Inspiralling compact binaries in quasi-elliptical orbits: The complete third post-Newtonian energy flux

K. G. Arun,[1,2,3,*] Luc Blanchet,[2,†] Bala R. Iyer,[1,‡] and Moh'd S. S. Qusailah[1,4,§]

[1]*Raman Research Institute, Bangalore 560 080, India*
[2]*GReCO, Institut d'Astrophysique de Paris—C.N.R.S., 98 bis Boulevard Arago, 75014 Paris, France*
[3]*LAL, Université Paris-Sud, IN2P3/CNRS, Orsay, France*
[4]*University of Sana, Yemen*

(Received 2 November 2007; published 31 March 2008)

The instantaneous contributions to the third post-Newtonian (3PN) gravitational wave luminosity from the inspiral phase of a binary system of compact objects moving in a quasi-elliptical orbit is computed using the multipolar post-Minkowskian wave generation formalism. The necessary inputs for this calculation include the 3PN accurate mass quadrupole moment for general orbits and the mass octupole and current quadrupole moments at 2PN. Using the recently obtained 3PN quasi-Keplerian representation of elliptical orbits, the flux is averaged over the binary's orbit. Supplementing this by the important hereditary contributions arising from tails, tails of tails, and tails-squared terms calculated in a previous paper, the complete 3PN energy flux is obtained. The final result presented in this paper would be needed for the construction of ready-to-use templates for binaries moving on noncircular orbits, a plausible class of sources not only for the space-based detectors like LISA but also for the ground-based ones.

DOI: 10.1103/PhysRevD.77.064035 PACS numbers: 04.25.Nx, 04.30.−w, 97.60.Jd, 97.60.Lf

I. INTRODUCTION

Inspiralling compact binaries, one of the prototype sources for laser interferometric gravitational wave (GW) detectors, are usually modeled as moving in quasicircular orbits. This is justified since gravitational radiation reaction, under which it inspirals, circularizes the orbit towards the late stages of inspiral [1,2]. This late phase of inspiral and the ensuing merger phase offer promises for the GW interferometric detectors. The recently discovered double pulsar system [3,4] has an eccentricity as low as 0.088 consistent with the circular-orbit assumption for the late inspiral and premerger phases, believed to be reasonable enough for most of the binary systems made of neutron stars or black holes (BHs).

The theoretical modeling of the binary's phase evolution to a very high precision is called the phasing formula. This is the basic theoretical ingredient used in the construction of search templates for GW using matched filtering [5]. The two key inputs required for the construction of templates for binaries moving in quasicircular orbits in the adiabatic approximation are the orbital energy and the GW luminosity (energy flux). These are computed using a cocktail of approximation schemes in general relativity. The schemes include the multipole decomposition, the post-Minkowskian expansion of the gravitational field or nonlinearity expansion in Newton's constant G, the post-Newtonian expansion in v/c, and the far-zone expansion in powers of $1/R$, where R is the distance from the source (see [6] for a recent review).

Though the garden variety binary sources of GWs for terrestrial laser interferometric GW detectors are those moving in quasicircular orbits, there is an increased recent interest in inspiralling binaries moving in *quasi-eccentric* orbits. Astrophysical scenarios currently exist which lead to binaries with nonzero eccentricity in the GW detector bandwidth, both terrestrial and space based. For instance, inner binaries of hierarchical triplets undergoing Kozai oscillations [7] could not only merge due to gravitational radiation reaction, but a good fraction ($\sim 30\%$) of them will have eccentricity greater than about 0.1 as they enter the sensitivity band of advanced ground-based interferometers [8]. Almost all of the above systems possess eccentricities below 0.2 at 40 Hz and below 0.02 at 200 Hz. The population of stellar mass binaries in globular clusters is expected to have a thermal distribution of eccentricities [9]. In a study on the growth of intermediate BHs [10] in globular clusters, it was found that the binaries have eccentricities between 0.1 and 0.2 in the LISA bandwidth. Though, supermassive black-hole binaries are powerful GW sources for LISA, it is not yet conclusive if they would be in quasicircular or quasi-eccentric orbits [11]. If a Kozai mechanism is at work, these supermassive BH binaries could be in highly eccentric orbits and merge within the Hubble time [12]. Sources of the kind discussed above provide the prime motivation to investigate higher post-Newtonian order modeling for quasi-eccentric binaries.

The GW energy flux or luminosity from a system of two point masses in elliptic motion was first computed by Peters and Mathews at Newtonian order [1,2]. The post-Newtonian (PN) corrections to the gravitational wave flux at 1PN and 1.5PN orders were provided in [13–17] and were used to study the associated evolution of orbital elements using the 1PN "quasi-Keplerian" (QK) represen-

*arun@iap.fr
†blanchet@iap.fr
‡bri@rri.res.in
§mssq@rri.res.in

tation of the binary's orbit [18]. Gopakumar and Iyer [19,20] further extended these results to 2PN order using the generalized quasi-Keplerian representation developed in Refs. [21–23]. The results for the energy flux and waveform presented in [19] were in perfect agreement with those obtained by Will and Wiseman using a different formalism [24]. Recently, Damour, Gopakumar, and Iyer [25] discussed an analytic method for constructing high accuracy templates for the GW signals from the inspiral phase of compact binaries moving on quasi-elliptical orbits. They used an improved "method of variation of constants" to combine the three time scales involved in the elliptical orbit case, namely, orbital period, periastron precession, and radiation reaction time scales, without making the usual approximation of treating the radiative time scale as an adiabatic process.

The generation problem for gravitational waves at any PN order requires the solution to two independent problems. The first relates to the equation of motion of the binary, and the second to the far-zone fluxes of energy, angular momentum, and linear momentum. The latter requires the computation of the relativistic mass and current multipole moments to appropriate PN orders. The 3PN equations of motion (EOM) required to handle gravitational wave phasing turned out to be technically very involved due to the issues related to the self-field regularization using Riesz or Hadamard regularizations [26,27]. Only by a deeper understanding of the origin of the ambiguities in Hadamard regularization, and the use of dimensional regularization, has the problem been uniquely resolved [28,29] and provided the value of the ambiguity parameter ω_s [26] or equivalently λ [27]. We thus have in hand the requisite 3PN EOM for compact binaries moving in general orbits. The computation of the GW luminosity at 3PN or $(v/c)^6$ beyond the leading Einstein quadrupole formula crucially requires the computation of the 3PN accurate mass quadrupole moment. For its completion, the same technique as in the EOM was successfully applied, namely, to compute using Hadamard's regularization all the terms except a few terms parametrized by ambiguity parameters (which turn out to be three, denoted ξ, κ, and ζ) [30,31], and then to determine the value of these parameters by computing the difference between the dimensional and Hadamard regularizations [31–34]. These works thus provide the fully determined 3PN accurate mass quadrupole for general orbits—the other important ingredient to compute the 3PN accurate energy and angular momentum fluxes for inspiralling compact binaries moving in general noncircular orbits. The 3.5PN phasing of inspiralling compact binaries moving in quasicircular orbits is now complete and available for use in GW data analysis [32,35]. Note that the 3PN contribution to the energy flux not only comes from the "instantaneous" terms discussed in this paper but also includes "hereditary" contributions arising from tails, tails of tails, and tails-squared terms. A semianalytical scheme is proposed and discussed in detail in a companion paper [36][1] to evaluate these history-dependent contributions.

In this paper, for binaries moving in elliptical orbits, we compute all the instantaneous contributions to the 3PN accurate GW energy flux. The orbital average of this flux will be obtained using the 3PN quasi-Keplerian parametrization of the binary's orbital motion recently constructed by Memmesheimer, Gopakumar, and Schäfer [37]. We shall supplement these by contributions from the hereditary terms computed in Paper I. The final expression will represent gravitational waves from a binary evolving negligibly under gravitational radiation reaction, including precisely up to 3PN order the effects of eccentricity and periastron precession during epochs of inspiral when the orbital parameters are essentially constant over a few orbital revolutions. It also represents the first step towards the discussion of the *quasi-elliptical* case: the evolution of the binary in an elliptical orbit under gravitational radiation reaction. The present work extends the circular-orbit results at 2.5PN [38] and 3PN [30,32] to the elliptical orbit case. Further, it extends earlier works on instantaneous contributions for binaries moving in elliptical orbits at 1PN [14,15] and 2PN [19] to 3PN order. Similarly, Paper I extends hereditary contributions at 1.5PN [16] to 2.5PN order and 3PN, where the 3PN hereditary contributions comprise the tails of tails and are extensions of Refs. [39,40] for circular orbits to the elliptical orbit case.

In Sec. II we begin with the structure of the far-zone flux of energy, use expressions relating the radiative moments to the source moments, and decompose the energy flux expression into its instantaneous and hereditary parts. Section III lists all the requisite multipole moments in standard harmonic coordinates for binaries moving in general (noncircular) orbits. Section IV introduces the 3PN equations of motion which are necessary to handle the time derivatives of the moments. Section V discusses the computation of the instantaneous terms in the energy flux, and Sec. VI recasts the flux in modified harmonic (MH) coordinates (without logarithms at 3PN order) and Arnowitt, Deser, and Misner (ADM) coordinates. Section VII summarizes the 3PN quasi-Keplerian representation required to average the flux expression over an orbit. Section VIII exhibits the orbital average of the energy flux in modified harmonic coordinates and ADM coordinates, and finally provides an expression of the complete energy flux in terms of gauge-invariant variables.

II. THE FAR-ZONE FLUX OF ENERGY

In this section, we discuss the computation of the 3PN accurate energy flux for general isolated sources. Starting from the expression for the far-zone flux in terms of the radiative multipole moments and using the relations connecting the radiative multipole moments to the source

[1] Hereafter, Ref. [36] will be called Paper I.

moments, we write the resultant structure of the GW energy flux.

Following Thorne [41], the expression for the 3PN accurate far-zone energy flux $\mathcal{F} \equiv (d\mathcal{E}/dt)^{\rm GW}$ in terms of symmetric trace-free (STF) radiative multipole moments reads as[2]

$$\mathcal{F} = \frac{G}{c^5}\left\{\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}U^{(1)}_{ijk}U^{(1)}_{ijk} + \frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right] + \frac{1}{c^4}\left[\frac{1}{9072}U^{(1)}_{ijkm}U^{(1)}_{ijkm} + \frac{1}{84}V^{(1)}_{ijk}V^{(1)}_{ijk}\right] + \frac{1}{c^6}\left[\frac{1}{594\,000}U^{(1)}_{ijkmn}U^{(1)}_{ijkmn} + \frac{4}{14\,175}V^{(1)}_{ijkm}V^{(1)}_{ijkm}\right] + \mathcal{O}(8)\right\}. \tag{2.1}$$

In the above, U_L and V_L (where $L = i_1 i_2 \cdots i_l$ represents a multi-index composed of l spatial indices) are the mass-type and current-type radiative multipole moments, respectively, and $U_L^{(l)}$ and $V_L^{(l)}$ denote their lth time derivatives. The moments are functions of retarded time $U \equiv T - R/c$ in radiative coordinates.

In the multipolar-post-Minkowskian (MPM) formalism, the radiative moments U_L and V_L can be reexpressed in terms of the source moments to an accuracy sufficient for the computation of the energy flux. For the flux to be complete up to 3PN approximation, one must compute the mass-type radiative quadrupole U_{ij} to 3PN accuracy, the mass octupole U_{ijk} and current quadrupole V_{ij} to 2PN accuracy, the mass hexadecapole U_{ijkm} and current octupole V_{ijk} to 1PN accuracy, and finally U_{ijkmn} and V_{ijkm} to Newtonian accuracy.

The relations connecting the different radiative moments U_L and V_L to the corresponding source moments I_L and J_L are given below. For the 3PN mass quadrupole moment we have [38–40,42]

$$U_{ij}(U) = I^{(2)}_{ij}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right] I^{(4)}_{ij}(U-\tau) + \frac{G}{c^5}\left\{-\frac{2}{7}\int_0^{+\infty} d\tau I^{(3)}_{a\langle i}(U-\tau) I^{(3)}_{j\rangle a}(U-\tau) + \frac{1}{7}I^{(5)}_{a\langle i}I_{j\rangle a} - \frac{5}{7}I^{(4)}_{a\langle i}I^{(1)}_{j\rangle a} - \frac{2}{7}I^{(3)}_{a\langle i}I^{(2)}_{j\rangle a} + \frac{1}{3}\varepsilon_{ab\langle i}I^{(4)}_{j\rangle a}J_b + 4[W^{(2)}I_{ij} - W^{(1)}I^{(1)}_{ij}]^{(2)}\right\} + 2\left(\frac{GM}{c^3}\right)^2 \times \int_0^{+\infty} d\tau I^{(5)}_{ij}(U-\tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124\,627}{44\,100}\right] + \mathcal{O}(7), \tag{2.2}$$

where the brackets $\langle\rangle$ surrounding indices denote the STF projection, and ε_{abi} is the usual Levi-Civita symbol such that $\varepsilon_{123} = +1$. The I_L's and J_L's are the mass-type and current-type source moments (and $I_L^{(p)}$, $J_L^{(p)}$ denote their pth time derivatives), and W is the monopole corresponding to the set of "gauge" moments W_L, using the same definitions as in [30]. In the above formula, M (which is in factor of the tail integral at 1.5PN order and the tail-of-tail integral at 3PN) is the total ADM mass of the source. The nonlinear memory integral at 2.5PN is a time antiderivative and will become instantaneous in the energy flux. The moments needed at 2PN order include only the dominant tails and are

$$U_{ijk}(U) = I^{(3)}_{ijk}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{97}{60}\right] \times I^{(5)}_{ijk}(U-\tau) + \mathcal{O}(5), \tag{2.3a}$$

$$V_{ij}(U) = J^{(2)}_{ij}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{7}{6}\right] \times J^{(4)}_{ij}(U-\tau) + \mathcal{O}(5). \tag{2.3b}$$

For all the other moments required in the computation, we need only the leading order accuracy in the relation between radiative and source moments, so that

$$U_L(U) = I^{(l)}_L(U) + \mathcal{O}(3), \tag{2.4a}$$

$$V_L(U) = J^{(l)}_L(U) + \mathcal{O}(3). \tag{2.4b}$$

The constant length r_0 scaling the logarithm is the one introduced in the general MPM formalism and has been chosen here to match with the choice made in the computation of tails of tails in [40]. It is a freely specifiable constant, entering the relation between the retarded time $U = T - R/c$ in radiative coordinates and the corresponding retarded time $t_{\rm H} - r_{\rm H}/c$ in harmonic coordinates (where $r_{\rm H}$ is the distance of the source in harmonic coordinates). More precisely, we have

$$U = t_{\rm H} - \frac{r_{\rm H}}{c} - \frac{2GM}{c^3}\ln\left(\frac{r_{\rm H}}{r_0}\right) + \mathcal{O}(5). \tag{2.5}$$

From Eqs. (2.2), (2.3), and (2.4), it is clear that the radiative moments have two distinct contributions. One part depends on the moments only at the retarded time, $U = T - R/c$; this part is referred to as the "instantaneous contribution" and forms the subject matter of the present paper. The second part, on the other hand, depends on the dynamics of the system in its entire past, i.e. at any $U - \tau < U$, and is referred to as the "hereditary contribution." Equally important but requiring a different treatment, the hereditary contribution is dealt with in Paper I as mentioned earlier. We are thus allowed to write down explicitly the different kinds of contributions to the far-zone energy flux up to 3PN. We have

$$\mathcal{F} = \mathcal{F}_{\rm inst} + \mathcal{F}_{\rm hered}, \tag{2.6}$$

[2] The shorthand $\mathcal{O}(n)$ is used throughout and indicates that the post-Newtonian remainder is of order of $\mathcal{O}(c^{-n})$.

where the instantaneous contribution of interest in this paper is explicitly given by

$$\begin{aligned}\mathcal{F}_{\rm inst} = \frac{G}{c^5}\bigg\{&\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij} + \frac{1}{c^2}\bigg[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\bigg] + \frac{1}{c^4}\bigg[\frac{1}{9072}I^{(5)}_{ijkm}I^{(5)}_{ijkm} + \frac{1}{84}J^{(4)}_{ijk}J^{(4)}_{ijk}\bigg] + \frac{G}{c^5}\bigg[\frac{8}{5}I^{(3)}_{ij}(I_{ij}W^{(5)} + 2I^{(1)}_{ij}W^{(4)}\\ &- 2I^{(3)}_{ij}W^{(2)} - I^{(4)}_{ij}W^{(1)}) + \frac{2}{5}I^{(3)}_{ij}\bigg(-\frac{4}{7}I^{(5)}_{ai}I^{(1)}_{aj} - I^{(4)}_{ai}I^{(2)}_{aj} - \frac{2}{7}I^{(3)}_{ai}I^{(3)}_{aj} + \frac{1}{7}I^{(6)}_{ai}I_{aj}\bigg) + \frac{2}{15}\varepsilon_{abi}I^{(5)}_{aj}J_bI^{(3)}_{ij}\bigg]\\ &+ \frac{1}{c^6}\bigg[\frac{1}{594\,000}I^{(6)}_{ijkmn}I^{(6)}_{ijkmn} + \frac{4}{14\,175}J^{(5)}_{ijkm}J^{(5)}_{ijkm}\bigg] + \mathcal{O}(8)\bigg\}. \end{aligned} \tag{2.7}$$

The hereditary contribution is given in Sec. III A of Paper I. We recall that it is decomposed as

$$\mathcal{F}_{\rm hered} = \mathcal{F}_{\rm tail} + \mathcal{F}_{\rm tail(tail)} + \mathcal{F}_{({\rm tail})^2}. \tag{2.8}$$

The quadratic-order (proportional to G^2) tails are given by

$$\begin{aligned}\mathcal{F}_{\rm tail} &= \frac{4G^2M}{5c^8}I^{(3)}_{ij}(U)\int_0^{+\infty}d\tau I^{(5)}_{ij}(U-\tau)\bigg[\ln\bigg(\frac{c\tau}{2r_0}\bigg) + \frac{11}{12}\bigg]\\ &+ \frac{4G^2M}{189c^{10}}I^{(4)}_{ijk}(U)\int_0^{+\infty}d\tau I^{(6)}_{ijk}(U-\tau)\bigg[\ln\bigg(\frac{c\tau}{2r_0}\bigg)\\ &+ \frac{97}{60}\bigg] + \frac{64G^2M}{45c^{10}}J^{(3)}_{ij}(U)\int_0^{+\infty}d\tau J^{(5)}_{ij}(U-\tau)\\ &\times\bigg[\ln\bigg(\frac{c\tau}{2r_0}\bigg) + \frac{7}{6}\bigg], \end{aligned}\tag{2.9}$$

and the cubic-order tails (proportional to G^3) read

$$\begin{aligned}\mathcal{F}_{\rm tail(tail)} &= \frac{4G^3M^2}{5c^{11}}I^{(3)}_{ij}(U)\int_0^{+\infty}d\tau I^{(6)}_{ij}(U-\tau)\bigg[\ln^2\bigg(\frac{c\tau}{2r_0}\bigg)\\ &+ \frac{57}{70}\ln\bigg(\frac{c\tau}{2r_0}\bigg) + \frac{124\,627}{44\,100}\bigg], \end{aligned}\tag{2.10a}$$

$$\mathcal{F}_{({\rm tail})^2} = \frac{4G^3M^2}{5c^{11}}\bigg(\int_0^{+\infty}d\tau I^{(5)}_{ij}(U-\tau)\bigg[\ln\bigg(\frac{c\tau}{2r_0}\bigg) + \frac{11}{12}\bigg]\bigg)^2. \tag{2.10b}$$

All the tail contributions are thoroughly computed in Paper I, and we shall use those results to obtain the complete GW energy flux in Sec. VIII.

