# Chapter 4

# The second post-Newtonian gravitational wave polarizations

## 4.1 Introduction

The basic aim of the present chapter is to obtain the instantaneous 2PN corrections to the 'plus' and 'cross' polarization waveforms for inspiraling compact binaries of arbitrary mass ratio moving in elliptical orbits starting from the corresponding 2PN contributions to $(h_{km}^{TT})_{\rm inst}$, given by Eqs.(2.42) and (2.43) presented in chapter 2. Most of the results presented in this chapter are based on Ref. [146]. As emphasised in [79], the gravitational wave observations of inspiraling compact binaries, is analogous to the high precision radio-wave observations of binary pulsars. The latter makes use of an accurate relativistic 'timing formula' based on the solution – in quasi-Keplerian parametrization – to the relativistic equation of motion for a compact binary in elliptical orbit [147]. In a similar manner, the former demands accurate 'phasing' *i.e.,* an accurate mathematical modeling of the continuous time evolution of the gravitational wave phase. This requires for elliptical binaries, a convenient solution to the 2PN accurate equations of motion. As mentioned in the last chapter, a very elegant 2PN accurate generalized quasi-Keplerian parametrization for elliptical orbits has been implemented by Damour, Schafer, and Wex [40, 41, 42]. This representation is thus the most natural and best suited for our purpose to parametrize the dynamical variables that enter the gravitational waveforms, given by Eqs.(2.42) and (2.43). It should be noted that the complete 2PN accurate expressions for $h_+$ and $h_\times$ require computations of the tail contributions at 1.5PN and 2PN orders. These are not considered here. We also explore the effects of the orbital inclination and the eccentricity on the Newtonian part of $h_+$ and $h$,. The orbital phase evolution for binaries in quasi-elliptical orbits, implicitly addressed in chapter **3** is explicitly discussed further in this chapter. Note that results of the present chapter will form the first step in the direction of obtaining 'ready to use' theoretical templates to search for gravitational waves from inspiraling compact binaries moving in ***quasi-elliptical*** orbits.

The chapter is organized as follows: In section 4.2 we present the details of the computation to obtain the 'instantaneous' 2PN corrections to $h_+$ and $h_\times$ for inspiraling compact binaries moving in elliptical orbits. Section 4.3 deals with the influence of the orbital parameters on the polarizations waveforms. In section 4.4 we derive the equations that determine the phasing formulae for the quasi-elliptic case while section 4.5 comprises our concluding remarks.

## 4.2 The 'plus' and 'cross' polarizations

The two independent polarization states of the gravitational wave $h_+$ and $h_\times$ are given by

$$h_+ = \frac{1}{2}\left(p_i\, p_j - q_i\, q_j\right) h_{ij}^{TT} \tag{4.1a}$$

$$h_\times = \frac{1}{2}\left(p_i\, q_j + p_j\, q_i\right) h_{ij}^{TT}\,, \tag{4.1b}$$

where p and q are the two polarization vectors, forming along with the unit vector N pointing from the source to the detector, an orthonormal right-handed triad [47]. From Eqs.(4.1) it is clear that the explicit computation of 2PN corrections to $h_+$ and $h_\times$ requires the following: a) The 2PN corrections to $h_{ij}^{TT}$, generally given in terms of the dynamical variables of the binary, namely $v^2$, $\frac{Gm}{r}$, r, $n_i$, $v_i$, N.n and **N.v**.

Here $\mathbf{r}$ and v are the relative position and velocity vectors for the two masses $m_1$ and $m_2$ in the center of mass frame of the binary. Also $r = |\mathbf{r}|$, $v = |\mathbf{v}|$, $n = \frac{\mathbf{r}}{r}$, $\dot{r} = \frac{dr}{dt}$ and $m = m_1 + m_2$; b) A 2PN accurate orbital representation for elliptical orbits to parametrize these dynamical variables.

To see why one needs a 2PN accurate orbital representation, let us consider the explicit computation of $h_\times$ at the Newtonian order. We have

$$(h^{TT}_{km})_N = \frac{4\,G}{c^4\,R}\mathcal{P}_{ijkm}(\mathrm{N})\Big(v_{ij} - \frac{G\,m}{r}\,n_{ij}\Big)\,, \tag{4.2}$$

where $\mathcal{P}_{ijkm}(\mathrm{N})$ is the usual transverse traceless projection operator projecting normal to N and $v_{ij} = v_i v_j$ , $n_{ij} = n_i n_j$. Employing the standard convention adopted in [47], gives $\mathrm{p} = (0,1,0)$, $\mathrm{q} = (-\cos\mathrm{i}, 0, \sin\mathrm{i})$, $\mathrm{N} = (\sin\mathrm{i}, 0, \cos\mathrm{i})$, $\mathrm{n} = (\sin\phi, -\cos\phi, 0)$, and $\mathrm{v} = (\dot{r}\sin\phi + r\,\dot{\phi}\cos\phi, -\dot{r}\cos\phi + r\,\dot{\phi}\sin\phi, 0)$, where $\phi$ is the orbital phase angle, $\dot{\phi} = d\phi/dt$ and i the inclination angle of the source. Using above convention we obtain, using Eq.(4.2)

$$h_\times = 2\frac{G\,m\,\eta\,C}{c^4\,R}\left\{\left(\frac{G\,m}{r} + r^2\,\dot{\phi}^2 - \dot{r}^2\right)\sin 2\phi - 2\dot{r}\,r\,\dot{\phi}\,\cos 2\phi\right\}\,, \tag{4.3}$$

where $\eta = \mu/m$. Here, as usual, $\mu$ is the reduced mass of the binary given by $m_1\,m_2/m$ and $C$ is a shorthand notation for cosi. When dealing with elliptical orbits, it is convenient and useful to use a representation to rewrite the dynamical variables $r$, $\dot{r}$, $\phi$ and $\dot{\phi}$ in terms of the parameters describing an elliptical orbit. For example, in Newtonian dynamics, the Keplerian representation in terms of angular velocity, eccentricity and eccentric anomaly is a convenient solution to the Newtonian equations of motion for two masses on elliptical orbits. Similarly, to compute $h_+$ and $h_\times$ to 2PN order, one needs a 2PN accurate orbital representation. In our computation here, we employ the most Keplerian-like solution to the 2PN accurate equations of motion. This solution was obtained by Damour, Schafer, and Wex [40, 41, 42], and is given in the usual polar representation associated with the Arnowit, Deser and Misner (ADM) coordinates. It is known as the generalized quasi-Keplerian parametrization and represents the 2PN motion of a binary containing two compact objects of arbitrary mass ratio, moving in elliptical orbits. The relevant details of the representation is summarized in what follows.

Let $r(t)$, $\phi(t)$ be the usual polar coordinates associated with the ADM coordinates in the plane of relative motion of the two compact objects. The radial motion $r$(t) is conveniently parametrized by

$$\begin{aligned} r &= \text{a, } (1 - \text{e, } \cos u) \,, && (4.4a) \\ n(t - t_0) &= u - e_t \sin u + \frac{\text{ft}}{c^4} \sin v + \frac{g_t}{\text{c}^4} (v - u) \,, && (4.4b) \end{aligned}$$

where $u$ is the 'eccentric anomaly' parametrizing the motion and the constants $a$,, e,, $e_t$, n and $t_0$ are some 2PN semi-major axis, radial eccentricity, time eccentricity, mean motion, and initial instant respectively. The angular motion $\phi(t)$ is given by

$$\begin{aligned} \phi - \phi_0 &= \left(1 + \frac{k}{c^2}\right) v + \frac{f_\phi}{c^4} \sin 2v + \frac{g_\phi}{c^4} \sin 3v \,, && (4.5a) \\ \text{where } v &= 2 \tan^{-1} \left\{ \left( \frac{1 + e_\phi}{1 - e_\phi} \right)^{\frac{1}{2}} \tan(\frac{u}{2}) \right\} . && (4.5b) \end{aligned}$$

. In the above v is some real anomaly, $\phi_0$, k, $e_\phi$ are some constant, periastron precession constant, and angular eccentricity respectively. The explicit expressions for the parameters n, k, a,, $e_t$, e,, $e_\phi$, $f_t$, $g_t$, $f_\phi$ and $g_\phi$ in terms of the 2PN conserved energy and angular momentum per unit reduced mass were obtained in [41, 42] and displayed in the last chapter as Eqs.(3.5). It is straightforward to obtain the 2PN accurate expressions for $r$, $\phi$, $\dot{r}$, $\dot{\phi}$, in terms of $\xi = G\,m\,n$, $e_r$ and $u$, using Eqs.(3.5) and the following relations,

$$\begin{aligned} (-2\,E) &= \xi^{2/3} \Big\{ 1 + \frac{\xi^{2/3}}{12\,c^2} \, (15 - \eta) \\ &\quad + \frac{\xi^{4/3}}{24\,c^4\,(1 - e_r^2)^{1/2}} \Big( (15 - 15\eta - \eta^2)\,(1 - e_r^2)^{1/2} + 120 - 48\,\eta \Big) \Big\} \end{aligned} \qquad (4.6a)$$

$$
\begin{aligned}
(-2\,E\,h^2) &= (1-e_r^2)\Big\{1+\frac{\xi^{2/3}}{4\,c^2\,(1-e_r^2)}\;(24-4\eta)-5\,(1-e_r^2)\;(3-\eta)\\
&\quad+\frac{\xi^{4/3}}{24\,c^4\,(1-ef)}((1-e_r^2)^2\;(23\,\eta^2-15-15\,\eta\Big)\\
&\quad+(1\;-e_r^2)\;(234\,\eta-22\,\eta^2-204)+(408-264\,\eta)\Big)\Big\} &\text{(4.6b)}\\
\cos v &= \frac{\cos u-e_\phi}{(1-e_\phi\;\cos u)} &\text{(4.6c)}\\
\sin v &= \frac{(1-\phi^2)^{(1/2)}\sin u}{(1-e_\phi\;\cos u)} &\text{(4.6d)}
\end{aligned}
$$

, Using the equations above, we can, for instance, write:

$$
\begin{aligned}
\phi &= v+3\frac{\xi^{2/3}}{c^2\,(1-e_r^2)}\,v\\
&\quad+\frac{\xi^{4/3}}{32\,(1-e_r^2)^2}\Big\{\Big(\;(360-80\,\eta)\,(1-e_r^2)+264-144\,\eta\Big)v\\
&\quad-((12\;\sin v\cos \mathrm{v}^2-3\;\sin v\Big)\;e_r{}^3+24\;\sin v\;\cos v\,e_r^2\Big)\;\eta^2\\
&\quad+8\;\sin v\;\cos v\,e_r^2\,\eta\Big\}\,. &\text{(4.7)}
\end{aligned}
$$

Proceeding along the above lines, we obtain expressions for $r$, $\dot{r}$, $\phi$ and $\dot{\phi}$, listed below:

$$
\begin{aligned}
r &= \left(\frac{G\,m}{n^2}\right)^{1/3}\Big\{1+\frac{\xi^{2/3}}{3\,c^2}\;(\eta-9)+\frac{\xi^{4/3}}{72\,c^4}\Big[\Big(8\,\eta^2+75\,\eta+72\Big)\\
&\quad+\frac{1}{(1-e_r^2)}\,(198\eta-306)+\frac{1}{(1-e_r^2)^{1/2}}\,(144\eta-360)\Big]\;\Big\}\,(1-e,\;\cos u) &\text{(4.8a)}
\end{aligned}
$$

$$
\begin{aligned}
\phi &= v+\frac{3\,\xi^{2/3}}{c^2(1-e_r^2)}\,v\\
&\quad+\frac{\xi^{4/3}}{128\,c^4}\;\frac{1}{(1-e_r^2)^2(1-e_r\;\cos u)^3}\Big\{\Big[(480\,\eta-2160)\;e_r{}^4\\
&\quad-(1024\,\eta-2304)\;e_r{}^2-896\,\eta+2496\\
&\quad+\Big(\Big(-240\,e_r{}^5\eta+1080\Big)\;e_r^5+\Big(-288\,e_r{}^3\eta+2448\Big)\;e_r{}^3\\
&\quad+2688\,e_r\eta-7488\,e,\Big)\cos u+\Big((480\eta-2160)\;e_r^4\\
&\quad-(1344\eta-3744)\;e_r^2\Big)\cos 2u+\Big((-80\eta+360)\;e_r^5\\
&\quad+(224\eta-624)\;e_r^3\Big)\cos 3u\Big]\;v
\end{aligned}
$$

$$+(1-e_r^2)^{1/2}e_r^2\Big[\Big(-45\,e_r^3\eta^2+(120\,\eta-40)\;e_r\eta\Big)\,sin\,u$$
$$+\Big((-12\,\eta+16)\;{e_r}^2\eta-48\,\eta^2+16\,\eta\Big)\sin 2u$$
$$+\Big(3\,\eta^2\,{e_r}^3+(12\,\eta-8)\;e_r\eta\Big)\sin 3u\Big]\Big\} \tag{4.8b}$$

$$\dot r = \frac{\xi^{1/3}e_r\sin u}{(1-e,\cos u)}\Big\{1$$
$$+\frac{\xi^{2/3}}{6\,c^2\,(1-e,\;\cos u)}\Big[(7\,\eta-6)\,e_r\cos u+2\,\eta-18\Big]$$
$$+\frac{\xi^{4/3}}{288\,(1-e_r^2)\,(1-e_r\;\cos u)^3\,c^4}\Big[\Big(204\,\eta^2-810\,\eta+1872)\;e_r^4$$
$$-(236q^2+42q-612)\;e_r^2+32\,\eta^2+1956\,\eta-3096$$
$$+\Big(\Big(105\,\eta^2-693\,\eta+216\Big)\;e_r^5-\Big(585\,\eta^2-4545\,\eta+6120\Big)\;e_r^3$$
$$+(480\,\eta^2-5436\,\eta+8352)\;e,)\;\cos u$$
$$+\Big(\Big(168\,\eta^2-954\,\eta+1872\Big)\;e_r^4$$
$$+(-168\eta^2+1350\,\eta-2484)\;e_r^2\Big)\;cos\,2u$$
$$+\Big(\Big(35\,\eta^2-231\,\eta+72\Big)\;e_r^5-\Big(35\,\eta^2-231\,\eta+72\Big)\;e_r^3\Big)\cos 3u$$
$$+(1-e_r^2)^{\frac{1}{2}}\Big((-432\eta+1080)\;e_r^2-288\,\eta+720$$
$$+\Big((216\,\eta-540)\;e_r^3+(864\,\eta-2160)\;e,)\;\;cosu$$
$$+\Big(-432\,\eta+1080\Big)\;e_r^2\;\cos 2u+\Big(72\,\eta-180\Big)\,e_r^3\;\cos 3u\Big)\Big]\Big\} \tag{4.8c}$$

$$\dot\phi = \frac{n(1-e_r^2)^{1/2}}{(1-e_r\;\cos u)^2}\Big\{1$$
$$+\frac{\xi^{2/3}}{6\,c^2(1-e_r\;\cos u)(1-e_r^2)}\;[((-9\eta+24)\;e_r^3+(12\eta-42)\;e,\Big)\cos u$$
$$+18-3\,e_r^2\,\eta\Big]+\frac{\xi^{4/3}}{48\,(1-e_r^2)^2(1-e_r\;\cos u)^3c^4}\Big[\Big(-\eta^2-142\,\eta+192\Big)\;{e_r}^6$$
$$+\Big(-30\,\eta^2+1238\,\eta-2436\Big)\;e_r^4+\Big(16\,\eta^2-1384\,\eta+2148\Big)\;e_r^2-192\,\eta+576$$
$$+\Big(\frac{1}{2}\;\Big(-63,\eta^2+363\,\eta-432\Big)\;e_r^7+\Big(178\,\eta^2-1637\,\eta+2406\Big)\;{e_r}^5$$
$$-(188\,\eta^2-802\,\eta-138)\;{e_r}^3+(64\,\eta^2+1192\,\eta-2832\Big)\;e,)\;\;cos\,u$$
$$-\Big(\Big(49\,\eta^2+118\,\eta-192\Big)\;{e_r}^6+\Big(-56\,\eta^2-962\,\eta+2076\Big)\;{e_r}^4$$

$$+\left(16\,\eta^2+1132\,\eta-2172)\;e_r{}^2\right)\cos 2u$$

$$+\left(-\frac{1}{2}\left(21\eta^2+121\,\eta-144\right)\,e_r{}^7+\left(32\,\eta^2-395\,\eta+618\right)\,e_r{}^5\right.$$

$$\left.+\,(-20\,\eta^2+322\,\eta-522)\;e_r{}^3\right)\cos 3u$$

$$+(1-e_r^2)^{\frac{1}{2}}\Big((216\,\eta-540)\;e_r{}^4+(-72\,\eta+180)\;e_r{}^2-144\,\eta+360$$

$$+\Big((-108\,\eta+270)\;e_r{}^5+(-324\,\eta+810)\;e_r{}^3+(432\,\eta-1080)\;e_r\Big)\cos u$$

$$+\Big((216\eta-540)\;e_r{}^4+(-216\eta+540)\;e_r{}^2\Big)\cos 2u$$

$$+\Big((-36\eta+90)\;e_r{}^5+(36\eta-90)\;e_r{}^3\Big)\cos 3u\Big)\Big]\Big\}\,. \tag{4.8d}$$

