Reprinted from "The Proceedings of the Indian Academy of Sciences", Vol. XXXV, No. 5, Sec. A, 1952

ON THE CHARACTERISTIC VIBRATIONS OF LINEAR LATTICES

By K. S. Viswanathan

(From the Raman Research Institute, Bangalore)

Received April 5, 1952

(Communicated by Sir C. V. Raman, F.R.S., N.L.)

Introduction

It is proposed in this paper to discuss the problem of the dynamical behaviour of a system which may be regarded as an one-dimensional analogue of a crystal, *viz.*, an extended chain of connected particles with a periodic structure, in other words, consisting of a great number of similar units each of which is made up of a finite number of particles with different masses held together by forces of interaction. To simplify the analysis, it is assumed that each particle interacts only with its immediate neighbours; otherwise, no restriction is laid upon the magnitude of the interacting forces.

The paper is divided into five parts. Parts I and II deal with the propagation of waves in a periodic linear lattice with p particles in each unit cell, and it is shown that the group velocities of the waves traversing the lattice vanish for $(2p-1)$ frequencies, which we call the characteristic frequencies of the lattice. In Parts III and IV, the spreading of an initial disturbance localised in a finite region of the lattice over its entire length is considered and an important result is established, *viz.*, that the resulting vibration of the particles of the lattice becomes a superposition of these $(2p-1)$ characteristic vibrations.[1] The application of the preceding results to crystal physics and the nature of the normal modes of vibration of periodic lattices are discussed in Part V.

I. Wave Propagation in Linear Lattices[2]

We shall consider a linear lattice of polyatomic molecules, consisting of p atoms in its unit cell. Each cell contains one period of the lattice and the masses of the particles in any cell are denoted by $m_1, m_2, \ldots . m_p$. We assume small displacements of the particles when a wave propagates along the lattice, but we restrict the interaction of the particles to their immediate neighbours alone. The force constant between the qth and the $(q+1)$th particles of any cell is represented by σ_q, σ_1 denoting the interaction constant between the last particle in any cell and its neighbour in the next cell. The notation adopted here is explained in Fig. 1, where the small circles indicate the equilibrium positions of the particles and the vertical lines the boundaries of any cell.

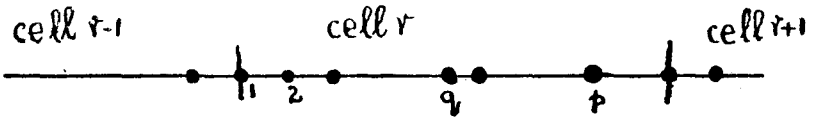

FIG. 1

If $x_{q,r}$ denotes the displacement of the qth particle in the rth cell about its equilibrium position, the equations of motion for the particles in the rth cell are given by

$$\begin{aligned}
-m_1\ddot{x}_{1,r} &= (\sigma_p + \sigma_1)\, x_{1,r} - \sigma_p x_{p,r-1} - \sigma_1 x_{2,r} \\
-m_q\,\ddot{x}_{q,r} &= (\sigma_{q-1} + \sigma_q)\, x_{q,r} - \sigma_{q-1} x_{q-1,r} - \sigma_q x_{q+1,r} \quad (q = 2, 3 \ldots . p-1) \\
-m_p\ddot{x}_{p,r} &= (\sigma_{p-1} + \sigma_p)\, x_{p,r} - \sigma_{p-1} x_{p-1,r} - \sigma_p\, x_{1,r+1} \qquad (1)
\end{aligned}$$

If we take the zero configuration for the potential energy as the energy of the lattice when all the particles are in their equilibrium positions, and assume that the displacements of the particles from their mean positions are due to an initial disturbance imparted to a finite region of the lattice, then the total energy of the lattice is a finite constant; in this case, the above equations can easily be obtained from a Lagrangian formulation. The expressions for the kinetic and potential energies of the lattice are given by

$$2\mathrm{T} = \sum_r \sum_q m_q \dot{x}_{q,r}^{\,2} \qquad (1a)$$

$$2\mathrm{V} = \sum_r \left[\sigma_p (x_{1,r} - x_{p,r-1})^2 + \sum_{q=1}^{p-1} \sigma_q (x_{q+1,r} - x_{q,r})^2 \right]$$

the summation extending over all the cells of the lattice. It can easily be seen that Lagrange's equations of motion for the particles of the rth cell of the lattice reduce to equations (1).

