# CHAPTER 2

# INTRODUCTION

## 2.1 Brief account of the structure of typical galaxies

The extragalactic objects that can be seen in any wide angle photograph of the sky are all galaxies. In regions of high galaxy density upto half the number of all galaxies are lenticular galaxies (SOs), about 40% are elliptical galaxies and only about 10% are spiral galaxies. The other 10% are classified as irregular galaxies. In low density regions nearly 80% of all galaxies are spirals and about 10% are ellipticals and less than 10% are SOs. Below we briefly describe these different types of galaxies. For a detailed review of the structure of galaxies see Kormendy (1982).

Spiral galaxies have a prominent stellar disc which includes young stars, gas and dust arranged in the form of spiral arms. The spiral arms are sites of star formation and they vary in size and structure from galaxy to galaxy. Matter in the disc seems to move in circles about the centre of the galaxy. A remarkable feature is that the circular speed is nearly independent of distance from the centre (except near the centre where the speed drops to zero). Typical rotation speeds are between 200 km/sec and 300 km/sec and the rotation speed remains constant even at distances far beyond the edge of the (optically) visible parts of the galaxy. This fact is conventionally interpreted to imply the presence of large amounts of matter (dark matter) whose presence is felt through its gravitational interaction with visible matter. Spirals also have old stars which are usually found in a halo surrounding the disc. The bulges of spirals are dominated by these old stars. Our Galaxy and our famous neighbour M31 are typical spirals. We give below a few properties of our Galaxy.

Disc Mass $\sim 6 \times 10^{10} M_\odot$

Gas Mass $\sim 1 \times 10^{10} M_\odot$

Mass in dark matter $\sim 4.5 \times 10^{11} M_\odot$

Disc scale length (characterising exponentially decaying surface density) $\sim 3.5$ kpc

Radius of Sun's orbit about the Galactic centre $\sim 8.5$ kpc

Circular speed $\sim$ 220 km/sec

where $1 M_\odot = 1.99 \times 10^{33}$ g and 1 kpc $= 3.086 \times 10^{21}$ cm.

Elliptical galaxies are apparently smooth and featureless containing old stars but very little gas or dust. Since we see only the projected surface brightness, it is not possible to determine directly whether they are axisymmetric or triaxial. Their rotation speeds are low and even highly flattened ellipticals balance self gravity by anisotropic pressure (Binney 1976, Illingworth 1977). These systems are "hot" and the dispersion of peculiar velocities is of the order of a few hundreds of km/sec. The relative simplicity of structure makes ellipticals ideal laboratories for studying the dynamics of a large number of stars divorced from the complications introduced by gas dynamics (this does not imply that gas dynamics was unimportant in the formation process for which both dissipational and dissipative scenarios have been explored). The surface brightness profile of an elliptical galaxy falls off smoothly with increasing distance from the centre and it is often impossible to detect the "edge" of the galaxy. Most ellipticals have surface brightness profiles that are well fit by the de Vaucouleurs (or $R^{1/4}$) law

$$I(R) = I(0)\exp(-kR^{1/4})$$

where $I$ = surface brightness

$R$ = radius

$k$ = constant

Lenticular galaxies (SOs) have smooth, featureless discs that contain no gas, young stars or spiral arms.

Irregular galaxies are low luminosity, gas rich systems. Many near neighbours of our Galaxy are irregulars, the most famous of them being the Large and Small Megallanic Clouds.

## 2.2 Dynamics of isolated galaxies

The structure and dynamics of a galaxy is determined mainly by gravitational interactions between its constituents. Gas and dust contribute to only about 10% of a galaxy's visible mass. So we expect that they will have only a small effect on the dynamics of "hot" systems like elliptical galaxies. The stellar content of galaxies is dominated by low mass stars whose lifetimes are about as long as the age of the galaxies. Ellipticals contain old (low mass) stars, so stellar evolution has practically no effect on the dynamics and stars may be treated as idealized point masses interacting with each other and with the other constituents through an inverse square attractive force. Spirals on the other hand have young stars as well. When these blow up as supernovae, star formation may be triggered and thus stellar evolution might play some role in the dynamics. Gas in spirals has small peculiar velocities and is hence gravitationally responsive. The effect on instabilities in the disc need not be small. Spirals also have large amounts of dark matter whose composition is as yet unknown. By and large, the internal dynamics of a galaxy is dominated by its stellar content and dark matter when present in sizable amounts. On an average a galaxy has $\sim 10^{11}$ stars. For essentially every proposed form of dark matter simple estimates predict that over times of the order of the age of the Universe ($\sim 10^{10}$ years), binary (star-star, dark matter - dark matter and star - dark matter) interactions have small effect on orbits calculated by assuming that the mass density distribution in the galaxy is smooth.

