rangle_{\rm Newt} + \langle\frac{dn}{dt}\rangle_{\rm 1PN} + \langle\frac{dn}{dt}\rangle_{\rm 2PN} + \langle\frac{dn}{dt}\rangle_{\rm 3PN}\right] \tag{6.59}$$

$$\langle\frac{dn}{dt}\rangle_{\rm Newt} = \frac{1}{\left(1-e_t^2\right)^{7/2}}\left\{\frac{96}{5} + \frac{292 e_t^2}{5} + \frac{37 e_t^4}{5}\right\}, \tag{6.60a}$$

$$\begin{aligned}\langle\frac{dn}{dt}\rangle_{\rm 1PN} = \frac{\zeta^{2/3}}{\left(1-e_t^2\right)^{9/2}}\Bigg\{&\frac{2546}{35} - \frac{264\,\nu}{5} + e_t^2\left[\frac{5497}{7} - 570\,\nu\right]\\ &+ e_t^4\left[\frac{14073}{20} - \frac{5061\,\nu}{10}\right] + e_t^6\left[\frac{11717}{280} - \frac{148\,\nu}{5}\right]\Bigg\},\end{aligned} \tag{6.60b}$$

$$
\begin{aligned}
\langle\frac{dn}{dt}\rangle_{2\mathrm{PN}} = {}& \frac{\zeta^{4/3}}{\left(1-e_t^2\right)^{11/2}}\Bigg\{\frac{393527}{945} + e_t^2\left[\frac{4098457}{945} - \frac{108047\nu}{15} + \frac{182387\nu^2}{90}\right] \\
& + e_t^4\left[\frac{1678961}{180} - \frac{2098263\nu}{140} + \frac{396443\nu^2}{72}\right] \\
& + e_t^6\left[\frac{1249229}{336} - \frac{76689\nu}{16} + \frac{192943\nu^2}{90}\right] \\
& + \sqrt{1-e_t^2}\Bigg[48 - \frac{47491\nu}{105} + \frac{944\nu^2}{15} \\
& + e_t^2\left[2134 - \frac{4268\nu}{5}\right] + e_t^4\left[2193 - \frac{4386\nu}{5}\right] \\
& + e_t^6\left[\frac{175}{2} - 35\nu\right] - \frac{96\nu}{5}\Bigg] \\
& + e_t^8\left[\frac{391457}{3360} - \frac{6037\nu}{56} + \frac{2923\nu^2}{45}\right]\Bigg]\Bigg\}\,, \qquad (6.60\mathrm{c})
\end{aligned}
$$