III. THE MULTIPOLE MOMENTS OF COMPACT BINARY SYSTEMS

We provide, in this section, the requisite multipole moments needed for the computation of the 3PN accurate energy flux for compact binaries in the *standard harmonic* coordinate system. By standard harmonic coordinates we refer to the specific coordinate system which has been used consistently in previous works [27,29–32,34,35,43]. We recall that these coordinates contain some *logarithms* at the 3PN level, both in the equations of motion of the binary [27,43] and in their multipole moments [30–32]. Later, we shall also define some *modified harmonic* coordinates which do not involve such logarithms at the 3PN order.

The multipole moments are generalizations to noncircular orbits of the expressions available in Ref. [30] for circular orbits. They are computed by implementing the detailed method described in Ref. [31]. Though algebraically long and involved, the procedure is fairly algorithmic, as explained in [30,31]. We thus skip all those details of computations and list the final results we need. The 3PN mass quadrupole I_{ij} is already given in Ref. [31] and its expression [valid in the frame of the center of mass (CM)] is

$$\begin{aligned}I_{ij} = \nu m\bigg\{&\bigg[A - \frac{24}{7}\frac{\nu}{c^5}\frac{G^2m^2}{r^2}\dot{r}\bigg]x_{\langle i}x_{j\rangle} + B\frac{r^2}{c^2}v_{\langle i}v_{j\rangle}\\ &+ 2\bigg[C\frac{r\dot{r}}{c^2} + \frac{24}{7}\frac{\nu}{c^5}\frac{G^2m^2}{r}\bigg]x_{\langle i}v_{j\rangle}\bigg\}, \end{aligned}\tag{3.1}$$

where the coefficients, up to 3PN order, are

$$\begin{aligned}A = 1 &+ \frac{1}{c^2}\bigg[v^2\bigg(\frac{29}{42} - \frac{29\nu}{14}\bigg) + \frac{Gm}{r}\bigg(-\frac{5}{7} + \frac{8}{7}\nu\bigg)\bigg] + \frac{1}{c^4}\bigg[\frac{Gm}{r}v^2\bigg(\frac{2021}{756} - \frac{5947}{756}\nu - \frac{4883}{756}\nu^2\bigg) + \frac{G^2m^2}{r^2}\bigg(-\frac{355}{252} - \frac{953}{126}\nu + \frac{337}{252}\nu^2\bigg)\\ &+ v^4\bigg(\frac{253}{504} - \frac{1835}{504}\nu + \frac{3545}{504}\nu^2\bigg) + \frac{Gm}{r}\dot{r}^2\bigg(-\frac{131}{756} + \frac{907}{756}\nu - \frac{1273}{756}\nu^2\bigg)\bigg] + \frac{1}{c^6}\bigg[v^6\bigg(\frac{4561}{11088} - \frac{7993}{1584}\nu + \frac{117\,067}{5544}\nu^2\\ &- \frac{328\,663}{11\,088}\nu^3\bigg) + v^4\frac{Gm}{r}\bigg(\frac{307}{77} - \frac{94\,475}{4158}\nu + \frac{218\,411}{8316}\nu^2 + \frac{299\,857}{8316}\nu^3\bigg) + \frac{G^3m^3}{r^3}\bigg(\frac{6\,285\,233}{207\,900} + \frac{15\,502}{385}\nu - \frac{3632}{693}\nu^2\\ &+ \frac{13\,289}{8316}\nu^3 - \frac{428}{105}\ln\bigg(\frac{r}{r_0}\bigg) - \frac{44}{3}\nu\ln\bigg(\frac{r}{r'_0}\bigg)\bigg) + \frac{G^2m^2}{r^2}\dot{r}^2\bigg(-\frac{8539}{20\,790} + \frac{52\,153}{4158}\nu - \frac{4652}{231}\nu^2 - \frac{54\,121}{5544}\nu^3\bigg)\\ &+ \frac{Gm}{r}\dot{r}^4\bigg(\frac{2}{99} - \frac{1745}{2772}\nu + \frac{16\,319}{5544}\nu^2 - \frac{311}{99}\nu^3\bigg) + \frac{G^2m^2}{r^2}v^2\bigg(\frac{187\,183}{83\,160} - \frac{605\,419}{16\,632}\nu + \frac{434\,909}{16\,632}\nu^2 - \frac{37\,369}{2772}\nu^3\bigg)\\ &+ \frac{Gm}{r}v^2\dot{r}^2\bigg(-\frac{757}{5544} + \frac{5545}{8316}\nu - \frac{98\,311}{16\,632}\nu^2 + \frac{153\,407}{8316}\nu^3\bigg)\bigg], \end{aligned}\tag{3.2a}$$

$$B = \frac{11}{21} - \frac{11}{7}\nu + \frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{106}{27} - \frac{335}{189}\nu - \frac{985}{189}\nu^2\right) + v^2\left(\frac{41}{126} - \frac{337}{126}\nu + \frac{733}{126}\nu^2\right) + \dot{r}^2\left(\frac{5}{63} - \frac{25}{63}\nu + \frac{25}{63}\nu^2\right)\right]$$
$$+ \frac{1}{c^4}\left[v^4\left(\frac{1369}{5544} - \frac{19\,351}{5544}\nu + \frac{45\,421}{2772}\nu^2 - \frac{139\,999}{5544}\nu^3\right) + \frac{G^2m^2}{r^2}\left(-\frac{40\,716}{1925} - \frac{10\,762}{2079}\nu + \frac{62\,576}{2079}\nu^2 - \frac{24\,314}{2079}\nu^3 + \frac{428}{105}\ln\left(\frac{r}{r_0}\right)\right) + \frac{Gm}{r}\dot{r}^2\left(\frac{79}{77} - \frac{5807}{1386}\nu + \frac{515}{1386}\nu^2 + \frac{8245}{693}\nu^3\right) + \frac{Gm}{r}v^2\left(\frac{587}{154} - \frac{67\,933}{4158}\nu + \frac{25\,660}{2079}\nu^2 + \frac{129\,781}{4158}\nu^3\right) + v^2\dot{r}^2\left(\frac{115}{1386} - \frac{1135}{1386}\nu + \frac{1795}{693}\nu^2 - \frac{3445}{1386}\nu^3\right)\right], \tag{3.2b}$$

$$C = -\frac{2}{7} + \frac{6}{7}\nu + \frac{1}{c^2}\left[v^2\left(-\frac{13}{63} + \frac{101}{63}\nu - \frac{209}{63}\nu^2\right) + \frac{Gm}{r}\left(-\frac{155}{108} + \frac{4057}{756}\nu + \frac{209}{108}\nu^2\right)\right] + \frac{1}{c^4}\left[\frac{Gm}{r}v^2\left(-\frac{2839}{1386} + \frac{237\,893}{16\,632}\nu - \frac{188\,063}{8316}\nu^2 - \frac{58\,565}{4158}\nu^3\right) + \frac{G^2m^2}{r^2}\left(-\frac{12\,587}{41\,580} + \frac{406\,333}{16\,632}\nu - \frac{2713}{396}\nu^2 + \frac{4441}{2772}\nu^3\right) + v^4\left(-\frac{457}{2772} + \frac{6103}{2772}\nu - \frac{13\,693}{1386}\nu^2 + \frac{40\,687}{2772}\nu^3\right) + \frac{Gm}{r}\dot{r}^2\left(\frac{305}{5544} + \frac{3233}{5544}\nu - \frac{8611}{5544}\nu^2 - \frac{895}{154}\nu^3\right)\right]. \tag{3.2c}$$

In the above equation r_0 is the length scale appearing in the definition of the source multipole moments [31] and is the same as in Eq. (2.5). On the other hand, the different constant r_0' is related to two other length scales, r_1' and r_2' (one for each particle), by $m \ln r_0' = m_1 \ln r_1' + m_2 \ln r_2'$, and is specific to the application of the formalism to point particle systems. It comes from regularizing the self-field of point particles in the standard harmonic coordinate system. It is very important to note that the two length scales r_1' and r_2' are the same as the two scales that appear in the final expression of the 3PN equations of motion in standard harmonic coordinates [27]. The requirement that these r_1' and r_2' should match with similar scales that appear in the equations of motion determines, using dimensional regularization, the values of Hadamard's regularization constants ξ, κ, and ζ that formerly appeared in the 3PN multipole moments [30,31]. The regularization constants are thus determined, and we have consistently replaced ξ, κ, and ζ by their values known from [32,34]. The constants r_1', r_2', and hence r_0' are "unphysical" in the sense that they can be arbitrarily changed by a coordinate transformation of the "bulk" metric outside the particles [27], or, more appropriately (when considering the renormalization which follows the dimensional regularization), by some shifts of the particles' world lines [29,34].

The 2PN mass octupole and current quadrupole moments for general orbits are the other nontrivial moments required. They are given by

$$I_{ijk} = \nu m\sqrt{1-4\nu}\left\{x_{\langle ijk\rangle}\left[-1 + \frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{5}{6} - \frac{13\nu}{6}\right) + v^2\left(-\frac{5}{6} + \frac{19}{6}\nu\right)\right] + \frac{1}{c^4}\left[v^4\left(-\frac{257}{440} + \frac{7319}{1320}\nu - \frac{5501}{440}\nu^2\right) + \frac{G^2m^2}{r^2}\left(\frac{47}{33} + \frac{1591}{132}\nu - \frac{235}{66}\nu^2\right) + \frac{Gm}{r}\dot{r}^2\left(\frac{247}{1320} - \frac{531}{440}\nu + \frac{1347}{440}\nu^2\right) + \frac{Gm}{r}v^2\left(-\frac{3853}{1320} + \frac{14\,257}{1320}\nu + \frac{17\,371}{1320}\nu^2\right)\right]\right] + x_{\langle ij}v_{k\rangle}\frac{r\dot{r}}{c^2}\left[1 - 2\nu + \frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{2461}{660} - \frac{8689}{660}\nu - \frac{1389}{220}\nu^2\right) + v^2\left(\frac{13}{22} - \frac{107}{22}\nu + \frac{102}{11}\nu^2\right)\right]\right] + x_{\langle i}v_{jk\rangle}\frac{r^2}{c^2}\left[-1 + 2\nu + \frac{1}{c^2}\left[v^2\left(-\frac{61}{110} + \frac{519}{110}\nu - \frac{504}{55}\nu^2\right) + \dot{r}^2\left(\frac{1}{11} - \frac{4}{11}\nu + \frac{3}{11}\nu^2\right) + \frac{Gm}{r}\left(-\frac{1949}{330} - \frac{62}{165}\nu + \frac{483}{55}\nu^2\right)\right]\right] + v_{\langle ijk\rangle}\frac{\dot{r}r^3}{c^4}\left(-\frac{13}{55} + \frac{52}{55}\nu - \frac{39}{55}\nu^2\right)\right\} + \mathcal{O}(6), \tag{3.3a}$$

$$J_{ij} = m\nu\sqrt{1-4\nu}\left\{\varepsilon_{ab\langle i}x_{j\rangle a}v_b\left[-1 + \frac{1}{c^2}\left[\frac{Gm}{r}\left(-\frac{27}{14} - \frac{15}{7}\nu\right) + v^2\left(-\frac{13}{28} + \frac{17}{7}\nu\right)\right] + \frac{1}{c^4}\left[v^4\left(-\frac{29}{84} + \frac{11}{3}\nu - \frac{505}{56}\nu^2\right) + \frac{G^2m^2}{r^2}\left(\frac{43}{252} + \frac{1543}{126}\nu - \frac{293}{84}\nu^2\right) + \frac{Gm}{r}\dot{r}^2\left(\frac{5}{252} + \frac{241}{252}\nu + \frac{335}{84}\nu^2\right) + \frac{Gm}{r}v^2\left(-\frac{671}{252} + \frac{1297}{126}\nu + \frac{121}{12}\nu^2\right)\right]\right] + \varepsilon_{ab\langle i}v_{j\rangle b}x_a\frac{r\dot{r}}{c^2}\left[-\frac{5}{28} + \frac{5}{14}\nu + \frac{1}{c^2}\left[v^2\left(-\frac{25}{168} + \frac{25}{24}\nu - \frac{25}{14}\nu^2\right) + \frac{Gm}{r}\left(-\frac{103}{63} - \frac{337}{126}\nu + \frac{173}{84}\nu^2\right)\right]\right]\right\} + \mathcal{O}(6). \tag{3.3b}$$

In the above and in what follows, $x_{ijk\ldots} \equiv x_ix_jx_k\cdots$ and $v_{ijk\ldots} \equiv v_iv_jv_k\cdots$, and the brackets $\langle\rangle$ denote the STF projection. The 1PN moments read as

$$\begin{aligned}I_{ijkl} &= \nu m\bigg\{x_{\langle ijkl\rangle}\bigg[1-3\nu+\frac{1}{c^2}\bigg[\bigg(\frac{103}{110}-\frac{147}{22}\nu+\frac{279}{22}\nu^2\bigg)v^2-\bigg(\frac{10}{11}-\frac{61}{11}\nu+\frac{105}{11}\nu^2\bigg)\frac{Gm}{r}\bigg]\bigg]\\&\quad-\frac{72}{55}v_{\langle i}x_{jkl\rangle}\frac{r\dot r}{c^2}(1-5\nu+5\nu^2)+\frac{78}{55}v_{\langle ij}x_{kl\rangle}\frac{r^2}{c^2}(1-5\nu+5\nu^2)\bigg\}+\mathcal{O}(4),\end{aligned}\tag{3.4a}$$

$$\begin{aligned}J_{ijk} &= \nu m\varepsilon_{ab\langle i}\bigg\{x_{jk\rangle a}v_b\bigg[1-3\nu+\frac{1}{c^2}\bigg[\bigg(\frac{41}{90}-\frac{77}{18}\nu+\frac{185}{18}\nu^2\bigg)v^2+\bigg(\frac{14}{9}-\frac{16}{9}\nu-\frac{86}{9}\nu^2\bigg)\frac{Gm}{r}\bigg]\bigg]\\&\quad+\frac{7}{45}v_{jk\rangle b}x_a\frac{r^2}{c^2}(1-5\nu+5\nu^2)+\frac{2}{9}x_{j\underline{a}}v_{k\rangle b}\frac{r\dot r}{c^2}(1-5\nu+5\nu^2)\bigg\}+\mathcal{O}(4).\end{aligned}\tag{3.4b}$$

(The underlined index $\underline{a}$ means that it should be excluded from the STF projection.) Finally, we also need

$$I_{ijklm} = -\nu m\sqrt{1-4\nu}(1-2\nu)x_{\langle ijklm\rangle}+\mathcal{O}(2),\tag{3.5a}$$

$$J_{ijkl} = -\nu m\sqrt{1-4\nu}(1-2\nu)\varepsilon_{ab\langle i}x_{jkl\rangle a}v_b+\mathcal{O}(2),\tag{3.5b}$$

as well as W, the monopole corresponding to the gauge moments W_L, which is given by

$$W=\tfrac{1}{3}\nu m r\dot r+\mathcal{O}(2).\tag{3.6}$$

IV. THE EQUATIONS OF MOTION OF COMPACT BINARY SYSTEMS

A. The equations of motion in standard harmonic coordinates

The computation of the flux will involve the time derivatives of the latter source moments. The 3PN accurate flux requires the 3PN equations of motion for compact binaries which are now complete [27–29,44,45]. For the present work, where the multipole moments are computed in standard harmonic coordinates and reduced to the CM frame, we require the 3PN accurate equation of motion (or acceleration) in the CM frame associated with the standard harmonic gauge. This was computed in [43] and given as

$$a^i=\frac{dv^i}{dt}=-\frac{Gm}{r^2}[(1+P)n^i+Qv^i]+\mathcal{O}(7),\tag{4.1}$$

where the coefficients P and Q are

$$\begin{aligned}P &= \frac{1}{c^2}\bigg\{-\frac{3\dot r^2\nu}{2}+v^2+3\nu v^2-\frac{Gm}{r}(4+2\nu)\bigg\}+\frac{1}{c^4}\bigg\{\frac{15\dot r^4\nu}{8}-\frac{45\dot r^4\nu^2}{8}-\frac{9\dot r^2\nu v^2}{2}+6\dot r^2\nu^2v^2+3\nu v^4-4\nu^2v^4\\&\quad+\frac{Gm}{r}\bigg(-2\dot r^2-25\dot r^2\nu-2\dot r^2\nu^2-\frac{13\nu v^2}{2}+2\nu^2v^2\bigg)+\frac{G^2m^2}{r^2}\bigg(9+\frac{87\nu}{4}\bigg)\bigg\}+\frac{1}{c^5}\bigg\{-\frac{24\dot r\nu v^2}{5}\frac{Gm}{r}-\frac{136\dot r\nu}{15}\frac{G^2m^2}{r^2}\bigg\}\\&\quad+\frac{1}{c^6}\bigg\{-\frac{35\dot r^6\nu}{16}+\frac{175\dot r^6\nu^2}{16}-\frac{175\dot r^6\nu^3}{16}+\frac{15\dot r^4\nu v^2}{2}-\frac{135\dot r^4\nu^2v^2}{4}+\frac{255\dot r^4\nu^3v^2}{8}-\frac{15\dot r^2\nu v^4}{2}+\frac{237\dot r^2\nu^2v^4}{8}-\frac{45\dot r^2\nu^3v^4}{2}\\&\quad+\frac{11\nu v^6}{4}-\frac{49\nu^2v^6}{4}+13\nu^3v^6+\frac{Gm}{r}\bigg(79\dot r^4\nu-\frac{69\dot r^4\nu^2}{2}-30\dot r^4\nu^3-121\dot r^2\nu v^2+16\dot r^2\nu^2v^2+20\dot r^2\nu^3v^2+\frac{75\nu v^4}{4}\\&\quad+8\nu^2v^4-10\nu^3v^4\bigg)+\frac{G^2m^2}{r^2}\bigg(\dot r^2+\frac{32\,573\dot r^2\nu}{168}+\frac{11\dot r^2\nu^2}{8}-7\dot r^2\nu^3+\frac{615\dot r^2\nu\pi^2}{64}-\frac{26\,987\nu v^2}{840}+\nu^3v^2-\frac{123\nu\pi^2v^2}{64}\\&\quad-110\dot r^2\nu\ln\bigg(\frac{r}{r_0'}\bigg)+22\nu v^2\ln\bigg(\frac{r}{r_0'}\bigg)\bigg)+\frac{G^3m^3}{r^3}\bigg(-16-\frac{437\nu}{4}-\frac{71\nu^2}{2}+\frac{41\nu\pi^2}{16}\bigg)\bigg\},\end{aligned}\tag{4.2a}$$

$$\begin{aligned}Q &= \frac{1}{c^2}\{-4\dot r+2\dot r\nu\}+\frac{1}{c^4}\bigg\{\frac{9\dot r^3\nu}{2}+3\dot r^3\nu^2-\frac{15\dot r\nu v^2}{2}-2\dot r\nu^2v^2+\frac{Gm}{r}\bigg(2\dot r+\frac{41\dot r\nu}{2}+4\dot r\nu^2\bigg)\bigg\}\\&\quad+\frac{1}{c^5}\bigg\{\frac{8\nu v^2}{5}\frac{Gm}{r}+\frac{24\nu}{5}\frac{G^2m^2}{r^2}\bigg\}+\frac{1}{c^6}\bigg\{-\frac{45\dot r^5\nu}{8}+15\dot r^5\nu^2+\frac{15\dot r^5\nu^3}{4}+12\dot r^3\nu v^2-\frac{111\dot r^3\nu^2v^2}{4}\\&\quad-12\dot r^3\nu^3v^2-\frac{65\dot r\nu v^4}{8}+19\dot r\nu^2v^4+6\dot r\nu^3v^4+\frac{Gm}{r}\bigg(\frac{329\dot r^3\nu}{6}+\frac{59\dot r^3\nu^2}{2}+18\dot r^3\nu^3-15\dot r\nu v^2-27\dot r\nu^2v^2-10\dot r\nu^3v^2\bigg)\\&\quad+\frac{G^2m^2}{r^2}\bigg(-4\dot r-\frac{18\,169\dot r\nu}{840}+25\dot r\nu^2+8\dot r\nu^3-\frac{123\dot r\nu\pi^2}{32}+44\dot r\nu\ln\bigg(\frac{r}{r_0'}\bigg)\bigg)\bigg\}.\end{aligned}\tag{4.2b}$$

Recall that there was initially a regularization ambiguity constant denoted λ in [27], which has been replaced here by its uniquely determined value $\lambda = -\frac{1987}{3080}$ [29]. On the other hand, the constant r_0' is the *same* as the one in the 3PN quadrupole moment (3.1) and (3.2).

