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GENERALIZED THEORY OF INTERFERENCE AND ITS APPLICATIONS

Part II. Partially Coherent Pencils

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§ 1. INTRODUCTION

Two beams in different states of polarisation are said to be incoherent when they cannot be made to interfere even after being resolved into the same state of vibration by the use of an analyser. Thus, if unpolarised light be incident on a transparent wedge of quartz the fact that the two oppositely polarised beams into which it is split are incoherent is experimentally demonstrated by the complete absence of interference effects using an analyser alone. Similarly we can speak of two polarised beams as being completely coherent only when, by the use of a suitable analyser, interference effects of maximum clarity can be produced—the interference minimum having zero intensity.

As was remarked in Part I (Pancharatnam, 1956) if an extended source of light be viewed through a plate of an absorbing biaxial crystal cut normal to an optic axis, faint interference rings can be seen by the use of an analyser alone behind the crystal plate—even with the incident light unpolarised. It follows that when a pencil of unpolarised light falls on such a medium, the two non-orthogonally polarised pencils into which it is split must be regarded as partially coherent—since they satisfy neither the test for incoherence nor that for coherence, given in the previous paragraph. Hence, a discussion of the interference phenomena presented with an analyser alone involves really the analysis of the interference of two partially coherent beams which are resolved to the same state of elliptic vibration by the use of an analyser (§ 4). Even when both polariser and analyser are absent, rudimentary traces of an interference pattern may be seen. This requires the analysis of the following problems: the direct interference of two partially coherent beams in different states of polarisation (§ 6); the composition of two such partially coherent beams to form a partially polarised beam—at the second face of the crystal (§ 6); and the converse process—occurring at the first face of the plate—of decomposing unpolarised light (or, more generally, incompletely polarised light) into two completely polarised vibrations which, we shall find, are partially coherent (§ 7). In § 8 we shall consider the addition of n partially coherent beams which are all completely polarised.

We may mention that without resort to the use of absorbing biaxial crystals, it is easily possible to produce two partially coherent polarised beams. For example, we will have two such completely polarised and physically separate pencils of light emerging from a rhomb of calcite if we allow a single narrow pencil of *partially* polarised light to fall on the first face. The test of their partial coherence is as before the fact that even after being resolved into the same state of vibration by the use of an analyser, their interference is only partial—as may be seen in a conoscopic arrangement. [Our study of such partially coherent beams in opposite states of polarisation (§ 5) has an interesting theoretical consequence: it reveals in a direct manner the equivalence of the Poincaré and the Stokes representation of an arbitrarily polarised light beam.]

The mutual interference characteristics of two incompletely coherent polarised beams which have been derived by the splitting of an incompletely polarised beam can no doubt be described using only the extreme concepts of coherence and incoherence. For example, at the end of § 7 we shall show that when unpolarised light is split into two non-orthogonally polarised elliptic vibrations, the partially coherent components obtained will behave as if a certain independent fraction f of the intensity of one beam were completely coherent with the whole of the second beam—having a definite phase advance over it; this result is obtained by regarding the original unpolarised beam as the sum of two incoherent beams in *specific* orthogonally polarised states. Similarly, if we consider the example of partial coherence quoted in the previous paragraph, the partially polarised pencil (falling on the calcite rhomb) may be regarded as composed of a polarised and an unpolarised portion which are incoherent: the calcite splits the former into two *coherent* and the latter into two incoherent orthogonal vibrations. Thus, the two orthogonally polarised pencils emerging from the calcite will behave as if an independent fraction f_1 of one pencil were completely coherent with an independent fraction f_2 of the second (having a definite phase advance over it), the remaining fractions being incoherent with one another. An unsatisfactory feature of such methods of analysis is that the result depends essentially on regarding the given incompletely polarised beam as the sum of two other incoherent beams (whose decomposition characteristics are known in terms of the ideas of coherence and incoherence alone). For example, we would have arrived at a completely different picture of the state of affairs existing between the partially coherent pencils emerging from the calcite rhomb, if we had regarded the incident partially polarised beam as the sum of two incoherent orthogonally polarised pencils of different intensity. A second unsatisfactory feature is that though we may be sure that these different pictures of the same state of affairs must lead to identical results, the reason why in fact they do so—*i.e.*, the invariant character underlying these and other possible representations of two partially coherent beams —becomes veiled in obscurity.

In order to effect a deeper analysis of the problem it becomes necessary to have (*a*) a method of defining directly the state of partial polarisation of any beam—without having to regard it as the sum of two other beams; and (*b*) a method of defining directly the mutual parameters (the degree of coherence and the effective phase difference) which will determine the consequence of the superposition of any two polarised beams—without having to regard each beam as the sum of several independent fractions. These two problems will accordingly be discussed in the two subsequent articles.

§ 2. THE REPRESENTATION OF PARTIALLY POLARISED LIGHT

Since, as has already been remarked, the addition of two partially coherent beams in different states of elliptic polarisation results in a beam which is necessarily incompletely polarised, we shall in this section digress on a method of extending the Poincaré method so that it may also be used to represent the state of a partially polarised beam. This extension (and its relation to the representation introduced by Stokes) has already been indicated by Fano (1949) and discussed in more detail by Ramachandran (1952) —but we shall introduce it in a somewhat different fashion which is more suited to our present requirement. The present section also constitutes in itself a presentation of the subject of the Stokes parameters of partially polarised radiation by a new procedure—through the Poincaré representation itself. The conventional presentation of the subject of Stokes parameters may be found in Chandrasekhar (1949) and Rayleigh (1902).

Hitherto (see Fig. 1) we have represented the state of polarisation of an elliptic vibration by a corresponding point P on a Poincaré sphere of unit radius whose centre is O. (The longitude 2λ gives the azimuth λ of the major axis, and the latitude 2ω gives the ellipticity $\tan\omega$.) Instead, if we draw in the direction OP a vector $\mathbf{s}$ whose length s is made equal to the intensity i of the elliptic vibration, then this vector represents not only the state of polarisation but also the intensity of the elliptic vibration. We shall refer to **s** as the '*Stokes vector*' defining the *ideal* elliptic vibration (see next paragraph).