Bearing all this in mind, an idealised but useful model of a galaxy is a fluid in 6 dimensional phase space described at any instant of time by a distribution function $f(\underline{x}, \underline{v}, t)$ where $f(\underline{x}, \underline{v}, t)\, d^3x\, d^3v$ is the mass of the galaxy contained in phase volume $d^3x\, d^3v$. This phase fluid moves under the action of self gravity. Since binary (and higher order) collisions are absent, the equations of motion for any point $(\underline{x}, \underline{v})$ in phase space are

$$\dot{\underline{x}} = \underline{v}$$

$$\dot{\underline{v}} = -\frac{\partial \varphi}{\partial \underline{x}}$$

where $\varphi(\underline{x}, t)$ is the gravitational potential at $\underline{x}$ (at time t). By Liouville's theorem phasevolumes ( $d^3x\, d^3v$ integrated over any 6 dimensional region of phase space) are conserved. This implies that $f$ is conserved along the trajectory of any point in phase space. This self consistent dynamics is described by the collisionless Boltzmann equation - CBE - which resembles the Vlasov-Poisson equations used commonly in plasma physics:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} - \frac{\partial \varphi}{\partial \underline{x}} \cdot \frac{\partial f}{\partial \underline{v}} = 0 \qquad (2.1a)$$

where $\varphi$ is self consistently determined by Poisson's equation

$$\nabla^2 \varphi = 4\pi G \rho = 4\pi G \int f\, d^3v \qquad (2.1b)$$

( $G$ is Newton's gravitational constant).

Galaxies are usually modelled as stationary solutions of the CBE ( $\frac{\partial f}{\partial t} = 0$) and models of rotating galaxies are assumed to be stationary in the rotating frame. It is difficult to test the assumption of stationarity, It is made for simplicity and convenience because (i) there is theoretical/numerical evidence for relaxation to a stationary state (see Section 2.3 below) (ii) a stationary model has fewer parameters and it is logical to try these first.

It is not difficult to build stationary galaxy models. In fact there are so many different models (i.e. different $f(\underline{x},\underline{v})$), one wonders if there is not some physical principle that restricts choice. The reason for the wide variety of stationary models is simple. Since the CBE is essentially $\frac{df}{dt} = 0$, any $f$ that is an arbitrary function of the constants of motion will solve (1.1a). To be self consistent, the assumed constants of motion should be consistent with the potential determined by (1.1b). Of course, f should be everywhere positive and give finite mass, energy etc. This principle is called Jeans' theorem and it is the most commonly used method of constructing galaxy models.

For example, $\frac{\partial f}{\partial t} = 0$ implies that $\varphi$ is independent of time. Then $E = \frac{v^2}{2} + \varphi$ (energy per unit mass of a particle) is constant along particle trajectories. Any $f(E)$ solves (1.1a). $\varphi$ itself is now determined by

$$\nabla^2\varphi = 4\pi G \int f\left(\frac{v^2}{2} + \varphi\right) d^3v = 4\pi G \rho(\varphi)$$

This nonlinear equation together with the boundary condition $\varphi \longrightarrow 0$ as $r \longrightarrow \infty$ determines $\varphi$. It is interesting to note that the only finite mass solutions are the spherically symmetric ones.

The distribution function provides the most detailed description of the galaxy. We can get a grosser description by taking moments of the CBE in velocity space. This gives us equations resembling the continuity equation, the Euler equation etc. of fluid dynamics and hence they are called stellar hydrodynamical equations (or Jeans' equations). The main trouble with this approach is that in the absence of collisions, the pressure tensor is not isotropic and infact there is usually no equation of state. The heirarchy of equations of motion for higher and higher velocity space moments does not close and one is forced to deal with the CBE in its pristine form.