To obtain the desired 2PN expressions for $\sin\phi$ and $\cos\phi$, we need to obtain sin v and cos v in terms of $\xi, e_r$ and u. Using Eqs.(4.6) and the relation connecting $e_\phi$ to e,, we obtain after some manipulations,

$$\cos v \;=\; \frac{1}{(1-e_r\,\cos u)}\Big\{\cos u-e_r-\frac{\xi^{2/3}}{2\,c^2}\,\frac{e_r\,\eta\,\sin^2 u}{(1-e_r\,\cos u)}$$

$$+\frac{\xi^{4/3}}{384\,c^4}\,\frac{1}{(1-e_r^2)\,(1-e_r\,\cos u)^2}\Big[\Big(66\,\eta^3-8\,\eta^2+240\,\eta\Big)\,e_r{}^3$$

$$+\Big(-66\,\eta^3+98\,\eta^2+96\,\eta-816\Big)\,e_r$$

$$+\Big(\Big(-33\,\eta^3+28\,\eta^2-120\,\eta\Big)\,e_r^4+(33\,\eta^3-73\eta^2-48\,\eta+408)\;e_r{}^2\Big)\cos u$$

$$+\Big(\Big(-66\,\eta^3+8\,\eta^2-240\,\eta\Big)\,e_r{}^3+(66\,\eta^3-98\eta^2-96\,\eta+816)\;e_r\Big)\cos 2u$$

$$+\Big(\Big(33\,\eta^3-28\eta^2+120\,\eta\Big)\,e_r{}^4-(33\,\eta^3-73\eta^2-48\,\eta+408)\;e_r{}^2\Big)\cos 3u\Big]\Big\} \tag{4.9a}$$

$$\sin v \;=\; \frac{(1-e_r^2)^{1/2}}{(1-e_r\,\cos u)}\Big\{1+\frac{\xi^{2/3}}{2\,c^2}\,\frac{1}{(1-e_r\,\cos u)\,(1-e_r^2)}\Big(-e_r{}^2\eta+\eta\,e_r\Big)\cos u$$

$$+\frac{\xi^{4/3}}{192\,c^4}\,\frac{1}{(1-e_r^2)^2(1-e_r\,\cos u)^2}\Big[\Big(99\,\eta^3-48\,\eta^2+360\,\eta\Big)\,e_r{}^4$$

$$+\,(-99\,\eta^3+147\,\eta^2+144\,\eta-1224)\;e_r{}^2$$

$$+\Big(\Big(-66\eta^3+56\,\eta^2-240\,\eta\Big)\,e_r{}^5+(-90\eta^2-336\,\eta+816)\;e_r{}^3$$

$$+\,(66\eta^3-98\eta^2-96\,\eta+816\Big)\,e_r\Big)\cos u+\Big(\Big(33\eta^3-40\,\eta^2+120\,\eta\Big)\,e_r{}^4$$

$$+\,(-33\,\eta^3+73\eta^2+48\,\eta-408)\;e_r{}^2)\;\cos 2u\Big]\Big\}\,. \tag{4.9b}$$

Eqs.(4.8) and (4.9) will be required to obtain $h_+$ and $h_\times$ in terms of $\xi, e_r$ and u from the expressions for 2PN corrections to $h_{ij}^{TT}$ in ADM coordinates.

The 2PN corrections to $h_{ij}^{TT}$, given by Eqs.(5.3) and (5.4) of [44], however, are available in the harmonic (De-Donder ) coordinates. Using, in a straightforward manner, the transformation equations of Damour and Schafer [142] to relate the dynamical variables in the harmonic and the ADM gauge, we obtain the 2PN accurate instantaneous contributions to $h_{ij}^{TT}$ in the ADM gauge. For completeness, we quote again the relevant transformation equations displayed in the previous chapter as Eqs.(3.8) relating the harmonic (De-Donder) variables to the corresponding ADM ones,

$$\begin{aligned}
\mathbf{r}_{\mathrm{D}} &= \mathbf{r}_{\mathrm{A}} + \frac{Gm}{8\,c^4\,r}\left\{\left[\left(5v^2 - \dot{r}^2\right)\eta + 2\,\left(1+12\eta\right)\frac{Gm}{r}\right]\mathbf{r}\right. \\
&\quad \left. -18\,\eta\, r\dot{r}\,\mathbf{v}\right\}\,, \qquad (4.10\mathrm{a})\\
t_{\mathrm{D}} &= t_{\mathrm{A}} - \frac{Gm}{c^4}\,\eta\,\dot{r}\,, \qquad (4.10\mathrm{b})\\
\mathbf{v}_{\mathrm{D}} &= \mathbf{v}_{\mathrm{A}} - \frac{Gm\dot{r}}{8\,c^4\,r^2}\left\{\left[7v^2 + 38\frac{Gm}{r} - 3\dot{r}^2\right]\eta + 4\,\frac{Gm}{r}\right\}\mathbf{r} \\
&\quad -\frac{Gm}{8\,c^4 r}\left\{\left[5v^2 - 9\dot{r}^2 - 34\frac{Gm}{r}\right]\eta - 2\,\frac{Gm}{r}\right\}\mathbf{v}\,, \qquad (4.10\mathrm{c})\\
r_{\mathrm{D}} &= r_{\mathrm{A}} + \frac{Gm}{8\,c^4}\left\{5\,\eta v^2 + 2\,\left(1+12\eta\right)\frac{Gm}{r} - 19\,\eta\dot{r}^2\right\}\,. \qquad (4.10\mathrm{d})
\end{aligned}$$

The subscripts ‘D’ and ‘A’ denote quantities in the De-Donder (harmonic) and in the ADM coordinates respectively. Note that in all the above equations the differences between the two gauges are of the 2PN order. As there is no difference between the harmonic and the ADM coordinates to 1PN accuracy, in Eqs.(4.10) no suffix is used for the 2PN terms.

Using Eqs.(4.10) the 2PN corrections to $h_{ij}^{TT}$ in ADM coordinates can easily be obtained from Eqs.(5.3) and (5.4) of [44]. For economy of presentation, we write $(h_{ij}^{TT})_{\mathrm{A}}$ in the following manner, $(h_{ij}^{TT})_{\mathrm{A}} = (h_{ij}^{TT})_{\mathrm{O}}+$ 'Corrections', where $(h_{ij}^{TT})_{\mathrm{A}}$ represent the metric perturbations in the ADM coordinates. $(h_{ij}^{TT})_{\mathrm{O}}$ is a short hand notation for expressions on the r.h.s of Eqs.(5.3) and (5.4) of [44], where N, n, v, $v^2, \dot{r}$, r are the ADM variables $\mathbf{N}_\mathrm{A}, \mathrm{n}_\mathrm{A}, \mathbf{v}_\mathrm{A}, v^2_\mathrm{A}, \dot{r}_\mathrm{A}, r_\mathrm{A}$ respectively. The 'Corrections' represent the differences at the 2PN order, that arise due to the change of the coordinate system, given by Eqs.(4.10). As the two coordinates are different only at the 2PN order, the 'Corrections' come only from the leading Newtonian terms in Eqs.(5.3) and (5.4) of [44].

$$\begin{aligned}(h^{TT}_{ij})^{\rm inst}_{\rm ADM} &= (h^{TT}_{ij})^{\rm inst}_{\rm O} + \frac{G}{c^4 R}\frac{G\,m}{2\,c^4\,r_\mathrm{A}}\Bigg\{\Bigg[5\,\eta\,v^2_\mathrm{A} - 55\,\eta\,\dot{r}^2_\mathrm{A} \\ &+2\,(1+12\,\eta)\,\frac{G\,m}{r_\mathrm{A}}\Bigg]\,\frac{G\,m}{r_\mathrm{A}}\,(n_{ij})^{TT}_\mathrm{A} \\ &+\Bigg[-14\,\eta\,v^2_\mathrm{A} + 6\,\eta\,\dot{r}^2_\mathrm{A} - 8\,(1+5\,\eta)\,\frac{G\,m}{r_\mathrm{A}}\Bigg]\,\frac{G\,m}{r_\mathrm{A}}\,(n_{(i}v_{j)})^{TT}_\mathrm{A} \\ &-\Bigg[10\,\eta\,v^2_\mathrm{A} - 18\,\eta\,\dot{r}^2_\mathrm{A} - (4+68\,\eta)\,\frac{G\,m}{r_\mathrm{A}}\Bigg]\,(v_{ij})^{TT}_\mathrm{A}\Bigg\}\,. \end{aligned} \tag{4.11}$$

To check the algebraic correctness of the above transformation, we compute the far-zone energy flux directly in the ADM coordinates using

$$\left(\frac{d\mathcal{E}}{dt}\right)_\mathrm{A} = \frac{c^3\,R^2}{32\pi G}\int\Big((\dot{h}^{TT}_{ij})_\mathrm{A}\,(\dot{h}^{TT}_{ij})_\mathrm{A}\Big)\,d\Omega(\mathbf{N})\,. \tag{4.12}$$

After a careful use of the transformation equations, the expression for $(d\mathcal{E}/dt)_\mathrm{A}$ calculated above, matches with the expression for the far-zone energy flux, Eq.(4.7a) of [44] obtained earlier. This provides a useful check on the transformation from $(h^{TT}_{ij})^{\rm inst}_{\rm D}$ to $(h^{TT}_{ij})^{\rm inst}_{\rm A}$.

As mentioned in [43, 44], there is no need to apply the TT projection to $(h^{TT}_{ij})$ given by Eq.(4.11) before contracting with p and **q,** as required by Eqs.(4.1). Thus, we schematically write,

$$h^{TT}_{ij} = \alpha_1\,v_{ij} + \alpha_2\,n_{ij} + \alpha_3\,n_{(i}v_{j)}\,. \tag{4.13}$$

The polarization states $h_+$ and h,, for Eqs.(4.13) are given by,

$$\begin{aligned} h_+ &= \frac{1}{2}\Big(p_i\,p_j - q_i\,q_j\Big)\,\Big(\alpha_1\,v_{ij} + \alpha_2\,n_{ij} + \alpha_3\,n_{(i}v_{j)}\Big)\,, \\ &= \frac{\alpha_1}{2}\Big((\mathbf{p}.\mathbf{v})^2 - (\mathbf{q}.\mathbf{v})^2\Big) + \frac{\alpha_2}{2}\Big((\mathbf{p}.\mathbf{n})^2 - (\mathbf{q}.\mathbf{n})^2\Big) \end{aligned}$$

$$
\begin{aligned}
&\quad +\frac{\alpha_3}{2}\Big((\mathbf{p}.\mathbf{n})(\mathbf{p}.\mathbf{v})-(\mathbf{q}.\mathbf{n})(\mathbf{q}.\mathbf{v})\Big)\,, && (4.14a)\\
h_\times &= \frac{1}{2}\Big(p_i\,q_j+p_j\,q_i\Big)\,\Big(\alpha_1\,v_{ij}+\alpha_2\,n_{ij}+\alpha_3\,n_{(i}v_{j)}\Big)\\
&= \alpha_1\,(\mathbf{p}.\mathbf{v})\,(\mathbf{q}.\mathbf{v})+\alpha_2\,(\mathbf{p}.\mathbf{n})\,(\mathbf{q}.\mathbf{n})\\
&\quad +\frac{\alpha_3}{2}\Big((\mathbf{p}.\mathbf{n})\,(\mathbf{q}.\mathbf{v})+(\mathbf{p}.\mathbf{v})\,(\mathbf{q}.\mathbf{n})\Big)\,. && (4.14b)
\end{aligned}
$$

Using Eqs.(4.13), (4.14), (4.8) and (4.9), we obtain after a lengthy but straightforward computation the instantaneous 2PN accurate polarizations $h_+$ and $h_\times$ in terms of $\xi$, e, and u. In order to compare with existing gauge independent circular limit results, we rewrite the expressions for $h_+$ and $h_\times$ in terms of the orbital angular freq'iency w, using a 2PN accurate relation connecting the mean motion n to w given by w$=n(1+$k). From Eqs.(39), (44) and (46) of [42], after some manipulation we obtain:

$$
\xi = \tau\left\{1-\frac{3\,\tau^{2/3}}{(1-e_\tau^2)}+\frac{\tau^{4/3}}{4\,(1-e_\tau^2)^2}\Big[(27+18\,\eta)-(45-10\,\eta)\,(1-e_\tau^2)\Big]\right\}, \tag{4.15}
$$

where $\tau=\frac{G\,m\,\omega}{c^3}$. All the computations are performed using MAPLE [134]. The final result for the two polarizations of the gravitational wave from an inspiraling, non-spinning, compact binary in elliptic orbit, is then written as,

$$
(h_{+,\times})_{\rm inst} = \frac{2\,G\,m\,\eta}{c^2\,R}\,\tau^{2/3}\left\{H^{(0)}_{+,\times}+\tau^{1/2}\,H^{(1/2)}_{+,\times}+\tau\,H^{(1)}_{+,\times}+\tau^{3/2}\,H^{(3/2)}_{+,\times}+\tau^2\,H^{(2)}_{+,\times}\right\}, \tag{4.16}
$$

where the curly brackets contain a post-Newtonian expansion. The explicit expressions for various post -Newtonian terms for the 'plus' polarization are given by

$$
\begin{aligned}
H_+^{(0)} &= \frac{1}{4\,(1-e_r\cos(u))^3}\Big\{-4\,e_\tau^2-e_r\Big((3\,e_\tau^2-3)\,C^2-7\Big)\cos u\\
&\quad +\Big((1-e_\tau^2)\,C^2+1\Big)\,\Big[-4\,\cos 2u+e_r\,\cos 3u\Big]\Big\} && (4.17a)\\
H_+^{(1/2)} &= \frac{6}{64}\,S\,\frac{(1-e_r^2)^{1/2}}{(1-e_r\cos u)^6}\Big\{20\,e_r\,\Big((e_r^4-2\,e_r^2+1)\,\mathbf{C}^2-5\,e_r^2+5\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+8\left((-10\,e_r^4+11\,e_r^2-1)\,C^2+6\,e_r^4+8\,e_r^2-5\right)\cos u\\
&+2\,e_r\left((15\,e_r^4+57\,e_r^2-72)\,C^2-7\,e_r^2-80\right)\cos 2u\\
&+4\left((-23\,e_r^4+5\,e_r^2+18)\,C^2-4\,e_r^4+27\,e_r^2+18\right)\cos 3u\\
&+e_r\left((12\,e_r^4+56\,e_r^2-68)\,C^2-12\,e_r^2-68\right)\cos 4u\\
&+20\,e_r^2\left((1-e_r^2)\,C^2+1\right)\cos 5u+2\,e_r^3\left((e_r^2-1)\,C^2-1\right)\cos 6u\Big\}
\end{aligned}
\tag{4.17b}
$$