We shall now describe in an explicit manner the picture—assumed implicitly in the usual presentation of Stokes parameters—of the vibration in an *actual* beam of sensibly monochromatic radiation. It is known that the existence of incompletely polarised beams and of beams not coherent with one another can be reconciled with the wave theory of light when the extremely short period of light vibrations is taken into account. The phenomena depending on the interference of light merely show that for a duration very long compared with the period of the light wave, the vibration of the light cannot depart sensibly from an ideal periodic vibration described in two dimensions—*i.e.*, an elliptic vibration constant in form, intensity and absolute phase. During this interval the vibration can be characterised by a definite temporary 'Stokes vector' **s**. In a light beam of the most general type we can conceive, with constant macroscopic properties, the vector **s** which specifies the temporary intensity and polarisation may yet fluctuate millions of times a second. The optical characteristics of the beam observed in usual experiments depend only on certain average quantities. These, we find, are the intensity I (which is the average of the temporary intensity *i*) and a vector **S** which is the average of the temporary 'Stokes vector' **s**. Thus

$$\mathrm{I} = \langle i \rangle ; \ \mathbf{S} = \langle \mathbf{s} \rangle \tag{1}$$

where the bent brackets denote 'the average value of'. The vector **S** may be called the three-component part of the Stokes vector of the actual light beam, but we shall merely refer to it as the Stokes vector. The Stokes vector may be specified by its components with respect to any co-ordinate system with origin at the centre of the sphere. In the particular case when we choose a right-handed co-ordinate system $OX_0Y_0Z_0$ with the X_0Y_0 plane coinciding with the equatorial plane, the components of **S** will be denoted by Q, U, V. The four quantities I, Q, U, V will be called the Stokes parameters of the beam (with reference to co-ordinate axes on the wave-front of the beam given by X_0 and X_0'). In presenting the subject in this fashion we are anticipating the fact (to be proved in § 5) that the parameters defined in this geometrical manner in the Poincaré representation are identical with those introduced by Stokes analytically in an entirely different manner.

In the special case of a completely polarised beam there are no fluctuations in the temporary polarisation but only in the temporary intensity *i*;

hence the temporary Stokes vector **s** does not fluctuate in direction but only in its length i. In this case we have obviously I = S, or

$$I = Q^2 + U^2 + V^2 \tag{2 a}$$

When there are also fluctuations in the temporary polarisation (*i.e.*, in the direction of **s**) the beam is partially polarised (or, in special cases, unpolarised). In such cases the specification of the intensity I in addition to the Stokes vector **S** is no longer redundant. If f_j denotes the fraction of the time for which the vibration is in the state $\mathbf{s}_j$ then the Stokes vector **S** is by definition the sum of the vectors $f_j\mathbf{s}_j$. Now it is an obvious geometrical fact that the resultant of a number of vectors not all in the same line must have a length S which is less than the sum of the lengths $f_j i_j$ of the individual vectors. Hence for any beam not completely polarised, I > S or

$$I > Q^2 + U^2 + V^2 \tag{2 b}$$

One may compare the simplicity of the above proof with that used in the usual treatment of Stokes parameters (Chandrasekhar, Rayleigh, *loc. cit.*).

We shall now show that I and **S** together completely determine the appearance presented when the beam is passed through any transparent double refracting crystal followed by a linear analyser—*i.e.*, when the beam is passed through *any* elliptic analyser which transmits completely light of some particular polarisation C (see Part I, § 8). The state of polarisation transmitted by the analyser, instead of being specified by the point C on the Poincare sphere may equally well be specified by a unit vector **C** drawn from the centre of the sphere to the point C. The **C**-component of the beam will then have a temporary intensity i_C which, according to a fundamental property of the Poincaré sphere, is equal to $i \cos^2 \frac{1}{2}\theta$, where θ is the angle between **C** and **s** (see Part I, § 2). Since $i = s$, we have

$$i_C = \tfrac{1}{2}(i + \mathbf{C}\cdot\mathbf{s})$$

where **s** is the temporary Stokes vector of length equal to the temporary intensity i of the beam. The intensity transmitted by an analyser **C** is obtained by averaging as:

$$I_C = \tfrac{1}{2}(I + \mathbf{C}.\mathbf{S}) \tag{3}$$

and is hence determined by I and S. This expression will also be of use later.

For unpolarised light we should expect [from our definition of the Stokes vector in (1)] that the Stokes vector should become a zero vector coinciding with the centre of the sphere. From (3) we see that this is necessarily so since unpolarised light has the property that *any* elliptic analyser **C** transmits always half the intensity of the beam. (See also Hurwitz, 1945.)

One of the most important properties of the Stokes parameters is the following. *When a number of incoherent beams are mixed, the Stokes vector* **S** *of the resultant beam is the vectorial sum of the individual Stokes vectors* $\mathbf{S}_j$ *of the separate beams* (the intensity I of the resultant beam being naturally the sum of the individual intensities I_j). This readily follows from the fact that the total intensity I_C transmitted by any analyser should be the sum of the transmitted portions of the separate beams—this being the experimental test of their incoherence. Or,

$$I_C = \tfrac{1}{2}(\Sigma I_j + \mathbf{C}.\Sigma \mathbf{S}_j)$$

Since I_C is also given by (3) the required result is obtained. We may also express the result by the statement that when a number of incoherent beams are combined, each Stokes parameter of the resultant beam is the sum of the corresponding Stokes parameters of the individual beams.

A particular consequence of the above result is that any partially polarised beam (I, **S**) may be looked upon as an incoherent combination of a completely polarised beam (S, **S**) and an unpolarised beam (I–S, O). This gives a second physical interpretation of the Stokes vector **S** of a partially polarised beam: *it is a vector whose length is equal to the intensity of the polarised portion of the beam, and whose orientation* (*i.e.*, point of intersection with the Poincare sphere) *gives the state of polarisation of this polarised portion.*

We shall not require any further properties of the Stokes representation than have been derived above.

