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GENERALISED THEORY OF INTERFERENCE AND ITS APPLICATIONS

Part III. Interference Figures in Transparent Crystals

BY S. PANCHARATNAM

(*Memoir No.* 97 *from the Raman Research Institute, Bangalore*-6)

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§ 1. INTRODUCTION

AMONG the most beautiful and interesting phenomena in crystal optics are the interference figures displayed by crystalline plates in convergent light. Figures of the most diverse descriptions are met with even in the realm of transparent crystals, and this is not a matter for surprise: for, firstly the crystal may possess optical activity—in which case the two waves propagated along any direction in the medium are in general elliptically polarised; and secondly the state of polarisation of the incident light as well as that of the light for which the analysing arrangement is completely transparent may be adjusted to be linear, circular or elliptic in form. By the use of the Poincaré sphere, the varied phenomena which fall under this head may be brought under a general method of analysis, which—without complicating the explanation of the simpler phenomena—helps in the understanding of more complex phenomena.

As specific applications of the method we shall consider some of the interference figures displayed by the optically active crystal, quartz. Using a single basal section, an interference figure consisting of two spirals coiled around each other may be obtained with a circular polariser and a linear analyser. This case has been theoretically discussed in the treatises of Walker,[1] Pockels[2] and Szivessy[3] (though, due to certain minor errors, the final description of the spirals given in these treatises differ as regards the handedness and orientation of the spirals). When a usual polarising microscope is used, the quarter-wave plate has be to inserted just prior to the analysing nicol, and hence the spirals have to be observed under an arrangement slightly different from that considered in the above-mentioned references. The spirals thus observed with a linear polariser and a 'circular analyser' are discussed in the treatise of Mascart.[4] The sense of description of the spirals is reversed when the quarter-wave plate is transferred from a position behind the quartz plate to a position in front of it (though Mascart states that the same results are obtained under both arrangements).

We shall also discuss from a novel point of view, the *Airy's spirals* observed when two basal sections of quartz of equal thickness and of opposite sign are superposed, and the combination observed between crossed nicols in convergent light.

§ 2. Discussion of the General Case

As is well known, each point in a convergent light figure is the focal point of a particular bundle of *parallel* rays emerging from the plate, of which each pair of rays which have been derived by the splitting of a common incident ray, can (after passing through the analysing arrangement) interfere at the focal point. Hence our first step is to discuss the interference effects presented in *parallel light* at normal incidence by an arbitrarily cut plate of any transparent material when examined between an elliptic polariser C_1 and an elliptic analyser C_2.† The crystal being transparent, the two waves propagated through it must necessarily be in two oppositely polarised states A and A′ as indicated in Fig. 1 (since only then will the intensity of the emerging beam be independant of the phase retardation introduced by the plate [Part I,[5] §2]).

When the incident light (of unit intensity) in the state of polarisation C_1 is split into two beams of opposite polarisation A and A′, these component beams will have the intensities $\cos^2 \frac{1}{2}a_1'$ and $\sin^2 \frac{1}{2}a_1'$ respectively, where a_1' is the angular distance between the states C_1 and A on the Poincaré sphere (Part I, §2). The analyser C_2 transmits the fractions $\cos^2 \frac{1}{2}a_2'$ and $\sin^2 \frac{1}{2}a_2'$ of these beams, since it transmits only the corresponding resolved components[5,6] of these beams in the state of polarisation C_2; here a_2' is the angular separation between the states C_2 and A on the Poincaré sphere. Taking the state of relative phase of the two oppositely polarised beams at the point of entry in the medium as the zero or standard, their phase difference at the point of emergence from the crystal will be equal to δ (which is the path retardation suffered by the slower beam of polarisation A′ relative to the other). The phase difference between the corresponding resolved components transmitted by the analyser will however be smaller than this value by an angle $\hat{A}$; according to Part I, eq. 9, $\hat{A}$ denotes now half the area of the lune $AC_1A'C_2A$, or $\hat{A} = \angle C_2AC_1 = \angle C_1A'C_2$. (The angles are measured with the usual sign convention, being reckoned positive in a counter-clockwise sense [see Fig. 1]).