$$
\begin{aligned}
\langle\frac{dn}{dt}\rangle_{3\mathrm{PN}} = {}& \frac{\zeta^{2}}{\left(1-e_t^2\right)^{13/2}}\Bigg\{\Bigg[\frac{6687854333}{1039500} - \frac{113898769\,\nu}{11340} + \frac{2337\pi^2\,\nu}{10} \\
& + \frac{564197\,\nu^2}{420} - \frac{1121\,\nu^3}{27} \\
& + e_t^2\Bigg[\frac{132891898933}{2079000} - \frac{1993945913\,\nu}{22680} + \frac{19207\pi^2\,\nu}{16} \\
& + \frac{5552087\,\nu^2}{168} - \frac{1287385\,\nu^3}{324}\Bigg] \\
& + e_t^4\Bigg[\frac{151872497839}{1188000} - \frac{2340827549\,\nu}{12960} + \frac{22723\pi^2\,\nu}{160} \\
& + \frac{28833055\,\nu^2}{224} - \frac{33769597\,\nu^3}{1296}\Bigg] \\
& + e_t^6\Bigg[\frac{63380900591}{792000} - \frac{2509038229\,\nu}{20160} - \frac{43937\pi^2\,\nu}{128} \\
& + \frac{236136203\,\nu^2}{2240} - \frac{3200965\,\nu^3}{108}\Bigg] \\
& + e_t^8\Bigg[\frac{93247526201}{7392000} - \frac{1814291\,\nu}{96} - \frac{12177\pi^2\,\nu}{640} \\
& + \frac{3251909\,\nu^2}{210} - \frac{982645\,\nu^3}{162}\Bigg] \\
& + e_t^{10}\left[\frac{33332681}{197120} - \frac{1874543\,\nu}{10080} + \frac{109733\,\nu^2}{840} - \frac{8288\,\nu^3}{81}\right]\Bigg] \\
& + \sqrt{1-e_t^2}\Bigg[\Bigg[-\frac{669319}{1125} - \frac{3670\,\nu}{21} - \frac{41\pi^2\,\nu}{10} + \frac{632\,\nu^2}{5} \\
& + e_t^2\left[\frac{11326954}{375} - \frac{14778121\,\nu}{315} + \frac{45961\pi^2\,\nu}{240} + \frac{125278\,\nu^2}{15}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+e_t^4\left[\frac{1534643951}{21000}-\frac{5720941\,\nu}{60}+\frac{6191\pi^2\,\nu}{32}+\frac{317273\,\nu^2}{15}\right]\\
&+e_t^6\left[\frac{775558207}{31500}-\frac{35318351\,\nu}{1260}+\frac{287\pi^2\,\nu}{960}+\frac{232177\,\nu^2}{30}\right]\\
&+e_t^8\left[\frac{56403}{112}-\frac{427733\,\nu}{840}+\frac{4739\,\nu^2}{30}\right]\Bigg]\Bigg]\\
&+\frac{107\left[3072+43520e_t^2+82736e_t^4+28016e_t^6+891e_t^8\right]\ln[X]}{1050\left[1-e_t^2\right]^{13/2}}\Bigg\}, \qquad (6.60d)
\end{aligned}
$$

where

$$
X=\left[\frac{c^2\left(\sqrt{1-e_t^2}+1\right)r_0\,\zeta^{2/3}}{2\left(1-e_t^2\right)G\,m}\right]. \qquad (6.61)
$$

Let us next consider the orbital average of $\frac{de_t}{dt}$. From Eq. (6.34d), it is evident that both energy and angular momentum fluxes are now required up to 3PN in order to compute the 3PN evolution of e_t. Using the procedure described above we obtain the 3PN accurate expression for the orbital average of the evolution of e_t which is given by

$$
\langle\frac{de_t}{dt}\rangle_{\rm inst}^{\rm ADM}=\frac{c^3\,e_t}{G\,m}\left[\langle\frac{de_t}{dt}\rangle_{\rm Newt}+\langle\frac{de_t}{dt}\rangle_{\rm 1PN}+\langle\frac{de_t}{dt}\rangle_{\rm 2PN}+\langle\frac{de_t}{dt}\rangle_{\rm 3PN}\right], \qquad (6.62)
$$

$$
\begin{aligned}
\langle\frac{de_t}{dt}\rangle_{\rm Newt} &= \frac{\zeta^{8/3}}{\left(1-e_t^2\right)^{5/2}}\left\{\frac{304}{15}+\frac{121e_t^2}{15}\right\}, && (6.63a)\\
\langle\frac{de_t}{dt}\rangle_{\rm 1PN} &= \frac{\zeta^{10/3}}{\left(1-e_t^2\right)^{7/2}}\left\{\frac{14207}{105}-\frac{4084\nu}{45}+e_t^2\left[\frac{12231}{35}-\frac{7753\nu}{30}\right]\right.\\
&\quad\left.+e_t^4\left[\frac{13929}{280}-\frac{1664\nu}{45}\right]\right\}, && (6.63b)\\
\langle\frac{de_t}{dt}\rangle_{\rm 2PN} &= \frac{\zeta^{4}}{\left(1-e_t^2\right)^{9/2}}\left\{\frac{257771}{378}-\frac{13271\,\nu}{14}+\frac{752\,\nu^2}{5}\right.\\
&\quad+e_t^2\left[\frac{7199837}{2520}-\frac{4133467\,\nu}{840}+\frac{64433\,\nu^2}{40}\right]\\
&\quad+e_t^4\left[\frac{34890643}{15120}-\frac{15971227\,\nu}{5040}+\frac{127411\,\nu^2}{90}\right]\\
&\quad+e_t^6\left[\frac{420727}{3360}-\frac{362071\,\nu}{2520}+\frac{821\,\nu^2}{9}\right]\\
&\quad+\sqrt{1-e_t^2}\left[\frac{1336}{3}-\frac{2672\,\nu}{15}+e_t^2\left[\frac{2321}{2}-\frac{2321\,\nu}{5}\right]\right.
\end{aligned}
$$