§ 3. The Degree of Coherence and the Effective Phase Difference Between Two Polarised Beams

We have already noted that in any beam which (for practical purposes) is completely polarised and monochromatic, the form of the elliptic vibration remains constant in time but the temporary intensity fluctuates. In order to explain the fact that it is possible for two polarised beams to be completely incoherent with one another, we must also add that the absolute phase of the elliptic vibration though remaining sensibly constant over successive durations very long compared with the period of light, also fluctuates very rapidly from a macroscopic standpoint (Stokes, 1852). Thus, any two beams of polarisation A and B travelling along the same direction may be characterised not only by temporary intensities i_1 and i_2, but also by the temporary phase advance δ_t which the vibration in one beam A has over that in the other. The phase difference between two ideal elliptic vibrations not in the same state of polarisation has already been defined in Part I, §§ 3 and 7. (The two vibrations are said to have zero phase difference, when the state of polarisation C obtained by their composition is represented by a point which lies on the great circular arc joining the points representing the states A and B on the Poincare sphere; for oppositely polarised vibrations a special arc AYB is chosen as the great circular arc of zero phase.)

We shall find that the observable characteristics (I, S) of the beam obtained on compounding the two beams depend on the following average quantities correlating the fluctuations in the two beams. These may be called the effective phase advance δ of one beam A over the other, and their mutual degree of coherence γ (defined to be a positive quantity). These are defined—in the same manner as is done in ordinary diffraction theory (Zernike, 1938), where a *scalar* wave theory of light is used—by the relation:

$$\left\langle \sqrt{i_1 i_2}\, e^{i\delta_t} \right\rangle = \sqrt{\mathrm{I}_1 \mathrm{I}_2}\, \gamma\, e^{i\delta} \tag{4}$$

where the sharp brackets are used to indicate the average value. The above relation (in which i represents $\sqrt{-1}$) is equivalent to the two relations

$$2\sqrt{\mathrm{I}_1\mathrm{I}_2}\,\gamma \cos\delta = 2\left\langle \sqrt{i_1 i_2}\cos\delta_t \right\rangle = \mathrm{U}', \text{ say,} \tag{5}$$

$$2\sqrt{\mathrm{I}_1\mathrm{I}_2}\,\gamma \sin\delta = 2\left\langle \sqrt{i_1 i_2}\sin\delta_t \right\rangle = \mathrm{V}', \text{ say,} \tag{6}$$

so that

$$\left.\begin{aligned} \gamma &= +\frac{1}{2\sqrt{\mathrm{I}_1\mathrm{I}_2}}\cdot\sqrt{\mathrm{U}'^2+\mathrm{V}'^2} \\ \tan\delta &= \frac{\mathrm{V}'}{\mathrm{U}'} \end{aligned}\right\} \tag{7}$$

From (4) we see that δ has the properties of a phase difference: if the instantaneous phase difference is altered by a constant amount, the effective phase difference alters by the same value; while, if the instantaneous intensities are respectively multiplied by constant factors, the effective phase difference is unaltered.

Regarding the degree of coherence γ (defined to be a positive quantity) it may be shown that it lies between the limits zero and unity. The proof is given in Linfoot (1955); it may also be obtained by representing the momentary 'mutual intensity' $\sqrt{i_1 i_2}\exp i\delta_t$ as a vector in an Argand diagram, applying the argument used in proving (2 *b*), and using the inequality $\sqrt{I_1 I_2} \geqslant \langle \sqrt{i_1 i_2} \rangle$. The beams will be said to be coherent when $\gamma = 1$, which occurs if there is complete correlation between the fluctuations in the two beams—the temporary phase difference as well as the ratio of the temporary intensities being absolutely constant in time. On the other hand when $\gamma = 0$ the beams will be said to be incoherent.

Instead of γ and δ, we shall sometimes find it more convenient to use the correlation parameters U′ and V′ defined in (5) and (6); or alternatively a single complex quantity I_{12} which has been termed the mutual intensity (Zernike, 1938).

$$I_{12} = \sqrt{I_1 I_2}\, \gamma\, e^{i\delta} = \tfrac{1}{2}(U' + iV')$$

§ 4. Interference of the Components of Two Partially Coherent Beams Transmitted by an Analyser

It is possible to experimentally determine the degree of coherence γ and the effective phase difference δ between two polarised beams of intensities I_1 and I_2, in states of polarisation A and B respectively. This can be done by observing the interference effects after resolving them to the same state of vibration by the use of an elliptic analyser which transmits light of polarisation C. Representing the states of polarisation A, B and C by corresponding points on the Poincaré sphere (see Part I, Fig. 1), the instantaneous intensity i_C transmitted by an analyser C can be expressed in terms of the sides *a*, *b*, *c* of the spherical triangle ABC, and its area E. According to the results of § 8, VII of Part I of this paper, the instantaneous intensities of the resolved components transmitted by the analyser will be $i_1 \cos^2 \frac{1}{2} b$ and $i_2 \cos^2 \frac{1}{2} a$ respectively, their instantaneous phase difference being $(\delta_t - \frac{1}{2} E)$.
Hence

$$i_C = i_1 \cos^2 \tfrac{1}{2} b + i_2 \cos^2 \tfrac{1}{2} a + 2\sqrt{i_1 i_2} \cos(\delta_t - \tfrac{1}{2} E) \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b$$

(It is to be remembered that this expression holds also in the limiting case when A and B represent orthogonal states of polarisation—see Part I, § 8.) The intensity I_C transmitted by the analyser C is obtained by taking the average of the above expression using (4).

$$I_C = I_1 \cos^2 \tfrac{1}{2} b + I_2 \cos^2 \tfrac{1}{2} a + 2\gamma\sqrt{I_1 I_2} \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos(\delta - \tfrac{1}{2} E) \quad (8)$$

This expression will be of much use later.

If we denote the intensities of the resolved components of the two beams by I_1' and I_2' the above result may be written:

$$I_C = I_1' + I_2' + 2\gamma\sqrt{I_1' I_2'} \cos(\delta - \tfrac{1}{2} E) \quad (8')$$

The interference effects will be most pronounced when the intensities of the resolved components (transmitted by the analyser) are equal in magnitude—this being secured by using any analyser C for which $I_1 \cos^2 \frac{1}{2} b = I_2 \cos^2 \frac{1}{2} a$. Under these conditions it can be easily shown that *the visibility of fringes* (as defined by Michelson) *is equal to the degree of coherence* γ. To experimentally determine δ we choose an analyser C for which the area E is zero, so that the point C lies on the great circular arc of zero phase joining A and B (see Part I, §§ 4 and 7). *The effective phase advance* δ *of one beam over the other can then be measured by the amount by which its path must be retarded in order that the intensity resulting from the interference of the resolved components becomes a maximum.*

From the expression (8) for the intensity transmitted by an analyser C when two partially coherent polarised beams are incident on it, we may deduce the following theorem.