B. The modified harmonic coordinates (without logarithms)

The standard harmonic (hereafter SH) coordinate system used up to now is useful for analytical algebraic checks, but contains gauge-dependent logarithmic terms that are not very convenient in numerical calculations. More importantly, in the presence of the logarithmic terms the simple generalized quasi-Keplerian representation (reviewed in Sec. VII) is not possible, impeding the process of averaging the flux over the orbital period. Consequently, it is useful to have the expression for the energy flux in a MH coordinate system without logarithms, like the one explicitly used in [46] (we shall alternatively use ADM-type coordinates which are also free of such logarithms at 3PN). This will require us to reexpress the instantaneous expressions for the energy flux [given by Eqs. (5.2) below] in terms of corresponding variables in the MH or ADM coordinate systems. We provide in this section the definition of the MH coordinate system.

Consider the coordinate transformation $x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)$ which removes the logarithms $\ln(r/r_0')$ at the level of the equations of motion as discussed in Ref. [27]. It is given by

$$\varepsilon_{\mu} = \frac{22}{3}\frac{G^2 m_1 m_2}{c^6}\partial_{\mu}\left[\frac{Gm_2}{r_1}\ln\left(\frac{r}{r_2'}\right) + \frac{Gm_1}{r_2}\ln\left(\frac{r}{r_1'}\right)\right], \quad (4.3)$$

where $r_1 = |\mathbf{x} - \mathbf{y}_1|$ and $r_2 = |\mathbf{x} - \mathbf{y}_2|$ are the distances to the two particles with trajectories $y_1^i(t)$ and $y_2^i(t)$, and where $r = |\mathbf{y}_1 - \mathbf{y}_2|$ is their relative distance. Following [29] the logarithms can be equivalently removed by the shifts (sometimes also called the "contact" transformations) of the particle world lines induced by the change of coordinates, namely,

$$y_1'^i = y_1^i + \xi_1^i, \quad (4.4a)$$
$$y_2'^i = y_2^i + \xi_2^i. \quad (4.4b)$$

The equalities here are *functional* relations; i.e. the two sides of the equations are evaluated at the *same* coordinate time, say, t. The spatial shifts ξ_1^i and ξ_2^i of the two world lines are related to the coordinate transformation restricted to the world lines [denoted $\varepsilon_1^{\mu}(t) \equiv \varepsilon^{\mu}(t, y_1(t))$ and $\varepsilon_2^{\mu}(t) \equiv \varepsilon^{\mu}(t, y_2(t))$] by

$$\xi_1^i = \varepsilon_1^i - \frac{v_1^i}{c}\varepsilon_1^0 + \mathcal{O}(\varepsilon^2), \quad (4.5a)$$

$$\xi_2^i = \varepsilon_2^i - \frac{v_2^i}{c}\varepsilon_2^0 + \mathcal{O}(\varepsilon^2), \quad (4.5b)$$

where $v_1^i = dy_1^i/dt$ and $v_2^i = dy_2^i/dt$ are the coordinate velocities. The latter relations are valid at linear order in ε_1^{μ} and ε_2^{μ}. Now the coordinate transformation (4.3) is at 3PN order, so we have $\varepsilon^0 = \mathcal{O}(7)$ and $\varepsilon^i = \mathcal{O}(6)$. Hence, we see from (4.5) that at the 3PN order the shifts simply agree with the spatial components of the coordinate transformation,

$$\xi_1^i = \varepsilon_1^i + \mathcal{O}(8), \quad (4.6a)$$
$$\xi_2^i = \varepsilon_2^i + \mathcal{O}(8). \quad (4.6b)$$

They are readily obtained from Eq. (4.3) as

$$\xi_1^i = -\frac{22}{3}\frac{G^3 m_1^2 m_2}{c^6 r^2} n^i \ln\left(\frac{r}{r_1'}\right) + \mathcal{O}(8), \quad (4.7a)$$

$$\xi_2^i = \frac{22}{3}\frac{G^3 m_1 m_2^2}{c^6 r^2} n^i \ln\left(\frac{r}{r_2'}\right) + \mathcal{O}(8). \quad (4.7b)$$

Under the shifts of world lines, the accelerations of the particles are changed by the amounts $\delta_{\xi} a_1^i$ and $\delta_{\xi} a_2^i$ (i.e. such that the functional equalities $a_{1,2}'^i = a_{1,2}^i + \delta_{\xi} a_{1,2}^i$ hold) given by

$$\delta_{\xi} a_1^i = \frac{d^2 \xi_1^i}{dt^2} - (\xi_1^j - \xi_2^j)\partial_{ij}\left(\frac{Gm_2}{r}\right) + \mathcal{O}(8), \quad (4.8a)$$

$$\delta_{\xi} a_2^i = \frac{d^2 \xi_2^i}{dt^2} + (\xi_1^j - \xi_2^j)\partial_{ij}\left(\frac{Gm_1}{r}\right) + \mathcal{O}(8), \quad (4.8b)$$

where the second terms come from reexpressing the gravitational force—gradient of the Newtonian potential—in terms of the new trajectories (4.4). The relative acceleration $a^i \equiv a_1^i - a_2^i$ is changed by the amount

$$\delta_{\xi} a^i = \frac{d^2 \xi_{12}^i}{dt^2} - \xi_{12}^j \partial_{ij}\left(\frac{Gm}{r}\right) + \mathcal{O}(8), \quad (4.9)$$

where $m \equiv m_1 + m_2$ and $\xi_{12}^i \equiv \xi_1^i - \xi_2^i$.[3] An easy calculation shows that the change in the relative acceleration associated with the shifts (4.7) is

$$\delta_{\xi} a^i = \frac{G^3 m^3 \nu}{c^6 r^4}\left\{\left[(-110\dot{r}^2 + 22v^2)n^i + 44\dot{r}v^i\right]\ln\left(\frac{r}{r_0'}\right) + \left(\frac{176}{3}\dot{r}^2 - \frac{22}{3}v^2 + \frac{22}{3}\frac{Gm}{r}\right)n^i - \frac{44}{3}\dot{r}v^i\right\} + \mathcal{O}(8). \quad (4.10)$$

Adding the above shift to the expression for the relative acceleration in SH coordinates, as given by Eqs. (4.1) and (4.2), yields the expression for the acceleration in MH coordinates. Since $a'^i = a^i + \delta_{\xi} a^i$ is a functional identity, the resulting MH acceleration is obtained as a function of the "dummy" variables denoted v^2, $\dot{r}$, and r. Evidently, these variables are to be interpreted as the natural variables describing the binary motion in MH coordinates.[4] As expected, the logarithms in Eq. (4.10) exactly cancel the

[3] This means that $x_{\mathrm{MH}}^i = x_{\mathrm{SH}}^i + \xi_{12}^i$.

[4] To avoid making the notation too heavy we do not add a subscript MH on the variables v^2, $\dot{r}$, and r. In the following our notation may not always be completely consistent but should be clear from the context.

logarithms in the SH acceleration (4.1) and (4.2). Some 3PN coefficients in the EOM are also modified and the final result agrees with that displayed in Ref. [46] (see also [6]).

For completeness we note also that the above shifts will modify the 3PN conserved energy of the binary (associated with the conservative part of the 3PN equations of motion) by the amount

$$\delta_\xi E = -m_1 v_1^i \frac{d\xi_1^i}{dt} - m_2 v_2^i \frac{d\xi_2^i}{dt} + \xi_{12}^i \partial_i\left(\frac{Gm_1m_2}{r}\right) + \mathcal{O}(8). \tag{4.11}$$

For the case at hand with the shifts (4.7) and in the center-of-mass frame, we find

$$\delta_\xi E = \frac{22}{3}\frac{G^3m^4\nu^2}{c^6r^3}\left\{\left[\frac{Gm}{r} - 3\dot{r}^2 + v^2\right]\ln\left(\frac{r}{r_0'}\right) + \dot{r}^2\right\} + \mathcal{O}(8). \tag{4.12}$$

Comparing with the 3PN energy in SH coordinates as given by Eq. (4.8) in [43], we see that the logarithms $\ln(r/r_0')$ are also canceled in the expression for the energy by going to the MH coordinates.

V. THE INSTANTANEOUS PART OF THE 3PN ENERGY FLUX

Using the multipole moments given in Eqs. (3.1), (3.2), (3.3), (3.4), and (3.5), one computes the required time derivatives with the help of the equations of motion (4.1) and obtains the instantaneous part of the energy flux as defined by Eq. (2.7). Here we are working in SH coordinates, in which the equations of motion are given by Eqs. (4.1) and (4.2). In the next section we consider the case of alternative coordinate systems. The hereditary part computed in Paper I will be added after the process of averaging over one orbit (this contribution is the same in all alternative coordinate systems considered in this paper). Though lengthy, the computation of the different parts constituting the instantaneous terms in the energy flux at 3PN order is straightforward.[5] We write the result as

$$\mathcal{F}_{\rm inst} = \mathcal{F}^{\rm N}_{\rm inst} + \mathcal{F}^{\rm 1PN}_{\rm inst} + \mathcal{F}^{\rm 2PN}_{\rm inst} + \mathcal{F}^{\rm 2.5PN}_{\rm inst} + \mathcal{F}^{\rm 3PN}_{\rm inst} + \mathcal{O}(7), \tag{5.1}$$

and find that the various PN pieces are given by

$$\mathcal{F}^{\rm N}_{\rm inst} = \frac{32}{5}\frac{G^3m^4\nu^2}{c^5r^4}\left\{v^2 - \frac{11}{12}\dot{r}^2\right\}, \tag{5.2a}$$

$$\begin{aligned}\mathcal{F}^{\rm 1PN}_{\rm inst} = \frac{32}{5}\frac{G^3m^4\nu^2}{c^7r^4}\bigg\{& v^4\left(\frac{785}{336} - \frac{71}{28}\nu\right) + \dot{r}^2v^2\left(-\frac{1487}{168} + \frac{58}{7}\nu\right) + \frac{Gm}{r}v^2\left(-\frac{170}{21} + \frac{10}{21}\nu\right) + \dot{r}^4\left(\frac{687}{112} - \frac{155}{28}\nu\right)\\ &+ \frac{Gm}{r}\dot{r}^2\left(\frac{367}{42} - \frac{5}{14}\nu\right) + \frac{G^2m^2}{r^2}\left(\frac{1}{21} - \frac{4}{21}\nu\right)\bigg\},\end{aligned} \tag{5.2b}$$

$$\begin{aligned}\mathcal{F}^{\rm 2PN}_{\rm inst} = \frac{32}{5}\frac{G^3m^4\nu^2}{c^9r^4}\bigg\{& v^6\left(\frac{47}{14} - \frac{5497}{504}\nu + \frac{2215}{252}\nu^2\right) + \dot{r}^2v^4\left(-\frac{573}{56} + \frac{1713}{28}\nu - \frac{1573}{42}\nu^2\right) + \frac{Gm}{r}v^4\left(-\frac{247}{14} + \frac{5237}{252}\nu - \frac{199}{36}\nu^2\right)\\ &+ \dot{r}^4v^2\left(\frac{1009}{84} - \frac{5069}{56}\nu + \frac{631}{14}\nu^2\right) + \frac{Gm}{r}\dot{r}^2v^2\left(\frac{4987}{84} - \frac{8513}{84}\nu + \frac{2165}{84}\nu^2\right) + \frac{G^2m^2}{r^2}v^2\left(\frac{281\,473}{9072} + \frac{2273}{252}\nu + \frac{13}{27}\nu^2\right)\\ &+ \dot{r}^6\left(-\frac{2501}{504} + \frac{10\,117}{252}\nu - \frac{2101}{126}\nu^2\right) + \frac{Gm}{r}\dot{r}^4\left(-\frac{5585}{126} + \frac{60\,971}{756}\nu - \frac{7145}{378}\nu^2\right)\\ &+ \frac{G^2m^2}{r^2}\dot{r}^2\left(-\frac{106\,319}{3024} - \frac{1633}{504}\nu - \frac{16}{9}\nu^2\right) + \frac{G^3m^3}{r^3}\left(-\frac{253}{378} + \frac{19}{7}\nu - \frac{4}{27}\nu^2\right)\bigg\},\end{aligned} \tag{5.2c}$$

$$\mathcal{F}^{\rm 2.5PN}_{\rm inst} = \frac{32}{5}\frac{G^3m^4\nu^2}{c^{10}r^4}\left\{\dot{r}\nu\left(-\frac{12\,349}{210}\frac{Gm}{r}v^4 + \frac{4524}{35}\frac{Gm}{r}v^2\dot{r}^2 - \frac{2753}{126}\frac{G^2m^2}{r^2}v^2 - \frac{985}{14}\frac{Gm}{r}\dot{r}^4 + \frac{13\,981}{630}\frac{G^2m^2}{r^2}\dot{r}^2 - \frac{1}{315}\frac{G^3m^3}{r^3}\right)\right\}, \tag{5.2d}$$

[5] In order to perform some independent checks on the long and involved algebra, we have found it expeditious to make two computations using the two harmonic coordinate systems: SH containing the (gauge-dependent) log terms *à la* [43] and MH without log terms as in Refs. [6,46].

$$
\begin{aligned}
\mathcal{F}^{\rm 3PN}_{\rm inst} = \frac{32}{5}\frac{G^3m^4\nu^2}{c^{11}r^4}\bigg\{ & v^8\left(\frac{80\,315}{14\,784} - \frac{694\,427}{22\,176}\nu + \frac{604\,085}{11\,088}\nu^2 - \frac{16\,985}{462}\nu^3\right) + \dot{r}^2v^6\left(-\frac{31\,499}{1008} + \frac{1\,119\,913}{5544}\nu - \frac{44\,701}{132}\nu^2\right.\\
& \left. + \frac{38\,725}{231}\nu^3\right) + \frac{Gm}{r}v^6\left(-\frac{61\,669}{3696} + \frac{95\,321}{1008}\nu - \frac{955\,013}{11\,088}\nu^2 + \frac{47\,255}{1386}\nu^3\right) + \dot{r}^4v^4\left(\frac{204\,349}{2464} - \frac{3\,522\,149}{7392}\nu\right.\\
& \left. + \frac{2\,354\,753}{3696}\nu^2 - \frac{109\,447}{462}\nu^3\right) + \frac{Gm}{r}\dot{r}^2v^4\left(\frac{136\,695}{1232} - \frac{202\,693}{336}\nu + \frac{744\,377}{1232}\nu^2 - \frac{931\,099}{5544}\nu^3\right)\\
& + \frac{G^2m^2}{r^2}v^4\left(\frac{598\,614\,941}{2\,494\,800} - \frac{856}{35}\ln\left(\frac{r}{r_0}\right) + \left[\frac{39\,896}{2079} - \frac{369}{64}\pi^2\right]\nu + \frac{1\,300\,907}{33\,264}\nu^2 - \frac{161\,783}{24\,948}\nu^3\right)\\
& + \dot{r}^6v^2\left(-\frac{1\,005\,979}{11\,088} + \frac{2\,589\,599}{5544}\nu - \frac{1\,322\,141}{2772}\nu^2 + \frac{90\,455}{693}\nu^3\right) + \frac{Gm}{r}\dot{r}^4v^2\left(-\frac{715\,157}{3696} + \frac{35\,158\,037}{33\,264}\nu\right.\\
& \left. - \frac{3\,672\,143}{3696}\nu^2 + \frac{871\,025}{4158}\nu^3\right) + \frac{G^2m^2}{r^2}\dot{r}^2v^2\left(-\frac{35\,629\,009}{37\,800} + \frac{3424}{35}\ln\left(\frac{r}{r_0}\right) + \left[-\frac{150\,739}{1232} + \frac{861}{32}\pi^2\right]\nu\right.\\
& \left. - \frac{453\,247}{1848}\nu^2 + \frac{496\,081}{8316}\nu^3\right) + \frac{G^3m^3}{r^3}v^2\left(-\frac{24\,608\,492}{155\,925} + \frac{856}{105}\ln\left(\frac{r}{r_0}\right) + \left[-\frac{6\,356\,291}{22\,680} + \frac{44}{3}\ln\left(\frac{r}{r'_0}\right)\right.\right.\\
& \left.\left. + \frac{451}{64}\pi^2\right]\nu + \frac{3725}{462}\nu^2 - \frac{841}{2268}\nu^3\right) + \dot{r}^8\left(\frac{1\,507\,925}{44\,352} - \frac{20\,365}{126}\nu + \frac{687\,305}{5544}\nu^2 - \frac{32\,755}{1386}\nu^3\right)\\
& + \frac{Gm}{r}\dot{r}^6\left(\frac{5\,476\,951}{55\,440} - \frac{671\,765}{1232}\nu + \frac{5\,205\,019}{11\,088}\nu^2 - \frac{860\,477}{11\,088}\nu^3\right) + \frac{G^2m^2}{r^2}\dot{r}^4\left(\frac{115\,627\,817}{166\,320} - \frac{214}{3}\ln\left(\frac{r}{r_0}\right)\right.\\
& \left. + \left[\frac{42\,671}{792} - \frac{697}{32}\pi^2\right]\nu + \frac{1\,099\,355}{4752}\nu^2 - \frac{825\,331}{16\,632}\nu^3\right) + \frac{G^3m^3}{r^3}\dot{r}^2\left(\frac{3\,202\,601}{23\,100} - \frac{1712}{315}\ln\left(\frac{r}{r_0}\right) + \left[\frac{6\,220\,199}{22\,680}\right.\right.\\
& \left.\left. - \frac{88}{9}\ln\left(\frac{r}{r'_0}\right) - \frac{1763}{192}\pi^2\right]\nu + \frac{57\,577}{1848}\nu^2 - \frac{43\,018}{6237}\nu^3\right) + \frac{G^4m^4}{r^4}\left(\frac{37\,571}{8316} - \frac{14\,962}{891}\nu - \frac{3019}{594}\nu^2 - \frac{866}{6237}\nu^3\right)\bigg\}.
\end{aligned}
\tag{5.2e}
$$

The new results are the instantaneous terms at 2.5PN and 3PN orders. Up to 2PN order, all the terms match with those obtained in Refs. [1,14,19]. As one may notice, the 2.5PN terms in the above equation are all proportional to $\dot{r}$ and hence are zero for the circular-orbit case, in agreement with the result of [38]. The $\dot{r}$ dependence of the 2.5PN terms is important when we discuss their orbital average in Sec. VIII. The 3PN terms provide the generalization of the circular-orbit results in Ref. [30]. As expected, the constant r_0 present in the expression of the mass quadrupole moment appears in the final expression for the 3PN flux (the presence of r'_0 is of a different type and is dealt with in the next section). The dependence of the instantaneous terms on the scale r_0 should exactly cancel a similar contribution coming from the tail terms as determined in Paper I. This cancellation has already been checked for circular orbits in [30], and we shall prove this cancellation for quasi-elliptical orbits in Sec. VIII.