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† An elliptic analyser C_2 appears completely transparent for light of *elliptic* polarisation C_2; a quarter-wave plate followed by a linear analyser together form a simple elliptic analyser (Part I §8).

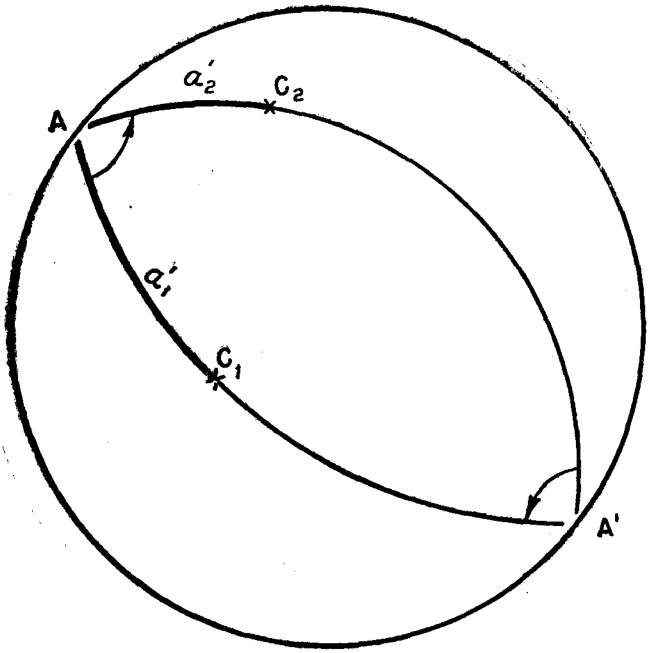

FIG. 1.

Thus the analyser C_2 transmits two beams having intensities I_1 and I_2 and phase difference $\triangle$ given by

$$\left.\begin{array}{l} I_1 = \cos^2 \tfrac{1}{2}a_1' \cos^2 \tfrac{1}{2}a_2'; \;\; I_2 = \sin^2 \tfrac{1}{2}a_1' \sin^2 \tfrac{1}{2}a_2' \\ \triangle = \delta - \hat{A} \end{array}\right\} \qquad (1)$$

These beams being in the same state of polarisation, the final intensity I transmitted by the analyser will be given by the usual interference formula

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \triangle \qquad (2)$$

The expression for the intensity I can also be written as the difference of two terms, the first of which is the intensity $\cos^2 \frac{1}{2} \widehat{C_1 C_2}$ which would be transmitted if the plate were absent ($\delta = 0$), the second term giving rise to subtraction colours in white light.

Equations (1) and (2) are the basic equations on which the further discussion of all interference phenomena shown by transparent crystals are based. Though they have been derived for the most general case they are practically identical with those customarily used[7] for the simplest case, *viz.*, when the plate possesses only linear birefringence, and the polarising and analysing elements are nicols—in which case $\hat{A}$ is equal to zero or π. In discussing any specific case, the values of the angles a_1', a_2' and A which obtain in that case have to be used; but their explicit values need be introduced not in the general expression (2), but only in certain subsidiary expressions determining those particular aspects of the phenomena which are to be investigated.