Suppose a number of independent streams of intensities $a_1, a_2 \ldots . a_n$ all in the state of polarisation A are combined with a number of independent streams of intensities $b_1, b_2 \ldots . b_n$ all in the state of polarisation B. Let γ_j and δ_j denote the degree of coherence and the effective phase relation between the *corresponding* pairs of beams a_j and b_j. Then the degree of coherence and the effective phase advance of the resultant beam of polarisation A over that of polarisation B will be given by

$$\sqrt{I_1 I_2}\, \gamma\, e^{i\delta} = \Sigma \sqrt{a_j b_j}\, \gamma_j\, e^{i\delta_j} \tag{9}$$

The above result follows from the fact that the intensity I_C transmitted by any analyser C given by (8), will also be the sum of the intensities $(I_C)_j$, where $(I_C)_j$ denotes the intensity transmitted due to the pair of beams a_j and b_j—this result being true for any value of E. The result may also be expressed by the statement that *the mutual intensity between the resultant beams is the sum of the mutual intensities of the individual pairs.*

As a particular case of the above theorem we note that if an independent fraction f_1 of the intensity of one beam is completely coherent with an independent fraction f_2 of the intensity of the second, having a phase advance δ over it, the remaining portions of the two beams being incoherent, then δ is also the effective phase advance of the first beam over the second, while $\sqrt{f_1 f_2}$ is their mutual degree of coherence. From the above result we obtain a still simpler method of regarding any two partially coherent beams; *an independent fraction* γ^2 *of the intensity of one beam may be regarded as coherent with the whole of the second beam, having a phase advance* δ *over it,*

§ 5. Addition of Two Partially Coherent Beams of Opposite Polarisation

Let i_1 and i_2 be the temporary intensities of two vibrations in the opposite states of polarisation X and X′, and let δ_t be the temporary phase advance of the first vibration over the second (Part I, § 7). Let **s** be the immediate value of the Stokes vector of the resultant vibration obtained, whose state of polarisation is represented by the point P (see Fig. 1).

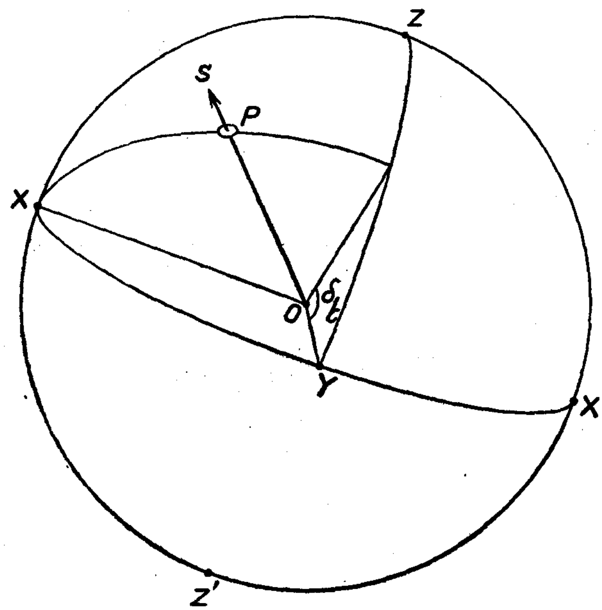

Fig. 1.

P– Momentary state of polarisation of partially polarised beam.

s– Momentary Stokes vector of length equal to the temporary intensity.

δ_t–Instantaneous phase difference between the X and X′ components.

We shall refer the temporary Stokes vector **s** to a right-handed system of co-ordinate axes OX, OY, OZ chosen such that the point Y represents the vibration whose X and X′ components are defined to be in the same phase (Part I, § 6). A fundamental property of the Poincare sphere (Part I, § 2, I) is that when a vibration of intensity s in the state of polarisation P is resolved into two oppositely polarised vibrations in the states of polarisation X and X′, the intensities of these components (which are equal to i_1 and i_2) are given by:

$$i_1 = s \cos^2 \tfrac{1}{2} \widehat{PX}; \quad i_2 = s \sin^2 \tfrac{1}{2} \widehat{PX}$$

or

$$i_1 - i_2 = s \cos \hat{\mathrm{PX}}; \quad 2\sqrt{i_1 i_2} = s \sin \hat{\mathrm{PX}}.$$

But $s \cos \hat{\mathrm{PX}}$ is the projection of the temporary Stokes vector in the OX direction; and $s \sin \hat{\mathrm{PX}}$ is the projection of the temporary Stokes vector on the YZ plane. Now according to Part I, § 6, VI, we have $\mathrm{P}\hat{\mathrm{X}}\mathrm{Y} = \delta_t$. Hence the temporary intensity i and the temporary Stokes vector of the resultant vibration are given by:

$$\left.\begin{aligned} i &= i_1 + i_2 \quad ; \quad s_x = (i_1 - i_2) \\ s_y &= 2\sqrt{i_1 i_2} \cos \delta_t; \quad s_z = 2\sqrt{i_1 i_2} \sin \delta_t \end{aligned}\right\} \tag{10}$$

We now take the average of the above expressions using (1), (5) and (6). We thus see that *on compounding two oppositely polarised beams of intensities I_1 and I_2 (for which the mutual degree of coherence and the effective phase advance of the first beam over the second are γ and δ respectively), the resultant partially polarised beam (I, S) is given by*

$$\left.\begin{aligned} \mathrm{I} &= \mathrm{I}_1 + \mathrm{I}_2 \quad ; \quad \mathrm{S}_x = \mathrm{I}_1 - \mathrm{I}_2 \\ \mathrm{S}_y &= 2\gamma\sqrt{\mathrm{I}_1\mathrm{I}_2} \cos \delta; \quad \mathrm{S}_z = 2\gamma\sqrt{\mathrm{I}_1\mathrm{I}_2} \sin \delta \end{aligned}\right\} \tag{11}$$

It may be noted that S_y and S_z are equal to the correlation parameters U′ and V′ respectively.