An approach that is more macroscopic leads to interesting predictions. This is the Virial Theorem (VT) for stellar systems

$$\frac{1}{2}\ddot{I} = 2T + W \tag{2.2}$$

where the second moment of Inertia $I = \int f r^2 d^3x$, $T$ = total kinetic energy and $W$ = total potential energy. In this form the VT is a dynamical statement about the overall dynamics. When the galaxy is in a steady state (Virial equilibrium), we have $\langle \ddot{I} \rangle = 0$ where $\langle \ \rangle$ is a time average. Hence $2\langle T \rangle + \langle W \rangle = 0$. With $T \sim \frac{1}{2} M\sigma^2$ and $W \sim -\frac{GM^2}{R}$ where $M$ = mass of the galaxy, $\sigma$ = typical velocity dispersion and $R$ = measure of radius, we have

$$\sigma^2 \sim \frac{2GM}{R} \qquad (2.3)$$

This is a striking feature of self gravitating systems that is independent of details. When dealing with neutral gases in the laboratory, we may specify the temperature, mass and volume of the gas as independent quantities. But for a stellar system in Virial equilibrium these are not independent quantities. They are related to one another through (2.3). Collisions (or encounters) between stars take the system through a sequence of states (in the "precollapse" phase) each of which is in Virial equilibrium. So this equilibrium is to be thought of as a long lived meta stable state. A11 stationary solutions of the CBE are equilibrium states of the system in this meta stable sense. For a recent discussion of these issues and the effect of collisions, see Padmanabhan (1990). Binney and Tremaine (1987) - hereafter BT - provide an excellent account of many of the issues treated rather sketchily here (the reader is referred to BT for a more detailed account of these and a lot more besides). We now go on to discuss some issues concerning the time dependent behaviour of collisionless stellar systems.

## 2.3 The relaxation conjecture

Galaxy formation is a difficult problem, undoubtedly involving gas dynamics and radiative processes. A simpler, idealized problem has therefore attracted attention; can a system of a large number of point masses interacting gravitationally with each other form objects resembling galaxies? We mentioned earlier that over time scales of the order of the age of the universe, the dynamics of a large number of stars (point masses) is described accurately by the CBE. The question then is about the behaviour of time dependent solutions of the CBE.

It has been conjectured (Lynden-Bell 1967) that over a few crossing times (crossing time = time taken for a typical star to travel a distance equal to the size of the system) the stellar system would reach a coarse grained steady state in phase space. Coarse graining is essential because the CBE (which governs evolution) is a reversible equation. So strict equilibrium is impossible to reach and it is clear that steady state can be reached by the system only in a coarse grained sense, while there is activity on finer scales.

Lynden-Bell's picture of the process may be summarised as follows. While the stellar system is in a time dependent state, the gravitational force experienced by a star is also time dependent leading to rapid changes in its energy. Thus energy is redistributed among the stars. The stars in their courses move with different periods and so get out of phase with each other in a few crossing times. Any structure depending on coherence in orbital phases such as a global oscillation is damped within a few crossing times. Lynden-Bell called this process violent relaxation because it occurs so much more rapidly than collisional relaxation. Perhaps this term is best reserved for the specific final distribution derived by Lynden-Bell, keeping "collisionless relaxation" for the general process. This conjecture of relaxation has been verified subsequently by numerical experiments (see eg. van Albada 1982).

The final (steady) state produced by collisional relaxation (for example, in neutral gases) has a Maxwellian distribution of velocities and the distribution function is unique. What is the corresponding end state of collisionless relaxation?

Lynden-Bell (1967) attempted to predict the end state through statistical arguments analogous to those given for the Maxwell-Boltzmann distribution for neutral gases. If the relaxation process is violent enough (he argued), the final distribution function is the one possessing maximum entropy constrained only by mass and energy conservation and the fact that no two different phase elements can occupy the same position in phase space at the same time (follows from Liouville's theorem on conservation of phase volumes). Lynden-Bell's distribution function resembles the distribution function of particles obeying the Fermi-Dirac statistics. When self consistency is imposed, the system turns out to have infinite mass and energy.

The belief now is that while Lynden-Bell's physical picture of phase mixing and energy redistribution is correct, relaxation is just not violent enough ("incomplete relaxation"). The system seems to have more memory of its initial state than just the mass, energy and microscopic phase volumes. For example, N-body simulations of the evolution of an initially cold clumpy distribution of stars produce a relaxed state with a surface density profile that is well fit by the $R^{1/4}$ law (van Albada 1982, Mc Glynn 1984, Villumsen 1984). The collapse is as violent as can be imagined because initially the system has very little kinetic energy. For gentler collapses (Smith and Miller 1986, Quinn et al. 1986, Frenk et al. 1985) the relaxed systems seem to have flat rotation curves typical of spiral galaxies. Perhaps maximising entropy a la Lynden-Bell with more constraints might explain the violent collapses.