The variation of I with the direction of propagation may be observed in convergent light, using a plate cut normal to an optic axis—each *point* in the convergent light figure corresponding to a definite *direction* of propagation. For a plate of moderate thickness the retardation δ introduced by the plate increases rapidly as we proceed outwards along directions normal to the curves of constant retardation. The rate of variation of I as we proceed outwards from the optic axis is therefore taken to be predominantly due to the change in $\triangle$. We should therefore expect the appearance of interference rings, the curves of minimum intensity occurring along directions for which the pairs of pencils transmitted by the analyser ' destructively ' interfere, *i.e.*, along the curves where $\triangle = (2n+1)\pi$. These do not however coincide with the curves where the retardation of the plate is an odd multiple of π. The physical reason for this is that the final phase difference $\triangle$ between the interfering pencils transmitted by the analyser is not identical with the retardation introduced by the plate but exceeds it by an amount $(-\hat{A})$; this latter may be correctly described as the contribution to the phase difference arising from the processes of dissolution and analysation, and it depends on the mutual relation between the state of incident polarisation C_1, the states of polarisation A and A$'$ of the beams into which it is split, and the analysing state C_2. The curves of minimum intensity are thus given by

$$\delta = (2n+1)\pi + \hat{A} \tag{3}$$

and from what has been said above, *these curves of minimum intensity do not in general follow the curves of constant retardation*, $\delta =$ const., because the angle A is not constant but is itself a function of the direction of propagation (since the states A and A$'$ vary with the direction of propagation). A fairly accurate method of plotting the curves of minimum intensity (for a uniaxial crystal or a biaxial crystal of not too small an axial angle) is obviously the following; the circular curve $\delta = (2n+1)\pi$ is first drawn and the radii vectors of this curve are increased by amounts which vary with the azimuth and which correspond to an additional retardation of $\hat{A}$. In this manner it can be shown that the interference rings exhibited by a basal section of quartz between inclined nicols take the form of squares with rounded corners.

The intensity at any point on a curve of minimum intensity may be obtained by substituting $\triangle = \pi$ in (2) and is given by

$$I_{min} = \cos^2 \tfrac{1}{2} (a_1' + a_2') \tag{4}$$

This is not constant along a curve of minimum intensity, since the states A and A′ vary with direction, and hence the rings appear darkest along particular zones which have to be determined for each specific case.

§ 3. Spiral Figures Due to a Single Basal Section of Quartz

In discussing the convergent light figures of quartz we will have to keep in mind the following features regarding the propagation of light near the optic axial direction. Referring to Fig. 2 *a*, consider any point P in the convergent light figure, whose polar co-ordinates are *r*, θ (the origin O re-