We shall now consider the converse problem, *viz.*, of resolving a given beam (I, **S**) into two oppositely polarised beams X and X′—a decomposition which occurs when the beam falls on any transparent crystalline plate. Obviously when the instantaneous vibration **s** of this beam is resolved into vibrations in the states of polarisation X and X′, these component vibrations will have intensities i_1 and i_2, and a phase relation δ_t which satisfy (10). Since the momentary vibration **s** fluctuates for an incompletely polarised beam, it is clear that i_1, i_2 and δ_t will also fluctuate—so that the component beams will, in general, be partially coherent. The intensities I_1 and I_2 of the component beams, their degree of coherence γ and the effective phase advance δ of the first beam over the second can be obtained from (11).

$$\left.\begin{aligned} \mathrm{I}_1 &= \tfrac{1}{2}(\mathrm{I} + \mathrm{S}_x) \quad ; \quad \mathrm{I}_2 = \tfrac{1}{2}(\mathrm{I} - \mathrm{S}_x) \\ \gamma &= +\frac{1}{2\sqrt{\mathrm{I}_1\mathrm{I}_2}} \cdot \sqrt{\mathrm{S}_y^2 + \mathrm{S}_z^2}\,; \quad \tan \delta = \frac{\mathrm{S}_z}{\mathrm{S}_y} \end{aligned}\right\} \tag{12}$$

It may be noted that the expression for the effective phase difference δ between the component beams does not involve the degree of polarisation of the given beam.

The first Stokes parameter of a beam is its intensity. We may now easily see that our method of defining the remaining Stokes parameters (as components of the average Stokes vector **S** with respect to a special co-ordinate system) is entirely equivalent to the usual method. To show this we consider the resolution of an arbitrarily polarised beam into two orthogonal *linearly* polarised beams. In this case the states X and X′ in Fig. 1 lie on the equator. The four Stokes parameters of the beam (with respect to axes on the wave-front given by X and X′) are then customarily *defined* as the average values of the expressions on the *right*-hand side of each of the four relations in (10). In our presentation (see § 2), the average values assumed by the quantities i, s_x, s_y, s_z, in the *particular* case when the XY plane of the co-ordinate system lies on the equatorial plane, are the Stokes parameters of the beam (with reference to axes on the wavefront given by X and X′). The relations (10) show that both methods of definition are equivalent. Our method of introducing the Stokes parameters (given in § 2) is more general, in that it does not at all involve the representation of an arbitrarily polarised beam as the sum of two other (partially coherent) linearly polarised beams. In fact such a decomposition is clearly seen to be merely a particular case of the problem which we shall discuss in § 7, *viz*., the decomposition of any partially polarised beam into two beams in non-orthogonal states of polarisation.

§ 6. Addition of Two Partially Coherent Beams in Non-Orthogonal States of Polarisation

When *any* two beams travelling along the same direction are combined, the instantaneous intensity of the resultant beam is obtained from Part I, § 3, III as

$$i = i_1 + i_2 + 2\sqrt{i_1 i_2}\cos\tfrac{1}{2}c\cos\delta_t$$

where the similarity factor $\cos^2\frac{1}{2}c$ will be absolutely constant in time if the two beams are completely polarised. Averaging the above equation using (5) we obtain the *generalized formula for the interference of two polarised beams of intensities* I_1 *and* I_2, *degree of coherence* γ *and effective phase difference* δ:

$$I = I_1 + I_2 + 2\gamma\sqrt{I_1 I_2}\cos\tfrac{1}{2}c\cos\delta \tag{13}$$

Thus γ and δ may be determined by *direct* interference experiments, though the method of using an analyser given in § 4, is to be preferred—to increase the visibility of interference effects. In relation (13), c is the angular separation between the states of polarisation on the Poincaré sphere. The above relation can also be easily obtained by regarding a fraction γ^2 of the first beam as coherent with the second beam, having a phase advance δ over it.

It remains to find the Stokes vector of the resultant beam. Let the states of polarisation of the two beams be given by the points A and B on the Poincaré sphere, or alternatively by unit vectors **A** and **B** joining the centre of the sphere to the points in question. The Stokes vectors of the two beams are then $\mathbf{S}_1 = I_1\mathbf{A}$ and $\mathbf{S}_2 = I_2\mathbf{B}$. When the beams are not completely coherent it is clear that there will be fluctuations in the temporary state of polarisation and intensity of the resultant vibration—so that the resultant beam will be incompletely polarised. We shall in this section prove that the Stokes vector **S** of the resultant beam may be obtained by the following procedure (see Fig. 2). It is obtained by adding to the sum of the given Stokes vector $\mathbf{S}_1$ and $\mathbf{S}_2$ (directed towards points A and B), a third vector $\mathbf{S}_{12}$ (directed towards a point C″). This last vector which arises because of the interference of the beams, may be specified in terms of the angles of the triangle ABC″ which is isosceles: *the base angles A and B are both equal to the effective phase difference* δ *between the beams, and the length of vector* $\mathbf{S}_{12}$ *is* $2\gamma\sqrt{I_1 I_2}\sin\frac{1}{2}\hat{C}''$.

The components of the Stokes vector of the resultant beam may be found by using the following proposition which follows from (3): *the component of the Stokes vector along any direction* **C** *is equal to the intensity transmitted by an analyser* **C**, minus *the intensity transmitted by the orthogonal analyser* (— **C**).

i.e.,

$$I_C - I_{-C} = \mathbf{S.C} \tag{14}$$

Since we have already (in eq. 8) derived the intensity transmitted by any analyser when it is introduced in the path of two partially coherent polarised beams, we may find the Stokes vector **S** of the resultant beam. (It may be remembered that in equation 8, the quantities *a*, *b*, *c* are the sides of the spherical triangle ABC, while E is the area of the triangle—measured with the usual sign convention.) The intensity I_{-C} transmitted by the orthogonal analyser (—**C**) may be obtained from (8) by changing *a* and *b* to their supplements, and E to E′, where E′ is the area of the triangle ABC′ columnar to ABC. We then have

$$\begin{aligned} I_C - I_{-C} &= I_1 \cos b + I_2 \cos a + 2\gamma\sqrt{I_1 I_2} \\ &\quad \times \{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos(\delta - \tfrac{1}{2}E) - \sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos(\delta - \tfrac{1}{2}E')\} \end{aligned}$$