$$
\begin{aligned}
H_+^{(1)} &= \frac{1}{(1-e_r^2)\;(1-\mathrm{e_r}\;\cos u)^3}\left\{\frac{1}{384}\,\frac{1}{(1-\mathrm{e_r}\;\cos u)^4}\,HP_{21}\right.\\
&\left.+\frac{(1+C^2)\,(1-e_r^2)^{1/2}}{2}\,v\,HP_{22}\right\}
\end{aligned}
\tag{4.17c}
$$

$$
\begin{aligned}
HP_{21} &= e_r^2\Big[24\left((13\,e_r^6-39\,e_r^4+39\,e_r^2-13)\,C^4\right.\\
&+(14\,e_r^6+266\,e_r^4-88\,e_r^2-192)\,C^2\\
&\left.-15\,e_r^6+147\,e_r^4-9\,e_r^2+273\right)\\
&+12\,\left(1-e_r^2\right)\left((78\,e_r^4-156\,e_r^2+78)\,C^4\right.\\
&\left.+(29\,e_r^4-223\,e_r^2+338)\,C^2-128\,e_r^4+493\,e_r^2-332\right)\eta\Big]\\
&+e_r\Big[\left((210\,e_r^8-958\,e_r^6+1614\,e_r^4-1194\,e_r^2+328)\,C^4\right.\\
&+(105\,e_r^8-5454\,e_r^6-8427\,e_r^4+12960\,e_r^2+816)\,C^2\\
&\left.-1009\,e_r^6-4965\,e_r^4-6774\,e_r^2-4280\right)\\
&+\left(1-e_r^2\right)\left((630\,e_r^6-2244\,e_r^4+2598\,e_r^2-984)\,C^4\right.\\
&+(-455\,e_r^6+1005\,e_r^4-1368\,e_r^2-2656)\,C^2\\
&\left.+573\,e_r^4-5406\,e_r^2+3480\right)\eta\Big]\cos u\\
&+\Big[4\left((-246\,e_r^8+770\,e_r^6-834\,e_r^4+342\,e_r^2-32)\,C^4\right.\\
&+(165\,e_r^8+2280\,e_r^6-1917\,e_r^4-672\,e_r^2+144)\,C^2\\
&\left.+168\,e_r^8+17\,e_r^6+2217\,e_r^4+330\,e_r^2+304\right)\\
&-2\,\left(1-e_r^2\right)\left((1476\,e_r^6-3144\,e_r^4+1860\,e_r^2-192)\,C^4\right.
\end{aligned}
$$

$$
\begin{aligned}
&+(-665\,e_r^6-995\,e_r^4+416\,e_r^2-352)\,C^2\\
&-704\,e_r^6+2205\,e_r^4-3220\,e_r^2+608\eta\Big)\Big]\ \cos 2u\\
&+3\,e_r\Big[\Big((42\,e_r^8+262\,e_r^6-1038\,e_r^4+1122\,e_r^2-388)\,C^4\\
&+(21\,e_r^8-1110\,e_r^6+97\,e_r^4+1456\,e_r^2-464)\,C^2\\
&-273\,e_r^6-621\,e_r^4-1370\,e_r^2+20\Big)\\
&+\left(1-e_r^2\right)\Big((126\,e_r^6+912\,e_r^4-2202\,e_r^2+1164)\,C^4\\
&+(-91\,e_r^6-1139\,e_r^4+372\,e_r^2+96)\,C^2\\
&-187\,e_r^4+806\,e_r^2-1356\Big)\eta\Big]\ \cos 3u\\
&+4\Big[2\Big((-51\,e_r^8+89\,e_r^6+39\,e_r^4-141\,e_r^2+64)\,C^4\\
&+(42\,e_r^8+186\,e_r^6-372\,e_r^4+144\,e_r^2)\,C^2\\
&+9\,e_r^8+111\,e_r^6+139\,e_r^4+153\,e_r^2-64\Big)\\
&-\left(1-e_r^2\right)\Big((306\,e_r^6-228\,e_r^4-462\,e_r^2+384)\,C^4\\
&+(-289\,e_r^6-277\,e_r^4+278\,e_r^2)\,C^2\\
&-16\,e_r^6+111\,e_r^4-52\,e_r^2-384\Big)\eta\Big]\ \cos 4u\\
&+e_r\Big[\Big((42\,e_r^8+190\,e_r^6-822\,e_r^4+906\,e_r^2-316)\,C^4\\
&+(21\,C^2e_r^8-438\,C^2e_r^6+849\,C^2{e_r}^4-432\,e_r^2)\,C^2\\
&-153\,e_r^6-261\,\mathrm{ef}-706\,e_r^2+316\Big)\\
&+\left(1-e_r^2\right)((126\,e_r^6+696\,e_r^4-1770\,e_r^2+948)\,C^4\\
&+(-91\,e_r^6-859\,\mathrm{ef}+572\,e_r^2)\,C^2-19\,\mathrm{ef}+446\,e_r^2-948\Big)\eta\Big]\ \cos 5u\\
&+6\,e_r^2\Big[2\Big((-6\,e_r^6+18\,e_r^4-18\,e_r^2+6)\,C^4\\
&+(e_r^4-8\,e_r^2+7)\,C^2e_r^2+5\,e_r^4+13\,e_r^2-6\Big)\\
&-\left(1-e_r^2\right)\Big((36\,e_r^4-72\,e_r^2+36)\,C^4\\
&+(-29\,C^2e_r^4+17)\,C^2e_r^2+17\,e_r^2-36\Big)\eta\Big]\ \cos 6u\\
&+e_r^3\Big[3\Big((2\,e_r^6-6\,e_r^4+6\,e_r^2-2)\,C^4
\end{aligned}
$$

$$+(e_r^4+2\,e_r^2-3)\,C^2e_r^2-e_r^4-5\,e_r^2+2\Big)$$
$$+\left(1-e_r^2\right)\Big((18\,e_r^4-36\,e_r^2+18)\,C^4$$
$$+(-13\,e_r^2+7)\,C^2e_r^2+7\,e_r^2-18\Big)\eta\Big]\ \cos 7u \qquad (4.17\text{d})$$

$$HP_{22} = \Big\{12\,\sin 2u-15\,e_r\sin u-3\,e_r\sin 3u\Big\} \qquad (4.17\text{e})$$

$$H_+^{(3/2)} = \frac{\delta}{(1-e_r^2)\,(1-e_r\,\cos u)^5}\Big\{\frac{1}{1536}\,\frac{(1-e_r^2)^{1/2}}{(1-e_r\,\cos u)^4}\,HP_{31}$$
$$+\frac{3}{16}\,v\,HP_{32}\Big\} \qquad (4.17\text{f})$$

$$HP_{31} = 2\,e_r\Big[2\,\Big((714\,e_r^8-2150\,e_r^6+2166\,e_r^4-738\,e_r^2+8)\,C^4$$
$$+(150\,e_r^8+12021\,e_r^6-171\,e_r^4-11472\,e_r^2-528)\,C^2$$
$$-624\,e_r^8+1152\,e_r^6-19245\,e_r^4+12978\,e_r^2-696\Big)$$
$$+\Big((-2856\,C^4e_r^8+8600\,C^4e_r^6-8664\,C^4e_r^4+2952\,C^4e_r^2-32)\,C^4$$
$$+(369\,e_r^8+6726\,e_r^6-20163\,e_r^4+13524\,e_r^2-456)\,C^2$$
$$+3360\,e_r^8-15953\,e_r^6+28545\,e_r^4-15444\,e_r^2+1208\Big)\eta\Big]$$
$$+2\,\Big[\Big((378\,e_r^{10}-3237\,e_r^8+7447\,e_r^6-6699\,e_r^4+2115\,e_r^2-4)\,C^4$$
$$+(567\,e_r^{10}-16137\,e_r^8-49161\,e_r^6+51051\,e_r^4+13440\,e_r^2+240)\,C^2$$
$$+129\,e_r^8+26292\,e_r^6+16788\,e_r^4-25419\,e_r^2+228\Big)$$
$$+\Big((-756\,e_r^{10}+6474\,e_r^8-14894\,e_r^6+13398\,e_r^4-4230\,e_r^2+8)\,C^4$$
$$+(378\,e_r^{10}-4491\,e_r^8+10518\,e_r^6+3891\,e_r^4-10392\,e_r^2+96)\,C^2$$
$$-2208\,e_r^8+11447\,e_r^6-27519\,e_r^4+14382\,e_r^2-392\Big)\,\eta\Big]\ \cos u$$
$$+4\,e_r\Big[\Big((-1422\,e_r^8+5065\,e_r^6-6663\,e_r^4+3819\,e_r^2-799)\,C^4$$
$$+(351\,e_r^8+19962\,e_r^6-2749\,e_r^4-18028\,e_r^2+464)\,C^2$$
$$+1038\,e_r^8-6119\,e_r^6+1031\,e_r^4-5189\,e_r^2+8303\Big)$$
$$+\Big((2844\,e_r^8-10130\,e_r^6+13326\,e_r^4-7638\,e_r^2+1598)\,C^4$$
$$+(-825\,e_r^8-4032\,e_r^6+4510\,e_r^4-2165\,e_r^2+2512)\,C^2$$

$$-1710\,e_r^8 + 8249\,e_r^6 - 14587\,e_r^4 + 15725\,e_r^2 - 4830\Big)\,\eta\Big]\ \cos 2u$$

$$+2\,\Big[\Big((252\,e_r^{10} + 3729\,e_r^8 - 13185\,e_r^6 + 14661\,e_r^4 - 5943\,e_r^2 + 486)\,C^4$$

$$+(378\,e_r^{10} - 10419\,e_r^8 - 25440\,e_r^6 + 30441\,e_r^4 + 7200\,e_r^2 - 2160)\,C^2$$

$$-1286\,e_r^8 + 2409\,e_r^6 - 3452\,e_r^4 - 3557\,e_r^2 - 3942\Big)$$

$$-3\,\Big((168\,e_r^{10} + 2486\,e_r^8 - 8790\,e_r^6 + 9774\,e_r^4 - 3962\,e_r^2 + 324)\,C^4$$

$$+(-84\,e_r^{10} - 3236\,e_r^8 + 1563\,e_r^6 + 1473\,e_r^4 - 4\,e_r^2 + 288)\,C^2$$

$$-728\,e_r^8 + 3826\,e_r^6 - 6077\,e_r^4 + 5790\,e_r^2 - 900\Big)\,\eta\Big]\ \cos 3u$$

$$+8\,e_r\Big[\Big((-309\,e_r^8 + 7\,e_r^6 + 1833\,e_r^4 - 2451\,e_r^2 + 920)\,C^4$$

$$+(33\,e_r^8 + 3492\,e_r^6 - 637\,e_r^4 - 3880\,e_r^2 + 992)\,C^2$$

$$-24\,e_r^8 + 891\,e_r^6 + 20\,e_r^4 + 2043\,e_r^2 - 32\Big)$$

$$+\Big((618\,e_r^8 - 14\,e_r^6 - 3666\,e_r^4 + 4902\,e_r^2 - 1840)\,C^4$$

$$+(-606\,e_r^8 - 1740\,e_r^6 + 2125\,e_r^4 + 307\,e_r^2 - 86)\,C^2$$

$$-70\,e_r^6 + 1076\,e_r^4 - 1939\,e_r^2 + 1962\Big)\,\eta\Big]\ \cos 4u$$

$$+2\,\Big[\Big((108\,e_r^{10} + 1591\,e_r^8 - 4171\,e_r^6 + 1887\,e_r^4 + 1835\,e_r^2 - 1250)\,C^4$$

$$+(162\,e_r^8 - 1481\,e_r^6 - 6164\,e_r^4 + 10075\,e_r^2 - 2592)\,C^2e_r^2$$

$$-366\,e_r^8 - 2993\,{e_r}^6 - 3472\,e_r^4 - 1799\,e_r^2 + 1250\Big)$$

$$+\Big((-216\,e_r^{10} - 3182\,e_r^8 + 8342\,e_r^6 - 3774\,e_r^4 - 3670\,e_r^2 + 2500)\,C^4$$

$$+(108\,e_r^8 + 3536\,e_r^6 + 41\,e_r^4 - 4849\,e_r^2 + 1164)\,C^2e_r^2$$

$$+24\ \ e_r^8 - 1198\,e_r^6 + 2005\,e_r^4 - 422\,e_r^2 - 2500\Big)\eta\Big]\ \cos 5u$$

$$+4\,e_r\Big[\Big((-186\,C^4e_r^8 + 71\,C^4e_r^6 + 903\,C^4e_r^4 - 1275\,C^4e_r^2 + 487)\,C^4$$

$$+(-159\,e_r^6 + 966\,e_r^4 - 1203\,e_r^2 + 396)\,C^2e_r^2$$

$$-30\,e_r^8 + 455\,e_r^6 + 881\,e_r^4 + 693\,e_r^2 - 487\Big)$$

$$+\Big((372\,e_r^8 - 142\,e_r^6 - 1806\,e_r^4 + 2550\,e_r^2 - 974)\,C^4$$

$$+(-231\,e_r^6 - 864\,e_r^4 + 1314\,e_r^2 - 219)\,C^2e_r^2$$

$$
\begin{aligned}
& +30\, e_r^8 - 41\, e_r^6 + 219\, e_r{}^4 - 813\, e_r^2 + 974\Big)\eta\Big] \cos 6u \\
& +e_r^2\Big[\Big((54\, e_r^8 + 460\, e_r^6 - 1704\, e_r^4 + 1812\, e_r^2 - 622)\, C^4 \\
& +(81\, e_r^6 + 10\, e_r^4 + 259\, e_r^2 - 350)\, C^2 e_r^2 \\
& -81\, e_r^6 - 1225\, e_r^4 - 918\, e_r^2 + 622\Big) \\
& +\Big((-108\, e_r^8 - 920\, e_r^6 + 3408\, e_r^4 - 3624\, e_r^2 + 1244)\, C^4 \\
& +(54\, e_r^6 + 929\, e_r^4 - 985\, e_r^2 + 2)\, C^2 e_r^2 \\
& -239\, e_r^4 + 1138\, e_r^2 - 1244\Big)\, \eta\Big] \cos 7u \\
& +6\, e_r^3\Big[2\Big((-8\, e_r^6 + 24\, e_r^4 - 24\, e_r^2 + 8)\, C^4 \\
& +(-8\, e_r^4 + e_r^2 + 7)\, C^2 e_r^2 + 14\, e_r^4 + 15\, e_r^2 - 8\Big) \\
& +\Big((32\, e_r^6 - 96\, e_r^4 + 96\, e_r^2 - 32)\, C^4 \\
& +(-19\, e_r^4 + 14\, e_r^2 + 5)\, C^2 e_r^2 + 3\, e_r^4 - 27\, e_r^2 + 32\Big)\eta\Big] \cos 8u, \\
& +3\, e_r^4\Big[\Big((2\, e_r^6 - 6\, e_r^4 + 6\, e_r^2 - 2\, C^4 \\
& +(3\, e_r^4 - 3)\, C^2 e_r^2 - 3\, e_r^4 - 5\, e_r^2 + 2\Big) \\
& +\Big((-4\, e_r^6 + 12\, e_r^4 - 12\, e_r^2 + 4)\, C^4 \\
& +(2\, e_r^4 - e_r^2 - 1)C^2 e_r^2 + 3\, e_r{}^2 - 4\Big)\, \eta\Big] \cos 9u
\end{aligned} \tag{4.17g}
$$