More recently, Tremaine, Henon and Lynden-Bell (1986) - here after THL - have suggested some constraints that any resonable, relaxing collisionless system should obey. They begin by defining a generalized entropy called by them an H-function:

$$H[f] = -\int C(f)\, d^3x\, d^3v$$

where $C(f)$ is any convex function of $f$.

$$\frac{dH}{dt} = -\int \frac{dC}{df}\frac{\partial f}{\partial t}\, d^3x\, d^3v$$

$H$ is conserved by the CBE. So at first sight the function looks useless. If a system relaxes, structures (produced in the course of evolution) on fine scales should be unimportant and one expects that repeated coarse graining (interspersed with time evolution) should not really affect the process of relaxation. THL prove the following. Suppose that at $t=0$ the system was in such an ordered state that the fine grained ($f$) and coarse grained ($f_c$) distribution functions are equal to each other. ($f_c = f$ averaged over some suitable cells in phase space; $f$ obeys the CBE while $f_c$ does not). Then at $t=0$

$$H[f(0)] = H[f_c(0)]$$

At later times. $H[f_c(t)] \geqslant H[f(t)]$

Since $\frac{dH[f]}{dt} = 0$ we have

$$H[f_c(t)] \geqslant H[f_c(0)]$$

Therefore the $H$- function has its lowest value at $t=0$. During violent relaxation, the system becomes more and more "mixed" and $H[f_c]$ increases with mixing. THL prove an interesting theorem (the Mixing theorem). They define

$$V(\eta) = \int d^3x\, d^3v\, \theta(f_c - \eta)$$

which is the phase space volume where $f_c > \eta$ ; $\theta(\eta)$ is defined by

$$\begin{aligned} \theta(\eta) &= 1 \quad \text{for} \quad > 0 \\ &= 0 \quad \text{for} \quad \eta \leq 0 \end{aligned}$$

They also define

$$M(\eta) = \int d^3x\, d^3v\, f_c\, \theta(f_c - \eta)$$

where $M(\eta)$ is the total mass of the system with $f_c > \eta$. Eliminating $\eta$, $M$ can be expressed as a function of $V$.

Theorem: For two distribution functions $f_1$ and $f_2$

$$H[f_1] > H[f_2] \text{ iff } M_1(V) < M_2(V)$$

for all $V$.

While THL haven't quite shown that $\frac{dH}{dt} \geqslant 0$ the mixing theorem is a useful inequality. If one knows from some theory of galaxy formation that such-and-such an initial state is a natural one, we can then decide if this-and-this final state is less or more mixed by computing $M(V)$. If the candidate for the final state turns out to be less mixed it can be ruled out as a possible relaxed galaxy.

## 2.4 The dynamics of time dependent solutions

While peripheral progress has been made, the details of the relaxation process are still unclear. No one has produced (an exact or even an approximate) analytic solution of the CBE that demonstrates relaxation. What little understanding exists comes from numerical simulations and Lynden-Bell's original picture. We give below Lynden-Bell's original example showing phase mixing in a time independent anharmonic potential ($\varphi(x)$) in one spatial dimension. Suppose that at $t = 0$, there is a distribution of a set of mutually noninteracting particles as shown in figure 2.1a.

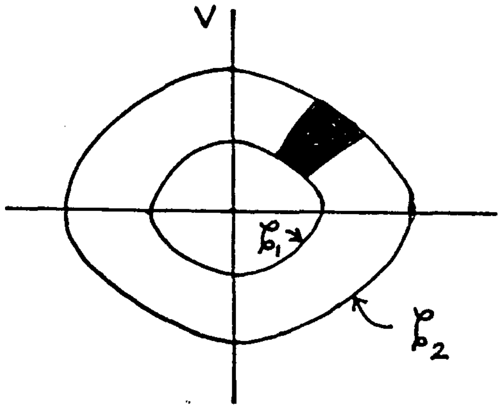



FIGURE 2.1a

The distribution function takes value 1 in the shaded region and zero outside. $\wp_1$ and $\wp_2$ are curves of constant energy equal to $E_1$ and $E_2$. The energy of any particle $E = \frac{v^2}{2} + \varphi$ is conserved under evolution. As time goes on, phase mixing, while conserving phase area occupied by the particles, stretches out the distribution because for an anharmonic potential, the period of oscillations varies with energy. After many (mean) oscillation times, the distribution function presumably looks like what is shown in figure 2.1b.