Since the two beams which are being combined are completely polarised, we have according to (2 *a*), $I_1 = S_1$ and $I_2 = S_2$, where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the Stokes vectors of these beams. The first two terms of the above expression are accordingly equal to $\mathbf{S}_1.\mathbf{C}$ and $\mathbf{S}_2.\mathbf{C}$ respectively. Comparing with (14) we see that the Stokes vector $\mathbf{S}$ of the resultant beam may be written as

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_{12} \tag{15}$$

where the term $\mathbf{S}_{12}$ which may be considered as arising from the mutual interference of the beams, may be determined from the relation

$$\mathbf{S}_{12}.\mathbf{C} = 2\gamma\sqrt{\mathrm{I}_1\mathrm{I}_2}\,\{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos(\delta - \tfrac{1}{2}\mathrm{E}) - \sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos(\delta - \tfrac{1}{2}\mathrm{E}')\} \tag{16}$$

It may be noted that $\mathbf{S}_{12}$ is γ times the value which it would have if the beams were completely coherent.

Since the last relation gives the component of $\mathbf{S}_{12}$ along *any* direction C, we may determine the vector $\mathbf{S}_{12}$ by finding its components with respect to the special co-ordinate system OX, OY, OZ given in Fig. 2. The positive x-axis is taken along the direction of the vector $(\mathbf{A} - \mathbf{B})$; the positive y-axis along the direction of the vector $(\mathbf{A} + \mathbf{B})$; and the positive z-axis along the direction of the vector $(\mathbf{A} \times \mathbf{B})$. (The definitions of the y and z directions would have to be slightly modified if we wish also to cover the limiting case, discussed in the previous section, when A and B are oppositely polarised.)

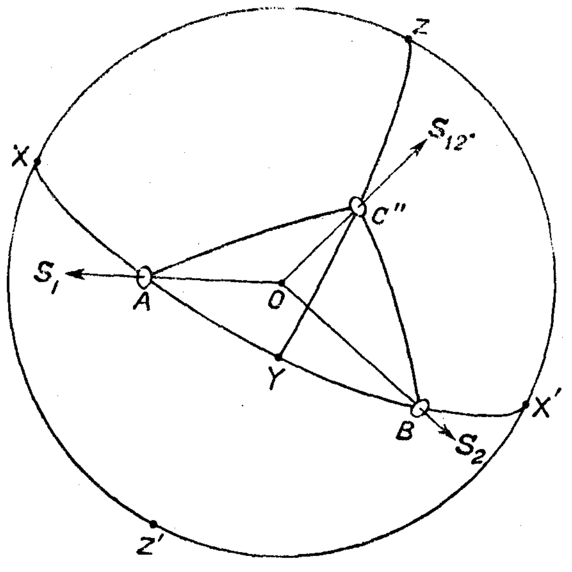

FIG. 2

Composition of non-orthogonally polarised beams S_1 and S_2. The vector for the resultant partially polarised beam is the sum of the three vectors drawn in the Figure; S_{12} has an orientation determined by $\mathrm{A} = \hat{\mathrm{B}} = \delta$, and a length equal to $2\gamma\sqrt{\mathrm{I}_1\mathrm{I}_2}\sin\frac{1}{2}\,\hat{\mathrm{C}}''$.

When the unit vector **C** is taken along the x direction (see Fig. 2) we will have in (16), $\cos \frac{1}{2} a \cos \frac{1}{2} b = \sin \frac{1}{2} a \sin \frac{1}{2} b$ and $E = E' = 0$. When the unit vector **C** coincides with the y-axis we have $E = 0$, $E' = \pi$ and $a = b = \frac{1}{2} c$. Lastly when the direction of **C** coincides with OZ we will have $a = b = \pi/2$ and $E = -E' = c$. Making these substitutions in (16) we obtain the components of $\mathbf{S}_{12}$ as

$$\left.\begin{array}{l}(S_{12})_x = 0 \\ (S_{12})_y = 2\gamma\sqrt{I_1 I_2}\cos\delta;\ (S_{12})_z = 2\gamma\sqrt{I_1 I_2}\sin\delta\sin\frac{1}{2}c\end{array}\right\} \quad (17)$$

We see that the vector $\mathbf{S}_{12}$ lies in the yz plane, and hence that the triangle ABC″ is isoscles. Also the inclination of this vector to the y-axis is equal to the arc YC″ and is given by

$$\tan \widehat{YC''} = S_z/S_y = \tan\delta \sin\tfrac{1}{2} c$$

Since the spherical triangle AYC″ is right angled at Y, it is seen from spherical trigonometry that the above relation implies that the angle at A is equal to δ. This locates the position of C or the orientation of the vector $\mathbf{S}_{12}$. Its length is given by

$$\begin{aligned} S_{12} &= \sqrt{S_y^2 + S_z^2} = 2\gamma\sqrt{I_1 I_2} \cdot \sqrt{(1 - \sin^2\delta\cos^2\tfrac{1}{2}c)} \\ &= 2\gamma\sqrt{I_1 I_2}\sin\tfrac{1}{2}C'' \end{aligned}$$

These are the relations for determining the vector $\mathbf{S}_{12}$ which were stated at the beginning of this section, and which we wished to prove. The superposition of oppositely polarised beams, discussed in the previous section, may be considered as a limiting case of the present discussion.

The result may also be expressed concisely in vector notation, the result being then independent of any co-ordinate system. Using the correlation parameters U′ and V′ introduced in (5) and (6) instead of γ and δ, we can then write (17) as

$$\mathbf{S}_{12} = \tfrac{1}{2}\sec\tfrac{1}{2}c\,\{U'(\mathbf{A}+\mathbf{B}) + V'(\mathbf{A}\times\mathbf{B})\} \quad (18$$

or

$$\mathbf{S}_{12} = \text{Real part of } I_{12}\,\mathbf{S}_{12}'\sec\tfrac{1}{2}c \quad (19)$$

where

$$\mathbf{S}_{12}' = (\mathbf{A}+\mathbf{B}) - i(\mathbf{A}\times\mathbf{B})$$

Hence when two non-orthogonally polarised beams with Stokes vectors $\mathbf{S}_1$ and $\mathbf{S}_2$ are combined, the Stokes vectors **S** of the resultant beam is given by (15) where $\mathbf{S}_{12}$ is given by (18), or (19). The intensity I of the resultant beam is given by (13).