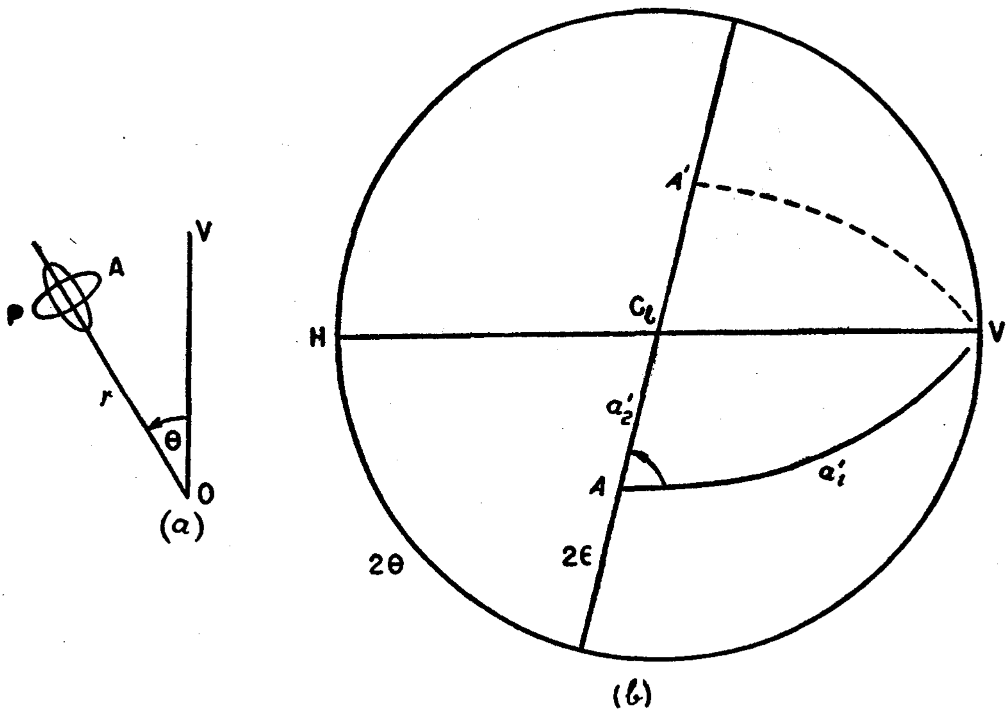

Fig. 2.

presenting the optic axial direction, and OV being a vertical line). Along the direction represented by P are propagated two crossed elliptic vibrations; and since quartz is a positive crystal, the major axis of the *slower* vibration coincides with the plane of the radius vector OP itself (so that θ is the azimuth of the major axis of the faster vibration A with respect to the horizontal). Hence if ϵ be the ellipticity of the faster wave, its state of polarisation will be represented on the Poincaré sphere (Fig. 2 *b*), by the point A of longitude 2θ and latitude 2ϵ, ϵ being positive or negative according as the crystal is optically left or right handed. The numerical value of the ellipticity decreases rapidly as the angular distance r from the optic axis increases, ($\tan 2\epsilon = 2\rho/\mathrm{A}r^2$), being a maximum along the optic axial direction where opposite circular vibrations are propagated. In what follows the main fact to remember is that for all points P on a circle of radius r, the ellipticity is constant—so that the latitude 2ϵ of the point A is constant, only the longitude 2θ being altered.

We shall first discuss the interference figure presented by a left-rotating basal section when a left-circular analyser (represented by the pole C_l in Fig. 2 *b*) is used, the vibration direction of the incident *plane* polarised light being vertical (represented by the point V). In this case we have $\hat{\mathrm{A}} = \angle C_l\mathrm{AV}$. It is seen immediately from Fig. 2 *b* that (keeping 2ϵ constant) as 2θ increases from $0 \rightarrow \frac{1}{2}\pi \rightarrow \pi \rightarrow 3\pi/2 \rightarrow 2\pi$, the $\angle C_l\mathrm{AV}$ also increases continuously and goes through the same range of values $0 \rightarrow \frac{1}{2}\pi \rightarrow \pi$; $-\pi \rightarrow -\frac{1}{2}\pi \rightarrow 0$. This means that along the circle $\delta = (2n+1)\pi$, the phase difference $\triangle$ between the interfering pencils transmitted by the analyser falls short of $(2n+1)\pi$ (condition for destructive interference) by an angle $C_l\mathrm{AV}$ which increases continuously with the azimuth θ — and which in fact becomes exactly equal to *twice* the azimuth at $\theta = m\pi/4$. It follows immediately (see Section 2) that the curves of minimum intensity consist of two mutually enwrapping *left-handed* spirals which are related to one another by a rotation of 180°. From the triangle $\mathrm{VA}C_l$ we have

$$\tan \angle C_l\mathrm{AV} = \tan 2\theta/\sin 2\epsilon \qquad (5)$$

In the usual treatment[4] equations (3) and (5) are derived by a more lengthy procedure, and these equations may be used to discuss the form of the spirals in more detail. Close to the optic axial direction we have $\angle C_l\mathrm{AV} \simeq 2\theta$, so that the spirals are given by:

$$\delta = \sqrt{(\mathrm{A}r^2 + 4\rho^2)} = (2n+1)\pi + 2\theta$$

If the two arms of the spiral be extrapolated to the origin (where they actually fade away) the common tangent at the origin will be at an azimuth $(\frac{1}{2}\pi + \rho)$ with respect to the vertical—where ρ is the total optical rotation along the optic axis. Towards the border the figure must approximate to the non-spiral form shown by inactive crystals. The transition occurs by way of the non-uniform rate of increase of the arm of the spiral—which manifests itself as kinks, which in turn assume the proportion of discontinuities towards the border of the figure (see Pockels, Plate VI, Figs. 3 and 4).