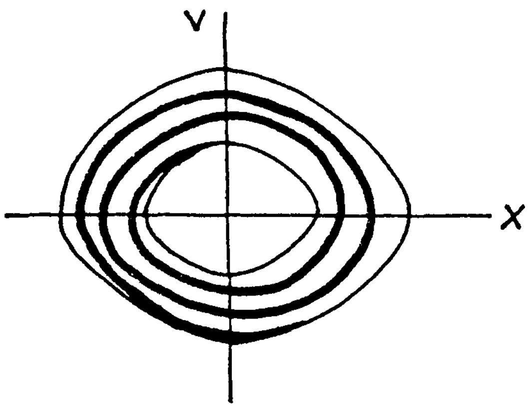



FIGURE 2.1b

At later times the distribution becomes more and more wound up. If we coarse grain (average over suitable fixed areas in the $X-V$ plane) the distribution function would seem to have reached a constant value (less than unity) everywhere in the annulus (figure 2.1c).

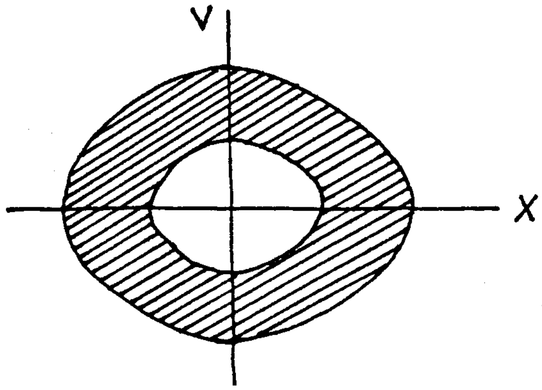



FIGURE 2.1c

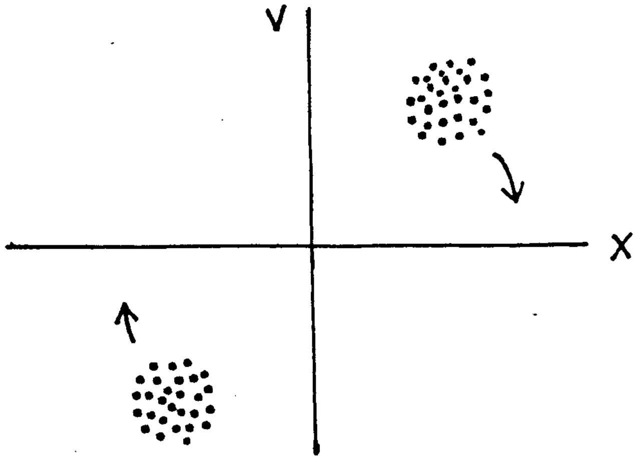
Similar mixing occurs during violent relaxation. Time dependence of the potential, .as noted earlier redistributes particle energies. The role of self consistency is unclear.

Not a11 solutions of the CBE relax to a steady state. Numerical simulations do turn up cases that show long lived oscillations (see eg. Henon 1968, Wilkinson and James 1982, Gerhard 1983). Recently Louis and Gerhard (1988) have numerically constructed a spherically symmetric galaxy model that has small but nonlinear radial oscillations.

One dimensional (ID) self gravitating systems have for years been used to study collisionless relaxation. The system may be thought of as N plane parallel sheets interacting with each other gravitationally. The sheets are constrained to move along the direction of their common normal and they can pass through each other. The interaction potential between two sheets is proportional to the distance between them. Since this potential is so much smoother than the corresponding 3-D case ($\varphi \sim -\frac{I}{r}$), we expect collisons to be operative only on very long time scales (see Reidl and

Miller 1988 and references therein). So the CBE should be a good description in 1-D too. In section 2.7 we describe a nonlinear homologous mode discovered by Kalnajs. Here we shall discuss an interesting numerical experiment performed by Reidl and Miller (1988). They put 100 particles into a "dumbell" like initial configuration which is schematically shown in figure 2.2.

FIGURE 2.2




The blobs began chasing each other in phase space and continued doing so for 600 crossing times. The blobs underwent very little structural change. Nityananda (private communication) has suggested that this is the same as the "phase locking" phenomenon (well known in driven nonlinear oscillators) in which the amplitude and phase of a particle vary but never get out of hand. Anharmonicity and time dependence seem to cancel each other. This phenomenon is perhaps related to results obtained by Henon (1968) for spherical shells. It is also the principle behind the construction of Louis and Gerhard's model (1988).