VI. THE 3PN ENERGY FLUX IN ALTERNATIVE COORDINATES

The dependence on r'_0 of the result (5.1) and (5.2) is due to our use of the SH coordinate system. For circular orbits, it was shown [30] that this r'_0 dependence disappears when the total flux is expressed in terms of the gauge-invariant parameter $x = (Gm\omega/c^3)^{2/3}$ related to the GW frequency. In the general orbit case we shall transform away the dependence on r'_0 by going to different coordinate systems such as the MH coordinates studied in Sec. IV B. Subsequently, we shall average the energy flux over an orbital period and exhibit alternative representations of the energy flux for elliptical orbits. In particular, some of these are in terms of gauge-invariant variables related to those suggested in Ref. [37].

A. The modified harmonic coordinates

We now provide the energy flux $\mathcal{F}$ in the MH system, avoiding the appearance of the logarithms $\ln(r/r'_0)$ at 3PN order and which has been introduced in Sec. IV B. First we notice that $\mathcal{F}$ is a function of the "natural" variables r, $\dot{r}$, and v^2 [see Eqs. (5.2)], and is a *scalar*, therefore it satisfies, under the shifts of these variables defined by (4.4),

$$\mathcal{F}[r, \dot{r}, v^2] = \mathcal{F}'[r', \dot{r}', v'^2]. \tag{6.1}$$

This means that we shall have the *functional* equality $\mathcal{F}' = \mathcal{F} + \delta_\xi\mathcal{F}$ in which

$$\delta_\xi\mathcal{F} = -\delta_\xi r\frac{\partial\mathcal{F}}{\partial r} - \delta_\xi\dot{r}\frac{\partial\mathcal{F}}{\partial\dot{r}} - \delta_\xi v^2\frac{\partial\mathcal{F}}{\partial v^2} + \mathcal{O}(\xi^2), \tag{6.2}$$

where

$$\left.\begin{array}{c}\delta_\xi r = n^i\xi_{12}^i\\ \delta_\xi \dot r = n^i\frac{d\xi_{12}^i}{dt} + \frac{v^i - \dot r n^i}{r}\xi_{12}^i\\ \delta_\xi v^2 = 2v^i\frac{d\xi_{12}^i}{dt}\end{array}\right\} + \mathcal{O}(\xi^2). \tag{6.3}$$

(Recall that $\xi_{12}^i \equiv \xi_1^i - \xi_2^i$.) Since the previous formulas are at linear order in the shifts and we are interested in the 3PN approximation, they are valid for any shifts at 3PN order (the case of the MH coordinates) and also at 2PN order like the ones associated with the passage to ADM coordinates—in the latter case, the error will be at 4PN order.

In the case of the MH shifts, which start at 3PN order, one can make an alternative computation of the modification of the energy flux. Indeed the only modification *vis-à-vis* the calculation in standard harmonic (SH) coordinates is the one related to the mass quadrupole moment which must be computed to 3PN accuracy. Under the shifts of the particles' trajectories $y_{1,2}^i$ as given by (4.4), the mass quadrupole moment I_{ij}, which equals $I_{ij} = m_1 y_1^{\langle ij\rangle} + m_2 y_2^{\langle ij\rangle} + \mathcal{O}(2)$ in the Newtonian approximation, is shifted by the amount ($1 \leftrightarrow 2$ meaning the same term but for the other particle)

$$\delta_\xi I_{ij} = 2m_1 y_1^{\langle i}\xi_1^{j\rangle} + 1 \leftrightarrow 2 + \mathcal{O}(8), \tag{6.4}$$

where the remainder $\mathcal{O}(8)$ comes from the 1PN corrections in the quadrupole moment coupled with the 3PN shifts. Using the explicit expressions of these shifts in (4.7), we find, in the center-of-mass frame,

$$\delta_\xi I_{ij} = -\frac{44}{3}\frac{G^3m^4\nu^2}{r^3}\ln\left(\frac{r}{r_0'}\right)x^{\langle ij\rangle} + \mathcal{O}(8), \tag{6.5}$$

where r_0' is given by $m\ln r_0' = m_1\ln r_1' + m_2\ln r_2'$. This modification of the quadrupole moment is seen to exactly cancel the $\ln(r/r_0')$ dependence of the mass quadrupole moment in SH coordinates as given by (3.1) and (3.2). Thus in the MH gauge the r_0' dependence of the mass quadrupole moment vanishes as expected. The rest of the expression of the moment remains exactly the same as in SH coordinates, Eqs. (3.1) and (3.2), and will not be repeated here.

Next we must take into account the fact that, when computing the third time derivative of the quadrupole moment, which is needed in the expression of the flux, the acceleration in MH coordinates is modified. We get

$$\delta_\xi(\dddot I_{ij}) = \frac{d^3}{dt^3}(\delta_\xi I_{ij}) - 2m_1\left[3v_1^{\langle i}\delta_\xi a_1^{j\rangle} + y_1^{\langle i}\frac{d\delta_\xi a_1^{j\rangle}}{dt}\right] + 1 \leftrightarrow 2 + \mathcal{O}(8), \tag{6.6}$$

where the dots mean the time derivative. The first term is the third time derivative of the direct modification of the quadrupole moment, Eqs. (6.4) and (6.5), and the extra terms come from the modification of the accelerations which are given by (4.8). On the other hand, all the other contributions coming from the higher multipole moments and their derivatives remain unchanged. We then find

$$\delta_\xi\mathcal{F} = -\frac{2G}{5c^5}[\dddot I_{ij}\delta_\xi(\dddot I_{ij}) + \mathcal{O}(8)]. \tag{6.7}$$

With the explicit expression of the shifts, one finally obtains the modification of the 3PN energy flux in the MH coordinates as (thus, $\mathcal{F}_{\rm MH} = \mathcal{F}_{\rm SH} + \delta_\xi\mathcal{F}$)

$$\delta_\xi\mathcal{F} = -\frac{1408}{15}\frac{G^6m^7\nu^3}{c^{11}r^7}\left[\left(v^2 - \frac{2}{3}\dot r^2\right)\ln\left(\frac{r}{r_0'}\right) - \frac{\dot r^2}{12} + \mathcal{O}(2)\right]. \tag{6.8}$$

[Of course the result agrees with the one we would obtain from directly using Eqs. (6.2) and (6.3).]

B. The ADM coordinates

Many related numerical relativity studies are in ADM (or ADM-type) coordinates, and hence for future applications we wish to provide the explicit expression for the 3PN energy flux in ADM coordinates. To transform the energy flux, we require the shift or contact transformation of the trajectories connecting the SH coordinates (with log terms) and the ADM coordinates. We recall that the ADM and SH coordinate systems agree at 1PN order inclusively, so that the contact transformation is composed of 2PN and 3PN terms. Hence the calculation is more involved than for the MH coordinates (for which only the modification of the quadrupole moment I_{ij} played a role), and we must come back to the general formulas (6.2) and (6.3). Note that the remainder $\mathcal{O}(\xi^2)$ in Eqs. (6.2) and (6.3) is of order 4PN, which is still negligible in the transformation to ADM-type coordinates.

The relative shift ξ_{12}^i linking SH and ADM coordinates, $x_{\rm ADM}^i = x_{\rm SH}^i + \xi_{12}^i$, is given in [43] as[6]

$$\begin{aligned}\xi_{12}^i = {} & \frac{Gm}{c^4}\left\{\left[\frac{\dot r^2\nu}{8} - \frac{5\nu v^2}{8} + \frac{Gm}{r}\left(-\frac{1}{4} - 3\nu\right)\right]n^i + \frac{9\dot r\nu}{4}v^i\right\}\\ & + \frac{Gm}{c^6}\left\{\left[-\frac{\dot r^4\nu}{16} + \frac{5\dot r^4\nu^2}{16} + \frac{5\dot r^2\nu v^2}{16} - \frac{15\dot r^2\nu^2v^2}{16} - \frac{\nu v^4}{2}\right.\right.\\ & + \frac{11\nu^2v^4}{8} + \frac{Gm}{r}\left(\frac{161\dot r^2\nu}{48} - \frac{5\dot r^2\nu^2}{2} - \frac{451\nu v^2}{48} - \frac{3\nu^2v^2}{8}\right)\\ & \left. + \frac{G^2m^2}{r^2}\left(\frac{2773\nu}{280} + \frac{21\nu\pi^2}{32} - \frac{22\nu}{3}\ln\left(\frac{r}{r_0'}\right)\right)\right]n^i\\ & + \left[-\frac{5\dot r^3\nu}{12} + \frac{29\dot r^3\nu^2}{24} + \frac{17\dot r\nu v^2}{8} - \frac{21\dot r\nu^2v^2}{4}\right.\\ & \left.\left. + \frac{Gm}{r}\left(\frac{43\dot r\nu}{3} + 5\dot r\nu^2\right)\right]v^i\right\},\end{aligned} \tag{6.9}$$

[6]For simplicity we use the same notation ξ_{12}^i as for the shift between the SH and MH coordinates.

from which we deduce, applying Eqs. (6.3), the transformation of variables necessary to compute the ADM energy flux:

$$\begin{aligned}\delta_\xi r &= \frac{Gm}{c^4}\left\{v^2\left(\frac{5}{8}\nu\right) + \dot{r}^2\left(-\frac{19}{8}\nu\right) + \frac{Gm}{r}\left(\frac{1}{4} + 3\nu\right)\right\} + \frac{Gm\nu}{c^6}\left\{v^4\left(\frac{1}{2} - \frac{11}{8}\nu\right) + \dot{r}^2 v^2\left(-\frac{39}{16} + \frac{99}{16}\nu\right) + \dot{r}^4\left(\frac{23}{48} - \frac{73}{48}\nu\right)\right.\\ &\quad \left. + \frac{Gm}{r}v^2\left(\frac{451}{48} + \frac{3}{8}\nu\right) + \frac{Gm}{r}\dot{r}^2\left(-\frac{283}{16} - \frac{5}{2}\nu\right) + \frac{G^2m^2}{r^2}\left(-\frac{2773}{280} + \frac{22}{3}\ln\left(\frac{r}{r_0'}\right) - \frac{21}{32}\pi^2\right)\right\}, \end{aligned} \tag{6.10a}$$

$$\begin{aligned}\delta_\xi \dot{r} &= \frac{Gm}{c^4 r}\dot{r}\left\{v^2\left(-\frac{19}{4}\nu\right) + \dot{r}^2\left(\frac{19}{4}\nu\right) + \frac{Gm}{r}\left(-\frac{1}{4} + \frac{1}{2}\nu\right)\right\} + \frac{Gm\nu}{c^6 r}\dot{r}\left\{v^4\left(-\frac{39}{8} + \frac{99}{8}\nu\right) + \dot{r}^2 v^2\left(\frac{163}{24} - \frac{443}{24}\nu\right)\right.\\ &\quad + \dot{r}^4\left(-\frac{23}{12} + \frac{73}{12}\nu\right) + \frac{Gm}{r}v^2\left(-\frac{1603}{48} - \frac{17}{4}\nu\right) + \frac{Gm}{r}\dot{r}^2\left(\frac{1777}{48} + \frac{131}{24}\nu\right)\\ &\quad \left. + \frac{G^2m^2}{r^2}\left(\frac{3121}{105} - \frac{44}{3}\ln\left(\frac{r}{r_0'}\right) + \frac{21}{16}\pi^2 - \frac{11}{4}\nu\right)\right\}, \end{aligned} \tag{6.10b}$$

$$\begin{aligned}\delta_\xi v^2 &= \frac{Gm}{c^4 r}\left\{v^4\left(-\frac{13}{4}\nu\right) + \dot{r}^2 v^2\left(\frac{5}{2}\nu\right) + \dot{r}^4\left(\frac{3}{4}\nu\right) + \frac{Gm}{r}v^2\left(\frac{1}{2} + \frac{21}{2}\nu\right) + \frac{Gm}{r}\dot{r}^2\left(-1 - \frac{19}{2}\nu\right)\right\}\\ &\quad + \frac{Gm\nu}{c^6 r}\left\{v^6\left(-\frac{13}{4} + \frac{31}{4}\nu\right) + \dot{r}^2 v^4\left(\frac{31}{8} - \frac{75}{8}\nu\right) + \dot{r}^4 v^2\left(-\frac{3}{2}\nu\right) + \dot{r}^6\left(-\frac{5}{8} + \frac{25}{8}\nu\right) + \frac{Gm}{r}v^4\left(-\frac{9}{8} - \frac{25}{4}\nu\right)\right.\\ &\quad + \frac{Gm}{r}\dot{r}^2 v^2\left(-\frac{131}{8} + \frac{121}{4}\nu\right) + \frac{Gm}{r}\dot{r}^4\left(\frac{99}{4} - \frac{259}{12}\nu\right) + \frac{G^2m^2}{r^2}v^2\left(-\frac{3839}{420} + \frac{44}{3}\ln\left(\frac{r}{r_0'}\right) - \frac{21}{16}\pi^2 + \nu\right)\\ &\quad \left. + \frac{G^2m^2}{r^2}\dot{r}^2\left(\frac{28\,807}{420} - 44\ln\left(\frac{r}{r_0'}\right) + \frac{63}{16}\pi^2 - \frac{13}{2}\nu\right)\right\}. \end{aligned} \tag{6.10c}$$

The above equations provide the 3PN generalization of Eq. (4.6) of [19]. They also incorporate the corrected transformation between ADM and harmonic coordinates at 2PN, as given in [25].

Using the latter expressions, one finds that the SH energy flux is changed by corrections at 2PN and 3PN relative orders given by (using a notation similar to that introduced above—i.e. in which the variables r, $\dot{r}^2$, and v^2 are considered as dummy variables[7])

$$\begin{aligned}\delta_\xi \mathcal{F} &= -\frac{G^4m^5\nu^2}{c^9 r^5}\left\{\frac{184}{5}v^4\nu - \frac{736}{5}\dot{r}^2 v^2\nu + \frac{Gm}{r}v^2\left(\frac{16}{5} + \frac{48}{5}\nu\right) + \frac{320}{3}\dot{r}^4\nu + \frac{Gm}{r}\dot{r}^2\left(-\frac{12}{5} - \frac{56}{15}\nu\right)\right\}\\ &\quad - \frac{G^4m^5\nu^2}{c^{11}r^5}\left\{v^6\left(\frac{5886}{35}\nu - \frac{1616}{7}\nu^2\right) + \dot{r}^2 v^4\left(-\frac{129\,866}{105}\nu + \frac{21\,598}{15}\nu^2\right) + \frac{Gm}{r}v^4\left(-\frac{22\,798}{105}\nu + \frac{7528}{35}\nu^2\right)\right.\\ &\quad + \dot{r}^4 v^2\left(\frac{689\,434}{315}\nu - \frac{714\,608}{315}\nu^2\right) + \frac{Gm}{r}\dot{r}^2 v^2\left(-\frac{936}{35} + \frac{16\,103}{21}\nu - \frac{14\,086}{21}\nu^2\right) + \frac{G^2m^2}{r^2}v^2\left(-\frac{272}{7} + \left[-\frac{31\,856}{75}\right.\right.\\ &\quad \left.\left. - \frac{42}{5}\pi^2\right]\nu + \frac{96}{35}\nu^2\right) + \dot{r}^6\left(-\frac{116\,138}{105}\nu + \frac{110\,986}{105}\nu^2\right) + \frac{Gm}{r}\dot{r}^4\left(\frac{328}{15} - \frac{198\,097}{315}\nu + \frac{143\,924}{315}\nu^2\right)\\ &\quad + \frac{G^2m^2}{r^2}\dot{r}^2\left(\frac{1612}{35} + \left(\frac{673\,544}{1575} + \frac{28}{5}\pi^2\right)\nu + \frac{828}{35}\nu^2\right) + \frac{G^3m^3}{r^3}\left(\frac{16}{35} + \frac{128}{35}\nu - \frac{768}{35}\nu^2\right)\\ &\quad \left. + \frac{G^2m^2}{r^2}\nu\left(\frac{1408}{15}v^2 - \frac{2816}{45}\dot{r}^2\right)\ln\left(\frac{r}{r_0'}\right)\right\}. \end{aligned} \tag{6.11}$$

The examination of the coefficient of $\ln(r/r_0')$, given by the last two terms of (6.11), reveals that this coefficient is the same as in the contact transformation from SH to MH, given by (6.8). Therefore, the contact transformation from SH to ADM exactly cancels out the logarithms of SH coordinates, and the final flux in ADM coordinates is free of $\ln r_0'$. This is consistent with the general understanding that the $\ln r_0'$ is a feature of a particular harmonic coordinate system and that ADM coordinates do not yield complications associated with such logarithms (the cancellation of the $\ln r_0'$ terms usually provides a useful internal check of the computations).

[7] $\mathcal{F}_{\rm ADM} = \mathcal{F}_{\rm SH} + \delta_\xi \mathcal{F}$.

VII. THE GENERALIZED QUASI-KEPLERIAN REPRESENTATION

Before we discuss the calculation of the orbital average of the energy flux in Sec. VIII, we must summarize the 3PN generalized quasi-Keplerian representation of the binary motion recently obtained by Memmesheimer, Gopakumar, and Schäfer [37]. Indeed, the main application of the present computation is the evolution of the orbital elements under GW radiation reaction to 3PN order. This requires one to average over an orbit the instantaneous expressions for the energy flux obtained in Sec. V. Averaging over an orbit is most conveniently accomplished by the use of an explicit solution of the equations of motion. The generalized QK representation of the motion at 3PN order [37] constitutes an essential input for the computations to follow.

The QK representation was introduced by Damour and Deruelle [18] to discuss the problem of binary pulsar timing at 1PN order, where relativistic periastron precession first appears and complicates the simpler Keplerian picture. This elegant formulation also played an important role in our computation of the hereditary terms in Paper I, where we provided a summary of it. The 2PN extension of this work in the ADM coordinates was next given in Refs. [21–23], and we shall now use the 3PN parametrization in ADM and MH coordinates [37].

The radial motion is given in parametric form as[8]

$$r = a_r(1 - e_r \cos u), \tag{7.1a}$$

$$\ell = u - e_t \sin u + f_t \sin V + g_t(V - u) + i_t \sin 2V + h_t \sin 3V, \tag{7.1b}$$

while the corresponding angular motion is

$$\frac{\phi - \phi_{\mathrm{P}}}{K} = V + f_\phi \sin 2V + g_\phi \sin 3V + i_\phi \sin 4V + h_\phi \sin 5V. \tag{7.2}$$

The four angles V, u, ℓ, and ϕ are, respectively, the true anomaly, the eccentric anomaly, the mean anomaly, and the orbital phase (V, u, and ℓ are measured from the periastron, and we denote by ϕ_{P} the value of ϕ at periastron). The mean anomaly is proportional to the time elapsed since the instant t_{P} of passage at periastron,

$$\ell = n(t - t_{\mathrm{P}}), \tag{7.3}$$

where $n = 2\pi/P$ is the mean motion and P is the orbital period. The true anomaly V is given by

$$V = 2\arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2} \tan\frac{u}{2}\right]. \tag{7.4}$$

In the above, a_r represents the semimajor axis of the orbit, and e_r, e_t, e_ϕ are three kinds of eccentricities, labeled after the coordinates t, r, and ϕ, and which differ from each other starting at 1PN order. The constant K is linked with the advance of periastron per orbital revolution, and is given by $K = \Phi/(2\pi)$ where Φ is the angle of return to the periastron. The notation $k \equiv K - 1$ for the relativistic precession is used in Paper I and will also be useful here. The orbital elements $f_t, f_\phi, g_t, g_\phi, \ldots$ parametrize the 2PN and 3PN relativistic corrections, as will be clear from their expressions below. (More precisely, f_t, f_ϕ, g_t, g_ϕ are composed of 2PN and 3PN terms, but i_t, i_ϕ, h_t, h_ϕ start only at 3PN order.)