$$
\begin{aligned}
HP_{32} = {} & 2\Big[(44\, e_r^4 - 45\, e_r^2 + 1)C^2 + 40\, e_r^4 - 66\, e_r^2 + 5\Big] \sin u \\
& +8\, e_r\Big[(-14\, e_r^2 + 14)C^2 - 9\, e_r^2 + 15\Big] \sin 2u \\
& +\Big[(19\, e_r^4 + 35\, e_r^2 - 54)\, C^2 + 10\, e_r^4 + 17\, e_r^2 - 54\Big] \sin 3u \\
& -8\, e,\Big[(3 e_r^2 - 3)C^2 + 2\, e_r^2 - 3\Big] \sin 4u \\
& +e_r^2\Big[(3\, e_r^2 - 3)\, C^2 + 2\, e_r^2 - 3\Big] \sin 5u
\end{aligned} \tag{4.17h}
$$

$$
\begin{aligned}
H_+^{(2)} = {} & \frac{1}{(1-e_r^2)^2\, (1-e, \cos u)^3}\Big\{\frac{(5-2\,\eta)}{4}\,(1-e_r^2)^{3/2}\, HP_{41} \\
& \frac{1}{737\,280}\, \frac{1}{(1-e,\ \cos u)^8}\, HP_{42} \\
& +\frac{1}{128}\, \frac{(1-e_r^2)^{1/2}}{(1-e_r\ \cos u)^4}\, v\, HP_{43} + \frac{9}{4}\,(1+C^2)\, v^2\, HP_{44}\Big\}
\end{aligned} \tag{4.17i}
$$

$$
\begin{aligned}
HP_{41} &= \Big\{-4\,e_r^2 - e_r\left((3\,e_r^2-3)\,C^2-7\right)\cos u + 4\left((e_r^2-1)\,C^2-1\right)\cos 2u \\
&\quad -e_r\left((e_r^2-1)\ -1\right)\cos 3u\Big\} \qquad (4.17j) \\
HP_{42} &= \Big\{HP_{421}+HP_{422}+HP_{423}+HP_{424}+HP_{425}+HP_{426} \\
&\quad +HP_{427}+HP_{428}+HP_{429}+HP_{4210}+HP_{4211}+HP_{4212}\Big\} \qquad (4.17k) \\
HP_{421} &= 20\,e_r^2\Big[12\Big((-11412\,e_r^{12}+58190\,e_r^{10}-119770\,e_r^8+125420\,e_r^6 \\
&\quad -68360\,e_r^4+17062\,e_r^2-1130)\,C^6 \\
&\quad +(11358\,e_r^{12}-200354\,e_r^{10}+397966\,e_r^8-121140\,e_r^6 \\
&\quad -191206\,e_r^4+87590\,e_r^2+15786)\,C^4 \\
&\quad +(-9888\,e_r^{12}-545279\,e_r^{10}-257984\,e_r^8+1302345\,e_r^6 \\
&\quad +250406\,e_r^4-994618\,e_r^2+255018)\,C^2 \\
&\quad -9075\,e_r^{12}-9858\,e_r^{10}-474666\,e_r^8+609319\,e_r^6 \\
&\quad -2119432\,e_r^4+183150\,e_r^2+56982\Big) \\
&\quad +4\Big((171180\,e_r^{12}-872850\,e_r^{10}+1796550\,e_r^8 \\
&\quad -1881300\,e_r^6+1025400\,e_r^4-255930\,e_r^2+16950)\,C^6 \\
&\quad +(-189582\,e_r^{12}+2027658\,e_r^{10}-3553282\,e_r^8+499372\,e_r^6 \\
&\quad +2388726\,e_r^4-1063238\,e_r^2-109654)\,C^4 \\
&\quad +(-136626\,e_r^{12}+225777\,e_r^{10}-2980566\,e_r^8+8658903\,e_r^6 \\
&\quad -13660176\,e_r^4+8143398\,e_r^2-250710)\,C^2 \\
&\quad +149025\,e_r^{12}-556978\,e_r^{10}+4505192\,e_r^8-12372133\,e_r^6 \\
&\quad +9879830\,e_r^4-6524422\,e_r^2+384566\Big)\,\eta \\
&\quad +3\,\left(1-e_r^2\right)^2\Big((-228240\,e_r^8+707320\,e_r^6-752520\,e_r^4 \\
&\quad +296040\,e_r^2-22600)\,C^6 \\
&\quad +(295272\,e_r^8-57304\,e_r^6-967968\,e_r^4+845640\,e_r^2-115640)\,C^4 \\
&\quad +(161642\,e_r^8-1867277\,e_r^6+3654850\,e_r^4
\end{aligned}
$$

$$
\begin{aligned}
& -2930376\, e_r^2 + 544296)\, C^2 \\
& -231200\, e_r^8 + 1072023\, e_r^6 - 2105622\, e_r^4 + 2119096\, e_r^2 - 261224\Big)\, \eta^2 \\
& -1881\, \left(1 + C^2\right) \left(1 - e_r^2\right)^2 \left(7\, e_r^6 + 70\, e_r^4 + 112\, e_r^2 + 32\right) \eta^3\Big] \qquad (4.171) \\
HP_{422} \;=\;& 2\, e_r \Big[6 \Big((-27720\, e_r^{14} + 542232\, e_r^{12} - 2306920\, e_r^{10} + 4371320\, e_r^8 \\
& -4290520\, e_r^6 + 2161480\, e_r^4 - 461432\, e_r^2 + 11560)\, C^6 \\
& +(-41580\, e_r^{14} + 1324848\, e_r^{12} + 5936800\, e_r^{10} - 23012760\, e_r^8 \\
& +21061540\, e_r^6 - 3475600\, e_r^4 - 1524408\, e_r^2 - 268840)\, C^4 \\
& +(31185\, e_r^{14} + 5180817\, e_r^{12} + 31023187\, e_r^{10} - 29441285\, e_r^8 \\
& -41684856\, e_r^6 + 37543688\, e_r^4 - 96280\, e_r^2 - 2556456)\, C^2 \\
& +367499\, e_r^{12} + 7465814\, e_r^{10} + 908535\, e_r^8 \\
& +27291860\, e_r^6 + 42451792\, e_r^4 - 12161208\, e_r^2 - 1071832\Big) \\
& +10 \Big((83160\, e_r^{14} - 1626696\, e_r^{12} + 6920760\, e_r^{10} - 13113960\, e_r^8 \\
& +12871560\, e_r^6 - 6484440\, e_r^4 + 1384296\, e_r^2 - 34680)\, C^6 \\
& +(36036\, e_r^{14} - 1235424\, e_r^{12} - 15180216\, e_r^{10} + 47999416\, e_r^8 \\
& -41545420\, e_r^6 + 6216528\, e_r^4 + 3266912\, e_r^2 + 442168)\, C^4 \\
& +(-68607\, e_r^{14} + 2131725\, e_r^{12} + 6417939\, e_r^{10} - 21510849\, e_r^8 \\
& +25180608\, e_r^6 + 12128928\, e_r^4 - 25011384\, e_r^2 + 731640)\, C^2 \\
& -721885\, e_r^{12} - 7520226\, e_r^{10} + 12056715\, e_r^8 + 25823916\, e_r^6 \\
& -16573992\, e_r^4 + 21606384\, e_r^2 - 1112504\Big)\, \eta \\
& -15\, \left(1 - e_r^2\right)^2 \Big((55440\, e_r^{10} - 973584\, e_r^8 + 2611232\, e_r^6 - 2546592\, e_r^4 \\
& +876624\, e_r^2 - 23120)\, C^6 + (-72072\, e_r^{10} + 1878192\, e_r^8 - 3982184\, e_r^6 \\
& +1266960\, e_r^4 + 978528\, e_r^2 - 69424)\, C^4 + (18942\, e_r^{10} - 613157\, e_r^8 \\
& -1216036\, e_r^6 + 5076104\, e_r^4 - 4793712\, e_r^2 + 527184)\, C^2 \\
& -432285\, e_r^8 + 1754916\, e_r^6 - 3265128\, e_r^4 + 3956544\, e_r^2 - 373200\Big)\, \eta^2
\end{aligned}
$$

$$
\begin{aligned}
& +495\ \left(1+C^2\right)\left(1-e_r^2\right)^2 (49\ e_r^8+1820\,\mathrm{ef}+6440\,e_r^4 \\
& +4032\,e_r^2+256\Big)\,\eta^3\Big]\ \mathrm{cosu} \qquad (4.17\mathrm{m}) \\
HP_{423} \;=\;& 4\Big[12\Big((85740\,\mathrm{ef}^4-542888\,\mathrm{ef}^2+1428980\,e_r^{10}-2002480\,e_r^8 \\
& +1576980\,e_r^6-663080\,e_r^4+117388\,e_r^2-640)\,C^6 \\
& +(-85080\ \mathrm{ef}^4-828832\,e_r^{12}+548980\,e_r^{10} \\
& +4078240\,e_r^8-5982640\,e_r^6+2207840\,e_r^4+42932\,e_r^2+18560)\,C^4 \\
& +(-220455\ e_r^{14}-3199870\,\mathrm{ef}^2-5009561\,\mathrm{ef}^0+16140398\,e_r^8 \\
& -8484236\,e_r^6+333800\,e_r^4+312436\,e_r^2+127488)\,C^2 \\
& +64920\,e_r^{14}-870755\,e_r^{12}+793\,e_r^{10}-6982190\,e_r^8 \\
& -4598448\,e_r^6-3370960\,e_r^4+2789324\,e_r^2+79616\Big) \\
& -20\,\Big((257220\,e_r^{14}-1628664\,e_r^{12}+4286940\,e_r^{10}-6007440\,e_r^8 \\
& +4730940\,e_r^6-1989240\,e_r^4+352164\,e_r^2-1920)\,C^6 \\
& +(-267594\,e_r^{14}-991092\,e_r^{12}-228440\,e_r^{10}+9076848\,e_r^8 \\
& -11967202\,e_r^6+4210772\,e_r^4+134580\,e_r^2+32128)\,C^4 \\
& +(-169419\,e_r^{14}+3114384\,e_r^{12}-7856703\,e_r^{10}+9976014\,e_r^8- \\
& 5977032\,e_r^6+4070232\,e_r^4-3203556\,e_r^2+46080)\,C^2 \\
& +188952\,{e_r}^{14}-1855201\,e_r^{12}+4118901\,e_r^{10} \\
& -6176586\,e_r^8+12809934\,e_r^6-5629764\,e_r^4+3243660\,e_r^2-71936\Big)\eta \\
& +15\,\left(1-e_r^2\right)^2\,\Big((342960\,e_r^{10}-1485632\,e_r^8 \\
& +2401696\,e_r^6-1720896\,e_r^4+464432\,e_r^2-2560)\,C^6 \\
& +(-405384\,e_r^{10}+748928\,e_r^8-1670008\,e_r^6 \\
& +1070928\,e_r^4+259120\,e_r^2-3584)\,C^4 \\
& +(-190954\,e_r^{10}+1878153\,e_r^8-3511504\,e_r^6 \\
& +4224904\,e_r^4-2171568\,e_r^2+55296)\,C^2
\end{aligned}
$$

$$
\begin{aligned}
& +250432\, e_r^{10} - 1314151\, e_r^8 + 2741592\, e_r^6 - 3120584\, e_r^4 \\
& +1494864\, e_r^2 - 50176\Big)\, \eta^2 \\
& +e_r^2 \left(495\, \left(C^2+1\right) \left(1 - e_r^2 +\right) (119\, e_r^8 + 777\, e_r^6 \right. \\
& \left. -168\, e_r^4 - 856\, e_r^2 + 128)\right) \eta^3 \Big] \cos 2u \qquad (4.17\mathrm{n}) \\
HP_{424} \;=\; & 2\, e_r \Big[\Big((6 - 19800\, e_r^{14} - 517616\, e_r^{12} + 3150304\, e_r^{10} - 7294280\, e_r^8 \\
& +8719400\, e_r^6 - 5715520\, e_r^4 + 1942736\, e_r^2 - 265224)\, C^6 \\
& +(-29700\, e_r^{14} + 1707296\, e_r^{12} + 1586376\, e_r^{10} - 8586280\, e_r^8 \\
& -1091780\, e_r^6 + 13943520\, e_r^4 - 8234336\, e_r^2 + 704904)\, C^4 \\
& +(22275\, e_r^{14} + 3547307\, e_r^{12} + 16757169\, e_r^{10} - 20002495\, e_r^8 \\
& -9132584\, e_r^6 + 16456512\, e_r^4 - 7334144\, e_r^2 - 314040)\, C^2 \\
& +1909\, e_r^{12} + 2104658\, e_r^{10} + 12650105\, e_r^8 + 24648924\, e_r^6 \\
& +4084144\, e_r^4 - 4611824\, e_r^2 - 4284616\Big) \\
& +10\, \Big((59400\, e_r^{14} + 1552848\, e_r^{12} - 9450912\, e_r^{10} + 21882840\, e_r^8 \\
& -26158200\, e_r^6 + 17146560\, e_r^4 - 5828208\, e_r^2 + 795672)\, C^6 \\
& +(25740\, e_r^{14} - 4391080\, e_r^{12} + 2203168\, e_r^{10} + 6599160\, e_r^8 \\
& +11829100\, e_r^6 - 31788280\, e_r^4 + 16843080\, e_r^2 - 1320888)\, C^4 \\
& +(-49005\, e_r^{14} + 1545015\, e_r^{12} + 2938569\, e_r^{10} - 15481563\, e_r^8 \\
& +20170296\, e_r^6 - 9285144\, e_r^4 + 4161408\, e_r^2 - 3999576)\, C^2 \\
& +257917\, e_r^{12} - 1895710\, e_r^{10} + 520525\, e_r^8 - 154964\, e_r^6 \\
& +23352832\, e_r^4 - 9469912\, e_r^2 + 5180152\Big)\, \eta \\
& -15\, \left(1 - e_r^2\right)^2 \Big((39600\, e_r^{10} + 1114432\, e_r^8 - 4111344\, e_r^6 \\
& +5251440\, e_r^4 - 2824576\, e_r^2 + 530448)\, C^6 \\
& +(-51480\, e_r^{10} - 2611360\, e_r^8 + 3821016\, e_r^6 \\
& -4268640\, e_r^4 + 3170320\, e_r^2 - 59856)\, C^4
\end{aligned}
$$

$$
\begin{aligned}
&+(13530\, e_r^{10} + 990001\, e_r^8 + 2135208\, e_r^6 \\
&-3797880\, e_r^4 + 4281920\, e_r^2 - 1925520)\, C^2 \\
&+443249\, e_r^8 - 2034432\, e_r^6 + 3309272\, e_r^4 - 4553232\, e_r^2 + 1435472\Big)\, \eta^2 \\
&+495\, \left(1 + C^2\right) \left(1 - e_r^2\right)^2 \Big(3\, e_r^8 - 168\, e_r^6 - 1960\, e_r^4 \\
&-2464\, e_r^2 - 256)\, \eta^3\Big] \cos 3u
\end{aligned}
\tag{4.17o}
$$