We shall assume wave solutions for equations (1) of the form

$$x_{q,r} = f_q e^{i(wt - r\theta)} \qquad (q = 1, 2, \ldots . p), \qquad (2)$$

where f and w are functions of the real variable θ.

Each of these p equations represents a wave propagating through particles of the same type in the lattice arrangement. The frequencies and wavelengths for a given disturbance must be equal, but the amplitudes of the waves are not necessarily equal; they may differ in phase as well as in magnitude. The amplitudes of the displacements of equivalent particles in successive cells are equal when a wave passes through them, but their phases change continuously.

Substituting the set of equations (2) in (1) we get

$$\begin{aligned} &a_1 f_1 - \sigma_1 f_2 - \sigma_p f_p e^{i\theta} = 0 \\ &-\sigma_{q-1} f_{q-1} + a_q f_q - \sigma_q f_{q+1} = 0 \quad (q = 2, 3, \ldots . p-1) \\ &-\sigma_p f_1 e^{-i\theta} - \sigma_{p-1} f_{p-1} + a_p f_p = 0 \end{aligned} \tag{3}$$

where

$$a_q = (\sigma_{q-1} + \sigma_q - m_q w^2), \qquad (q = 2, 3, \ldots . p)$$

and

$$a_1 = (\sigma_p + \sigma_1 - m_1 w^2)$$

Eliminating the f's from the equations (3), we get

$$\begin{vmatrix} a_1 & -\sigma_1 & \cdot & \cdot & \cdot & & & -\sigma_p e^{i\theta} \\ -\sigma_1 & a_2 & -\sigma_2 & \cdot & & & & \\ \cdot & -\sigma_2 & a_3 & -\sigma_3 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot \\ & & & & & & & \\ & & & & & & & \\ \cdot & & & & & -\sigma_{p-2} & a_{p-1} & -\sigma_{p-1} \\ -\sigma_p e^{-i\theta} & & & & & & -\sigma_{p-1} & a_p \end{vmatrix} = 0 \tag{4}$$

Expanding this determinant along the first row and column by Cauchy's method, we get

$$a_1 A_{11} - \sigma_1^2 |\, A_{11,22} | - \sigma_1 \sigma_p e^{-i\theta} |\, A_{11,2p} | - \sigma_1 \sigma_p e^{i\theta} |\, A_{11,p2} | - \sigma_p^2 |\, A_{11,pp} | = 0 \tag{5}$$

If we denote the (ij)th element of the above determinant by a_{ij} and the cofactor of a_{ij} by A_{ij}, then $-|\, A_{11,ij} \,|$ represents the cofactor of a_{i1} in $|\, A_{1j} \,|$. The notation is adopted from Aitken's *Determinants and Matrices.*[3]

It can easily be seen that $|\, A_{11,2p} \,| = |\, A_{11,p2} \,|$ and each is equal to $\sigma_2 \sigma_3 \ldots . \sigma_{p-1}$.

Hence (5) becomes

$$\Delta_p (w^2) - \sigma_p^2 \,|\, A_{11,pp}^{(w^2)} \,| = 2\sigma_1 \sigma_2 \ldots . \sigma_p \cos \theta \tag{6}$$

where

$$\Delta_p (w^2) = \begin{vmatrix} a_1 & -\sigma_1 & \cdot & \cdot & & & \cdot \\ -\sigma_1 & a_2 & -\sigma_1 & \cdot & & & \\ \cdot & -\sigma_2 & a_3 & -\sigma_3 & & & \\ & & & & & & \\ & & & & & & \\ \cdot & & & & -\sigma_{p-2} & a_{p-1} & -\sigma_{p-1} \\ \cdot & & & & \cdot & -\sigma_{p-1} & a_p \end{vmatrix}$$

$|A_{11,pp}|$ can be obtained by suppressing the first and the last rows and columns of the above determinant.

From the difference equations satisfied by the continuants, it can easily be proved that

$$\Delta_p(0) = \sigma_1\sigma_2\ldots\ldots\sigma_p\left(2 + \sum_{r=1}^{p-1}\frac{\sigma_p}{\sigma_r}\right) \tag{7}$$

and

$$|A_{11,pp}^{(0)}| = \sigma_1\sigma_2\ldots\ldots\sigma_{p-1}\sum_{r=1}^{p-1}\frac{1}{\sigma_r}. \tag{8}$$

Hence the term independent of w in (6) is equal to $2\sigma_1\sigma_2\ldots\sigma_p(1-\cos\theta)$.