$$
+e_t^4\left[\frac{565}{6}-\frac{113\,\nu}{3}\right]\Bigg]\Bigg\}\,, \tag{6.63c}
$$

$$
\begin{aligned}
\langle\frac{de_t}{dt}\rangle_{3\mathrm{PN}} \;=\;& \frac{\zeta^{14/3}}{\left(1-e_t^2\right)^{11/2}}\Bigg\{\frac{81933388819}{6237000}-\left(\frac{378365677}{22680}-\frac{10081\pi^2}{30}\right)\nu\\
&+\frac{23512757\nu^2}{7560}-\frac{61001\nu^3}{486}\\
&+e_t^2\Bigg[\frac{248490452963}{6237000}-\frac{25758469\nu}{504}+\frac{183233\pi^2\nu}{960}\\
&+\frac{92123261\nu^2}{3024}-\frac{86910509\nu^3}{19440}\Bigg]\\
&+e_t^4\Bigg[\frac{582802520423}{16632000}-\frac{1510921939\nu}{25920}-\frac{43043\pi^2\nu}{384}\\
&+\frac{2933299781\nu^2}{60480}-\frac{2223241\nu^3}{180}\Bigg]\\
&+e_t^6\Bigg[\frac{32454140779}{3696000}-\frac{145492421\nu}{10080}-\frac{6519\pi^2\nu}{640}\\
&+\frac{23529761\nu^2}{1890}-\frac{11792069\nu^3}{2430}\Bigg]\\
&+e_t^8\left[\frac{302322169}{1774080}-\frac{1921387\nu}{10080}+\frac{41179\nu^2}{216}-\frac{193396\nu^3}{1215}\right]\\
&+\sqrt{1-e_t^2}\Bigg[\frac{74222951}{15750}-\frac{1570691\nu}{189}+\frac{8323\pi^2\nu}{180}+\frac{54332\nu^2}{45}\\
&+e_t^2\left[\frac{214371937}{7875}-\frac{46294037\nu}{1260}+\frac{94177\pi^2\nu}{960}+\frac{681989\nu^2}{90}\right]\Bigg]\\
&+e_t^4\left[\frac{5992521613}{378000}-\frac{137760163\nu}{7560}+\frac{2501\pi^2\nu}{2880}+\frac{225106\nu^2}{45}\right]\\
&+e_t^6\left[\frac{186961}{336}-\frac{289691\nu}{504}+\frac{3197\nu^2}{18}\right]\\
&+\frac{730168}{23625\left(1+\sqrt{1-e_t^2}\right)}\\
&+\frac{107\left[24608+89024e_t^2+42884e_t^4+1719e_t^6\right]}{3150}\log\left[X\right]\Bigg\}\,,
\end{aligned} \tag{6.63d}
$$

where X is defined, similar to the earlier case, by

$$
X=\left[\frac{c^2\left(\sqrt{1-e_t^2}+1\right)r_0\,\zeta^{2/3}}{2\left(1-e_t^2\right)G\,m}\right]. \tag{6.64}
$$

Finally we compute the orbital average of the time derivative of semi-major axis a_r. Similar to the case of n, one requires a 3PN energy flux expression for its evaluation but only 1PN angular momentum flux. The final result reads

$$\langle\frac{da_r}{dt}\rangle^{\rm ADM}_{\rm inst} \;=\; \nu\, c\, \zeta^2 \left[\langle\frac{da_r}{dt}\rangle_{\rm Newt} + \langle\frac{da_r}{dt}\rangle_{\rm 1PN} + \langle\frac{da_r}{dt}\rangle_{\rm 2PN} + \langle\frac{da_r}{dt}\rangle_{\rm 3PN}\right] \tag{6.65}$$