$$
\begin{aligned}
HP_{425} \;=\;& 16\, \Big[12\, \Big((8010\, e_r^{14} + 3358\, e_r^{12} - 141036\, e_r^{10} + 374460\, e_r^8 \\
&-434990\, e_r^6 + 249990\, e_r^4 - 63888\, e_r^2 + 4096)\, C^6 \\
&+(-2700\, e_r^{14} - 174398\, e_r^{12} + 212856\, e_r^{10} + 356100\, e_r^8 \\
&-538940\, e_r^6 + 24330\, e_r^4 + 144768\, e_r^2 - 22016)\, C^4 \\
&+(-22110\, e_r^{14} - 494551\, e_r^{12} + 137510\, e_r^{10} + 495473\, e_r^8 \\
&-268856\, e_r^6 - 66890\, e_r^4 + 231712\, e_r^2 - 12288)\, C^2 \\
&-3735\, e_r^{14} + 30788\, e_r^{12} - 412830\, e_r^{10} - 955031\, e_r^8 \\
&-640062\, e_r^6 + 498570\, e_r^4 + 231152\, e_r^2 + 30208\Big) \\
&-20\, \Big((24030\, e_r^{14} + 10074\, e_r^{12} - 423108\, e_r^{10} + 1123380\, e_r^8 \\
&-1304970\, e_r^6 + 749970\, e_r^4 - 191664\, e_r^2 + 12288)\, C^6 \\
&+(-22806\, e_r^{14} - 353296\, e_r^{12} + 654564\, e_r^{10} + 169272\, e_r^8 \\
&-517654\, e_r^6 - 229944\, e_r^4 + 342360\, e_r^2 - 42496)\, C^4 \\
&+(8382\, e_r^{14} + 42855\, e_r^{12} + 142104\, e_r^{10} - 1027287\, e_r^8 \\
&+1451856\, e_r^6 - 686190\, e_r^4 + 105144\, e_r^2 - 36864)\, C^2 \\
&-8685\, e_r^{14} + 108194\, e_r^{12} - 362384\, e_r^{10} + 245511\, e_r^8 \\
&+180480\, e_r^6 + 513644\, e_r^4 - 115920\, e_r^2 + 67072\Big)\, \eta \\
&+15\, \left(1 - e_r^2\right)^2 \Big((32040\, e_r^{10} + 77512\, e_r^8 - 441160\, e_r^6 \\
&+538008\, e_r^4 - 222784\, e_r^2 + 16384)\, C^6
\end{aligned}
$$

$$
\begin{aligned}
&+(-68328\, e_r^{10} - 301120\, e_r^{8} + 488152\, e_r^{6} - 353328\, e_r^{4} \\
&+244864\, e_r^{2} - 10240)\, C^4 + (38794\, e_r^{10} + 166415\, e_r^{8} \\
&+148738\, e_r^{6} - 333748\, e_r^{4} + 252648\, e_r^{2} - 49152)\, C^2 \\
&-2752\, e_r^{10} + 43067\, e_r^{8} - 187750\, e_r^{6} + 151908\, e_r^{4} \\
&-278616\, e_r^{2} + 43008\Big)\, \eta^2 \\
&+495\ \left(1 + C^2\right)\left(1 - e_r^2\right)^2 (23\, \mathrm{ef} + 294\, e_r^4 + 532\, e_r^2 + 120)\ \eta^3 e_r^2\Big]\ \cos 4u
\end{aligned}
\tag{4.17p}
$$

$$
\begin{aligned}
HP_{426} \;=\;& 5\, e_r \Big[6 \Big((-3960\, e_r^{14} - 102296\, e_r^{12} + 393072\, e_r^{10} - 292320\, e_r^{8} \\
&-576920\, e_r^{6} + 1171560\, e_r^{4} - 766944\, e_r^{2} + 177808)\, C^6 \\
&+(-5940\, e_r^{14} + 124904\, e_r^{12} + 826824\, e_r^{10} - 2702352\, e_r^{8} \\
&+1978892\, e_r^{6} + 260376\, e_r^{4} .- 489984\, e_r^{2} + 7280)\, \mathrm{C}^4 \\
&+(4455\, e_r^{14} + 709031\, \mathrm{ef}^2 + 1376269\, e_r^{10} - 2582315\, e_r^{8} \\
&+1550280\, e_r^{6} - 116968\, e_r^{4} - 756160\, e_r^{2} - 184592)\, C^2 \\
&+13385\, e_r^{12} + 189506\, e_r^{10} + 3585485\, e_r^{8} + 3156500\, e_r^{6} \\
&-1048376\, e_r^{4} - 2233184\, e_r^{2} - 496\Big) \\
&+2 \Big((59400\, e_r^{14} + 1534440\, e_r^{12} - 5896080\, e_r^{10} + 4384800\, e_r^{8} \\
&+8653800\, e_r^{6} - 17573400\, e_r^{4} + 11504160\, e_r^{2} - 2667120)\, C^6 \\
&+(25740\, e_r^{14} - 2244152\, e_r^{12} - 5403864\, e_r^{10} + 26021856\, e_r^{8} \\
&-24584900\, e_r^{6} + 4783944\, e_r^{4} + 403440\, e_r^{2} + 997936)\, C^4 \\
&+(-49005\, e_r^{14} - 239049\, e_r^{12} + 3911673\, e_r^{10} - 13313283\, e_r^{8} \\
&+5234568\, e_r^{6} + 13510536\, e_r^{4} - 11824320\, e_r^{2} + 2768880)\, C^2 \\
&-73979\, e_r^{12} + 2019186\, e_r^{10} - 9275659\, e_r^{8} + 8708964\, e_r^{6} \\
&+2469096\, e_r^{4} + 6670768\, e_r^{2} - 1099696\Big)\, \eta \\
&-3\ \left(1 - e_r^2\right)^2\ \Big((39600\, e_r^{10} + 1102160\, e_r^{8} - 1766000\, e_r^{6}
\end{aligned}
$$

$$
\begin{aligned}
& -1710960\,e_r^4 + 4113280\,e_r^2 - 1778080)\,C^6 \\
& +(-51480\,e_r^{10} - 2245760\,e_r^8 - 963640\,e_r^6 \\
& +4621776\,e_r^4 - 3046432\,e_r^2 + 1685536)\,C^4 \\
& +(13530\,e_r^{10} + 1120041\,e_r^8 + 2429824\,e_r^6 \\
& -1635904\,e_r^4 - 1367936\,e_r^2 + 1845920)\,C^2 \\
& -39191\,e_r^8 + 57544\,e_r^6 - 1444480\,e_r^4 + 248224\,e_r^2 - 1753376\Big)\,\eta^2 \\
& -99\,\left(1 + C^2\right)\left(1 - e_r^2\right)^2 (53\,e_r^6 + 2128\,e_r^4 + 7280\,e_r^2 + 3136)\;\eta^3 e_r^2\Big]\,\cos 5u
\end{aligned}
\tag{4.17q}
$$

$$
\begin{aligned}
HP_{427} \;=\; & 2\Big[12\,\Big((25680\,e_r^{14} + 25096\,e_r^{12} - 448472\,e_r^{10} + 967120\,e_r^8 - 784480\,e_r^6 \\
& +119720\,e_r^4 + 157544\,e_r^2 - 62208)\,C^6 \\
& +(25500\,e_r^{14} - 369496\,e_r^{12} + 529712\,e_r^{10} + 459600\,e_r^8 \\
& -1306820\,e_r^6 + 872680\,e_r^4 - 273384\,e_r^2 + 62208)\,C^4 \\
& +(-95175\,e_r^{14} - 944590\,e_r^{12} + 1077719\,e_r^{10} - 281682\,e_r^8 \\
& -888416\,e_r^6 + 1095640\,e_r^4 - 25704\,e_r^2 + 62208)\,C^2 \\
& -8880\,e_r^{14} - 15019\,e_r^{12} - 1015099\,e_r^{10} - 2719906\,e_r^8 \\
& -98252\,e_r^6 + 2030200\,e_r^4 + 141544\,e_r^2 - 62208\Big) \\
& -20\,\Big((77040\,e_r^{14} + 75288\,e_r^{12} - 1345416\,e_r^{10} + 2901360\,e_r^8 \\
& -2353440\,e_r^6 + 359160\,e_r^4 + 472632\,e_r^2 - 186624)\,C^6 \\
& +(7662\,e_r^{14} - 847368\,e_r^{12} + 1797868\,e_r^{10} - 262264\,e_r^8 \\
& -1828290\,e_r^6 + 1629392\,e_r^4 - 683624\,e_r^2 + 186624)\,C^4 \\
& +(-62799\,e_r^{14} + 104784\,e_r^{12} - 179151\,e_r^{10} - 1626474\,e_r^8 \\
& +3067176\,e_r^6 - 1413048\,e_r^4 - 77112\,e_r^2 + 186624)\,C^2 \\
& -20400\,e_r^{14} + 122267\,e_r^{12} - 226623\,e_r^{10} \\
& -532166\,e_r^8 + 602362\,e_r^6 + 851856\,e_r^4 + 288104\,e_r^2 - 186624\Big)\,\eta
\end{aligned}
$$

$$
\begin{aligned}
&+15\left(1-e_r^2\right)^2\Big((102720\,e_r^{10}+305824\,e_r^8-1284960\,e_r^6\\
&+992736\,e_r^4+132512\,e_r^2-248832)\,C^6\\
&+(-142824\,e_r^{10}-1020208\,e_r^8+1280376\,e_r^6\\
&-176640\,e_r^4-189536\,e_r^2+248832)\,C^4\\
&+(55342\,e_r^{10}+573957\,e_r^8+171264\,e_r^6-877168\,e_r^4\\
&+394848\,e_r^2+248832)\,C^2-17536\,e_r^{10}+60389\,e_r^8-302200\,e_r^6\\
&+29424\,e_r^4-337824\,e_r^2-248832\Big)\,\eta^2\\
&+495\,e_r^4\,\left(1+C^2\right)\left(1-e_r^2\right)^2(229\,e_r^4+1792\,e_r^2+1456)\,\eta^3\Big]\cos 6u
\end{aligned}
\tag{4.17r}
$$

$$
\begin{aligned}
HP_{428} \;=\;& e_r\Big[6\Big((-6600\,e_r^{14}-138848\,e_r^{12}+560552\,e_r^{10}-489040\,e_r^8\\
&-641400\,e_r^6+1474240\,e_r^4-991592\,e_r^2+232688)\,C^6\\
&+(-9900\,\mathrm{ef}^4+13928\,\mathrm{ef}^2+552928\,e_r^{10}-1982160\,e_r^8\\
&+2870820\,e_r^6-2249800\,e_r^4+1036872\,e_r^2-232688)\,\mathrm{C}^4\\
&+(7425\,e_r^{14}+873065\,\mathrm{ef}^2-155341\,e_r^{10}-1909885\,e_r^8+4307064\,e_r^6\\
&-3411632\,e_r^4+521992\,e_r^2-232688)\,\mathrm{C}^2\\
&+53907\,\mathrm{ef}^2+503710\,e_r^{10}+3405455\,e_r^8+2686636\,e_r^6\\
&-4200424\,e_r^4-567272\,e_r^2+232688\Big)\\
&+10\,\Big((19800\,\mathrm{ef}^4+416544\,e_r^{12}-1681656\,\mathrm{ef}^0+1467120\,e_r^8\\
&+1924200\,e_r^6-4422720\,e_r^4+2974776\,e_r^2-698064)\,C^6\\
&+(8580\,e_r^{14}-286864\,e_r^{12}-425088\,e_r^{10}+3814096\,e_r^8\\
&-6849884\,e_r^6+6038736\,e_r^4-2997640\,e_r^2+698064)\,C^4\\
&+(-16335\,\mathrm{ef}^4-348483\,\mathrm{ef}^2+627315\,e_r^{10}-2704137\,e_r^8\\
&+3130512\,e_r^6+179040\,e_r^4-1565976\,e_r^2+698064)\,\mathrm{C}^2\\
&-72837\,e_r^{12}+396174\,e_r^{10}-1126909\,e_r^8-146316\,e_r^6
\end{aligned}
$$

$$+1146672\, e_r^4 + 1588840\, e_r^2 - 698064\Big)\, \eta$$
$$-15\, \left(1 - e_r^2\right)^2 \Big((13200\, e_r^{10} + 304096\, e_r^8 - 526112\, e_r^6 - 378240\, e_r^4$$
$$+1052432\, e_r^2 - 465376)\, C^6 + (-17160\, e_r^{10} - 531952\, e_r^8 - 55384\, e_r^6$$
$$+1130592\, e_r^4 - 991472\, e_r^2 + 465376)\, C^4 + (4510\, e_r^{10} + 201395\, e_r^8$$
$$+447860\, e_r^6 - 624368\, e_r^4 - 113232\, e_r^2 + 465376)\, C^2$$
$$-10405\, e_r^8 - 32628\, e_r^6 - 203152\, e_r^4 + 52272\, e_r^2 - 465376\Big)\eta^2$$
$$+e_r^4\, (-495\, \left(1 + C^2\right)\, \left(1 - e_r^2\right)^2\, (39\, e_r^4 + 1036\, e_r^2 + 1680)\Big)\Big)\eta^3\Big]\, \cos 7u \tag{4.17s}$$

$$HP_{429} = 4\, e_r^2\, \Big[12\, \Big((2940\, e_r^{12} - 2942\, e_r^{10} - 29390\, e_r^8 + 88180\, e_r^6 - 102880\, e_r^4$$
$$+55850\, e_r^2 - 11758)\, C^6 + (2970\, e_r^{12} - 13278\, e_r^{10} + 40730\, e_r^8$$
$$-83980\, e_r^6 + 95790\, e_r^4 - 53990\, e_r^2 + 11758)\, C^4$$
$$+(-9960\, e_r^{12} - 38761\, e_r^{10} + 147624\, e_r^8 - 254449\, e_r^6 + 176098\, e_r^4$$
$$-32310\, e_r^2 + 11758)\, C^2 - 165\, e_r^{12} - 11334\, e_r^{10} - 55542\, e_r^8 - 193895\, e_r^6$$
$$+181344\, e_r^4 + 30450\, e_r^2 - 11758\Big) - 20\, \Big((8820\, e_r^{12} - 8826\, e_r^{10} - 88170\, e_r^8$$
$$+264540\, e_r^6 - 308640\, e_r^4 + 167550\, e_r^2 - 35274)\, C^6 + (906\, e_r^{12} - 28790\, e_r^{10}$$
$$+132974\, e_r^8 - 270516\, e_r^6 + 288014\, e_r^4 - 157862\, e_r^2 + 35274)\, C^4$$
$$+(-8874\, e_r^{12} - 3819\, e_r^{10} - 28278\, e_r^8 + 10995\, e_r^6$$
$$+91632\, e_r^4 - 96930\, e_r^2 + 35274)\, C^2$$
$$-363\, e_r^{12} - 1338\, e_r^{10} + 15592\, e_r^8 - 82113\, e_r^6$$
$$+47574\, e_r^4 + 87242\, e_r^2 - 35274\Big)\, \eta$$
$$+15\, \left(1 - e_r^2\right)^2\, \Big((11760\, e_r^8 + 11752\, e_r^6 - 105816\, e_r^4$$
$$+129336\, e_r^2 - 47032)\, C^6$$
$$+(-16584\, e_r^8 - 55336\, e_r^6 + 133824\, e_r^4 - 108936\, e_r^2 + 47032)\, C^4$$
$$+(4206\, e_r^8 + 33489\, e_r^6 - 27082\, e_r^4 - 35176\, e_r^2 + 47032)\, C^2$$