The subject of galaxy interactions has grown in importance owing to recent observational developments. On the theoretical side the problem involves modelling the time dependent behaviour of two or more interacting collisionless stellar systems (if gas is ignored). The only genera1 approach to this difficult problem is large scale numerical simulation (see eg. the impressive simulations of Barnes (1989)). Limiting cases have been treated analytically. For example (i) the passage of a small galaxy through the halo of a much larger (and more massive) system perturbs the larger system. The back reaction on the smaller galaxy produces a drag ("dynamical friction"). Chandrasekhar's (1942) formula for the dynamical friction is very often used. (ii) When the time scale of encounter between two galaxies is significantly smaller than the time scale for stellar motions in the individual galaxies, the impulse approximation (Spitzer 1958) has been used. In chapter 5 we derive the consequences of the impulse approximation in some detail for a specific model. This is preliminary to going beyond the impulse approximation in this case.

The general approach taken in this thesis is described in the next section.

## 2.5 General Strategy

In this thesis we shall use constants of motion in some special time dependent potentials to construct exact time dependent solutions of the CBE. In fact this is a straightforward generalization of the use of Jeans' theorem to construct stationary models. All the time dependent models constructed here share the property that at any instant of time the real space density is uniform over some (in general ellipsoidal) region of space and zero outside. Let us imagine that the shape and size of the region changes with time. Then, the gravitational force within the region is linear in the spatial coordinates, though time dependent. The equations of motion of any particle in the ellipsoidal region are therefore linear. Under these conditions time dependent quadratic forms in the coordinates and momenta can be found which are constants of the motion. The phase space distribution function (f) is then chosen to be a function of these and solves the CBE by Jeans' theorem. The functional form of $f$ is so chosen that integrating over velocities gives uniform density ellipsoids. Self consistency governs the evolution of the models and introduces nonlinearity into the equations governing the evolution. Below we shall derive an invariant for a time dependent harmonic oscillator in one dimension originally due to Lewis (1968). Then we use this to construct a time dependent model in one spatial dimension. This turns out to be the homolgous oscillation mode of a homogeneous stellar system stratified in plane parallel layers discovered by Kalnajs (1973) using a different method.

2.6 The Lewis Invariant for the 1-D oscillator

The equations of motion for the time dependent harmonic oscillator in one dimension are

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega^2(t)x \end{aligned} \tag{2.4}$$

The linearity of the equations (2.4) guarantees the existence of a quadratic invariant which we write as

$$I = \frac{A}{2}v^2 + Bvx + \frac{C}{2}x^2 \tag{2.5}$$

where $A$, $B$ and $C$ are time dependent coefficients

$$\dot{I} = \left(\frac{\dot{A}}{2} + B\right)v^2 + \left(\dot{B} + C - A\omega^2\right)vx + \left(\frac{\dot{C}}{2} - B\omega^2\right)x^2 \tag{2.6}$$

Requiring $\dot{I} = 0$ gives us

$$\frac{\dot{A}}{2} + B = 0 \tag{2.7a}$$

$$\dot{B} + C - A\omega^2 = 0 \tag{2.7b}$$

$$\frac{\dot{C}}{2} - B\omega^2 = 0 \tag{2.7c}$$

We shall be using $I$ to construct a time dependent model of a collisionless stellar system. So we need to choose $A, B, C$ such that $\mathcal{I}$ is positive definite (if $\mathcal{I}$ is indefinite, lines of constant $\mathcal{I}$ in the X-V plane at any instant of time are infinite in extent and hence unsuitable for building stellar systems). Positive definiteness of $\mathcal{I}$ requires $AC - B^2 > 0$. It is straightforward to verify from (2.7) that $(AC - B^2)$ is a constant. Let us choose

$$AC - B^2 = 1 \tag{2.8}$$

Using (2.8) and (2.7a) in (2.7b) we get

$$\frac{\ddot{A}}{2} - \frac{1 + \left(\frac{\dot{A}^2}{4}\right)}{A} + \omega^2 A = 0 \tag{2.9}$$

This can be written in a neater form in the variable $\xi$, where $A = \xi^2$:

$$\ddot{\xi} + \omega^2(t)\,\xi - \frac{1}{\xi^3} = 0 \tag{2.10}$$

Therefore, the invariant $\mathcal{I}$ can be written as

$$\mathcal{I} = \frac{x^2}{2\xi^2} + \frac{1}{2}\left(\xi v - \dot{\xi} x\right)^2 \tag{2.11}$$

where $\xi(t)$ is any solution of (2.10).