Equation (6) can be expressed as a polynomial in w^2 and we can rewrite (6) as

$$\phi(w^2) = s_0w^{2p} + s_1w^{2p-2} + \cdot + s_{p-t}w^{2t} + \cdot + s_p(1-\cos\theta) = 0 \tag{9}$$

The coefficients $s_0, s_1, \ldots\ldots s_p$ are expressed below as symmetric functions in the masses and force constants of the lattice.

$$s_0 = (-1)^p m_1m_2\ldots\ldots m_p$$

$$s_1 = (-1)^{p-1}m_1m_2\ldots\ldots m_p\left(\sum_{r=1}^{p}\frac{\sigma_{r-1}+\sigma_r}{m_r}\right), \text{ where } \sigma_0 = \sigma_p.$$

$$s_{p-t} = (-1)^t\sigma_1\sigma_2\ldots\ldots\sigma_p{}^2\,\Sigma\, m_{i_1}m_{i_2}\ldots\ldots m_{i_t}(A-B). \tag{10}$$

where

$$A = \left(\frac{1}{\sigma_p}+\frac{1}{\sigma_1}+\ldots+\frac{1}{\sigma_{i_1-1}}\right)\left(\frac{1}{\sigma_{i_1}}+\frac{1}{\sigma_{i_1+1}}+\cdot+\frac{1}{\sigma_{i_2}}\right)\ldots\ldots$$
$$\left(\frac{1}{\sigma_{i_t}}+\frac{1}{\sigma_{i_t+1}}+\cdot+\frac{1}{\sigma_p}\right)$$

and

$$B = \left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}+\cdot+\frac{1}{\sigma_{i_1-1}}\right)\left(\frac{1}{\sigma_{i_1}}+\frac{1}{\sigma_{i_1+1}}+\ldots+\frac{1}{\sigma_{i_2}}\right)\cdots$$
$$\left(\frac{1}{\sigma_{i_t}}+\frac{1}{\sigma_{i_t+1}}+\cdot+\frac{1}{\sigma_{p-1}}\right).$$

The summation above extends over all the different t combinations of the symbols $i_1, i_2, \ldots\ldots i_p$ which represent the first p natural numbers in some order or other. We assume without loss of generality that the i's in any term of the summation are arranged in order of increasing magnitude. The two different sums involving A and B in (10) represent the coefficient of w^{2t} in $\Delta_p(w^2)$ and $\sigma_p{}^2|A_{11,pp}|$ respectively; clearly $(A-B)$ is positive and B vanishes when $i_1 = 1$ or $i_t = p$.

$$s_{p-1} = -\sigma_1\sigma_2\ldots\ldots\sigma_p(m_1+m_2+\cdot+m_p)\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}+\ldots\ldots+\frac{1}{\sigma_p}\right)$$

$$s_p = 2\sigma_1, \sigma_2\ldots\ldots\sigma_p.$$

s_q and s_{q-i} being always of the opposite signs, it follows from Descarte's rule of signs that equation (9) cannot have any negative roots for w^2 and hence all its roots are real and positive. $\frac{\partial \phi}{\partial w^2}$ is a polynomial of degree $(p-1)$ in w^2 with constant coefficients and hence all the roots of $\frac{\partial \phi}{\partial w^2} = 0$ are independent of θ. It follows that (9) cannot have any multiple root which is a function of θ. In this paper, we consider only the case of non-degenerate vibrations in which all the roots of (9) are different for all values of θ in $(0, 2\pi)$. Since (9) is an equation of degree p in w^2, for each value of θ we get p values for w^2 which are the roots of (9).

From (3) and (9) we can easily see that f and w are periodic functions of θ with period 2π and hence the values of θ outside the range $(-\pi, \pi)$ are mere repetitions of those inside the closed interval $-\pi \leqslant \theta \leqslant \pi$ for $x_{q,r}$ in (2). Therefore, we write

$$-\frac{1}{2d} \leqslant \frac{1}{\lambda} \leqslant \frac{1}{2d}, \text{ where } \frac{1}{\lambda} = \frac{\theta}{2\pi d} \tag{11}$$

$\therefore |\lambda| \geqslant 2d$ and hence the shortest wavelength equals twice the lattice spacing.