$$
\begin{aligned}
& +32\, e_r^8 - 1667\, e_r^6 - 13650\, e_r^4 + 14776\, e_r^2 - 47032\Big)\, \eta^2 \\
& +495\, e_r^4\, (1 + C^2 + 1)\, (1 - e_r^2)^2\, (41\, e_r^2 + 154)\, \eta^3\Big]\, \cos 8u \qquad (4.17\mathrm{t}) \\
HP_{4210} \;=\; & e_r^3\, [-6\, ((1320\, e_r^{12} + 13904\, e_r^{10} - 89320\, e_r^8 + 191840\, \mathrm{ef} \\
& -198440\, e_r^4 + 101200\, e_r^2 - 20504)\, C^6 + (1980\, e_r^{12} + 5896\, e_r^{10} \\
& \$12000\;\; e_r^8 - 111680\, e_r^6 + 174380\, e_r^4 - 103080\, e_r^2 + 20504)\, C^4 \\
& +(-1485\, e_r^{12} - 76517\, e_r^{10} + 284361\, e_r^8 - 492695\, e_r^6 + 326024\, e_r^4 \\
& -60192\, e_r^2 + 20504)\, C^2 - 7815\, e_r^{10} - 26518\, e_r^8 - 380915\, e_r^6 \\
& +293460\, e_r^4 + 62072\, e_r^2 - 20504\Big) + 10\, \Big((3960\, \mathrm{ef}^2 + 41712\, e_r^{10} \\
& -267960\, e_r^8 + 575520\, e_r^6 - 595320\, e_r^4 + 303600\, e_r^2 - 61512)\, C^6 \\
& +(1716\, \mathrm{ef}^2 - 15856\, e_r^{10} + 144112\, e_r^8 - 415872\, e_r^6 + 515764\, e_r^4 \\
& -291376\, e_r^2 + 61512)\, C^4 + (-3267\, \mathrm{ef}^2 - 47655\, e_r^{10} + 37431\, e_r^8 \\
& -85605\, \mathrm{ef} + 218160\, e_r^4 - 180576\, e_r^2 + 61512)\, C^2 \\
& -5649\, e_r^{10} + 47382\, e_r^8 - 158633\, e_r^6 + 51316\, e_r^4 \\
& +168352\, e_r^2 - 61512\Big)\, \eta \\
& -15\, \left(1 - e_r^2\right)^2\, ((2640\, e_r^8 + 33088\, e_r^6 - 115104\, e_r^4 \\
& +120384\, e_r^2 - 41008)\, C^6 \\
& +(-3432\, e_r^8 - 57328\, e_r^6 + 113064\, e_r^4 - 93312\, e_r^2 + 41008)\, \mathrm{C}^4 \\
& +(902\, e_r^8 + 20007\, e_r^6 - 9484\, e_r^4 - 38368\, e_r^2 + 41008)\, C^2 \\
& -977\, e_r^6 - 6580\, e_r^4 + 11296\, e_r^2 - 41008\Big)\, \eta^2 \\
& -495\, e_r^4\, (1 + C^2 + 1)\, (1 - e_r^2)^2\, (11\, e_r^2 + 140)\, \eta^3\Big]\, \cos 9u \qquad (4.17\mathrm{u}) \\
HP_{4211} \;=\; & 30\, e_r^4\, \Big[12\, \Big((40\, e_r^{10} - 200\, e_r^8 + 400\, e_r^6 - 400\, e_r^4 + 200\, e_r^2 - 40)\, C^6 \\
& +(44\, e_r^{10} - 24\, e_r^8 - 232\, e_r^6 + 400\, e_r^4 - 228\, e_r^2 + 40)\, C^4 \\
& +(-101\, e_r^{10} + 534\, e_r^8 - 1035\, e_r^6 + 682\, e_r^4 - 120\, e_r^2 + 40)\, C^2 \\
& -e_r^8 - 793\, e_r^6 + 546\, e_r^4 + 148\, e_r^2 - 40\Big)
\end{aligned}
$$

$$
\begin{aligned}
& -4\Big((600\,e_r^{10}-3000\,e_r^{8}+6000\,e_r^{6}-6000\,e_r^{4}+3000\,e_r^{2}-600)\,C^6 \\
& +(98\,e_r^{10}+976\,e_r^{8}-4116\,e_r^{6}+5512\,e_r^{4}-3070\,e_r^{2}+600)\,C^4 \\
& +(-521\;e_r^{10}+432\,e_r^{8}-1041\,e_r^{6}+2330\,\text{elf}-1800\,e_r^{2}+600)\,C^2 \\
& +269\,e_r^{8}-1209\,e_r^{6}+30\,e_r^{4}+1870\,e_r^{2}-600\Big)\eta \\
& +\left(1-e_r^2\right)^2\Big((2400\,e_r^{6}-7200\,e_r^{4}+7200\,e_r^{2}-2400)\,C^6 \\
& +(-3336\,e_r^{6}+6192\,e_r^{4}-5256\,e_r^{2}+2400)\,C^4 \\
& +(934\,e_r^{6}-279\,e_r^{4}-2400\,e_r^{2}+2400)\,C^2 \\
& -279\,\text{elf}+456\,e_r^{2}-2400\Big)\eta^2 \\
& +e_r^4\,(297\,\left(1+C^2\right)\left(1-e_r^2\right)^2\eta^3\Big]\;\cos 10u \qquad (4.17\text{v}) \\
HP_{4212} \;=\;& 15\,e_r^5\Big[-6\Big((8\,e_r^{10}-40\,e_r^{8}+80\,e_r^{6}-80\,e_r^{4}+40\,e_r^{2}-8)\,C^6 \\
& +(12\,e_r^{10}-8\,e_r^{8}-56\,e_r^{6}+96\,\text{elf}-52\,e_r^{2}+8)\,C^4 \\
& +(-9\;e_r^{10}+103\,\text{ef}-227\,e_r^{6}+149\,\text{elf}-24\,\text{ef}+8)\,C^2 \\
& +9\,e_r^{8}-174\,e_r^{6}+109\,e_r^{4}+36\,{e_r}^2-8\Big) \\
& +2\Big((120\,e_r^{10}-600\,e_r^{8}+1200\,e_r^{6}-1200\,e_r^{4}+600\,e_r^{2}-120)\,C^6 \\
& +(52\,e_r^{10}+152\,e_r^{8}-888\,e_r^{6}+1232\,\text{elf}-668\,e_r^{2}+120)\,\mathrm{C}^4 \\
& +(-99\;e_r^{10}+57\,e_r^{8}-201\,e_r^{6}+483\,\text{elf}-360\,e_r^{2}+120)\,C^2 \\
& +27\,e_r^{8}-178\,e_r^{6}-85\,e_r^{4}+428\,e_r^{2}-120\Big)\eta \\
& -\left(1-e_r^2\right)^2\Big((240\,e_r^{6}-720\,e_r^{4}+720\,e_r^{2}-240)\,C^6 \\
& +(-312\,e_r^{6}+576\,e_r^{4}-504\,e_r^{2}+240)\,C^4 \\
& +(82\,e_r^{6}-27\,e_r^{4}-240\,e_r^{2}+240)\,C^2-27\,\text{elf}+24\,e_r^{2}-240\Big)\,\eta^2 \\
& -33\,e_r^4\left(1+C^2\right)\left(1-e_r^2\right)^2\eta^3\Big]\;\cos 11u \qquad (4.17\text{w}) \\
HP_{43} \;=\;& e_r\Big[(4392\,\text{ef}-9488\,\text{elf}+5800\,e_r^{2}-704)\,C^4 \\
& +(279\,\text{ef}+5916\,\text{elf}+336\,e_r^{2}-9600)\,\mathrm{C}^2 \\
& -2841\text{ef}+22804\,\text{elf}-17144\,e_r^{2}-5888
\end{aligned}
$$

$$
\begin{aligned}
&-2\Big(\left(6588\,e_r^6-14232\,e_r^4+8700\,e_r^2-1056\right)C^4\\
&+\left(-905\,e_r^6-5115\,e_r^4+6120\,e_r^2-5248\right)C^2\\
&-5585\,e_r^6+20217\,e_r^4-20100\,e_r^2+320\Big)\,\eta\Big]\sin u\\
&+4\Big[\left(-1688\,e_r^6+3440\,e_r^4-1816\,e_r^2+64\right)C^4\\
&+\left(-975\,e_r^6-2472\,e_r^4+3696\,e_r^2+576\right)C^2\\
&-1151\,e_r^6-4912\,e_r^4+6632\,e_r^2+256\\
&+\Big(\left(5064\,e_r^6-10320\,e_r^4+5448\,e_r^2-192\right)C^4\\
&+\left(-2491\,e_r^6+1195\,e_r^4-1336\,e_r^2-800\right)C^2\\
&-1963\,e_r^6+8515\,e_r^4-10144\,e_r^2+160\Big)\,\eta\Big]\sin 2u\\
&+3\,e_r\Big[\left(240\,e_r^6+1456\,e_r^4-3632\,e_r^2+1936\right)C^4\\
&+\left(261\,e_r^6+3372\,e_r^4-2464\,e_r^2-1664\right)C^2\\
&+509\,e_r^6+3460\,e_r^4-672\,e_r^2-3792\\
&-2\Big(\left(360\,e_r^6+2184\,e_r^4-5448\,e_r^2+2904\right)C^4\\
&+\left(-311\,e_r^6-729\,e_r^4-100\,e_r^2-576\right)C^2\\
&+61\,e_r^6-597\,e_r^4+2588\,e_r^2-3768\Big)\,\eta\Big]\sin 3u\\
&+16\Big[\left(-122\,e_r^6+116\,e_r^4+134\,e_r^2-128\right)C^4\\
&+\left(-249\,e_r^4-36\,e_r^2+288\right)C^2\,e_r^2\\
&-267\,e_r^6-144\,e_r^4+286\,e_r^2+128\\
&+\Big(\left(366\,e_r^6-348\,e_r^4-402\,e_r^2+384\right)C^4\\
&+\left(-110\,e_r^4-117\,e_r^2-85\right)C^2e_r^2\\
&-56\,e_r^6+207\,e_r^4-79\,e_r^2-384\Big)\,\eta\Big]\sin 4u\\
&+e_r\Big[\left(192\,e_r^6+880\,e_r^4-2336\,e_r^2+1264\right)C^4\\
&+\left(555\,e_r^4+1860\,e_r^2-2160\right)C^2e_r^2\\
&+579\,e_r^6+2028\,e_r^4-1088\,e_r^2-1264
\end{aligned}
$$

$$
\begin{aligned}
&-2\Big((288\,e_r^6+1320\,e_r^4-3504\,e_r^2+1896)\,C^4\\
&+(-33\,e_r^4-535\,e_r^2-212)\,C^2e_r^2\\
&+3\,e_r^6-283\,e_r{}^4+1396\,e_r^2-1896\Big)\,\eta\Big]\ \sin 5u\\
&+12\,e_r^2\Big[(-24\,e_r^4+48\,e_r^2-24)\,C^4\\
&+(-49\,e_r^2+40)\,C^2\,e_r^2-49\,e_r^4+16\,e_r^2+24\\
&+\Big((72\,e_r^4-144\,e_r^2+72)\,C^4+(-13\,e_r^4-11)\,C^2e_r^2\\
&-13\,e_r^4+61\,e_r^2-72\Big)\,\eta\Big]\ \sin 6u\\
&+e_r^3\Big[24\,e_r^4-48\,e_r^2+24)\,C^4+(51\,e_r^2-36)\,C^2e_r^2\\
&+51\,e_r^4-12\,e_r^2-24-2\,\Big((36\,e_r^4-72\,e_r^2+36)\,C^4\\
&+(-5\,e_r^4-7)\,C^2e_r^2-5\,e_r^4+29\,e_r^2-36\Big)\eta\Big]\ \sin 7u
\end{aligned}
\tag{4.17x}
$$

$$
\begin{aligned}
HP_{44} \;=\;& \Big\{4\,e_r^2+\mathrm{e}_r\,(3\;e_r^2-10\Big)\cos\mathrm{u}+4\,\Big(-e_r^2+2)\ \cos 2u\\
&+e_r\Big(e_r^2-2)\ \cos 3\mathrm{u})\ .
\end{aligned}
\tag{4.17y}
$$

Similarly, for the 'cross' polarization we have

$$
H_\times^{(0)} = \frac{C}{2}\,\frac{(1-e_r^2)^{(1/2)}}{(1-e_r\,\cos u)^3}\Big\{5\,e_r\sin u-4\,\sin 2u+e_r\sin 3u\Big\} \tag{4.18a}
$$

$$
\begin{aligned}
H_\times^{(1/2)} =\;& \frac{\delta}{8}\,S\,C\,\frac{(1-e_r^2)}{(1-e_r\,\cos u)^5}\,\Big\{2\,\Big(14\,e_r^2-3\Big)\sin u-32\,e_r\sin 2u\\
&+\Big(5\,e_r^2+18\Big)\sin 3u-8\,\mathrm{e}_r\,\sin 4u+e_r^2\sin 5u\Big\}
\end{aligned}
\tag{4.18b}
$$

$$
H_\times^{(1)} = \frac{C}{192}\,\frac{1}{\Big(1-e_r\cos u\Big)^3\Big(1-e_r^2\Big)}\Big\{\frac{(1-e_r^2)^{1/2}}{(1-e_r\,\cos u)^4}\,HX_{21}-96\,\mathrm{v}\,HX_{22}\Big\} \tag{4.18c}
$$

$$
\begin{aligned}
HX_{21} =\;& e_r\Big\{\Big[\Big(-1014\,e_r^6+2708\,e_r^4-2374\,e_r^2+680\Big)\ C^2\\
&+849\,e_r^6-3187\,e_r^4+4190\,e_r^2-2248\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[\big(3042\,e_r^6 - 8124\,e_r^4 + 7122\,e_r^2 - 2040\big)\ C^2 \\
&-3058\,e_r^6 + 7701\,e_r^4 - 7890\,e_r^2 + 1960\Big]\eta\Big\}\ \sin \mathrm{u} \\
&+2\Big\{2\Big[\big(314\,\mathrm{ef} - 692\,e_r^4 + 442\,e_r^2 - 64\big)\ C^2 \\
&-58\,\mathrm{ef} + 355\,e_r^4 - 338\,e_r^2 + 272\Big] \\
&-\Big[\big(1884\,\mathrm{ef} - 4152\,e_r^4 + 2652\,e_r^2 - 384\big)\ C^2 \\
&-1889\,e_r^6 + 3755\,e_r^4 - 3140\,e_r^2 + 416\Big]\eta\Big\}\ \sin 2u \\
&-3\,e_r\Big\{\Big[\big(30\,e_r^6 + 352\,e_r^4 - 794\,e_r^2 + 412\big)\ C^2 \\
&+65\,e_r^6 - 71\,e_r^4 + 398\,e_r^2 + 4\Big] \\
&-\Big[\big(90\,e_r^6 + 1056\,e_r^4 - 2382\,e_r^2 + 1236\big)\ C^2 \\
&-70\,e_r^6 - 969\,e_r^4 + 1894\,e_r^2 - 1284\Big]\eta\Big\}\ \sin 3u \\
&+8\Big\{2\Big[\big(26\,e_r^6 - 20\,e_r^4 - 38\,e_r^2 + 32\big)\ C^2 \\
&+21\,e_r^6 - 9\,e_r^4 + 77\,e_r^2 - 32\Big] \\
&-\Big[\big(156\,e_r^6 - 120\,e_r^4 - 228\,e_r^2 + 192\big)\ C^2 \\
&-106\,e_r^6 + 51\,e_r^4 + 169\,e_r^2 - 192\Big]\eta\Big\}\ \sin 4u \\
&+e_r\Big\{-\Big[\big(42\,e_r^6 + 232\,e_r^4 - 590\,e_r^2 + 316\big)\ C^2 \\
&+87\,e_r^6 - 57\,e_r^4 + 706\,e_r^2 - 316\Big] \\
&+\Big[\big(126\,e_r^6 + 696\,e_r^4 - 1770\,e_r^2 + 948\big)\ C^2 \\
&-42\,e_r^6 - 599\,e_r^4 + 1394\,e_r^2 - 948\Big]\eta\Big\}\ \sin 5u \\
&+6\,e_r^2\Big\{2\Big[\big(e_r^4 - 12\,e_r^2 + 6\big)\ C^2 + 2\,e_r^4 + 13\,e_r^2 - 6\Big] \\
&-\Big[\big(36\,e_r^4 - 72\,e_r^2 + 36\big)\ C^2 - 23\,e_r^4 + 53\,e_r^2 - 36\Big]\eta\Big\}\ \sin 6u \\
&+e_r^3\Big\{-3\Big[\big(2\,e_r^4 - 4\,e_r^2 + 2\big)\ C^2 + e_r^4 + 5\,e_r^2 - 2\Big] \\
&+\Big[\big(18\,e_r^4 - 36\,e_r^2 + 18\big)\ C^2 - 10\,e_r^4 + 25\,e_r^2 - 18\Big]\eta\Big\}\ \sin 7u
\end{aligned}
\tag{4.18d}
$$