The components of the Stokes vector of the resultant beam (with respect o the special co-ordinate axes chosen) may now be written down from (15) nd (17) by noting that the x-components of $\mathbf{S}_1$ and $\mathbf{S}_2$ will be $I_1 \sin \frac{1}{2} c$ and $-I_2 \sin \frac{1}{2} c$ respectively, while the y-components will be $I_1 \cos \frac{1}{2} c$ and $_2 \cos \frac{1}{2} c$ respectively. Hence the Stokes vector of the resultant beam is ;iven by:

$$\left.\begin{aligned} S_x &= (I_1 - I_2) \sin \tfrac{1}{2} c \\ S_y &= (I_1 + I_2) \cos \tfrac{1}{2} c + 2\gamma \sqrt{I_1 I_2} \cos \delta \\ S_z &= 2\gamma \sqrt{I_1 I_2} \sin \delta \sin \tfrac{1}{2} c \end{aligned}\right\} \tag{20}$$

§ 7. Decomposition of any Beam into Two Non-Orthogonally Polarised Pencils

We now consider a problem which is the converse of that treated in the previous section, *viz.*, the resolution of a given arbitrarily polarised beam $(\mathrm{I}, \mathbf{S})$ into two non-orthogonally polarised beams in given states of polarisation $\mathbf{A}$ and $\mathbf{B}$. Such a process occurs for example when the beam falls on a plate of an absorbing biaxial crystal.

If $\mathbf{S}$ denotes the instantaneous Stokes vector of the given beam, then according to the results proved in Part I, § 4, IV, the temporary intensities i_1 and i_2 of the component beams will be given by

$$i_1 = i \frac{\sin^2 \frac{1}{2} \theta_2}{\sin^2 \frac{1}{2} c}; \quad i_2 = i \frac{\sin^2 \frac{1}{2} \theta_1}{\sin^2 \frac{1}{2} c}$$

where θ_1 and θ_2 are the the (momentary) angles that the vector $\mathbf{s}$ makes with the vectors $\mathbf{A}$ and $\mathbf{B}$ respectively.

Now, since $i = s$, this may be re-written thus:

$$i_1 = \tfrac{1}{2} (i - \mathbf{s} \cdot \mathbf{B}) \operatorname{cosec}^2 \tfrac{1}{2} c; \quad i_2 = \tfrac{1}{2} (i - \mathbf{s} \cdot \mathbf{A}) \operatorname{cosec}^2 \tfrac{1}{2} c$$

The average intensities of the non-orthogonally polarised component beams will therefore be

$$I_1 = \tfrac{1}{2} (I - \mathbf{S} \cdot \mathbf{B}) \operatorname{cosec}^2 \tfrac{1}{2} c; \quad I_2 = \tfrac{1}{2} (I - \mathbf{S} \cdot \mathbf{A}) \operatorname{cosec}^2 \tfrac{1}{2} c \tag{21}$$

It remains to determine the degree of coherence and the effective phase difference between the component beams, or alternatively the correlation parameters U' and V' defined in (5) and (6). The first parameter is obtained by eliminating $(I_1 + I_2)$ from the expression for S_y in (20) by using (13). We thus obtain

$$U' = (S_y - I\cos\tfrac{1}{2}c)\operatorname{cosec}^2\tfrac{1}{2}c \tag{22-a}$$

$$= \tfrac{1}{2}\{\mathbf{S}\cdot(\mathbf{A}+\mathbf{B}) - 2I\cos^2\tfrac{1}{2}c\}\operatorname{cosec}^2\tfrac{1}{2}c\sec\tfrac{1}{2}c \tag{22-b}$$

The parameter V′ is given by the last relation in (17)

$$V' = S_z\operatorname{cosec}\tfrac{1}{2}c \tag{23-a}$$

$$= \tfrac{1}{2}\mathbf{S}\cdot(\mathbf{A}\times\mathbf{B})\operatorname{cosec}^2\tfrac{1}{2}c\sec\tfrac{1}{2}c \tag{23-b}$$

or

$$2I_{12} = \tfrac{1}{2}(\mathbf{S}\cdot\mathbf{S}_{21}' - 2I\cos^2\tfrac{1}{2}c\}\operatorname{cosec}^2\tfrac{1}{2}c\sec\tfrac{1}{2}c$$

As a particular example of much interest we may consider the decomposition of *unpolarised* light into two non-orthogonally polarised beams. Since for unpolarised light $\mathbf{S} = 0$, the intensities I_1 and I_2 of the component beams are, according to (21), both equal to $\tfrac{1}{2}I\operatorname{cosec}^2\tfrac{1}{2}c$. The effective phase difference and the degree of coherence could be determined by finding the parameters U′ and V′ from (22) and (23). But it is more instructive to go back to relation (15) from which it may be noted that for $\mathbf{S}$ to be a zero vector, $\mathbf{S}_{12}$ must be coplanar with $\mathbf{S}_1$ and $\mathbf{S}_2$. Since $\mathbf{S}_{12}$ must also be in the *yz* plane, it is clear that the point C″ towards which it points must be the mid-point of the greater segment of the great circular arc through A and B. The length of the vector $\mathbf{S}_{12}$ being equal to that of $(\mathbf{S}_1 + \mathbf{S}_2)$ will be given by $(I_1 + I_2)\cos\tfrac{1}{2}c$. The effective phase difference between the component beams is π (being equal to the angle A of the isosceles triangle ABC″). The degree of coherence between the two beams is $\cos\tfrac{1}{2}c$ (since the length of the vector $\mathbf{S}_{12}$ is also given by $2\gamma\sqrt{I_1I_2}\sin\pi/2$).

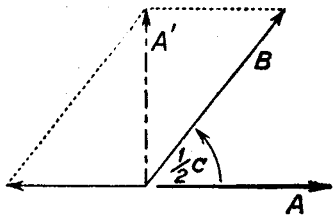

FIG. 3.

Decomposition of unpolarised light into two non-orthogonally polarised beams with vibration-directions along A and B. The resultant beams have a degree of coherence $\cos\tfrac{1}{2}c$ and an effective phase difference of π.