We note that $\mathcal{I}$ is a generalization of the adiabatic invariant for a harmonic oscillator. To see this, we observe that if $\omega^2(t)$ is a slowly varying function of time, higher derivatives of $\xi$ can be ignored and $\xi(t)$ can itself be approximated as $\omega^{-1/2}$. Using this approximation for $\xi$ in (2.11) and dropping the terms containing $\dot{\xi}$, we get

$$I \approx \frac{\omega x^2}{2} + \frac{v^2}{2\omega} = \frac{1}{\omega}\left(\frac{v^2}{2} + \frac{\omega^2 x^2}{2}\right)$$

which is the well known adiabatic invariant for a harmonic oscillator. Goldstein's (1980) text book on Classical Mechanics has an alternative derivation of the Lewis Invariant. This derivation is analogous to Gauss' trick for evaluating definite integrals named after him - embed the given problem in a higher dimension and exploit symmetry. Although it is not mentioned in the text book, the derivation also shows that the Lewis invariant is the square of the Wronskian which, of course, is conserved for the linear second order system in (2.4). We thank Prof. M.V.Berry for pointing this out.

## 2.7 Kalnajs' homologous mode

In 1-D, the CBE describes the evolution of a system of plane parallel sheets each of which is infinite in extent. The sheets are allowed movement only along the direction of their common normal. All equilibrium distribution functions in 1-D are functions of energy alone since in 1-D, energy is the only integral of motion. The distribution function for an equilibrium model with uniform density within some interval and zero outside is

$$f_o(E) = K\,(E_m - E)^{-1/2} \quad \text{for} \quad E < E_m$$
$$= 0 \quad \text{for} \quad E > E_m \tag{2.12}$$

where $K$, $E_m$ are constants and

$$E = \frac{v^2}{2} + \varphi \tag{2.13}$$

We can construct the time dependent model by replacing $E$ by the Lewis Invariant (2.11), $K$ and $E_m$ by some other constants $K'$ and $I_m$ respectively:

$$f(x,v,t) = K'(I_m - I)^{-1/2} \quad \text{for} \quad I < I_m \tag{2.14}$$
$$= 0 \quad \text{for} \quad I > I_m$$

The density

$$\rho(x,t) = \int f\,dv = \frac{\pi\sqrt{2}\,K'}{\xi} \quad \text{for} \quad |x| < \xi\sqrt{2I_m}$$

$$= 0 \quad \text{for} \quad |x| > \xi\sqrt{2I_m} \tag{2.15}$$

The potential in the region $|x| < \xi\sqrt{2I_m}$ is

$$\varphi(x,t) = \omega^2(t)\,\frac{x^2}{2} \tag{2.16}$$

where $\omega^2(t)$ is determined from Poisson's equation:

$$\omega^2(t) = 4\pi G\rho = \frac{4\pi^2 G\sqrt{2}\,K'}{\xi} = \mathcal{A}^2/\xi \qquad (2.17)$$

We recall that $\xi$ is any solution of (2.10) with $\omega^2(t)$ given by (2.17). Therefore

$$\ddot{\xi} + \mathcal{A}^2 - \frac{1}{\xi^3} = 0 \qquad (2.18)$$

From (2.15) we see that $\xi(t)$ is proportional to the size of the model. Equation (2.18) describes the oscillations of the size of the uniform density region and we can see immediately that **all** solutions are periodic functions of time. We thus have a one parameter family of oscillating models (with given total mass and energy) where the parameter may be taken as the first integral of (2.18). In the limit $\mathcal{A}^2 \to \infty$ we recover cold homologous collapse of a system of plane parallel sheets. When $\xi = \mathcal{A}^{-2/3}$, $\ddot{\xi} = 0$ and the system is in equilibrium. From (2.9) we see that the orbital angular frequency of individual sheets is $\omega_0 = \mathcal{A}^{4/3}$. From (2.10) we can easily compute the angular frequency of small oscillations of the model about the equilibrium state as $\omega_{\text{small osc.}} = \sqrt{3}\,\omega_0$.