$$
HX_{22} = \{\mathrm{e},\ \ (9\,e_r^2 - 30)\ e_r \cos \mathrm{u} - 12\left(e_r^2 - 2\right)\ \cos 2u
$$

$$
\begin{aligned}
& +3\, e_r \left(e_r^2 - 2\right) \cos 3u + 12\, e_r^2 \Big\} && (4.18\text{e}) \\
H_{\times}^{(3/2)} &= \frac{\delta}{(1 - e_r \cos u)^5}\, C\, S \left\{ \frac{1}{768\,(1 - e_r \cos u)^4}\, HX_{31} \right. \\
& \left. + \frac{3}{8} \frac{v}{(1 - e_r^2)^{1/2}}\, HX_{32} \right\} && (4.18\text{f}) \\
HX_{31} &= 2 \left\{ \left[ \left(-3228\, e_r^8 + 10661\, e_r^6 - 11658\, e_r^4 + 4245\, e_r^2 - 20\right) C^2 \right. \right. \\
& \left. +2508\, e_r^8 - 9015\, e_r^6 + 41118\, e_r^4 - 11697\, e_r^2 + 252 \right] \\
& + \left[ \left(6456\, e_r^8 - 21322\, e_r^6 + 23316\, e_r^4 - 8490\, e_r^2 + 40\right) C^2 \right. \\
& \left. \left. -6636\, e_r^8 + 17569\, e_r^6 - 28581\, e_r^4 + 8394\, \mathrm{ef} - 184 \right] \eta \right\} \sin u \\
& +4\, e_r \left\{ \left[ \left(1959\, e_r^6 - 5323\, e_r^4 + 4769\, e_r^2 - 1405\right) C^2 \right. \right. \\
& \left. +26\, e_r^6 - 6449\, e_r^4 - 13474\, e_r^2 + 5389 \right] \\
& - \left[ \left(3918\, e_r^6 - 10646\, e_r^4 + 9538\, e_r^2 - 2810\right) C^2 \right. \\
& \left. \left. -4404\, e_r^6 + 7127\, e_r^4 - 13069\, e_r^2 + 3170 \right] \eta \right\} \sin 2u \\
& +2 \left\{ \left[ \left(48\, e_r^8 - 5207\, e_r^6 + 11080\, e_r^4 - 6731\, e_r^2 + 810\right) C^2 \right. \right. \\
& \left. -1263\, e_r^8 + 4838\, e_r^6 + 14586\, e_r^4 + 4177\, e_r^2 - 3618 \right] \\
& - \left[ \left(96\, e_r^8 - 10414\, e_r^6 + 22160\, e_r^4 - 13462\, e_r^2 + 1620\right) C^2 \right. \\
& \left. \left. -231\, e_r^8 + 10696\, e_r^6 - 10979\, e_r^4 + 15982\, e_r^2 - 2052 \right] \eta \right\} \sin 3u \\
& +4\, e_r \left\{ 2 \left[ \left(306\, e_r^6 + 334\, e_r^4 - 1586\, e_r^2 + 946\right) C^2 \right. \right. \\
& \left. +539\, e_r^6 - 1939\, e_r^4 - 88\, e_r^2 - 6 \right] \\
& - \left[ \left(1224\, e_r^6 + 1336\, e_r^4 - 6344\, e_r^2 + 3784\right) C^2 \right. \\
& \left. \left. -423\, e_r^6 - 3277\, e_r^4 + 3056\, e_r^2 - 3820 \right] \eta \right\} \sin 4u \\
& +2 \left\{ \left[ \left(-108\, e_r^8 - 1731\, e_r^6 + 2536\, e_r^4 + 553\, e_r^2 - 1250\right) C^2 \right. \right. \\
& \left. -447\, e_r^8 - 402\, e_r^6 + 1070\, e_r^4 - 1831\, e_r^2 + 1250 \right] \\
& + \left[ \left(216\, e_r^8 + 3462\, e_r^6 - 5072\, e_r^4 - 1106\, e_r^2 + 2500\right) C^2 \right. \\
& \left. \left. +135\, e_r^8 - 2628\, \mathrm{ef} + 1031\, e_r^4 - 358\, e_r^2 - 2500 \right] \eta \right\} \sin 5u
\end{aligned}
$$

$$
\begin{aligned}
&+4\,e_r\Big\{\Big(187\,e_r^6+113\,e_r^4-787\,e_r^2+487\Big)\;C^2\\
&+274\,e_r^6+131\,e_r^4+694\,e_r^2-487\\
&-\Big[\Big(374\,e_r^6+226\,e_r^4-1574\,e_r^2+974\Big)\;C^2\\
&-116\,e_r^6-469\,e_r^4+815\,e_r^2-974\Big]\eta\Big]\sin 6u\\
&+e_r^2\Big\{\Big(-54\,e_r^6-514\,e_r^4+1190\,e_r^2-622)\;C^2\\
&-81\,e_r^6-685\,e_r^4-918\,e_r^2+622\\
&+\Big[\Big(108\,e_r^6+1028\,e_r^4-2380\,e_r^2+1244)\;C^2\\
&-27\,e_r^6-557\,e_r^4+1138\,e_r^2-1244\Big]\eta\Big]\sin 7u\\
&+6\,e_r^3\Big\{2\,\Big[\Big(8\,e_r^4-16\,e_r^2+8\Big)\;C^2+11\,e_r^4+15\,e_r^2-8\Big]\\
&-\Big[\Big(32\,e_r^4-64\,e_r^2+32\Big)\;C^2-11\,e_r^4+27\,e_r^2-32\Big]\eta\Big]\sin 8u\\
&+3\,e_r^4\Big\{\Big(-2\,e_r^4+4\,e_r^2-2\Big)\;C^2-3\,e_r^4-5\,e_r^2+2\\
&+\Big[\Big(4\,e_r^4-8\,e_r^2+4\Big)\;C^2-e_r^4+3\,e_r^2-4\Big]\eta\Big\}\sin 9u
\end{aligned}
\tag{4.18g}
$$

$$
\begin{aligned}
HX_{32} \;=\;& 4\,e_r\Big(-11\,e_r^2+3\Big)+2\,\Big(-5\,e_r^4+46\,e_r^2-3\Big)\cos u\\
&+4\,e_r\Big(9\,e_r^2-29\Big)\cos 2u+\Big(-5\,e_r^4+e_r^2+54\Big)\cos 3u\\
&+8\,e_r\Big(e_r^2-3\Big)\cos 4u+e_r^2\Big(-e_r^2+3\Big)\cos 5u
\end{aligned}
\tag{4.18h}
$$

$$
\begin{aligned}
H_\times^{(2)} \;=\;& \frac{C}{(1-e_r^2)^{3/2}\,(1-e_r\,\cos u)^3}\Big\{\frac{1}{737\,280\,(1-e_r\,\cos u)^8}\,HX_{44}\\
&+(1-e_r^2)^{3/2}\,HX_{43}\\
&+\frac{1}{128}\,\frac{1}{(1-e_r^2)^{1/2}\,(1-e_r\,\cos u)^4}\,v\,HX_{41}+v^2\,HX_{42}\Big\}
\end{aligned}
\tag{4.18i}
$$

$$
\begin{aligned}
HX_{41} \;=\;& 24\,e_r^2\Big\{\Big[\Big(194\,e_r^6-528\,e_r^4+474\,e_r^2-140)\;C^2\\
&-305\,e_r^6+1978\,e_r^4-158\,e_r^2-1020\Big]\\
&-3\,\Big[\Big(194\,e_r^6-528\,e_r^4+474\,e_r^2-140\Big)\;C^2\\
&-211\,e_r^6+641\,e_r^4-516\,e_r^2-112\Big]\eta\Big\}\\
&+e_r\Big\{\Big(420\,e_r^8-9896\,e_r^6+21444\,e_r^4-14880\,e_r^2+2912\Big)\;C^2
\end{aligned}
$$

$$+1785\,e_r^8-3866\,e_r^6-98244\,e_r^4+65760\,e_r^2+13280$$

$$-2\left[\left(630\,e_r^8-14844\,e_r^6+32166\,e_r^4-22320\,e_r^2+4368\right)C^2\right.$$

$$\left.-280\,e_r^8+13843\,e_r^6-42120\,e_r^4+39712\,e_r^2+1616\right]\eta\Big\}\cos u$$

$$+4\Big\{\left(-924\,e_r^8+5656\,e_r^6-8796\,e_r^4+4320\,e_r^2-256\right)C^2$$

$$-207\,e_r^8+5374\,e_r^6+11652\,e_r^4-12384\,e_r^2-640$$

$$+\left[\left(2772\,e_r^8-16968\,e_r^6+26388\,e_r^4-12960\,e_r^2+768\right)C^2\right.$$

$$\left.-2375\,e_r^8+13375\,e_r^6-25086\,e_r^4+18576\,e_r^2+64\right]\eta\Big\}\cos 2u$$

$$+3\,e_r\Big\{\left(84\,e_r^8+224\,e_r^6-4668\,e_r^4+8328\,e_r^2-3968\right)C^2$$

$$+357\,e_r^8-1494\,e_r^6-5836\,e_r^4-3320\,e_r^2+7488$$

$$-2\left[\left(126\,e_r^8+336\,e_r^6-7002\,e_r^4+12492\,e_r^2-5952\right)C^2\right.$$

$$\left.-56\,e_r^8-733\,e_r^6+5332\,e_r^4-10252\,\mathrm{ef}+7392\right]\eta\Big\}\cos 3u$$

$$+8\Big\{\left(-102\,e_r^8+568\,e_r^6-318\,e_r^4-660\,e_r^2+512\right)C^2$$

$$-213\,e_r^8+762\,e_r^6+1154\,e_r^4-756\,e_r^2-512$$

$$+\left[\left(306\,e_r^8-1704\,e_r^6+954\,e_r^4+1980\,e_r^2-1536\right)C^2\right.$$

$$\left.-235\,e_r^8+1073\,e_r^6-1004\,e_r^4-848\,e_r^2+1536\right]\eta\Big\}\cos 4u$$

$$+e_r\Big\{\left(84\,e_r^8+80\,e_r^6-2940\,e_r^4+5304\,e_r^2-2528\right)C^2$$

$$+357\,e_r^8+450\,e_r^6-4620\,e_r^4+280\,e_r^2+2528$$

$$-2\left[\left(126\,e_r^8+120\,e_r^6-4410\,e_r^4+7956\,e_r^2-3792\right)C^2\right.$$

$$\left.-56\,e_r^8-245\,e_r^6+2748\,e_r^4-5636\,\mathrm{ef}+3792\right]\eta\Big\}\cos 5u$$

$$+12\,e_r^2\Big\{\left(-12\,e_r^6+72\,e_r^4-108\,e_r^2+48\right)C^2$$

$$-43\,e_r^6+102\,e_r^4+4\,e_r^2-48$$

$$+\left[\left(36\,e_r^6-216\,e_r^4+324\,e_r^2-144\right)C^2\right.$$

$$\left.-19\,e_r^6+123\,e_r^4-230\,e_r^2+144\right]\eta\Big\}\cos 6u$$

$$+e_r^3\Big\{3\left[\left(4\,e_r^6-24\,e_r^4+36\,e_r^2-16\right)C^2\right.$$

$$
\begin{aligned}
& +17\,e_r^6 - 34\,e_r^4 - 4\,e_r^2 + 16\Big] \\
& -2\left[\left(18\,e_r^6 - 108\,e_r^4 + 162\,e_r^2 - 72\right)\,C^2\right. \\
& \left.-8\,e_r^6 + 57\,e_r^4 - 112\,e_r^2 + 72\right]\eta\Big\}\,\cos 7u \qquad (4.18j)
\end{aligned}
$$

$$HX_{42} = -45\,\mathrm{e}_r\,\sin u + 36\,\sin 2u - 9\,\mathrm{e}_r\,\sin 3u \qquad (4.18k)$$

$$HX_{43} = (5 - 2\,\eta)\,\sin u\Big\{3\,e_r - 4\,\cos u + e_r\cos 2u\Big\} \qquad (4.18l)$$

$$
\begin{aligned}
HX_{44} = {} & 2\,e_r\Big\{12\Big[\Big(427440\,e_r^{12} - 2575928\,e_r^{10} + 6064152\,e_r^8 \\
& -7046128\,e_r^6 + 4101152\,e_r^4 - 1005528\,e_r^2 + 34840\Big)\,C^4 \\
& +\Big(-749940\,e_r^{12} + 4203844\,e_r^{10} - 18484828\,e_r^8 \\
& +30713476\,e_r^6 - 19449520\,e_r^4 + 4178344\,e_r^2 - 411376\Big)\,C^2 \\
& +350505\,e_r^{12} - 1279820\,e_r^{10} + 9107149\,e_r^8 \\
& -28400076\,e_r^6 + 18333032\,e_r^4 - 6748112\,e_r^2 - 1566248\Big] \\
& -20\Big[\Big(282320\,e_r^{12} - 7727784\,e_r^{10} + 18192456\,e_r^8 \\
& -21138384\,e_r^6 + 12303456\,e_r^4 - 3016584\,e_r^2 + 104520\Big)C^4 \\
& +\Big(-2379600\,e_r^12 + 13416444\,e_r^{10} - 48065648\,e_r^8 \\
& +74076284\,e_r^6 - 46627608\,e_r^4 + 10484336\,e_r^2 - 904208\Big)C^2 \\
& +1109523\,e_r^{12} - 5624018\,e_r^{10} + 28069747\,e_r^8 - 55934684\,e_r^6 \\
& +32750224\,e_r^4 - 8715368\,e_r^2 + 786376\Big]\,\eta \\
& +15\Big[\Big(1709760\,e_r^{12} - 10303712\,e_r^{10} + 24256608\,e_r^8 \\
& -28184512\,e_r^6 + 16404608\,e_r^4 - 4022112\,e_r^2 + 139360\Big)\,C^4 \\
& +\Big(-3518880\,e_r^{12} + 18838480\,e_r^{10} - 46637408\,e_r^8 + 58796176\,e_r^6 \\
& -35967712\,e_r^4 + 9017408\,e_r^2 - 528064\Big)\,C^2 \\
& +1812789\,e_r^{12} - 8420599\,e_r^{10} + 22403662\,e_r^8 \\
& -31362120\,e_r^6 + 19667568\,e_r^4 - 4781280\,e_r^2 + 327264\Big]\,\eta^2 \\
& +495\,\Big(1 - \mathrm{e}_r^2\Big)\Big[21\,e_r^{10} + 350\,e_r^8 - 2520\,e_r^6 - 8400\,e_r^4
\end{aligned}
$$

$$-2560\,e_r^2 + 512\Big]\eta^3\Big\}\ \sin u$$

$$+4\,\Big\{24\,\Big[\Big(-122076\,e_r^{12} + 679796\,e_r^{10} - 1500344\,e_r^8$$