$$\begin{aligned}
\langle\frac{da_r}{dt}\rangle_{\rm Newt} &= \frac{1}{\left(1-e_t^2\right)^{9/2}}\left\{-\frac{64}{5} - \frac{392e_t^2}{15} + 34e_t^4 + \frac{74e_t^6}{15}\right\},\\
\langle\frac{da_r}{dt}\rangle_{\rm 1PN} &= \frac{\zeta^{2/3}}{\left(1-e_t^2\right)^{11/2}}\left\{-\frac{5092}{105} + \frac{176\nu}{5} + e_t^2\left[-\frac{16626}{35} + \frac{1724\nu}{5}\right]\right.\\
&\quad + e_t^4\left[\frac{11429}{210} - \frac{213\nu}{5}\right] + e_t^6\left[\frac{37061}{84} - \frac{953\nu}{3}\right]\\
&\quad \left.+ e_t^8\left[\frac{11717}{420} - \frac{296\nu}{15}\right]\right\}, \qquad (6.66)\\
\langle\frac{da_r}{dt}\rangle_{\rm 2PN} &= \frac{\zeta^{4/3}}{(1-e_t^2)^{13/2}}\left\{-\frac{180998}{567} + \frac{22054\,\nu}{63} - \frac{608\,\nu^2}{15}\right.\\
&\quad + e_t^2\left[-\frac{7080622}{2835} + \frac{154921\,\nu}{35} - 1309\,\nu^2\right]\\
&\quad + e_t^4\left[-\frac{19396577}{5670} + \frac{2153051\,\nu}{420} - \frac{27935\,\nu^2}{12}\right]\\
&\quad + e_t^6\left[\frac{28278521}{7560} - \frac{5582839\,\nu}{840} + \frac{81053\,\nu^2}{36}\right]\\
&\quad + e_t^8\left[\frac{814607}{336} - \frac{8012201\,\nu}{2520} + \frac{12449\,\nu^2}{9}\right]\\
&\quad + e_t^{10}\left[\frac{366593}{5040} - \frac{9703\,\nu}{126} + \frac{1924\,\nu^2}{45}\right]\\
&\quad + \sqrt{1-e_t^2}\left[-96 + \frac{192\,\nu}{5} + e_t^2\left[-1356 + \frac{2712\,\nu}{5}\right]\right.\\
&\quad + e_t^4\left[99 - \frac{198\,\nu}{5}\right] + e_t^6\left[1279 - \frac{2558\,\nu}{5}\right]\\
&\quad \left.\left.+ e_t^8\left[74 - \frac{148\,\nu}{5}\right]\right]\right\}, \qquad (6.67)\\
\langle\frac{da_r}{dt}\rangle_{\rm 3PN} &= \frac{\zeta^2}{\left(1-e_t^2\right)^{15/2}}\left\{-\frac{7894936583}{1559250} - \frac{\left(-72118997 + 1600641\pi^2\right)\nu}{8505}\right.\\
&\quad - \frac{412199\,\nu^2}{378} + \frac{122\,\nu^3}{5}\\
&\quad + e_t^2\left[-\frac{38484254989}{1039500} - \frac{\left(-224390800 + 2612169\pi^2\right)\nu}{4536}\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{27073451\,\nu^2}{1260}+\frac{15541\,\nu^3}{6}\Bigg]\\
&+e_t^4\Bigg[-\frac{174073001497}{4158000}+\frac{\left(766493776+8997615\pi^2\right)\nu}{12960}\\
&-\frac{44679677\,\nu^2}{720}+\frac{355105\,\nu^3}{24}\Bigg]\\
&+e_t^6\Bigg[\frac{147582276487}{4989600}+\frac{\left(-17756552672+146897793\pi^2\right)\nu}{544320}\\
&+\frac{12433661\,\nu^2}{864}+\frac{171605\,\nu^3}{72}\Bigg]\\
&+e_t^8\Bigg[\frac{27778935839}{604800}-\frac{\left(2152262158+5663259\pi^2\right)\nu}{30240}\\
&+\frac{599347477\,\nu^2}{10080}-\frac{31569\,\nu^3}{2}\Bigg]+\\
&e_t^{10}\Bigg[\frac{185160951727}{22176000}-\frac{\left(796530586+773073\pi^2\right)\nu}{60480}\\
&+\frac{5381443\,\nu^2}{504}-\frac{59156\,\nu^3}{15}\Bigg]\\
&+e_t^{12}\left[\frac{81086491}{887040}-\frac{109847\,\nu}{864}+\frac{66209\,\nu^2}{630}-\frac{592\,\nu^3}{9}\right]\\
&+\sqrt{1-e_t^2}\Bigg[-\frac{14502034}{23625}-\frac{1}{105}\left(-163948+861\pi^2\right)\nu-240\,\nu^2\\
&+e_t^2\left[-\frac{491783909}{23625}-\frac{1}{280}\left(-8716320+32431\pi^2\right)\nu-\frac{28764\,\nu^2}{5}\right]\\
&+e_t^4\left[-\frac{268462759}{10500}+\frac{\left(95404256+28413\pi^2\right)\nu}{3360}-\frac{38949\,\nu^2}{5}\right]\\
&+e_t^6\left[\frac{5907794003}{189000}+\frac{\left(-432327416+1101219\pi^2\right)\nu}{10080}+\frac{44566\,\nu^2}{5}\right]\\
&+e_t^8\left[\frac{2894124703}{189000}+\frac{\left(-89427908+31857\pi^2\right)\nu}{5040}+\frac{23681\,\nu^2}{5}\right]\\
&+e_t^{10}\left[\frac{17933}{42}-\frac{47459\,\nu}{105}+\frac{666\,\nu^2}{5}\right]\\
&-\frac{107\left(891e_t^8+28016e_t^6+82736e_t^4+43520e_t^2+3072\right)\log(X)}{1575\left(1-e_t^2\right)^{13/2}}\Bigg\},\quad (6.68)
\end{aligned}
$$