Crucial to the formalism are the explicit formulas for all the orbital elements and all the coefficients in Eqs. (7.1) above in terms of the 3PN conserved orbital energy E and angular momentum J (divided by the binary's reduced mass). Recall that the construction of a generalized quasi-Keplerian representation exploits the fact that the radial equation—which is given by Eq. (2.1a) in Paper I—is a *polynomial* in $1/r$ (of seventh degree at 3PN order). Therefore the presence of logarithmic terms in the SH coordinates at 3PN order obstructs the construction of the QK parametrization (at least by this method), and Ref. [37] obtained it in coordinates avoiding such logarithms, namely, the MH and ADM coordinates. In both ADM and MH coordinates the QK representation takes the same form given by Eqs. (7.1) and (7.2), but of course the equations linking the orbital elements to E and J are different. These have been obtained as post-Newtonian series up to 3PN order in Ref. [37]. Since they form the basis for our computation of the average energy flux, we provide the complete relations here.

For convenience, in the present paper we introduce a PN parameter which is directly linked to the energy E and defined by

$$\varepsilon \equiv -\frac{2E}{c^2}. \tag{7.5}$$

(Recall that $E < 0$ for gravitationally bound orbits.) The equations to follow will then appear as PN expansions in terms of $\varepsilon = \mathcal{O}(2)$. Also, we find it useful to define, in place of the angular momentum J,

$$j \equiv -\frac{2EJ^2}{(Gm)^2}. \tag{7.6}$$

We have $j = -2Eh^2$ in terms of the more usual definition $h \equiv J/(Gm)$. This parameter is at Newtonian order, $j = c^2\varepsilon h^2 = \mathcal{O}(0)$. The point is now to give all the orbital elements as PN series in powers of ε with coefficients depending on j (and the dimensionless, reduced mass ratio ν). In ADM coordinates these are given by [37]

[8]For convenience, in this paper we adapt somewhat the notation with respect to the one in Ref. [37].

$$n^{\mathrm{ADM}} = \frac{\varepsilon^{3/2}c^3}{Gm}\bigg\{1 + \frac{\varepsilon}{8}(-15+\nu) + \frac{\varepsilon^2}{128}\bigg[555 + 30\nu + 11\nu^2 + \frac{192}{j^{1/2}}(-5+2\nu)\bigg] + \frac{\varepsilon^3}{3072}\bigg[-29\,385 - 4995\nu - 315\nu^2 + 135\nu^3 - \frac{16}{j^{3/2}}(10\,080 + 123\nu\pi^2 - 13\,952\nu + 1440\nu^2) + \frac{5760}{j^{1/2}}(17 - 9\nu + 2\nu^2)\bigg]\bigg\}, \tag{7.7a}$$

$$K^{\mathrm{ADM}} = 1 + \frac{3\varepsilon}{j} + \frac{\varepsilon^2}{4}\bigg[\frac{3}{j}(-5+2\nu) + \frac{15}{j^2}(7-2\nu)\bigg] + \frac{\varepsilon^3}{128}\bigg[\frac{24}{j}(5 - 5\nu + 4\nu^2) - \frac{1}{j^2}(10\,080 - 13\,952\nu + 123\nu\pi^2 + 1440\nu^2) + \frac{5}{j^3}(7392 - 8000\nu + 123\nu\pi^2 + 336\nu^2)\bigg], \tag{7.7b}$$

$$a_r^{\mathrm{ADM}} = \frac{Gm}{\varepsilon c^2}\bigg\{1 + \frac{\varepsilon}{4}(-7+\nu) + \frac{\varepsilon^2}{16}\bigg[1 + 10\nu + \nu^2 + \frac{1}{j}(-68 + 44\nu)\bigg] + \frac{\varepsilon^3}{192}\bigg[3 - 9\nu - 6\nu^2 + 3\nu^3 + \frac{1}{j}(864 + (-3\pi^2 - 2212)\nu + 432\nu^2) + \frac{1}{j^2}(-6432 + (13\,488 - 240\pi^2)\nu - 768\nu^2)\bigg]\bigg\}, \tag{7.7c}$$

$$e_r^{\mathrm{ADM}} = \bigg[1 - j + \frac{\varepsilon}{4}\bigg\{24 - 4\nu + 5j(-3+\nu)\bigg\} + \frac{\varepsilon^2}{8}\bigg\{52 + 2\nu + 2\nu^2 - j(80 - 55\nu + 4\nu^2) - \frac{8}{j}(-17 + 11\nu)\bigg\} + \frac{\varepsilon^3}{192}\bigg\{-768 - 6\nu\pi^2 - 344\nu - 216\nu^2 + 3j(-1488 + 1556\nu - 319\nu^2 + 4\nu^3) - \frac{4}{j}(588 - 8212\nu + 177\nu\pi^2 + 480\nu^2) + \frac{192}{j^2}(134 - 281\nu + 5\nu\pi^2 + 16\nu^2)\bigg\}\bigg]^{1/2}, \tag{7.7d}$$

$$e_t^{\mathrm{ADM}} = \bigg[1 - j + \frac{\varepsilon}{4}\{-8 + 8\nu - j(-17 + 7\nu)\} + \frac{\varepsilon^2}{8}\bigg\{8 + 4\nu + 20\nu^2 - j(112 - 47\nu + 16\nu^2) - 24j^{1/2}(-5+2\nu) + \frac{4}{j} \times (17 - 11\nu) - \frac{24}{j^{1/2}}(5 - 2\nu)\bigg\} + \frac{\varepsilon^3}{192}\bigg\{24(-2+5\nu)(-23 + 10\nu + 4\nu^2) - 15j(-528 + 200\nu - 77\nu^2 + 24\nu^3) - 72j^{1/2}(265 - 193\nu + 46\nu^2) - \frac{2}{j}(6732 + 117\nu\pi^2 - 12\,508\nu + 2004\nu^2) + \frac{2}{j^{1/2}}(16\,380 - 19\,964\nu + 123\nu\pi^2 + 3240\nu^2) - \frac{2}{j^{3/2}}(10\,080 + 123\nu\pi^2 - 13\,952\nu + 1440\nu^2) + \frac{96}{j^2}(134 - 281\nu + 5\nu\pi^2 + 16\nu^2)\bigg\}\bigg]^{1/2}, \tag{7.7e}$$

$$e_\phi^{\mathrm{ADM}} = \bigg[1 - j + \frac{\varepsilon}{4}\{24 + j(-15+\nu)\} + \frac{\varepsilon^2}{16}\bigg\{-32 + 176\nu + 18\nu^2 - j(160 - 30\nu + 3\nu^2) + \frac{1}{j}(408 - 232\nu - 15\nu^2)\bigg\} + \frac{\varepsilon^3}{384}\bigg\{-16\,032 + 2764\nu + 3\nu\pi^2 + 4536\nu^2 + 234\nu^3 - 36j(248 - 80\nu + 13\nu^2 + \nu^3) - \frac{6}{j}(2456 - 26\,860\nu + 581\nu\pi^2 + 2689\nu^2 + 10\nu^3) + \frac{3}{j^2}(27\,776 - 65436\nu + 1325\nu\pi^2 + 3440\nu^2 - 70\nu^3)\bigg\}\bigg]^{1/2}, \tag{7.7f}$$

$$f_t^{\mathrm{ADM}} = -\frac{\varepsilon^2}{8j^{1/2}}\{(4+\nu)\nu\sqrt{1-j}\} + \frac{\varepsilon^3}{192}\bigg\{\frac{1}{j^{3/2}\sqrt{1-j}}(1728 - 4148\nu + 3\nu\pi^2 + 600\nu^2 + 33\nu^3) + 3\frac{j^{1/2}}{\sqrt{1-j}}\nu(-64 - 4\nu + 23\nu^2) + \frac{1}{\sqrt{j(1-j)}}(-1728 + 4232\nu - 3\nu\pi^2 - 627\nu^2 - 105\nu^3)\bigg\}, \tag{7.7g}$$

$$g_t^{\mathrm{ADM}} = \frac{3\varepsilon^2}{2}\bigg(\frac{5-2\nu}{j^{1/2}}\bigg) + \frac{\varepsilon^3}{192}\bigg\{\frac{1}{j^{3/2}}(10\,080 + 123\nu\pi^2 - 13\,952\nu + 1440\nu^2) + \frac{1}{j^{1/2}}(-3420 + 1980\nu - 648\nu^2)\bigg\}, \tag{7.7h}$$

$$i_t^{\mathrm{ADM}} = \frac{\varepsilon^3}{32}\nu\frac{1-j}{j^{3/2}}(23 + 12\nu + 6\nu^2), \tag{7.7i}$$

$$h_t^{\mathrm{ADM}} = \frac{13\varepsilon^3}{192}\nu^3\bigg(\frac{1-j}{j}\bigg)^{3/2}, \tag{7.7j}$$

$$f_\phi^{\mathrm{ADM}} = \frac{\varepsilon^2}{8}\frac{1-j}{j^2}\nu(1-3\nu) + \frac{\varepsilon^3}{256}\bigg\{\frac{4\nu}{j}(-11 - 40\nu + 24\nu^2) + \frac{1}{j^2}(-256 + 1192\nu - 49\nu\pi^2 + 336\nu^2 - 80\nu^3) + \frac{1}{j^3}(256 + 49\nu\pi^2 - 1076\nu - 384\nu^2 - 40\nu^3)\bigg\}, \tag{7.7k}$$

$$g_\phi^{\rm ADM} = -\frac{3\varepsilon^2}{32}\frac{\nu^2}{j^2}(1-j)^{3/2} + \frac{\varepsilon^3}{768}\sqrt{1-j}\bigg\{-\frac{3}{j}\nu^2(9-26\nu) - \frac{1}{j^2}\nu(220+3\pi^2+312\nu+150\nu^2) + \frac{1}{j^3}\nu(220+3\pi^2+96\nu+45\nu^2)\bigg\}, \quad (7.7l)$$

$$i_\phi^{\rm ADM} = \frac{\varepsilon^3}{128}\frac{(1-j)^2}{j^3}\nu(5+28\nu+10\nu^2), \quad (7.7m)$$

$$h_\phi^{\rm ADM} = \frac{5\varepsilon^3}{256}\frac{\nu^3}{j^3}(1-j)^{5/2}. \quad (7.7n)$$

The latter expressions are specific to the ADM coordinates, and we now want to give the corresponding expressions in MH coordinates. However, we recall first an important point related to the use of gauge-invariant variables in the elliptical orbit case as stressed by Ref. [37]. Indeed, Damour and Schäfer [21] showed that the functional forms of n and $K = \Phi/(2\pi)$ as functions of gauge-invariant variables like ε and j are identical in different coordinate systems. Hence the expressions in MH coordinates of these two parameters are the same as in ADM coordinates,

$$n \equiv n^{\rm MH} = n^{\rm ADM}, \quad (7.8a)$$

$$K \equiv K^{\rm MH} = K^{\rm ADM}. \quad (7.8b)$$

This prompted Ref. [37] to suggest the use of n and $k = K - 1$ as two gauge-invariant variables in the general orbit case.[9] In the present work we propose to use a variant of the former variables. Namely, instead of working with the mean motion n, we shall systematically use the orbital frequency $\omega = Kn$ as defined in a general context in Sec. II A of Paper I, and define as a gauge-invariant post-Newtonian parameter

$$x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}. \quad (7.9)$$

This choice constitutes the obvious generalization of the gauge-invariant variable x used in the circular-orbit case and will thus facilitate the straightforward reading out and check of the circular-orbit limit. The parameter x is related to the energy and angular momentum variables ε and j up to 3PN order by

$$x = \varepsilon\bigg\{1 + \varepsilon\left(-\frac{5}{4} + \frac{1}{12}\nu + 2\frac{1}{j}\right) + \varepsilon^2\left(\frac{5}{2} + \frac{5}{24}\nu + \frac{1}{18}\nu^2 + \frac{1}{j^{1/2}}(-5+2\nu) + \frac{1}{j}\left(-5 + \frac{7}{6}\nu\right) + \frac{1}{j^2}\left(\frac{33}{2} - 5\nu\right)\right) + \varepsilon^3\bigg(-\frac{235}{48} - \frac{25}{24}\nu - \frac{25}{576}\nu^2 + \frac{35}{1296}\nu^3 + \frac{1}{j}\left(\frac{35}{4} - \frac{5}{3}\nu + \frac{25}{36}\nu^2\right) + \frac{1}{j^{1/2}}\left(\frac{145}{8} - \frac{235}{24}\nu + \frac{29}{12}\nu^2\right) + \frac{1}{j^{3/2}}\left(-45 + \left(\frac{472}{9} - \frac{41}{96}\pi^2\right)\nu - 5\nu^2\right) + \frac{1}{j^2}\left(-\frac{565}{8} + \left(\frac{1903}{24} - \frac{41}{64}\pi^2\right)\nu - \frac{95}{12}\nu^2\right) + \frac{1}{j^3}\left(\frac{529}{3} + \left(-\frac{610}{3} + \frac{205}{64}\pi^2\right)\nu + \frac{35}{4}\nu^2\right)\bigg)\bigg\}. \quad (7.10)$$

The other orbital elements are not gauge invariant, and therefore their expressions in MH coordinates differ at 2PN and 3PN orders from those in ADM coordinates. We conclude by giving here all the needed differences [37],

$$a_r^{\rm MH} - a_r^{\rm ADM} = Gm\varepsilon\left(-\frac{5}{8}\nu + \frac{1}{j}\left(\frac{1}{4} + \frac{17}{4}\nu\right)\right) + Gm\varepsilon^2\bigg(\frac{1}{32}\nu + \frac{1}{32}\nu^2 + \frac{1}{j}\left(-\frac{1}{2} + \left(-\frac{11\,499}{560} + \frac{21}{32}\pi^2\right)\nu + \frac{19}{4}\nu^2\right) + \frac{1}{j^2}\left(\frac{3}{2} + \left(\frac{14\,501}{420} - \frac{21}{16}\pi^2\right)\nu - 5\nu^2\right)\bigg), \quad (7.11a)$$

$$e_r^{\rm MH} - e_r^{\rm ADM} = \frac{\varepsilon^2}{\sqrt{1-j}}\left(\frac{1}{2} + \frac{73}{8}\nu - j\frac{5}{8}\nu + \frac{1}{j}\left(-\frac{1}{2} - \frac{17}{2}\nu\right)\right) + \frac{\varepsilon^3}{\sqrt{1-j}}\bigg(\frac{13}{16} + \left(-\frac{5237}{1680} + \frac{21}{32}\pi^2\right)\nu + \frac{19}{16}\nu^2 + j\left(-\frac{143}{64}\nu + \frac{37}{64}\nu^2\right) + \frac{1}{j}\left(\frac{13}{8} + \left(\frac{3667}{56} - \frac{105}{32}\pi^2\right)\nu - \frac{51}{4}\nu^2\right) + \frac{1}{j^2}\left(-3 + \left(-\frac{14\,501}{210} + \frac{21}{8}\pi^2\right)\nu + 10\nu^2\right)\bigg), \quad (7.11b)$$

[9] Actually, Ref. [37] used $x_{\rm MGS} = (Gmn/c^3)^{2/3}$ together with $k' = k/3$.

$$e_t^{\mathrm{MH}} - e_t^{\mathrm{ADM}} = \frac{\varepsilon^2}{\sqrt{1-j}}\left(\frac{1}{4}+\frac{17}{4}\nu\right)\left(1-\frac{1}{j}\right) + \frac{\varepsilon^3}{\sqrt{1-j}}\left(-\frac{19}{32}-\frac{52}{3}\nu+\frac{225}{32}\nu^2+\frac{1}{j}\left(\frac{29}{16}+\left(\frac{79\,039}{1680}-\frac{21}{16}\pi^2\right)\nu-\frac{201}{16}\nu^2\right)\right.$$
$$\left. + \frac{1}{j^2}\left(-\frac{3}{2}+\left(-\frac{14\,501}{420}+\frac{21}{16}\pi^2\right)\nu+5\nu^2\right)\right), \tag{7.11c}$$

$$e_\phi^{\mathrm{MH}} - e_\phi^{\mathrm{ADM}} = \frac{\varepsilon^2}{\sqrt{1-j}}\left(-\frac{1}{4}-\frac{71}{16}\nu+j\frac{1}{32}\nu+\frac{1}{j}\left(\frac{1}{4}+\frac{141}{32}\nu\right)\right) + \frac{\varepsilon^3}{\sqrt{1-j}}\left(-\frac{13}{32}+\left(\frac{36\,511}{8960}-\frac{21}{128}\pi^2\right)\nu\right.$$
$$-\frac{1723}{256}\nu^2+j\left(\frac{17}{256}\nu+\frac{33}{256}\nu^2\right)+\frac{1}{j}\left(-\frac{13}{16}+\left(-\frac{21\,817}{480}+\frac{147}{64}\pi^2\right)\nu+\frac{169}{8}\nu^2\right)$$
$$\left.+\frac{1}{j^2}\left(\frac{3}{2}+\left(\frac{621\,787}{13\,440}-\frac{273}{128}\pi^2\right)\nu-\frac{1789}{128}\nu^2\right)\right), \tag{7.11d}$$

$$f_t^{\mathrm{MH}} - f_t^{\mathrm{ADM}} = \varepsilon^2\frac{19}{8}\frac{\sqrt{(1-j)}}{\sqrt{j}}\nu + \frac{\varepsilon^3}{\sqrt{j(1-j)}}\left(-1+\left(-\frac{296\,083}{6720}+\frac{21}{32}\pi^2\right)\nu+\frac{989}{64}\nu^2+j\left(\frac{361}{64}\nu-\frac{171}{64}\nu^2\right)\right.$$
$$\left.+\frac{1}{j}\left(1+\left(\frac{276\,133}{6720}-\frac{21}{32}\pi^2\right)\nu-\frac{799}{64}\nu^2\right)\right), \tag{7.11e}$$

$$g_t^{\mathrm{MH}} - g_t^{\mathrm{ADM}} = 0, \tag{7.11f}$$

$$h_t^{\mathrm{MH}} - h_t^{\mathrm{ADM}} = -\frac{\varepsilon^3}{192}(1-j)^{3/2}j^{-(3/2)}\nu(-23+73\nu), \tag{7.11g}$$

$$i_t^{\mathrm{MH}} - i_t^{\mathrm{ADM}} = -\frac{11}{32}\varepsilon^3(1-j)j^{-(3/2)}\nu(-19+10\nu), \tag{7.11h}$$

$$f_\phi^{\mathrm{MH}} - f_\phi^{\mathrm{ADM}} = -\varepsilon^2\left(\frac{1}{j}-\frac{1}{j^2}\right)\left(\frac{1}{8}+\frac{9}{4}\nu\right)+\frac{\varepsilon^3}{j}\left(\frac{1}{32}+\frac{1045}{192}\nu-\frac{99}{32}\nu^2+\frac{1}{j}\left(-\frac{5}{4}+\left(-\frac{139\,633}{3360}+\frac{21}{16}\pi^2\right)\nu+\frac{117}{8}\nu^2\right)\right.$$
$$\left.+\frac{1}{j^2}\left(\frac{3}{2}+\left(\frac{92\,307}{2240}-\frac{21}{16}\pi^2\right)\nu-\frac{351}{32}\nu^2\right)\right), \tag{7.11i}$$

$$g_\phi^{\mathrm{MH}} - g_\phi^{\mathrm{ADM}} = \varepsilon^2\frac{1}{32}\frac{(1-j)^{3/2}}{j^2}\nu+\varepsilon^3\sqrt{1-j}\left(\frac{1}{j}\left(\frac{7}{128}\nu-\frac{5}{32}\nu^2\right)+\frac{1}{j^2}\left(\left(-\frac{49\,709}{13\,440}+\frac{21}{128}\pi^2\right)\nu+\frac{445}{128}\nu^2\right)\right.$$
$$\left.+\frac{1}{j^3}\left(\left(\frac{100\,783}{26\,880}-\frac{21}{128}\pi^2\right)\nu-\frac{847}{256}\nu^2\right)\right), \tag{7.11j}$$

$$h_\phi^{\mathrm{MH}} - h_\phi^{\mathrm{ADM}} = -\frac{\varepsilon^3}{256}(1-j)^{5/2}j^{-3}\nu(-1+5\nu), \tag{7.11k}$$

$$i_\phi^{\mathrm{MH}} - i_\phi^{\mathrm{ADM}} = -\frac{\varepsilon^3}{384}(-1+j)^2j^{-3}\nu(-149+198\nu). \tag{7.11l}$$

Finally, we note that in the case of a circular-orbit the angular momentum variable, say $j_\odot$, is related to the constant of energy ε by the 3PN gauge-invariant expansion

$$j_\odot = 1+\frac{\varepsilon}{4}(9+\nu)+\frac{\varepsilon^2}{16}(81-32\nu+\nu^2)+\frac{\varepsilon^3}{192}(2835 - 7699\nu+246\nu\pi^2+96\nu^2+3\nu^3), \tag{7.12}$$

which is easily deduced using either MH or ADM coordinates. This expression can be used to compute all the orbital elements for circular orbits, and we can check that all of the eccentricities e_r, e_t, or e_ϕ are zero.