The zones along which the successive arcs of the spiral appear darkest may be determined with sufficient accuracy by studying the variation of (4) along a circle described about the optic axis ($a_2' =$ constant). Referring to Fig. 2 *b*, the arc a_1' acquires its maximum value of $(\pi - 2\epsilon)$ at $2\theta = 0$, and its minimum value of 2ϵ at $2\theta = \pi$. Along directions close to the optic axis (where $2\epsilon > \pi/4$) the sum $(a_1' + a_2')$ lies between the limits $\frac{1}{2}\pi$ to π. Hence the arcs of the spiral appear darkest along the vertical radius of the field of view. This result has also been derived in the usual presentation. We must emphasize however, that for a plate of moderate thickness (say 3 mm.) the above result holds only for points on the first convolution of the spiral, since at greater angular distances $2\epsilon < \pi/4$ (see Pockels, Plate VI, Figs. 3 and 4).

When the analysing state C_2 is changed to its opposite state C_2', the result (as is seen from Fig. 1) is to diminish the value of $\hat{A}$ by π and to change the length of the arc a_2' to its supplement. The same alteration in the expression (2) may be produced if the polarising state C_1 is changed to its orthogonal state. Hence, when a right-circular analyser is used, the entire spiral figure is rotated around by a right angle compared to the previous case (when a left-circular analyser was used).

We may obviously summarise the above results in a form applicable to both right and left-rotating basal sections. The handedness of the double spirals exhibited with a linear polariser and circular analyser is always the same as the handedness of the quartz. When the handedness of the circular analyser is opposite to that of the quartz, the tangent to the spiral at the origin coincides with the vibration direction of the light *emerging* along the optic axial direction; and, close to the optic axis, the spirals appear darkest along the diameter perpendicular to the plane of vibration of the incident light. A change in the handedness of the circular analyser causes a rotation of the entire figure, as such, through a right angle.

We see from Fig. 1 that when the state of polarisation of the incident light, and the state of polarisation C_2 of the light for which the analyser is transparent are interchanged, the sign of $\hat{A}$ is reversed, but the remaining factors in (1) remain unchanged. In Fig. 2 *b* the same result is obtained by change of the sign of 2θ. Hence *the entire spiral figure exhibited with a circular polariser and linear analyser is merely the reflection about the plane of vibration of the analyser, of the figure obtained when the polariser and* *analyser are interchanged.* This statement holds with regard to all the details of the figure, and hence we need not discuss this case further.

The results of this section were qualitatively verified using a plate of right-quartz. The general arguments may be applied *mutatis mutandis* for discussing the spiral figures in negative uniaxial crystals and in biaxial crystals. In the latter case only a *single* spiral is formed because, as the polar angle of the point P increases, the azimuth of the major axis of the faster elliptic vibration increases at only *half* the rate.

§ 4. Airy's Spirals Due to Two Superposed Basal Sections

Our general treatment can also be applied to the case of superposed transparent plates which often exhibit curious interference figures (see *e.g.*, Walker, *loc. cit.*, p. 293). For this purpose it is necessary to replace the effect of the passage through a succession of plates by a passage through a single 'equivalent plate'. This can be done by combining successive rotations of the Poincaré sphere.

As an example, let us suppose that a left-handed and a right-handed section of quartz (both of the same thickness and cut at the same angle to the optic axis) are superposed (in that order) such that the corresponding principal planes of the two plates are in coincidence, the combination being then viewed in parallel light at normal incidence.

Referring to Fig. 3, let the point A_1 of latitude 2ϵ represent the state of polarisation of the faster elliptic vibration (of ellipticity ϵ) propagated in the first plate; the state of the faster elliptic vibration of ellipticity $(-\epsilon)$ propagated in the second plate is then represented by a point A_2 of the same longitude as A_1, but of latitude (-2ϵ). Let us construct an isosceles triangle A_1XA_2 as indicated, such that the base angles A_2A_1X and XA_2A_1 are both equal to $\delta/2$ where δ is the retardation of each plate.