where

$$X = \left[\frac{c^2 \left(\sqrt{1 - e_t^2} + 1 \right) r_0 \, \zeta^{2/3}}{2 \left(1 - e_t^2 \right) G \, m} \right] \tag{6.69}$$

The three expressions obtained here are the 3PN generalizations of the expressions given in Ref. [139] which are at the lowest quadrupolar order. They could be used to provide 3PN extensions of $n(e)$ and $a(e)$ relations of Ref. [139] in the future.

6.10 Conclusions and future directions

In this chapter we have computed the *instantaneous* contribution to the angular momentum flux from inspiralling compact binaries on eccentric orbits in the standard harmonic and ADM coordinates up to 3PN order. Using the 3PN quasi-Keplerian representation in ADM coordinates from [227], we have obtained the orbital average of the *instantaneous* part of the angular momentum flux. This together with the results of a similar calculation for the energy flux [190, 114] is used to compute the 3PN *instantaneous* terms in the evolution of the three main orbital elements e_t, n and a_r. It should be emphasized that results of this chapter are *partial* at present.

The above results have to be supplemented by the computation of *hereditary* terms at 2.5PN and 3PN for completion. These hereditary terms include the tails at 2.5PN and tail of tails and tail-square terms at 3PN. The 1.5PN tail terms in the angular momentum flux was computed in Ref. [110] by Rieth and Schäfer. The complication one encounters while generalizing it is the need to use a 1PN quasi-Keplerian parametrization. One has to worry about the complexities arising from the double periodicity, which first appear at 1PN order. For the energy flux, Refs. [190, 114] have provided a method to deal with this and computed numerically the 'enhancement factors' in the energy flux at different PN orders arising from different effects. This method could also be used for the computation of the hereditary terms in the angular momentum flux. Further, one would need to use the expressions for the hereditary parts of the energy and angular momentum flux and use them to obtain numerical expressions for the evolution of orbital elements completing the work described here. We plan to take this up in the near future.

Implications of these results should be examined along the lines pursued by Refs [112, 113] for the construction of templates for the eccentric binaries. Also, the orbital evolution of the other parameters such as 1PN precession parameter k under gravitational radiation reaction should be investigated in future.

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