II. Group Velocity of the Waves

Differentiating (9) with respect to θ, we arrive at the following expression for the group velocity of the waves traversing the lattice

$$\frac{d\nu}{d\left(\frac{1}{\lambda}\right)} = \mathrm{U}(\theta) = \frac{-s_p d \sin\theta}{2w\left(ps_0 w^{2p-2} + (p-1)s_1 w^{2p-4} + \ldots + s_{p-1}\right)}. \tag{12}$$

When $\theta = \pi$, $\frac{d\nu}{d\left(\frac{1}{\lambda}\right)}$ vanishes and thus for each of the p frequencies corresponding to this value of θ, the group velocity of the waves vanishes. When however $\theta = 0$, one of the roots of (9) becomes zero and the expression (12) tends to a definite limit $d\sqrt{\frac{-s_p}{s_{p-1}}}$ corresponding to this zero frequency. For the remaining $(p-1)$ roots of (9) which are non-zero, the expression (12) is equal to zero. *Thus, for a lattice with* p *particles in its unit cell, there are* (2p -1) *characteristic frequencies for which the group velocities of the waves are zero.*

The group velocity is the velocity with which the maximum amplitude of a train of waves of a given frequency travels and for an observer moving with this velocity in the direction of propagation of the waves, the disturbance will turn into an approximately simple harmonic wave-train of an assigned wave-length λ. The fact that the group velocity of the waves corresponding

to the $(2p-1)$ characteristic frequencies vanishes is highly significant and implies that wave clusters corresponding to these characteristic frequencies do not move at all while individual waves may spread along the lattice with their wave velocities. Hence if a finite region of the lattice suffers an initial disturbance, all wave clusters other than those that correspond to the limiting frequencies would have spread along both sides of the lattice so that after a certain time, the motion of the particles near the domain of initial disturbance will be predominantly influenced by wave-trains with zero group velocity. It follows that the state of movements of these particles is one of a stationary vibration, consisting of a superposition of the $(2p-1)$ characteristic vibrations of the lattice—a result that is rigorously proved in Section III.

III. EFFECT OF AN INITIAL DISTURBANCE

In this Section, we discuss the spreading of an initial disturbance in the form of small displacements of a group of particles in any cell into those regions which were previously undisturbed. Due to the interaction of the particles with their neighbours, the energy concentrated in a limited region of the lattice is progressively transferred to the entire length of the lattice and thus all the particles are set in agitation and execute small oscillations about their mean positions. The general expressions for the displacements of the particles from their mean positions can be obtained by superposing wave solutions of the type (2) for all values of θ lying in one period of w, *i.e.*, within the closed interval $(0, 2\pi)$. Hence, if we denote the roots of (9) by $w_1, w_2 \ldots w_p$; $-w_1, -w_2, \ldots -w_p$ and the corresponding values of f_q by $f_{q,1}$; $f_{q,2}$; $\ldots f_{q,p}$; $f_{q,p+1}; \ldots f_{q,2p}$ respectively, the general expressions are given by

$$x_{q,r} = \frac{1}{4\pi} \sum_{s=1}^{p} \int_0^{2\pi} [f_{q,s}(\theta)\, e^{iw_s t} + f_{q,s+p}(\theta)\, e^{-iw_s t}]\, e^{-ir\theta} d\theta$$

$$(q = 1, 2, \ldots p) \qquad (13)$$

Let us now suppose that initially all the particles in the cell with index zero are displaced by small amounts and the velocities and displacements of all other particles are zero.

$$(i.e.)\ x_{q,r}(0) = \alpha_q \delta_{r0} \text{ and } \dot{x}_{q,r}(0) = 0 \qquad (14)$$

for all q and r.

Then from (13) we get

$$x_{q,r}(0) = \frac{1}{4\pi} \sum_{s=1}^{p} \int_0^{2\pi} (f_{q,s} + f_{q,s+p})\, e^{-ir\theta}\, d\theta, \qquad (15)$$

Let us now expand $\sum_{s=1}^{p}(f_{q,s}+f_{q,s+p})$ as a Fourier series in the form $\sum_{s=0}^{p}(f_{q,s}+f_{q,s+p})=\sum_{-\infty}^{+\infty}A_k\,e^{ik\theta}$, so that $A_k=\frac{1}{2\pi}\int_0^{2\pi}\sum_{s=1}^{p}(f_{q,s}+f_{q,s+p})\,e^{-ik\theta}\,d\theta$.