The decomposition of unpolarised light into two non-orthogonal vibrations in the states of polarisation A and B (separated by an angle c on the Poincaré sphere) may be more simply analysed by replacing the unpolarised light by two *incoherent* beams each of intensity $\tfrac{1}{2}I$ in the orthogonally polarised states A and A′. (In order that the general method may be perfectly clear, the particular case when A and B are linear vibrations inclined at the angle $\frac{1}{2}c$ is drawn in Fig. 3 and may be followed simultaneously.) The beam of polarisation A′ may now be replaced by two coherent beams in the non-orthogonal states of polarisation B and A. These latter two vibrations will have a phase difference of π, since A′ lies on the greater segment of the great circle through A and B (Part I, § 4, V). Their intensities will be respectively $\frac{1}{2}\mathrm{I}\operatorname{cosec}^2\frac{1}{2}c$ and $\frac{1}{2}\mathrm{I}\cot^2\frac{1}{2}c$ (as is given by the parallelogram law in the case of Fig. 3, and in the general case by substituting $b=\pi$ and $a=\pi-c$ in the results of Part I, § 4, IV). Thus in the state of polarisation B we have a beam of intensity $\frac{1}{2}\mathrm{I}\operatorname{cosec}^2\frac{1}{2}c$; while in the state of polarisation A we have two incoherent vibrations which add to give a beam of the same intensity $\frac{1}{2}\mathrm{I}\operatorname{cosec}^2\frac{1}{2}c$. Of the latter beam, however, an independent fraction comprising an intensity $\frac{1}{2}\mathrm{I}\cot^2\frac{1}{2}c$ is completely coherent with the other beam and is opposed in phase to it. In other words, the degree of coherence between the beams is $\cos\frac{1}{2}c$ and the effective phase difference is π (according to the result proved at the end of § 4).

§ 8. The Addition of n Partially Coherent Beams

We shall now consider the addition of n polarised beams whose states of polarisation are represented by the points $P_1, P_2, \ldots P_n$ on the Poincaré sphere, or alternatively by the unit vectors $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \ldots \mathbf{P}_n$ joining the centre of the sphere to these points. The instantaneous intensity i of the resultant beam will be given by equation (14) of Part I, § 9

$$i=\Sigma i_j+\sum_{j\neq k} i_{jk}\cos\tfrac{1}{2}c_{jk}$$

where i_j denotes the temporary intensity of the jth beam, i_{jk} the temporary mutual intensity of the jth beam with respect to the kth and c_{jk} is the angle between the vectors $\mathbf{P}_j$ and $\mathbf{P}_k$. Averaging the above equation we obtain the following expression for the intensity I of the resultant beam:

$$\mathrm{I}=\Sigma \mathrm{I}_j+\sum_{j\neq k}\mathrm{I}_{jk}\cos\tfrac{1}{2}c_{jk} \tag{24}$$

where I_{jk} is the mutual intensity of the jth beam with respect to the kth. The second term in (24) arises from the mutual interference of the different pairs of beams.

The form of (24) suggests that the Stokes vector **S** of the resultant beam may be obtained by a similar generalisation of relation (15) obtained for two beams:

$$\mathbf{S}=\Sigma\mathbf{S}_j+\tfrac{1}{2}\sum_{j\neq k}\mathbf{S}_{jk} \tag{25}$$

or alternatively

$$\left.\begin{aligned} \mathbf{S} &= \Sigma\, \mathbf{S}_j + \tfrac{1}{2}\Sigma\, \mathrm{I}_{jk}\cdot \mathbf{S}_{jk}' \sec \tfrac{1}{2}\, c_{jk} \\ \mathbf{S}_{jk}' &= (\mathbf{P}_j + \mathbf{P}_k) - i\,(\mathbf{P}_j \times \mathbf{P}_k) \end{aligned}\right\} \qquad (26)$$

where

That the relations (25 and 26) do indeed give the Stokes vector of the resultant beam may be verified by taking recourse to the intensity transmitted by any analyser **C** when the n beams are incident on it. This intensity $\mathrm{I_C}$ will be the average of the instantaneous transmitted intensity [which may be obtained by using equation (13) of Part I, § 9]:

$$\mathrm{I_C} = \Sigma\, \mathrm{I}_j \cos^2 \theta_j + \Sigma\, \gamma_{jk} \sqrt{\mathrm{I}_j \mathrm{I}_k} \cos \theta_j \cos \theta_k \cos (\delta_{jk} - \tfrac{1}{2}\, \mathrm{E}_{jk})$$

where $2\theta_j$ denotes the angle between $\mathbf{P}_j$ and **C**, and E_{jk} the area of the spherical triangle $\mathrm{CP}_j\mathrm{P}_k$. The component of the Stokes vector of the resultant beam along any direction C is obtained by writing the value of $(\mathrm{I_C} - \mathrm{I_{-C}})$. Since the $\mathbf{S}_{jk}$ satisfy relations of the type of (16) it may be easily shown that (26) gives the Stokes vector of the resultant beam.

It is a pleasure to acknowledge the encouragement given by Prof. Sir C. V. Raman, F.R.S., N.L., and the keen interest he took in this investigation.

§ 9. Summary

The superposition of two partially coherent but completely polarised beams is discussed. The formula for the intensity of the resultant beam is obtained from the interference formula for coherent beams by multiplying the third interference term by the degree of coherence γ (defined statistically). The states of the two given polarised beams and that of the resultant incompletely polarised beam may be characterised by respective vectors drawn from the centre of the Poincaré sphere: the length of each vector and its orientation (*i.e.*, point of intersection with the sphere) may be regarded as giving respectively the intensity and state of polarisation, of the polarised fraction of the corresponding beam. The vector for the resultant beam is obtained by adding to the sum of the two given vectors (which are directed towards points A and B), a third vector directed towards a point C″ on the Poincaré sphere. This last vector which arises because of the interference of the beams, is specified in terms of the angles of the triangle ABC″, which is isoscles: the base angles A and B are both equal to the effective phase difference δ and the length of the vector is equal to $2\gamma\sqrt{\mathrm{I_1 I_2}} \sin \frac{1}{2}\, \hat{\mathrm{C}}''$.

The converse problem is discussed and also the addition of n partially coherent polarised beams. The paper also presents the subject of the Stokes

parameters of partially polarised radiation through an extension of the Poincaré representation.

§ 10. References

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