The action of the first plate is equivalent to an anti-clockwise rotation of the sphere about A_1 through twice the internal angle at A_1, while the action of the second plate is equivalent to an anti-clockwise rotation about A_2 through twice the internal angle at A_2 (see *e.g.*, Part I, § 6). By a well-known theorem for compounding rotations,[8] the two successive rotations are equivalent to a single rotation about the equatorial diameter through X, through twice the external angle at X. We have thus proved that the *combination is equivalent to a single optically inactive birefringent plate* of retardation $\triangle$, the faster vibration direction of which makes an angle $-\alpha$ with the common principal plane of the two quartz plates (which corres-

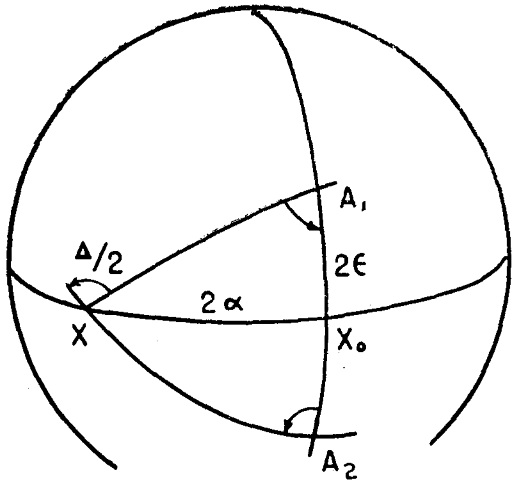

FIG. 3.

ponds to the major axis of the faster ellipse). Here α and $\triangle$ have the values indicated in Fig. 3. From the spherical triangle A_1XX_0 we have

$$\tan 2\alpha = \tan \tfrac{1}{2}\delta \sin 2\epsilon \tag{6}$$

Since the equivalent plate does not possess optical activity we should expect the appearance of 'isogyres' when two equal basal sections of left- and right-quartz are superposed, and the combination observed between crossed nicols. The 'isogyres' should occur along the directions where the 'equivalent' principal planes of the combination coincide with the vibration directions of the polariser and analyser. Since, as we have seen, the equivalent principal planes for any direction of propagation do not coincide with the principal planes of the individual plates the 'isogyres' will *not* take the form of a uniaxial cross. If θ be the azimuth of a point in the field of view with respect to the plane of the polariser, the dark 'isogyres' obviously occur where θ is equal to α or $(\frac{1}{2}\pi + \alpha)$, where α is given by (6). This equation has been derived by an entirely different method of analysis in the usual presentation (Walker,[1] p. 368, eq. 39). According to this equation it may be shown by following the usual treatment that the dark 'isogyres' take the form of four mutually enwrapping left-handed spirals known as Airy's spirals. Besides these spirals we will have dark curves, where the retardation $\triangle$ of the equivalent plate is a whole multiple of 2π, and from Fig. 3 these occur along the circles $\delta = 2n\pi$.

The sense of description of Airy's spirals is reversed when the right-handed plate is placed first because the sign of α is then changed.

5. Summary

The geometric method of specifying states of polarisation by points on the Poincaré sphere is used for giving a unified and physically intelligible approach to the interference phenomena displayed by crystalline plates in parallel or convergent light—under general conditions when the polarising and analysing states are linear, circular or elliptic in form. Examples discussed are the spiral figures exhibited in convergent light (*a*) by a basal section of quartz between a circular polariser and linear analyser (or *vice-versa*), and (*b*) by two superposed basal sections of left- and right-quartz between crossed nicols. The Airy's spirals observed in the latter case are interpreted as the 'isogyres' of the optically inactive plate to which the combination is equivalent.

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