Then from (13) and (14) we get

$$a_q=\frac{1}{4\pi}\int_0^{2\pi}(f_{q,s}+f_{q,s+p})\,d\theta$$

$$0=\frac{1}{4\pi}\int_0^{2\pi}(f_{q,s}+f_{q,s+p})\,e^{-ir\theta}\,d\theta \text{ for all } r\neq 0.$$

Since $\frac{1}{2\pi}\int_0^{2\pi}e^{i(k-r)\theta}\,d\theta=\delta_{kr}$, we get

$$\sum_{s=1}^{p}(f_{q,s}+f_{q,s+p})=2a_q\ (q=1,2,\ldots.p) \tag{16}$$

Similarly, from (14) we get

$$0=\frac{1}{4\pi}\sum_{s=1}^{p}\int_0^{2\pi}w_s\,(f_{q,s}-f_{q,s+p})e^{-ir\theta}\,d\theta \text{ for all } r.$$

$$\therefore\ \sum_{s=1}^{p}w_s(f_{q,s}-f_{q,s+p})=0\ (q=1,2,\ldots.p) \tag{17}$$

Now, from the set of equations (3), the ratios of the quantities $f_1, f_2\ldots.f_p$ can be uniquely determined. Let us suppose that when $w=w_s$, the ratios are given by

$$f_{1,s}:f_{2,s}:\ldots.:f_{p,s}=g_1(w_s^2):g_2(w_s^2):\quad :g_p(w_s^2).$$

Also, since the equations (3) are unaffected by writing $-w$ for w, we have

$$f_{1,s+p}:f_{2,s+p}:\ldots.:f_{p,s+p}=g_1(w_s^2):g_2(w_s^2):\ldots.:g_p(w_s^2).$$

We shall write $f_{q,s}=\lambda_s g_q(w_s^2)$ and $f_{q,s+p}=\lambda_{s+p}g_q(w_s^2)$ (18)

The g's are known from equations (3) and the λ's are the unknown quantities. By substituting (18) in (16) and (17) we get $2p$ equations in the λ's which enable us to determine them in terms of the g's and a's. Thus the equations (3) together with the initial conditions are sufficient to determine the f's. To obtain the actual values of the f's as functions of the frequencies of the waves, the following procedure may be adopted. From the initial conditions (14) and the set of equations (1), the initial values of the higher derivatives of the displacements of the particles can be determined and hence from the initial values of the derivatives of order up to $(2p-1)$ of the displacements of the particles, we get the following $2p$ linearly independent equations

$$\sum_{s=1}^{p} w_s^{2r} (f_{q,s} + f_{q,s+p}) = h_{q,r} \; (r = 0, 1, \ldots . p - 1) \tag{19}$$

$$\sum_{s=1}^{p} w_s^{2r+1} (f_{q,s} - f_{q,s+p}) = 0 \quad (r = 0, 1, \ldots . p - 1). \tag{20}$$

where $h = h(m, \alpha, \sigma, \theta)$. For example, $h_{q,o} = 2\alpha_q$, etc. Since by assumption the roots of (9) are all different, the square matrix of the coefficients of $(f_{q,s} - f_{q,s+p})$ in (20) is non-singular and hence it follows that

$$f_{q,s} = f_{q,s+p} \; (q, s = 1, 2, \ldots . p) \tag{21}$$

The equations (13) can now be written as

$$x_{q,r} = \frac{1}{4\pi} \sum_{s=1}^{p} \int_{0}^{2\pi} f_{q,s} (e^{iw_s t} + e^{-iw_s t}) e^{-ir\theta} \, d\theta$$

$$= \frac{1}{4\pi} \sum_{s=1}^{p} \int_{-\beta}^{2\pi-\beta} f_{q,s} (e^{iw_s t} + e^{-iw_s t}) e^{-ir\theta} \, d\theta \quad (q = 1, 2, \ldots . p) \tag{22}$$

where $0 < \beta < \pi$ (since f and w are periodic functions of θ).

We shall find the asymptotic value of the above integral for large values of t by employing Kelvin's method of stationary phase.[4] When $w_s t$ is large compared to r, $\cos w_s t$ is a rapidly oscillating function and hence the maximum contribution of the above integral arises from those points at which w_s is stationary (*i.e.*) at the points where $\frac{dw_s}{d\theta} = 0$. $\therefore \quad \theta = 0$ or π.