VIII. ORBITAL AVERAGE OF THE 3PN ENERGY FLUX

To average the energy flux over an orbit, we will require the use of the previous 3PN quasi-Keplerian representation of the motion. Consequently, the averaging is only possible in MH or ADM coordinates without the logarithms as discussed before. The average of the (instantaneous part of the) energy flux is defined by

$$\langle\mathcal{F}_{\mathrm{inst}}\rangle = \frac{1}{P}\int_0^P dt\mathcal{F}_{\mathrm{inst}} = \frac{1}{2\pi}\int_0^{2\pi} du\frac{d\ell}{du}\mathcal{F}_{\mathrm{inst}}. \tag{8.1}$$

As we have seen, the energy flux (2.6) is made of instantaneous terms and hereditary (tail) terms. The hereditary terms have already been computed and averaged in Paper I.

Using the QK representation of the orbit discussed in Sec. VII, we can reexpress the energy flux $\mathcal{F}_{\mathrm{inst}}$ [or, more exactly, $(d\ell/du)\mathcal{F}_{\mathrm{inst}}$], which is a function of its natural variables r, $\dot{r}$, and v^2, as a function of the frequency-related parameter x defined by Eqs. (7.9) and (7.10), the "time" eccentricity e_t, and the eccentric anomaly u.[10] We note that in the expression of the energy flux at 3PN order there are some logarithmic terms of the type $\ln(r/r_0)$ even in MH coordinates. Indeed, we recall that the MH coordinates permit the removal of the log terms $\ln(r/r_0')$, where r_0' is the scale associated with Hadamard's self-field regularization, but there are still the terms $\ln(r/r_0)$ which involve the constant r_0 entering the definition of the multipole moments for general sources. As a result, we find that the general structure of $\mathcal{F}_{\rm inst}$ (in MH or ADM coordinates) is

$$\frac{d\ell}{du}\mathcal{F}_{\rm inst} = \sum_{N=3}^{11}\bigg\{\alpha_N(e_t)\frac{1}{(1-e_t\cos u)^N} + \beta_N(e_t) \times \frac{\sin u}{(1-e_t\cos u)^N} + \gamma_N(e_t)\frac{\ln(1-e_t\cos u)}{(1-e_t\cos u)^N}\bigg\}, \tag{8.2}$$

where the coefficients α_N, β_N, γ_N so defined are straightforwardly computed using the QK parametrization (they are too long to be listed here). It is worth noting that the β_N's correspond to all the 2.5PN terms while the γ_N's represent the logarithmic terms at order 3PN. The dependence on the constant $\ln r_0$ has been included into the coefficients α_N. To compute the average, we have at our disposal some integration formulas. First of all,

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin u}{(1-e\cos u)^N}du = 0, \tag{8.3}$$

which shows that in the final result there will be no terms (of the instantaneous type) at 2.5PN order. The 2.5PN instantaneous contribution is proportional to $\dot{r}$ and vanishes after averaging since it includes only odd functions of u. Next, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{du}{(1-e\cos u)^N} = \frac{(-)^{N-1}}{(N-1)!}\left(\frac{d^{N-1}}{dy^{N-1}} \times \left[\frac{1}{\sqrt{y^2-e^2}}\right]\right)_{y=1}, \tag{8.4}$$

which can also be formulated with the help of the standard Legendre polynomial P_{N-1} as

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{du}{(1-e\cos u)^N} = \frac{1}{(1-e^2)^{N/2}}P_{N-1}\left(\frac{1}{\sqrt{1-e^2}}\right). \tag{8.5}$$

Finally, for the log terms we have a less trivial formula but which takes a structure similar as in Eq. (8.4), namely

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\ln(1-e\cos u)}{(1-e\cos u)^N}du = \frac{(-)^{N-1}}{(N-1)!}\left(\frac{d^{N-1}Y(y,e)}{dy^{N-1}}\right)_{y=1}, \tag{8.6}$$

in which

$$Y(y,e) = \frac{1}{\sqrt{y^2-e^2}}\bigg\{\ln\left[\frac{\sqrt{1-e^2}+1}{2}\right] + 2\ln\left[1+\frac{\sqrt{1-e^2}-1}{y+\sqrt{y^2-e^2}}\right]\bigg\}. \tag{8.7}$$

A. Orbital average in MH coordinates

The expression for the instantaneous energy flux in MH coordinates is given by Eqs. (5.1) and (5.2) together with the modification (6.8) for transforming to MH coordinates. Implementing all the above integrations, the flux can be averaged over an orbit to order 3PN, extending the results of [19] at 2PN.[11] The result is presented in the form

$$\langle\mathcal{F}_{\rm inst}\rangle = \frac{32c^5}{5G}\nu^2x^5(\mathcal{I}_{\rm N}^{\rm MH} + x\,\mathcal{I}_{\rm 1PN}^{\rm MH} + x^2\,\mathcal{I}_{\rm 2PN}^{\rm MH} + x^3\,\mathcal{I}_{\rm 3PN}^{\rm MH}), \tag{8.8}$$

where the instantaneous post-Newtonian pieces $\mathcal{I}_{n\rm PN}^{\rm MH}$ depend on ν and the time eccentricity e_t in MH coordinates (note that $e_t \equiv e_t^{\rm MH}$ here), and read[12]

$$\mathcal{I}_{\rm N}^{\rm MH} = \frac{1}{(1-e_t^2)^{7/2}}\bigg\{1+\frac{73}{24}e_t^2+\frac{37}{96}e_t^4\bigg\}, \tag{8.9a}$$

$$\mathcal{I}_{\rm 1PN}^{\rm MH} = \frac{1}{(1-e_t^2)^{9/2}}\bigg\{-\frac{1247}{336}-\frac{35}{12}\nu + e_t^2\left(\frac{10\,475}{672}-\frac{1081}{36}\nu\right) + e_t^4\left(\frac{10\,043}{384}-\frac{311}{12}\nu\right) + e_t^6\left(\frac{2179}{1792}-\frac{851}{576}\nu\right)\bigg\}, \tag{8.9b}$$

[10]Reference [19] uses Gm/a_r and e_r while [25] employs Gmn/c^3 and e_t. We propose the use of $x=(Gm\omega/c^3)^{2/3}$ for reasons outlined in the previous section. The choice of e_t rather than, say, e_r is a matter of convenience since it appears in the Kepler equation which is directly dealt with when averaging over an orbit.

[11]Results of [19] are given in ADM coordinates.

[12]The Newtonian coefficient $\mathcal{I}_{\rm N}^{\rm MH}$ is nothing but the Peters & Mathews [1] "enhancement" function of eccentricity $f(e_t) \equiv (1+\frac{73}{24}e_t^2+\frac{37}{96}e_t^4)/(1-e_t^2)^{7/2}$, so called because it enhances the numerical value of the orbital decay of the binary pulsar by gravitational radiation (viz. the orbital $\dot{P}$).

$$\mathcal{I}_{\rm 2PN}^{\rm MH} = \frac{1}{(1-e_t^2)^{11/2}}\bigg\{-\frac{203\,471}{9072} + \frac{12\,799}{504}\nu + \frac{65}{18}\nu^2 + e_t^2\bigg(-\frac{3\,807\,197}{18\,144} + \frac{116\,789}{2016}\nu + \frac{5935}{54}\nu^2\bigg) + e_t^4\bigg(-\frac{268\,447}{24\,192} - \frac{2\,465\,027}{8064}\nu + \frac{247\,805}{864}\nu^2\bigg) + e_t^6\bigg(\frac{1\,307\,105}{16\,128} - \frac{416\,945}{2688}\nu + \frac{185\,305}{1728}\nu^2\bigg) + e_t^8\bigg(\frac{86\,567}{64\,512} - \frac{9769}{4608}\nu + \frac{21\,275}{6912}\nu^2\bigg) + \sqrt{1-e_t^2}\bigg[\frac{35}{2} - 7\nu + e_t^2\bigg(\frac{6425}{48} - \frac{1285}{24}\nu\bigg) + e_t^4\bigg(\frac{5065}{64} - \frac{1013}{32}\nu\bigg) + e_t^6\bigg(\frac{185}{96} - \frac{37}{48}\nu\bigg)\bigg]\bigg\}, \tag{8.9c}$$

$$\mathcal{I}_{\rm 3PN}^{\rm MH} = \frac{1}{(1-e_t^2)^{13/2}}\bigg\{\frac{2\,193\,295\,679}{9\,979\,200} + \bigg[\frac{8\,009\,293}{54\,432} - \frac{41\pi^2}{64}\bigg]\nu - \frac{209\,063}{3024}\nu^2 - \frac{775}{324}\nu^3 + e_t^2\bigg(\frac{20\,506\,331\,429}{19\,958\,400} + \bigg[\frac{649\,801\,883}{272\,160} + \frac{4879\pi^2}{1536}\bigg]\nu - \frac{3\,008\,759}{3024}\nu^2 - \frac{53\,696}{243}\nu^3\bigg) + e_t^4\bigg(-\frac{3\,611\,354\,071}{13\,305\,600} + \bigg[\frac{755\,536\,297}{136\,080} - \frac{29\,971\pi^2}{1024}\bigg]\nu - \frac{179\,375}{576}\nu^2 - \frac{10\,816\,087}{7776}\nu^3\bigg) + e_t^6\bigg(\frac{4\,786\,812\,253}{26\,611\,200} + \bigg[\frac{1\,108\,811\,471}{1\,451\,520} - \frac{84\,501\pi^2}{4096}\bigg]\nu + \frac{87\,787\,969}{48\,384}\nu^2 - \frac{983\,251}{648}\nu^3\bigg) + e_t^8\bigg(\frac{21\,505\,140\,101}{141\,926\,400} + \bigg[-\frac{32\,467\,919}{129\,024} - \frac{4059\pi^2}{4096}\bigg]\nu + \frac{79\,938\,097}{193\,536}\nu^2 - \frac{4\,586\,539}{15\,552}\nu^3\bigg) + e_t^{10}\bigg(-\frac{8\,977\,637}{11\,354\,112} + \frac{9287}{48\,384}\nu + \frac{8977}{55\,296}\nu^2 - \frac{567\,617}{124\,416}\nu^3\bigg) + \sqrt{1-e_t^2}\bigg[-\frac{14\,047\,483}{151\,200} + \bigg[-\frac{165\,761}{1008} + \frac{287\pi^2}{192}\bigg]\nu + \frac{455}{12}\nu^2 + e_t^2\bigg(\frac{36\,863\,231}{100\,800} + \bigg[-\frac{14\,935\,421}{6048} + \frac{52\,685\pi^2}{4608}\bigg]\nu + \frac{43\,559}{72}\nu^2\bigg) + e_t^4\bigg(\frac{759\,524\,951}{403\,200} + \bigg[-\frac{31\,082\,483}{8064} + \frac{41\,533\pi^2}{6144}\bigg]\nu + \frac{303\,985}{288}\nu^2\bigg) + e_t^6\bigg(\frac{1\,399\,661\,203}{2\,419\,200} + \bigg[-\frac{40\,922\,933}{48\,384} + \frac{1517\pi^2}{9216}\bigg]\nu + \frac{73\,357}{288}\nu^2\bigg) + e_t^8\bigg(\frac{185}{48} - \frac{1073}{288}\nu + \frac{407}{288}\nu^2\bigg)\bigg] + \bigg(\frac{1712}{105} + \frac{14\,552}{63}e_t^2 + \frac{553\,297}{1260}e_t^4 + \frac{187\,357}{1260}e_t^6 + \frac{10\,593}{2240}e_t^8\bigg)\ln\bigg[\frac{x}{x_0}\frac{1+\sqrt{1-e_t^2}}{2(1-e_t^2)}\bigg]\bigg\}. \tag{8.9d}$$

For ease of presentation we have not put a label on e_t to indicate that it is the time eccentricity in MH coordinates. Of course, since x is gauge invariant, no such label is required on it. It is important to keep track of this fact when comparing formulas in different gauges, as we will eventually do.

The last term in the 3PN coefficient $\mathcal{I}_{\rm 3PN}^{\rm MH}$ given by Eq. (8.9d) is proportional to some logarithm which directly arises from the integration formula (8.6) and (8.7). Inside the logarithm we posed

$$x_0 \equiv \frac{Gm}{c^2 r_0}, \tag{8.10}$$

exhibiting the dependence of the instantaneous part of the 3PN energy flux upon the arbitrary constant length scale r_0. Only after computing the complete energy flux can one discuss the structure of the logarithmic term in the energy flux and the required cancellation of $\ln r_0$. Therefore we now add the hereditary contribution to the 3PN flux, which has been computed in Paper I. From Eq. (6.2) in Paper I, we write the result as

$$\langle \mathcal{F}_{\rm hered}\rangle = \frac{32c^5}{5G}\nu^2 x^5 (x^{3/2}\mathcal{H}_{\rm 1.5PN}^{\rm MH} + x^{5/2}\mathcal{H}_{\rm 2.5PN}^{\rm MH} + x^3\mathcal{H}_{\rm 3PN}^{\rm MH}), \tag{8.11}$$

where the hereditary post-Newtonian coefficients (starting at 1.5PN order) read

$$\mathcal{H}_{\rm 1.5N}^{\rm MH} = 4\pi\varphi(e_t), \tag{8.12a}$$

$$\mathcal{H}_{\rm 2.5PN}^{\rm MH} = -\frac{8191}{672}\pi\psi(e_t) - \frac{583}{24}\nu\pi\zeta(e_t), \tag{8.12b}$$

$$\mathcal{H}_{\rm 3PN}^{\rm MH} = -\frac{116761}{3675}\kappa(e_t) + \bigg[\frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{1712}{105}\ln\bigg(\frac{4x^{3/2}}{x_0}\bigg)\bigg]F(e_t). \tag{8.12c}$$

The function $F(e_t)$ in factor of the logarithm in the 3PN coefficient does admit a closed analytic form which was determined in Paper I as

$$F(e_t) = \frac{1}{(1-e_t^2)^{13/2}}\bigg[1 + \frac{85}{6}e_t^2 + \frac{5171}{192}e_t^4 + \frac{1751}{192}e_t^6 + \frac{297}{1024}e_t^8\bigg]. \tag{8.13}$$

On the other hand, Paper I found that the four enhancement functions of eccentricity $\varphi(e_t)$, $\psi(e_t)$, $\zeta(e_t)$, and $\kappa(e_t)$ very likely do not admit any analytic closed-form expressions. Numerical plots of the four enhancement factors $\varphi(e_t)$, $\psi(e_t)$, $\theta(e_t)$, and $\kappa(e_t)$ as functions of eccentricity e_t have been presented in Paper I. The coefficients in Eqs. (8.12) have been introduced in such a way that the circular-orbit limit of all the functions $F(e_t)$ and $\varphi(e_t), \cdots, \kappa(e_t)$ is 1.

Finally, the PN coefficients in the total averaged energy flux $\mathcal{F}$ in MH coordinates are given by the sum of the instantaneous and hereditary contributions, say

$$\mathcal{K}^{\mathrm{MH}}_{n\mathrm{PN}} = \mathcal{I}^{\mathrm{MH}}_{n\mathrm{PN}} + \mathcal{H}^{\mathrm{MH}}_{n\mathrm{PN}}. \tag{8.14}$$

We notice that up to 2.5PN order there is a clean separation between the instantaneous terms which are at even PN orders (recall that there is no 2.5PN term in the averaged flux) and the hereditary terms which appear at odd PN orders and are specifically due to tails (i.e. $\mathcal{H}^{\mathrm{MH}}_{1.5\mathrm{PN}}$ and $\mathcal{H}^{\mathrm{MH}}_{2.5\mathrm{PN}}$). On the contrary, at 3PN order—and, indeed, at any higher PN order—there is a mixture of instantaneous and hereditary terms. The 3PN hereditary term $\mathcal{H}^{\mathrm{MH}}_{3\mathrm{PN}}$ is due to the so-called GW tails of tails (see Paper I).

The analytical result (8.13) is crucial for checking that the arbitrary constant x_0 disappears from the final result, namely, from the 3PN coefficient $\mathcal{K}^{\mathrm{MH}}_{3\mathrm{PN}}$. Indeed, we immediately verify from comparing the last term in Eq. (8.9d) with Eq. (8.12c) and the explicit expression (8.13) of $F(e_t)$ that x_0 cancels out from the sum of the instantaneous and hereditary contributions, extending to noncircular orbits this fact which was already observed for the circular case in Ref. [30]. Finally, the complete 3PN coefficient (independent of x_0) reads

$$\begin{aligned}
\mathcal{K}^{\mathrm{MH}}_{3\mathrm{PN}} = {}& \frac{1}{(1-e_t^2)^{13/2}}\bigg\{\frac{2\,193\,295\,679}{9\,979\,200} + \bigg[\frac{8\,009\,293}{54\,432} - \frac{41\pi^2}{64}\bigg]\nu - \frac{209\,063}{3024}\nu^2 - \frac{775}{324}\nu^3 \\
&+ e_t^2\bigg(\frac{20\,506\,331\,429}{19\,958\,400} + \bigg[\frac{649\,801\,883}{272\,160} + \frac{4879\pi^2}{1536}\bigg]\nu - \frac{3\,008\,759}{3024}\nu^2 - \frac{53\,696}{243}\nu^3\bigg) \\
&+ e_t^4\bigg(-\frac{3\,611\,354\,071}{13\,305\,600} + \bigg[\frac{755\,536\,297}{136\,080} - \frac{29\,971\pi^2}{1024}\bigg]\nu - \frac{179\,375}{576}\nu^2 - \frac{10\,816\,087}{7776}\nu^3\bigg) \\
&+ e_t^6\bigg(\frac{4\,786\,812\,253}{26\,611\,200} + \bigg[\frac{1\,108\,811\,471}{1\,451\,520} - \frac{84\,501\pi^2}{4096}\bigg]\nu + \frac{87\,787\,969}{48\,384}\nu^2 - \frac{983\,251}{648}\nu^3\bigg) \\
&+ e_t^8\bigg(\frac{21\,505\,140\,101}{141\,926\,400} + \bigg[-\frac{32\,467\,919}{129\,024} - \frac{4059\pi^2}{4096}\bigg]\nu + \frac{79\,938\,097}{193\,536}\nu^2 - \frac{4\,586\,539}{15\,552}\nu^3\bigg) + e_t^{10} \\
&\times\bigg(-\frac{8\,977\,637}{11\,354\,112} + \frac{9287}{48\,384}\nu + \frac{8977}{55\,296}\nu^2 - \frac{567\,617}{124\,416}\nu^3\bigg) + \sqrt{1-e_t^2}\bigg[-\frac{14\,047\,483}{151\,200} \\
&+ \bigg[-\frac{165\,761}{1008} + \frac{287\pi^2}{192}\bigg]\nu + \frac{455}{12}\nu^2 + e_t^2\bigg(\frac{36\,863\,231}{100\,800} + \bigg[-\frac{14\,935\,421}{6048} + \frac{52\,685\pi^2}{4608}\bigg]\nu + \frac{43\,559}{72}\nu^2\bigg) \\
&+ e_t^4\bigg(\frac{759\,524\,951}{403\,200} + \bigg[-\frac{31\,082\,483}{8064} + \frac{41\,533\pi^2}{6144}\bigg]\nu + \frac{303\,985}{288}\nu^2\bigg) + e_t^6\bigg(\frac{1\,399\,661\,203}{2\,419\,200} \\
&+ \bigg[-\frac{40\,922\,933}{48\,384} + \frac{1517\pi^2}{9216}\bigg]\nu + \frac{73\,357}{288}\nu^2\bigg) + e_t^8\bigg(\frac{185}{48} - \frac{1073}{288}\nu + \frac{407}{288}\nu^2\bigg)\bigg] + \bigg(\frac{1712}{105} + \frac{14\,552}{63}e_t^2 \\
&+ \frac{553\,297}{1260}e_t^4 + \frac{187\,357}{1260}e_t^6 + \frac{10\,593}{2240}e_t^8\bigg)\bigg[-C + \frac{35}{107}\pi^2 - \frac{1}{2}\ln(16x) + \ln\bigg(\frac{1+\sqrt{1-e_t^2}}{2(1-e_t^2)}\bigg)\bigg]\bigg\} \\
&- \frac{116\,761}{3675}\kappa(e_t).
\end{aligned} \tag{8.15}$$

The 1.5PN and 2.5PN coefficients are only due to tails, and thus

$$\mathcal{K}^{\mathrm{MH}}_{1.5\mathrm{N}} = 4\pi\varphi(e_t), \tag{8.16a}$$

$$\mathcal{K}^{\mathrm{MH}}_{2.5\mathrm{PN}} = -\frac{8191}{672}\pi\psi(e_t) - \frac{583}{24}\nu\pi\zeta(e_t). \tag{8.16b}$$

The Newtonian, 1PN, and 2PN coefficients reduce to their instantaneous contributions $\mathcal{I}^{\mathrm{MH}}_{\mathrm{N}}$, $\mathcal{I}^{\mathrm{MH}}_{1\mathrm{PN}}$, and $\mathcal{I}^{\mathrm{MH}}_{2\mathrm{PN}}$ already given in Eqs. (8.9).

Since the enhancement functions $\varphi(e_t)$, $\psi(e_t)$, $\zeta(e_t)$, and $\kappa(e_t)$ reduce to 1 in the circular case, when $e_t = 0$, the circular-orbit limit of the energy flux is immediately deduced from inspection of Eqs. (8.9) and (8.16) as

$$\langle\mathcal{F}\rangle_{\odot} = \frac{32c^5}{5G}x^5\nu^2\bigg\{1 + x\bigg(-\frac{1247}{336} - \frac{35}{12}\nu\bigg) + 4\pi x^{3/2} + x^2\bigg(-\frac{44\,711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\bigg) + \pi x^{5/2}\bigg(-\frac{8191}{672} - \frac{583}{24}\nu\bigg) + x^3\bigg(\frac{6\,643\,739\,519}{69\,854\,400} - \frac{1712}{105}C + \frac{16}{3}\pi^2 - \frac{856}{105}\ln(16x) + \bigg[-\frac{134\,543}{7776} + \frac{41}{48}\pi^2\bigg]\nu - \frac{94\,403}{3024}\nu^2 - \frac{775}{324}\nu^3\bigg)\bigg\}. \tag{8.17}$$

This limiting case is in exact agreement with Eq. (12.9) of [30] (after taking into account the values of the ambiguity parameters $\lambda = -\frac{1987}{3080}$ and $\theta = -\frac{11\,831}{9240}$ computed in Refs. [32–34]). Notice that the flux in the circular-orbit limit (8.17) depends only on the parameter x, and hence its expression becomes gauge invariant.

B. Orbital average in ADM coordinates

We start from the expression for the instantaneous energy flux in ADM coordinates as given by (6.11), employ the appropriate 3PN QK representation, and follow the procedure for performing the average as outlined in the previous section. We find that the β_N's and γ_N's in ADM coordinates [cf. Eq. (8.2)] are exactly the same as in MH coordinates; the α_N's, however, are different in general (except for α_{11}). The result for the average energy flux in ADM coordinates is of the form

$$\langle\mathcal{F}_{\rm inst}\rangle = \frac{32c^5}{5G}\nu^2x^5(I_{\rm N}^{\rm ADM} + xI_{\rm 1PN}^{\rm ADM} + x^2I_{\rm 2PN}^{\rm ADM} + x^3I_{\rm 3PN}^{\rm ADM}), \tag{8.18}$$

where the coefficients depend on the time eccentricity in ADM coordinates (hence $e_t \equiv e_t^{\rm ADM}$ here) and on ν, and read

$$I_{\rm N}^{\rm ADM} = \frac{1}{(1-e_t^2)^{7/2}}\bigg\{1 + \frac{73}{24}e_t^2 + \frac{37}{96}e_t^4\bigg\}, \tag{8.19a}$$

$$I_{\rm 1PN}^{\rm ADM} = \frac{1}{(1-e_t^2)^{9/2}}\bigg\{-\frac{1247}{336} - \frac{35}{12}\nu + e_t^2\bigg(\frac{10\,475}{672} - \frac{1081}{36}\nu\bigg) + e_t^4\bigg(\frac{10\,043}{384} - \frac{311}{12}\nu\bigg) + e_t^6\bigg(\frac{2179}{1792} - \frac{851}{576}\nu\bigg)\bigg\}, \tag{8.19b}$$

$$I_{\rm 2PN}^{\rm ADM} = \frac{1}{(1-e_t^2)^{11/2}}\bigg\{-\frac{203\,471}{9072} + \frac{12\,799}{504}\nu + \frac{65}{18}\nu^2 + e_t^2\bigg(-\frac{3\,866\,543}{18\,144} + \frac{4691}{2016}\nu + \frac{5935}{54}\nu^2\bigg) + e_t^4\bigg(-\frac{369\,751}{24\,192} - \frac{3\,039\,083}{8064}\nu + \frac{247\,805}{864}\nu^2\bigg) + e_t^6\bigg(\frac{1\,302\,443}{16\,128} - \frac{215\,077}{1344}\nu + \frac{185\,305}{1728}\nu^2\bigg) + e_t^8\bigg(\frac{86\,567}{64\,512} - \frac{9769}{4608}\nu + \frac{21\,275}{6912}\nu^2\bigg) + \sqrt{1-e_t^2}\bigg[\frac{35}{2} - 7\nu + e_t^2\bigg(\frac{6425}{48} - \frac{1285}{24}\nu\bigg) + e_t^4\bigg(\frac{5065}{64} - \frac{1013}{32}\nu\bigg) + e_t^6\bigg(\frac{185}{96} - \frac{37}{48}\nu\bigg)\bigg]\bigg\}, \tag{8.19c}$$

$$\begin{aligned} I_{\rm 3PN}^{\rm ADM} = \frac{1}{(1-e_t^2)^{13/2}}\bigg\{&\frac{2\,193\,295\,679}{9\,979\,200} + \bigg[\frac{8\,009\,293}{54\,432} - \frac{41\pi^2}{64}\bigg]\nu - \frac{209\,063}{3024}\nu^2 - \frac{775}{324}\nu^3 + e_t^2\bigg(\frac{2\,912\,411\,147}{2\,851\,200} + \bigg[\frac{249\,108\,317}{108\,864} \\ &+ \frac{31\,255}{1536}\pi^2\bigg]\nu - \frac{3\,525\,469}{6048}\nu^2 - \frac{53\,696}{243}\nu^3\bigg) + e_t^4\bigg(-\frac{4\,520\,777\,971}{13\,305\,600} + \bigg[\frac{473\,750\,339}{108\,864} - \frac{7459\pi^2}{1024}\bigg]\nu + \frac{697\,997}{576}\nu^2 \\ &- \frac{10\,816\,087}{7776}\nu^3\bigg) + e_t^6\bigg(\frac{3\,630\,046\,753}{26\,611\,200} + \bigg[-\frac{8\,775\,247}{145\,152} - \frac{78\,285\pi^2}{4096}\bigg]\nu + \frac{31\,147\,213}{12\,096}\nu^2 - \frac{983\,251}{648}\nu^3\bigg) \\ &+ e_t^8\bigg(\frac{21\,293\,656\,301}{141\,926\,400} + \bigg[-\frac{36\,646\,949}{129\,024} - \frac{4059\pi^2}{4096}\bigg]\nu + \frac{85\,830\,865}{193\,536}\nu^2 - \frac{4\,586\,539}{15\,552}\nu^3\bigg) + e_t^{10}\bigg(-\frac{8\,977\,637}{11\,354\,112} \\ &+ \frac{9287}{48\,384}\nu + \frac{8977}{55\,296}\nu^2 - \frac{567\,617}{124\,416}\nu^3\bigg) + \sqrt{1-e_t^2}\bigg[-\frac{14\,047\,483}{151\,200} + \bigg[-\frac{165\,761}{1008} + \frac{287\pi^2}{192}\bigg]\nu + \frac{455}{12}\nu^2 \\ &+ e_t^2\bigg(\frac{36\,863\,231}{100\,800} + \bigg[-\frac{14\,935\,421}{6048} + \frac{52\,685\pi^2}{4608}\bigg]\nu + \frac{43\,559}{72}\nu^2\bigg) + e_t^4\bigg(\frac{759\,524\,951}{403\,200} + \bigg[-\frac{31\,082\,483}{8064} + \frac{41\,533\pi^2}{6144}\bigg]\nu \\ &+ \frac{303\,985}{288}\nu^2\bigg) + e_t^6\bigg(\frac{1\,399\,661\,203}{2\,419\,200} + \bigg[-\frac{40\,922\,933}{48\,384} + \frac{1517\pi^2}{9216}\bigg]\nu + \frac{73\,357}{288}\nu^2\bigg) + e_t^8\bigg(\frac{185}{48} - \frac{1073}{288}\nu + \frac{407}{288}\nu^2\bigg)\bigg] \\ &+ \bigg(\frac{1712}{105} + \frac{14\,552}{63}e_t^2 + \frac{553\,297}{1260}e_t^4 + \frac{187\,357}{1260}e_t^6 + \frac{10\,593}{2240}e_t^8\bigg)\ln\bigg[\frac{x}{x_0}\frac{1+\sqrt{1-e_t^2}}{2(1-e_t^2)}\bigg]\bigg\}. \end{aligned} \tag{8.19d}$$

We recall that the Newtonian and 1PN orders are the same in MH and ADM coordinates (the coefficients $I_{\rm N}^{\rm ADM}$ and $I_{\rm 1PN}^{\rm ADM}$ agree with their MH counterparts). On the other hand, adding up the hereditary contribution (8.11) and (8.12) (which is the same in MH and ADM coordinates), we obtain the total 3PN coefficient $\mathcal{K}_{\rm 3PN}^{\rm ADM}$, analogous to Eq. (8.15) but in ADM coordinates,

$$\begin{aligned}\mathcal{K}_{\rm 3PN}^{\rm ADM} &= \frac{1}{(1-e_t^2)^{13/2}}\bigg\{\frac{2\,193\,295\,679}{9\,979\,200} + \bigg[\frac{8\,009\,293}{54\,432} - \frac{41\pi^2}{64}\bigg]\nu - \frac{209\,063}{3024}\nu^2 - \frac{775}{324}\nu^3 + e_t^2\bigg(\frac{2\,912\,411\,147}{2\,851\,200}\\ &\quad + \bigg[\frac{249\,108\,317}{108\,864} + \frac{31\,255}{1536}\pi^2\bigg]\nu - \frac{3\,525\,469}{6048}\nu^2 - \frac{53\,696}{243}\nu^3\bigg) + e_t^4\bigg(-\frac{4\,520\,777\,971}{13\,305\,600} + \bigg[\frac{473\,750\,339}{108\,864}\\ &\quad - \frac{7459\pi^2}{1024}\bigg]\nu + \frac{697\,997}{576}\nu^2 - \frac{10\,816\,087}{7776}\nu^3\bigg) + e_t^6\bigg(\frac{3\,630\,046\,753}{26\,611\,200} + \bigg[-\frac{8\,775\,247}{145\,152} - \frac{78\,285\pi^2}{4096}\bigg]\nu\\ &\quad + \frac{31\,147\,213}{12\,096}\nu^2 - \frac{983\,251}{648}\nu^3\bigg) + e_t^8\bigg(\frac{21\,293\,656\,301}{141\,926\,400} + \bigg[-\frac{36\,646\,949}{129\,024} - \frac{4059\pi^2}{4096}\bigg]\nu + \frac{85\,830\,865}{193\,536}\nu^2\\ &\quad - \frac{4\,586\,539}{15\,552}\nu^3\bigg) + e_t^{10}\bigg(-\frac{8\,977\,637}{11\,354\,112} + \frac{9287}{48\,384}\nu + \frac{8977}{55\,296}\nu^2 - \frac{567\,617}{124\,416}\nu^3\bigg) + \sqrt{1-e_t^2}\\ &\quad \times\bigg[-\frac{14\,047\,483}{151\,200} + \bigg[-\frac{165\,761}{1008} + \frac{287\pi^2}{192}\bigg]\nu + \frac{455}{12}\nu^2 + e_t^2\bigg(\frac{36\,863\,231}{100\,800} + \bigg[-\frac{14\,935\,421}{6048} + \frac{52\,685\pi^2}{4608}\bigg]\nu\\ &\quad + \frac{43\,559}{72}\nu^2\bigg) + e_t^4\bigg(\frac{759\,524\,951}{403\,200} + \bigg[-\frac{31\,082\,483}{8064} + \frac{41\,533\pi^2}{6144}\bigg]\nu + \frac{303\,985}{288}\nu^2\bigg) + e_t^6\bigg(\frac{1\,399\,661\,203}{2\,419\,200}\\ &\quad + \bigg[-\frac{40\,922\,933}{48\,384} + \frac{1517\pi^2}{9216}\bigg]\nu + \frac{73\,357}{288}\nu^2\bigg) + e_t^8\bigg(\frac{185}{48} - \frac{1073}{288}\nu + \frac{407}{288}\nu^2\bigg)\bigg] + \bigg(\frac{1712}{105} + \frac{14\,552}{63}e_t^2\\ &\quad + \frac{553\,297}{1260}e_t^4 + \frac{187\,357}{1260}e_t^6 + \frac{10\,593}{2240}e_t^8\bigg)\bigg[-C + \frac{35}{107}\pi^2 - \frac{1}{2}\ln(16x) + \ln\bigg(\frac{1+\sqrt{1-e_t^2}}{2(1-e_t^2)}\bigg)\bigg]\bigg\}\\ &\quad - \frac{116\,761}{3675}\kappa(e_t),\end{aligned} \tag{8.20}$$

in which again $e_t = e_t^{\rm ADM}$. A useful internal consistency check of the algebraic correctness of different coordinate representations of the energy flux is the verification that the equality of Eqs. (8.9) and (8.19) holds if and only if we have the transformation between the time eccentricities $e_t^{\rm MH}$ and $e_t^{\rm ADM}$ given by

$$\begin{aligned}\frac{e_t^{\rm MH}}{e_t^{\rm ADM}} &= 1 + \frac{x^2}{1-e_t^2}\bigg(-\frac{1}{4} - \frac{17}{4}\nu\bigg) + \frac{x^3}{(1-e_t^2)^2}\\ &\quad\times\bigg(-\frac{1}{2} + \bigg[-\frac{16\,739}{1680} + \frac{21}{16}\pi^2\bigg]\nu\\ &\quad + \frac{83}{24}\nu^2 + e_t^2\bigg(-\frac{1}{2} - \frac{249}{16}\nu + \frac{241}{24}\nu^2\bigg)\bigg].\end{aligned} \tag{8.21}$$

(There is no ambiguity in not having a label on the e_t in the 2PN and 3PN terms above.) We find that the relation (8.21) is perfectly equivalent to what is predicted from using different QK representations of the motion, namely, Eq. (7.11c) together with (7.10).

C. Gauge-invariant formulation

In the previous section, the averaged energy flux was represented using x—a gauge-invariant variable defined by (7.9)—and the eccentricity e_t, which, however, is coordinate dependent (but is useful in extracting the circular limit of the result). In the present section we provide a gauge-invariant formulation of the energy flux.