Now the asymptotic value of the integral[4] $I = \int_a^b f(x) e^{iw(x)t} \, dx$ for large values of t is given by

$$I \sim \sum_{x_0} \frac{\sqrt{2\pi} \, f(x_0) \, e^{i\left(w(x_0)t \pm \frac{\pi}{4}\right)}}{| w''(x_0) \, t |^{\frac{1}{2}}}$$

where the positive or negative sign in the exponent is to be taken according as $w''(x_0)$ is positive or negative and the summation extends over all the points in the interval (a, b) which are solutions of the equation $\frac{dw}{dx} = 0$.

Applying this result to (22), we find that the asymptotic value of $x_{q,r}$ for large values of t is given by

$$x_{q,r} = \left(\frac{1}{2\pi t}\right)^{\frac{1}{2}} \sum_{s=2}^{p} \frac{f_{q,s}(0)}{| w_s''(0) |^{\frac{1}{2}}} \cos\left(v_s t \pm \frac{\pi}{4}\right)$$

$$+ (-1)^r \left(\frac{1}{2\pi t}\right)^{\frac{1}{2}} \sum_{s=1}^{p} \frac{f_{q,s}(\pi)}{| w_s''(\pi) |^{\frac{1}{2}}} \cos\left(v_{s+p} t \pm \frac{\pi}{4}\right),$$

$$(q = 1, 2, \ldots . p) \tag{23}$$

wherethe positive or negative sign in the cosine terms is to be taken according as $w_s''(\theta_0)$ $(\theta_0=0, \pi)$ is positive or negative. v_s and v_{s+p} are the values of $w_s(\theta)$ when $\theta = 0$ and π. $f(0)$ and $f(\pi)$ are real and $v_1 = 0$. *Hence, asymptotically*

(1) *the state of vibration of the particles tends to a superposition of the* $(2p-1)$ *characteristic vibrations of the lattice and*

(2) *the amplitudes of vibrations of the particles vary inversely as the square root of the time elapsed.*

IV. Particular Cases

As a particular case of the above general theory, we shall consider the vibrations of a monatomic linear lattice which was first studied by Hamilton in connection with his researches on the theory of the dispersion of light.[5] The equation of motion for the rth particle is given by $m\ddot{x}_r = \sigma(x_{r-1} + x_{r+1} - 2x_r)$, where m is the mass of any particle and σ is a force constant.

If we assume wave solutions for the above equations of the form

$$x_r = f(\theta)\, e^{i(wt - r\theta)} \tag{24}$$

we can easily see that w is given by the roots of the equation

$$mw^2 = 4\sigma \sin^2 \theta/2. \tag{25}$$

If the initial conditions are $x_r(0) = \alpha\delta_{r0}$

$$\text{and } \dot{x}_r(0) = 0 \tag{26}$$

for all r, then the general expressions (22) for the displacements of the particles about their equilibrium positions reduce to

$$\begin{aligned} x_r &= \frac{\alpha}{4\pi}\int_0^{2\pi} (e^{iwt} + e^{-iwt})\, e^{-ir\theta}\, d\theta \\ &= \frac{\alpha}{\pi}\int_0^{\pi} \cos(vt \sin\theta) \cos 2r\theta d\theta \\ &= \alpha J_{2r}(vt), \end{aligned} \tag{27}$$

where $v = \sqrt{4\sigma/m}$—we arrive at the same result obtained by Hamilton in a different way.

Hence when vt is large compared to r, the asymptotic value of (27) becomes

$$x_r = \left(\frac{2}{\pi vt}\right)^{\frac{1}{2}} (-1)^r \cos\left(vt - \frac{\pi}{4}\right) \tag{28}$$

showing that the motion is periodic with a frequency that is precisely the characteristic frequency of the lattice.

It is interesting to consider the way in which the nth particle begins its motion. The Bessel function attains its maximum value when its argument is slightly greater than its order so that the maximum amplitude of this particle is attained only after a time given by the inequality $vt > 2n$. Also since $J_{2n}(z)$ approaches zero as z^{2n} and the first value of z for which $J_{2n}(z)$ vanishes lies between $\sqrt{2n(2n+2)}$ and $\sqrt{2(2n+1)(2n+3)}$, we can easily see that the larger n is, the slower is the beginning of the motion and a longer time does this particle take to reach its maximum in one direction. Finally, when vt is large compared to n so that the asymptotic approximation for the Bessel function can be applied to (28), all the masses within the belt $\pm nd$ and those in the neighbourhood of the nth particle vibrate with the limiting frequency of the lattice, consecutive particles always oscillating with opposite phases.

The case $p = 2$ corresponds to the vibrations of a linear lattice of diatomic molecules which was recently discussed by Nagendra Nath and S. K. Roy.[6,7] By evaluating the values of the f's as functions of the frequencies of the waves, we can easily see that (23) reduces to the expressions obtained by these authors by using the method of steepest descents.

V. SOME GENERAL REMARKS

The dependence of the amplitudes of the particles on time is instructive and indicates that the vibrations decay slowly, the law of decrease being $t^{-\frac{1}{2}}$. This is due to the fact that the initial energy is progressively transferred to the neighbouring cells and the amplitude of any particle is proportional to the square root of its vibrational energy. The damping of the vibrations suggests the possibility of restricting our discussion to a finite but long linear lattice, with its ends far from the region of initial disturbance. In such a case, the outgoing cluster of waves would have almost died out before it reaches the end and gets reflected, and hence the boundary exerts practically an insignificant influence on the subsequent vibrations of the particles of the lattice. Again for all observations confined within the time-limit T which is the time taken by the fastest group to reach the end, the above analysis holds good rigorously and T can be made sufficiently large by taking the length of the lattice great enough.

Restricting ourselves to the region in which the disturbance has settled into a stationary mode of oscillation, the asymptotic form (23) suggests a striking similarity of the vibrations of the particles with the normal modes

of vibrations of polyatomic molecules characterised by a common frequency and phase for the vibrations of all the atoms of the molecule. The displacements of the particles of the lattice from their equilibrium positions arise due to the transmission of the initial energy by waves of all possible wavelengths and amplitudes to them, but the asymptotic state of their movements is built up of a superposition of $(2p-1)$ normal modes of vibration which differ entirely in their behaviour from a wave-motion involving a continuous change in the phases of particles. The $(2p-1)$ characteristic modes of vibration fall into two classes, one consisting of $(p-1)$ modes in which equivalent particles in successive cells vibrate with the same phase and amplitude at any instant, while in the other p modes due to the dependence of the amplitudes on the factor $(-1)^r$, equivalent particles in successive cells vibrate with opposite phases. Thus there is a complete agreement of these results with the general theory for three-dimensional lattices proposed by Sir C. V. Raman[1] that a crystal with p atoms in its unit cell has $(24p-3)$ normal modes of vibration of which $21p$ modes belong to a set in which vibration occurs with alternate layers moving with opposite phases while in the remaining $(3p-3)$ modes, equivalent particles in consecutive cells vibrate with the same amplitudes and phases.

The author feels deeply grateful to Professor Sir C. V. Raman for the many useful discussions he had with him and for his valuable suggestions and guidance in the preparation of this paper.

Summary

The nature of wave propagation in a linear lattice consisting of p particles in its unit cell has been studied and it is shown that the group velocity of the waves corresponding to each of the $(2p-1)$ characteristic frequencies of the lattice is identically equal to zero. It has been further proved that the state of movements of the particles resulting from an initial disturbance localised in a finite region of the lattice tends asymptotically to a superposition of the $(2p-1)$ characteristic vibrations of the lattice. In $(p-1)$ modes of these vibrations, equivalent particles in successive cells vibrate with the same phases and amplitudes, while in the remaining p normal modes, vibration occurs with equivalent particles moving alternately with opposite phases. These results form an one-dimensional analogue of the general theory of vibrations of crystal lattices formulated by Sir C. V. Raman and are generalisations of the results corresponding to the cases $p=1$ and $p=2$ which were first obtained respectively by Hamilton and by Nagendra Nath and S. K. Roy. In view of the decay of the amplitudes of vibrations of the particles with time, these results hold good for a finite lattice

also, provided its length is sufficiently large and the domain of initial disturbance is far from its ends.

REFERENCES

1. Raman, C. V. .. *Proc. Ind. Acad. Sci.*, 1943, **18**, 237.
2. Brillouin, L. .. *Wave Propagation in Periodic Structures* (McGraw Hill Book Co., 1946).
3. Aitken, A. C. .. *Determinants and Matrices* (Pp. 74 and 87).
4. Lamb .. *Hydrodynamics.*
5. Hamilton .. *Collected Papers*, p. 452.
6. Nagendra Nath and Roy, S. K. *Proc. Ind. Acad. Sci.*, 1948, **28**, 289.
7. ———— .. *Ibid.*, 1950, **31**, 330.