Perhaps the most natural choice is to express the result in terms of the conserved energy E and angular momentum J (per unit of reduced mass), or, rather, in terms of the pair of rescaled variables (ε, j) defined by Eqs. (7.5) and (7.6). However, there are other possible choices for a couple of gauge-invariant quantities. As we have seen in Eqs. (7.8) the mean motion n and the periastron precession K are gauge invariant, so we may define as our first choice the pair of variables (x, ι), where we recall that x is related to the orbital frequency $\omega = Kn$ by Eq. (7.9), and where we define

$$\iota \equiv \frac{3x}{k}, \tag{8.22}$$

with $k \equiv K - 1$. Here we have introduced a factor 3 so that ι reduces to j in the first approximation (i.e. when $\varepsilon \longrightarrow 0$). To 3PN order this parameter is related to the energy and angular momentum variables ε and j by

$$\iota = j + \varepsilon\left\{-\frac{27}{4} + \frac{5}{2}\nu - j\frac{5}{12}\nu\right\} + \varepsilon^2\left\{\frac{205}{16} + \left[-\frac{1201}{48} + \frac{41}{128}\pi^2\right]\nu + \frac{35}{24}\nu^2 + j^{1/2}(-5+2\nu) + j\left(\frac{35}{16} + \frac{1}{72}\nu^2\right) + \frac{1}{j}\left(-\frac{331}{16} + \left[\frac{725}{12} - \frac{205}{128}\pi^2\right]\nu + \frac{15}{8}\nu^2\right)\right\} + \varepsilon^3\left\{\frac{495}{64} + \left[-\frac{1145}{24} + \frac{205}{512}\pi^2\right]\nu + \left(\frac{2341}{72} - \frac{451}{1536}\pi^2\right)\nu^2 - \frac{415}{144}\nu^3 + j^{1/2}\left(\frac{95}{8} - \frac{115}{24}\nu + \frac{17}{12}\nu^2\right) + j\left(-\frac{415}{192} - \frac{385}{192}\nu - \frac{5}{32}\nu^2 + \frac{161}{1296}\nu^3\right) + \frac{1}{j^{1/2}}\left(-\frac{5}{4} + \left[\frac{202}{9} - \frac{41}{96}\pi^2\right]\nu\right) + \frac{1}{j}\left(-\frac{12345}{32} + \left[\frac{147283}{192} - \frac{3895}{512}\pi^2\right]\nu + \left[-\frac{77945}{288} + \frac{4715}{1536}\pi^2\right]\nu^2 + \frac{445}{32}\nu^3\right) + \frac{1}{j^2}\left(\frac{193351}{192} + \left[-\frac{165835}{96} + \frac{7175}{256}\pi^2\right]\nu + \left[\frac{1300}{3} - \frac{1025}{128}\pi^2\right]\nu^2 - \frac{25}{4}\nu^3\right)\right\}. \tag{8.23}$$

We have performed two calculations of the gauge-invariant result, in terms of the variables (x, ι), starting from the expression of the averaged flux in either MH or ADM coordinates. The instantaneous part of the flux takes the form

$$\langle \mathcal{F}_{\mathrm{inst}} \rangle = \frac{32c^5}{5G}\nu^2 x^5 \iota^{-13/2}(I_{\mathrm{N}} + xI_{1\mathrm{PN}} + x^2 I_{2\mathrm{PN}} + x^3 I_{3\mathrm{PN}}), \tag{8.24}$$

in which the PN coefficients are polynomials of ι and the mass ratio ν, and are given by

$$I_{\mathrm{N}} = \frac{425}{96}\iota^3 - \frac{61}{16}\iota^4 + \frac{37}{96}\iota^5, \tag{8.25a}$$

$$I_{1\mathrm{PN}} = \left(-\frac{289}{3} + \frac{3605}{384}\nu\right)\iota^2 + \left(\frac{1865}{24} + \frac{3775}{384}\nu\right)\iota^3 + \left(-\frac{5297}{336} - \frac{2725}{384}\nu\right)\iota^4 + \left(\frac{139}{112} + \frac{259}{1152}\nu\right)\iota^5, \tag{8.25b}$$

$$I_{2\mathrm{PN}} = \left(\frac{267725837}{258048} + \left[\frac{1440583}{2304} - \frac{609875}{24576}\pi^2\right]\nu + \frac{24395}{1024}\nu^2\right)\iota + \left(-\frac{51894953}{82944} + \left[-\frac{583921}{512} + \frac{497125}{24576}\pi^2\right]\nu + \frac{1625}{48}\nu^2\right)\iota^2 + \left(\frac{49183667}{387072} + \left[\frac{14718145}{32256} - \frac{32595}{8192}\pi^2\right]\nu + \frac{37145}{4608}\nu^2\right)\iota^3 + \left(-\frac{305}{16} + \frac{61}{8}\nu\right)\iota^{7/2} + \left(-\frac{2145781}{64512} + \left[-\frac{505639}{10752} + \frac{1517}{8192}\pi^2\right]\nu - \frac{105}{16}\nu^2\right)\iota^4 + \left(\frac{185}{48} - \frac{37}{24}\nu\right)\iota^{9/2} + \left(\frac{744545}{258048} + \frac{19073}{32256}\nu + \frac{2849}{27648}\nu^2\right)\iota^5, \tag{8.25c}$$

$$I_{3\mathrm{PN}} = \frac{149899221067}{7741440} + \left[-\frac{186950547065}{3096576} + \frac{46739713}{32768}\pi^2\right]\nu + \left[\frac{66297815}{6144} - \frac{8315825}{32768}\pi^2\right]\nu^2 - \frac{415625}{12288}\nu^3 - \frac{161249}{192}\iota^{1/2} + \left(-\frac{66998702987}{2073600} + \left[\frac{71728525525}{1032192} - \frac{117241181}{98304}\pi^2\right]\nu + \left[-\frac{24611099}{2304} + \frac{6633185}{49152}\pi^2\right]\nu^2 + \frac{4346075}{12288}\nu^3\right)\iota + \frac{3727559}{2880}\iota^{3/2} + \left(\frac{4774135897}{322560} + \left[-\frac{332003303819}{13934592} + \frac{29862965}{114688}\pi^2\right]\nu + \left[\frac{15103071}{7168} - \frac{3258475}{294912}\pi^2\right]\nu^2 - \frac{2249695}{18432}\nu^3\right)\iota^2 + \left(-\frac{928043}{5760} + \left[-\frac{1879}{1152} - \frac{2501}{1536}\pi^2\right]\nu - \frac{5605}{192}\nu^2\right)\iota^{5/2} + \left(-\frac{2740721737}{1290240} + \left[\frac{225135517}{73728} - \frac{8351167}{688128}\pi^2\right]\nu + \left[\frac{6154165}{64512} - \frac{615}{1024}\pi^2\right]\nu^2 + \frac{298895}{6144}\nu^3\right)\iota^3 + \left(-\frac{3913177}{37800} + \left[-\frac{351499}{12096} + \frac{1517}{4608}\pi^2\right]\nu + \frac{1153}{32}\nu^2\right)\iota^{7/2} + \left(\frac{1758850201}{141926400} + \left[-\frac{186455099}{1032192} + \frac{68757}{229376}\pi^2\right]\nu + \left[-\frac{5900711}{387072} - \frac{1517}{16384}\pi^2\right]\nu^2 - \frac{2568655}{331776}\nu^3\right)\iota^4 + \left(\frac{51335}{2688} - \frac{10951}{2688}\nu - \frac{481}{192}\nu^2\right)\iota^{9/2} + \left(\frac{2635805}{405504} + \frac{891535}{3096576}\nu + \frac{4537}{27648}\nu^2 + \frac{106375}{995328}\nu^3\right)\iota^5 + \left(\frac{161249}{192} - \frac{125939}{80}\iota + \frac{263113}{288}\iota^2 - \frac{168953}{1008}\iota^3 + \frac{10593}{2240}\iota^4\right)\ln\left[\frac{x}{x_0}\frac{1+\sqrt{\iota}}{2\iota}\right]. \tag{8.25d}$$

Similarly, we can also obtain the equivalent expression of the flux in terms of the rescaled variables (ε, j) defined by Eqs. (7.5) and (7.6).

The hereditary part of the flux given by (8.11) and (8.12) is straightforwardly added. In this part we simply have to replace e_t by its expression in terms of x and ι at 1PN order, namely,

$$e_t = \left[1 - \iota + x\left\{-\frac{35}{4} + \frac{9}{2}\nu + \iota\left(\frac{17}{4} - \frac{13}{6}\nu\right)\right\}\right]^{1/2}. \tag{8.26}$$

(At this order there is no difference between MH and ADM coordinates.) Note also that with the latter choice of gauge-invariant variables the circular-orbit limit is not directly readable from the expressions. However, it can be easily obtained by using the expression for the variable $j_\odot$ as reduced to circular orbits in terms of ε, Eq. (7.12).

IX. THE TEST PARTICLE LIMIT OF THE 3PN ENERGY FLUX

An important check on our result is the test particle limit for which the energy flux in the eccentric orbit case is available (to second order in the eccentricity) from computations based on perturbation theory around a Schwarzschild background. We compare the end result of our computation—composed of the instantaneous terms and the hereditary terms computed in Paper I—with the result obtained in Ref. [47]. Thus, we take the test particle limit of our result (i.e. $\nu \equiv \mu/m \to 0$), say, in the form given by Eqs. (8.8) and (8.9) in which $e_t \equiv e_t^{\rm MH}$, and expand it in powers of e_t retaining only terms up to e_t^2. The instantaneous contribution to the energy flux in the test-mass limit is then given by

$$\begin{aligned}\langle \mathcal{F}_{\rm inst}\rangle = \frac{32c^5}{5G}\nu^2 x^5\biggl\{&1 - \frac{1247}{336}x - \frac{44711}{9072}x^2 \\ &+ \left[\frac{1\,266\,161\,801}{9\,979\,200} + \frac{1712}{105}\ln\left(\frac{x}{x_0}\right)\right]x^3 \\ &+ e_t^2\biggl(\frac{157}{24} - \frac{187}{168}x - \frac{84\,547}{756}x^2 \\ &+ \left[\frac{22\,718\,275\,589}{9\,979\,200} + \frac{106\,144}{315}\ln\left(\frac{x}{x_0}\right)\right]x^3\biggr) \\ &+ \mathcal{O}(\nu)\biggr\} + \mathcal{O}(e_t^4).\end{aligned} \tag{9.1}$$

On the other hand, the hereditary contribution has been reported in Eqs. (8.11) and (8.12) and admits the test-mass limit

$$\begin{aligned}\langle \mathcal{F}_{\rm hered}\rangle = \frac{32c^5}{5G}\nu^2 x^5\biggl\{&4\pi x^{3/2}\varphi(e_t) - \frac{8191}{672}\pi x^{5/2}\psi(e_t) \\ &+ x^3\biggl[-\frac{116\,761}{3675}\kappa(e_t) + \biggl[\frac{16}{3}\pi^2 - \frac{1712}{105}C \\ &- \frac{1712}{105}\ln\left(\frac{4x^{3/2}}{x_0}\right)\biggr]F(e_t)\biggr] + \mathcal{O}(\nu)\biggr\}.\end{aligned} \tag{9.2}$$

To proceed further, all the enhancement functions should be expanded up to power e_t^2. This is easy for $F(e_t)$ which is known analytically from Eq. (8.13), and we have

$$F(e_t) = 1 + \frac{62}{3}e_t^2 + \mathcal{O}(e_t^4). \tag{9.3}$$

The other enhancement functions are only known numerically for general eccentricity. We have, however, succeeded in obtaining analytically their leading correction term e_t^2 by implementing our calculation of the tails in Paper I at order e_t^2 from the start. The results we thereby obtained [Eqs. (6.8) of Paper I] are

$$\varphi(e_t) = 1 + \frac{2335}{192}e_t^2 + \mathcal{O}(e_t^4), \tag{9.4a}$$

$$\psi(e_t) = 1 - \frac{22\,988}{8191}e_t^2 + \mathcal{O}(e_t^4), \tag{9.4b}$$

$$\begin{aligned}\kappa(e_t) = 1 + &\left(\frac{62}{3} - \frac{4\,613\,840}{350\,283}\ln 2 + \frac{24\,570\,945}{1\,868\,176}\ln 3\right) \\ &\times e_t^2 + \mathcal{O}(e_t^4).\end{aligned} \tag{9.4c}$$

[We do not need $\zeta(e_t)$ here, since it is in factor of a ν-dependent term.] Our final result to $\mathcal{O}(\nu)$ and $\mathcal{O}(e_t^4)$ is therefore

$$\begin{aligned}\langle \mathcal{F}\rangle = \frac{32c^5}{5G}\nu^2 x^5\biggl\{&1 - \frac{1247}{336}x + 4\pi x^{3/2} - \frac{44\,711}{9072}x^2 - \frac{8191}{672}\pi x^{5/2} + \biggl[\frac{6\,643\,739\,519}{69\,854\,400} + \frac{16}{3}\pi^2 - \frac{1712}{105}C \\ &- \frac{856}{105}\ln(16x)\biggr]x^3 + e_t^2\biggl(\frac{157}{24} - \frac{187}{168}x + \frac{2335}{48}\pi x^{3/2} - \frac{84\,547}{756}x^2 + \frac{821}{24}\pi x^{5/2} + \biggl[\frac{113\,160\,471\,971}{69\,854\,400} + \frac{18\,832}{45}\ln 2 \\ &- \frac{234\,009}{560}\ln 3 + \frac{992}{9}\pi^2 - \frac{106\,144}{315}C - \frac{53\,072}{315}\ln(16x)\biggr]x^3\biggr) + \mathcal{O}(e_t^4) + \mathcal{O}(\nu)\biggr\}.\end{aligned} \tag{9.5}$$

The above expression is in terms of our chosen eccentricity e_t. One should note that the "Schwarzschild" eccentricity e appearing in the black-hole perturbation theory [47] is *a priori* different from e_t; therefore, the above result can only be compared modulo a transformation of these eccentricities. We find that, indeed, Eq. (9.5) is equivalent to the black-hole perturbation result given by Eq. (180) of [47], if and only if the two eccentricities are linked together by

$$e_t^2 = e^2(1 - 6x + 4x^2 - 8x^3). \tag{9.6}$$

(Recall that $e_t = e_t^{\mathrm{MH}}$ here.)

X. CONCLUDING REMARKS

The instantaneous contributions to the 3PN gravitational wave luminosity from the inspiral phase of a binary system of compact objects moving in an elliptical orbit are computed using the multipolar post-Minkowskian wave generation formalism.[13] The nontrivial inputs for this calculation include the mass octupole and current quadrupole at 2PN order for general orbits and the 3PN accurate mass quadrupole. Using the 3PN quasi-Keplerian representation of elliptical orbits obtained recently, the flux is averaged over the binary's orbit. The instantaneous part of the energy flux is computed in the standard harmonic coordinate system (with logarithms). For technical reasons, the average over an orbit of the instantaneous contributions is presented in other coordinate systems: modified harmonic coordinates (without logarithms) and ADM coordinates. Alternative *gauge-invariant* expressions are also provided. Supplementing the instantaneous contributions of this paper by the important hereditary contributions arising from tails, tails of tails, and tails-squared terms calculated in Paper I [36], the complete energy flux has been obtained.

For binaries moving on circular orbits the 3PN energy flux agrees with that computed in [30]. However, the circular-orbit results are known to the higher 3.5PN order [30]. The extension of the 3.5PN term to eccentric orbits would be interesting, but some uncomputed modules remain in the general formalism to compute the multipole moments for general sources required for the 3.5PN generation in the eccentric orbit case. We leave this to a future investigation.

[13] The instantaneous part of the 3PN gravitational wave flux of angular momentum and linear momentum from inspiralling compact binaries moving on elliptical orbits has been computed [48,49].

ACKNOWLEDGMENTS

L. B. and B. R. I. thank the Indo-French Collaboration (IFCPAR) under which this work has been carried out. M. S. S. Q. acknowledges the Indo-Yemen cultural exchange programme. B. R. I. acknowledges the hospitality of the Institut Henri Poincaré and Institut des Hautes Etudes Scientifiques during the final stages of the writing of the paper. Almost all algebraic calculations leading to the results of this paper were done with the software MATHEMATICA.

[1] P. Peters and J. Mathews, Phys. Rev. **131**, 435 (1963).
[2] P. Peters, Phys. Rev. **136**, B1224 (1964).
[3] M. Burgay *et al.*, Nature (London) **426**, 531 (2003).
[4] V. Kalogera, C. Kim, D. Lorimer, M. Burgay, N. D'Amico, A. Possenti, R. Manchester, A. Lyne, B. Joshi, M. McLaughlin *et al.*, Astrophys. J. **601**, L179 (2004); **614**, L137(E) (2004).
[5] L. A. Wainstein and V. D. Zubakov, *Extraction of Signals from Noise* (Prentice-Hall, Englewood Cliffs, 1962).
[6] L. Blanchet, Living Rev. Relativity **9**, 4 (2006).
[7] Y. Kozai, Astron. J. **67**, 591 (1962).
[8] L. Wen, Astrophys. J. **598**, 419 (2003).
[9] M. J. Benacquista, Living Rev. Relativity **5**, 2 (2002), http://www.livingreviews.org/lrr-2002-2.
[10] K. Gültekin, M. C. Miller, and D. P. Hamilton, Astrophys. J. **616**, 221 (2004).
[11] K. S. Thorne and V. B. Braginskii, Astrophys. J. **204**, L1 (1976).
[12] O. Blaes, M. H. Lee, and A. Socrates, Astrophys. J. **578**, 775 (2002).
[13] R. Wagoner and C. Will, Astrophys. J. **210**, 764 (1976).
[14] L. Blanchet and G. Schäfer, Mon. Not. R. Astron. Soc. **239**, 845 (1989).
[15] W. Junker and G. Schäfer, Mon. Not. R. Astron. Soc. **254**, 146 (1992).
[16] L. Blanchet and G. Schäfer, Classical Quantum Gravity **10**, 2699 (1993).
[17] R. Rieth and G. Schäfer, Classical Quantum Gravity **14**, 2357 (1997).
[18] T. Damour and N. Deruelle, Ann. Inst. Henri Poincare Phys. Theor. **43**, 107 (1985).
[19] A. Gopakumar and B. R. Iyer, Phys. Rev. D **56**, 7708 (1997).
[20] A. Gopakumar and B. R. Iyer, Phys. Rev. D **65**, 084011 (2002).
[21] T. Damour and G. Schäfer, Nuovo Cimento Soc. Ital. Fis. B **101**, 127 (1988).
[22] G. Schäfer and N. Wex, Phys. Lett. A **174**, 196 (1993); **177**, 461(E) (1993).
[23] N. Wex, Classical Quantum Gravity **12**, 983 (1995).
[24] C. Will and A. Wiseman, Phys. Rev. D **54**, 4813 (1996).
[25] T. Damour, A. Gopakumar, and B. R. Iyer, Phys. Rev. D **70**, 064028 (2004).
[26] P. Jaranowski and G. Schäfer, Phys. Rev. D **60**, 124003 (1999).
[27] L. Blanchet and G. Faye, Phys. Rev. D **63**, 062005 (2001).
[28] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Lett. B **513**, 147 (2001).
[29] L. Blanchet, T. Damour, and G. Esposito-Farèse, Phys. Rev. D **69**, 124007 (2004).

[30] L. Blanchet, B. R. Iyer, and B. Joguet, Phys. Rev. D **65**, 064005 (2002); **71**, 129903(E) (2005).
[31] L. Blanchet and B. R. Iyer, Phys. Rev. D **71**, 024004 (2005).
[32] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer, Phys. Rev. Lett. **93**, 091101 (2004).
[33] L. Blanchet, T. Damour, and B. R. Iyer, Classical Quantum Gravity **22**, 155 (2005).
[34] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer, Phys. Rev. D **71**, 124004 (2005).
[35] L. Blanchet, G. Faye, B. R. Iyer, and B. Joguet, Phys. Rev. D **65**, 061501(R) (2002); **71**, 129902(E) (2005).
[36] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. Qusailah, preceding article, Phys. Rev. D **77**, 064034 (2008).
[37] R. Memmesheimer, A. Gopakumar, and G. Schäfer, Phys. Rev. D **70**, 104011 (2004).
[38] L. Blanchet, Phys. Rev. D **54**, 1417 (1996); **71**, 129904(E) (2005).
[39] L. Blanchet, Classical Quantum Gravity **15**, 89 (1998).
[40] L. Blanchet, Classical Quantum Gravity **15**, 113 (1998).
[41] K. Thorne, Rev. Mod. Phys. **52**, 299 (1980).
[42] L. Blanchet and T. Damour, Phys. Rev. D **46**, 4304 (1992).
[43] L. Blanchet and B. R. Iyer, Classical Quantum Gravity **20**, 755 (2003).
[44] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **63**, 044021 (2001); **66**, 029901(E) (2002).
[45] V. de Andrade, L. Blanchet, and G. Faye, Classical Quantum Gravity **18**, 753 (2001).
[46] T. Mora and C. M. Will, Phys. Rev. D **69**, 104021 (2004).
[47] M. Sasaki and H. Tagoshi, Living Rev. Relativity **6**, 6 (2003).
[48] K. G. Arun, Ph.D. thesis, Jawaharlal Nehru University, New Delhi, 2006.
[49] K. G. Arun, L. Blanchet, and B. R. Iyer (unpublished).