$$+1644936\,e_r^6 - 899564\,e_r^4 + 199172\,e_r^2 - 1920\Big)\,C^4$$

$$+\Big(112408\,e_r^{12} + 436040\,e_r^{10} + 1485516\,e_r^8 - 5551896\,e_r^6$$

$$+4369972\,e_r^4 - 880968\,e_r^2 + 28928\Big)\,C^2$$

$$-68910\,e_r^{12} - 884141\,e_r^{10} + 2289422\,e_r^8 + 3196984\,e_r^6$$

$$-3136176\,e_r^4 + 2369092\,e_r^2 + 85504\Big]$$

$$+40\,\Big[\Big(366228\,e_r^{12} - 2039388\,e_r^{10} + 4501032\,e_r^8$$

$$-4954808\,e_r^6 + 2698692\,e_r^4 - 597516\,e_r^2 + 5760\Big)\,C^4$$

$$+\Big(-500622\,e_r^{12} + 613566\,e_r^{10} - 6027710\,e_r^8$$

$$+14234186\,e_r^6 - 10399116\,e_r^4 + 2142416\,e_r^2 - 62720\Big)\,C^2$$

$$+164401\,e_r^{12} + 1220351\,e_r^{10} + 308090\,e_r^8 - 9929638\,e_r^6$$

$$+7014656\,e_r^4 - 1809412\,e_r^2 + 54784\Big]\,\eta$$

$$-15\,\Big[\Big(976608\,e_r^{12} - 5438368\,e_r^{10} + 12002752\,e_r^8$$

$$-13159488\,e_r^6 + 7196512\,e_r^4 - 1593376\,e_r^2 + 15360\Big)\,C^4$$

$$+\Big(-2267664\,e_r^{12} + 9755312\,e_r^{10} - 22386960\,e_r^8$$

$$+27989584\,e_r^6 - 16572896\,e_r^4 + 3554304\,e_r^2 - 71680\Big)\,C^2$$

$$+1313393\,e_r^{12} - 4226511\,e_r^{10} + 10008014\,e_r^8$$

$$-15045368\,e_r^6 + 9631120\,e_r^4 - 2009312\,e_r^2 + 57344\Big]\,\eta^2$$

$$-495\,\left(1 - e_r^2\right)\,e_r^2\Big[133\,e_r^8 - 42\,e_r^6 - 4872\,e_r^4 - 4976\,e_r^2 - 256\Big]\,\eta^3\Big\}\ \sin 2u$$

$$+2\,e_r\Big\{12\,\Big[\Big(-48000\,e_r^{12} + 958512\,e_r^{10} - 3770664\,e_r^8$$

$$+6457536\,e_r^6 - 5613744\,e_r^4 + 2432976\,e_r^2 - 416616\Big)\,C^4$$

$$+\Big(267300\,e_r^{12} - 2603300\,e_r^{10} + 578476\,e_r^8 + 95548\,e_r^6$$

$$+9173200\,e_r^4-9534248\,e_r^2+2023024\Big)\,C^2$$
$$-161595\,e_r^{12}+2203588\,e_r^{10}-2495743\,e_r^8-8589636\,e_r^6$$
$$+1396888\,e_r^4-3057256\,e_r^2-3685896\Big]$$
$$+20\,\Big[\Big(144000\,e_r^{12}-2875536\,e_r^{10}+11311992\,e_r^8$$
$$-19372608\,e_r^6+16841232\,e_r^4-7298928\,e_r^2+1249848\Big)\,C^4$$
$$+\Big(-621552\,e_r^{12}+7022684\,e_r^{10}-8557640\,e_r^8+12893788\,e_r^6$$
$$-29888336\,e_r^4+23941760\,e_r^2-4790704\Big)\,C^2$$
$$+450705\,e_r^{12}-4258038\,e_r^{10}+590857\,e_r^8+10013092\,e_r^6$$
$$+14890984\,e_r^4-13625808\,e_r^2+3868536\Big]\,\eta$$
$$-15\,\Big[\Big(192000\,e_r^{12}-3834048\,e_r^{10}+15082656\,e_r^8$$
$$-25830144\,e_r^6+22454976\,e_r^4-9731904\,e_r^2+1666464\Big)\,C^4$$
$$+\Big(-403488\,e_r^{12}+8164304\,e_r^{10}-25149824\,e_r^8+42210448\,e_r^6$$
$$-43388288\,e_r^4+22343936\,e_r^2-3777088\Big)\,C^2$$
$$+206835\,e_r^{12}-4414329\,e_r^{10}+10443154\,e_r^8-15699680\,e_r^6$$
$$+20404528\,e_r^4-12508416\,e_r^2+2091168\Big]\,\eta^2$$
$$+495\,\Big(1-e_r^2\Big)\Big[45\,e_r^{10}+806\,e_r^8-3584\,e_r^6-12432\,e_r^4$$
$$-4672\,e_r^2-512\Big]\,\eta^3\Big\}\,\sin 3u$$
$$+32\,\Big\{12\,\Big[\Big(-7047\,e_r^{12}-18341\,e_r^{10}+149978\,e_r^8$$
$$-275562\,e_r^6+215933\,e_r^4-71105\,e_r^2+6144\Big)\,C^4$$
$$+\Big(-17408\,e_r^{12}+199683\,e_r^{10}-227147\,e_r^8$$
$$+16327\,e_r^6-97209\,e_r^4+160058\,e_r^2-34304\Big)\,C^2$$
$$+4533\,e_r^{12}-109220\,e_r^{10}+332451\,e_r^8-198585\,e_r^6$$
$$+370212\,e_r^4+182919\,e_r^2+28160\Big]$$
$$+20\,\Big[\Big(21141\,e_r^{12}+55023\,e_r^{10}-449934\,e_r^8+826686\,e_r^6$$

$$-647799\, e_r^4 + 213315\, e_r^2 - 18432 \Big)\, C^4$$
$$+\Big(25775\, e_r^{12} - 446396\, e_r^{10} + 732016\, e_r^8 - 547062\, e_r^6$$
$$+634145\, e_r^4 - 477838\, e_r^2 + 79360 \Big)\, C^2$$
$$-37999\, e_r^{12} + 336742\, e_r^{10} - 497517\, e_r^8 - 297939\, e_r^6$$
$$-195066\, e_r^4 + 194563\, e_r^2 - 60928 \Big]\, \eta$$
$$-15 \Big[\Big(28188\, e_r^{12} + 73364\, e_r^{10} - 599912\, e_r^8 + 1102248\, e_r^6$$
$$-863732\, e_r^4 + 284420\, e_r^2 - 24576 \Big)\, C^4$$
$$+\Big(-20044\, e_r^{12} - 347160\, e_r^{10} + 1173752\, e_r^8 - 1729248\, e_r^6$$
$$+1485492\, e_r^4 - 622184\, e_r^2 + 59392 \Big)\, C^2$$
$$-7252\, e_r^{12} + 263814\, e_r^{10} - 611009\, e_r^8$$
$$+652395\, e_r^6 - 626096\, e_r^4 + 339708\, e_r^2 - 34816 \Big]\, \eta^2$$
$$+495\Big(1 - e_r^2\Big)\, e_r^2 \Big[-20\, \mathrm{ef} - 30\, e_r^6 + 427\, e_r^4 + 472\, e_r^2 + 120 \Big]\, \eta^3 \Big\} \sin 4u$$
$$+5\, e_r \Big\{12 \Big[\Big(4440\, e_r^{12} + 106688\, e_r^{10} - 291648\, e_r^8 + 10912\, e_r^6$$
$$+583672\, e_r^4 - 593568\, e_r^2 + 179504 \Big)\, C^4$$
$$+\Big(20556\, e_r^{12} - 48068\, e_r^{10} - 528492\, e_r^8 + 1085412\, e_r^6$$
$$-673168\, e_r^4 + 321072\, e_r^2 - 177312 \Big)\, C^2$$
$$-7305\, e_r^{12} + 41676\, e_r^{10} - 86749\, e_r^8 + 738156\, e_r^6$$
$$-838776\, e_r^4 - 1850640\, e_r^2 - 2192 \Big]$$
$$-4 \Big[\Big(66600\, e_r^{12} + 1600320\, e_r^{10} - 4374720\, e_r^8$$
$$+163680\, e_r^6 + 8755080\, e_r^4 - 8903520\, e_r^2 + 2692560 \Big)\, C^4$$
$$+\Big(195360\, e_r^{12} - 1441844\, e_r^{10} - 3225800\, e_r^8 + 11019244\, e_r^6$$
$$-11337328\, e_r^4 + 8557184\, e_r^2 - 3766816 \Big)\, C^2$$
$$-213615\, e_r^{12} + 40714\, e_r^{10} + 4092545\, e_r^8 - 11657292\, e_r^6$$
$$-701352\, e_r^4 - 3030688\, e_r^2 + 1074256 \Big]\, \eta$$

$$+3\Big[\Big(88800\,\mathrm{ef}^2 + 2133760\,e_r^{10} - 5832960\,e_r^8$$
$$+218240\,e_r^6 + 11673440\,e_r^4 - 11871360\,e_r^2 + 3590080\Big)C^4$$
$$+\Big(75360\,e_r^{12} - 3946480\,e_r^{10} + 4792960\,e_r^8 + 6487632\,e_r^6$$
$$-19007936\,e_r^4 + 18661376\,e_r^2 - 7062912\Big)C^2$$
$$-161685\,e_r^{12} + 1729535\,e_r^{10} + 284930\,e_r^8 - 6420208\,e_r^6$$
$$+7424768\,e_r^4 - 6737152\,e_r^2 + 3472832\Big]\eta^2$$
$$+99\left(1-e_r^2\right)e_r^2\left[75\,e_r^8 + 1450\,e_r^6 - 2240\,e_r^4 - 11424\,e_r^2 - 6272\right]\eta^3\Big\}\ \sin 5u$$
$$+2\Big\{24\Big[\Big(-25776\,e_r^{12} - 52248\,e_r^{10} + 404544\,e_r^8$$
$$-580176\,e_r^6 + 222384\,e_r^4 + 93480\,e_r^2 - 62208\Big)C^4$$
$$+\Big(-43032\,e_r^{12} + 140628\,e_r^{10} - 151880\,e_r^8 + 65956\,e_r^6$$
$$+100792\,e_r^4 - 236880\,e_r^2 + 124416\Big)C^2$$
$$-4206\,e_r^{12} + 138189\,e_r^{10} - 978650\,e_r^8 + 4796\,e_r^6$$
$$+1735944\,e_r^4 + 143400\,e_r^2 - 62208\Big]$$
$$+40\Big[\Big(77328\,e_r^{12} + 156744\,e_r^{10} - 1213632\,e_r^8 + 1740528\,e_r^6$$
$$-667152\,e_r^4 - 280440\,e_r^2 + 186624\Big)C^4$$
$$+\Big(50306\,e_r^{12} - 420214\,e_r^{10} + 954142\,e_r^8 - 1021250\,e_r^6$$
$$+236152\,e_r^4 + 574112\,e_r^2 - 373248\Big)C^2$$
$$-81125\,e_r^{12} + 102313\,e_r^{10} + 6610\,e_r^8 - 820022\,e_r^6$$
$$-282680\,e_r^4 - 293672\,e_r^2 + 186624\Big]\eta$$
$$-15\Big[\Big(206208\,e_r^{12} + 417984\,e_r^{10} - 3236352\,e_r^8 + 4641408\,e_r^6$$
$$-1779072\,e_r^4 - 747840\,e_r^2 + 497664\Big)C^4$$
$$+\Big(-195728\,e_r^{12} - 1371056\,e_r^{10} + 4922576\,e_r^8$$
$$-5857360\,e_r^6 + 2414528\,e_r^4 + 1082368\,e_r^2 - 995328\Big)C^2$$
$$-11497\,e_r^{12} + 822839\,e_r^{10} - 1760854\,e_r^8 + 1304000\,e_r^6$$

$$
\begin{aligned}
&-603808\, e_r^4 - 334528\, e_r^2 + 497664\Big]\, \eta^2 \\
&+495\, \left(1 - e_r^2\right) e_r^4 \left[-195\, e_r^6 - 570\, e_r^4 + 2128\, e_r^2 + 2912\right] \eta^3 \Big\} \sin 6u \\
&+e_r \Big\{ 12\, \Big[ \Big( 6600\, e_r^{12} + 146504\, e_r^{10} - 419328\, e_r^8 \\
&+80272\, e_r^6 + 711112\, e_r^4 - 757848\, e_r^2 + 232688 \Big)\, C^4 \\
&+\Big( 5460\, e_r^{12} + 171684\, e_r^{10} - 267860\, e_r^8 + 394252\, e_r^6 \\
&-1164336\, e_r^4 + 1326176\, e_r^2 - 465376 \Big)\, C^2 \\
&+23295\, e_r^{12} - 229748\, e_r^{10} + 1105395\, e_r^8 + 2040132\, e_r^6 \\
&-3740584\, e_r^4 - 568328\, e_r^2 + 232688 \Big] \\
&-20\, \Big[ \Big( 19800\, e_r^{12} + 439512\, e_r^{10} - 1257984\, e_r^8 \\
&+240816\, e_r^6 + 2133336\, e_r^4 - 2273544\, e_r^2 + 698064 \Big)\, C^4 \\
&+\Big( 1056\, e_r^{12} + 103060\, e_r^{10} - 32752\, e_r^8 + 825396\, e_r^6 \\
&-3366184\, e_r^4 + 3865552\, e_r^2 - 1396128 \Big)\, C^2 \\
&+4605\, e_r^{12} - 421902\, e_r^{10} + 723509\, e_r^8 - 830900\, e_r^6 \\
&-238016\, e_r^4 - 1592008\, e_r^2 + 698064 \Big]\, \eta \\
&+15\, \Big[ \Big( 26400\, e_r^{12} + 586016\, e_r^{10} - 1677312\, e_r^8 + 321088\, e_r^6 \\
&+2844448\, e_r^4 - 3031392\, e_r^2 + 930752 \Big)\, C^4 \\
&+\Big( -32736\, e_r^{12} - 773136\, e_r^{10} + 1426400\, e_r^8 + 876208\, e_r^6 \\
&-4636896\, e_r^4 + 5001664\, e_r^2 - 1861504 \Big)\, C^2 \\
&+8195\, e_r^{12} + 164559\, e_r^{10} + 87074\, e_r^8 \\
&-1143784\, e_r^6 + 1867616\, e_r^4 - 1970272\, e_r^2 + 930752 \Big]\, \eta^2 \\
&+3465\, \left(1 - e_r^2\right) e_r^4 \left[5\, e_r^6 + 94\, e_r^4 - 56\, e_r^2 - 480\right] \eta^3 \Big\} \sin 7u \\
&+16\, e_r^2 \Big\{ 12\, \Big[ \Big( -1473\, e_r^{10} + 13\, e_r^8 + 14678\, e_r^6 \\
&-29382\, e_r^4 + 22043\, e_r^2 - 5879 \Big)\, C^4
\end{aligned}
$$

$$+\left(-2072\,e_r^{10} - 1987\,e_r^{8} - 3565\,e_r^{6} + 33137\,e_r^{4} - 37271\,e_r^{2} + 11758\right)C^2$$

$$-261\,e_r{}^{10} + 284\,e_r^{8} - 78355\,e_r^{6} + 83833\,e_r^{4} + 15228\,e_r^{2} - 5879\Big]$$

$$+20\left[\left(4419\,e_r^{10} - 39\,e_r^{8} - 44034\,e_r^{6} + 88146\,e_r^{4} - 66129\,e_r^{2} + 17637\right)C^4\right.$$

$$+\left(1969\,e_r^{10} + 1604\,e_r^{8} + 30096\,e_r^{6} - 108154\,e_r^{4} + 109759\,e_r^{2} - 35274\right)C^2$$

$$\left.-2409\,e_r^{10} - 7422\,e_r^{8} + 18389\,e_r^{6} - 9637\,e_r^{4} - 43630\,e_r^{2} + 17637\right]\eta$$

$$-15\left[\left(5892\,e_r^{10} - 52\,e_r^{8} - 58712\,e_r^{6} + 117528\,e_r^{4} - 88172\,e_r^{2} + 23516\right)C^4\right.$$

$$+\left(-6452\,e_r^{10} - 9800\,e_r^{8} + 87496\,e_r^{6} - 166816\,e_r^{4} + 142604\,e_r^{2} - 47032\right)C^2$$

$$\left.+1072\,e_r^{10} + 7150\,e_r^{8} - 30615\,e_r^{6} + 52469\,e_r^{4} - 54432\,e_r^{2} + 23516\right]\eta^2$$

$$\left.-495\left(1 - e_r^2\right)e_r^4\left[8\,e_r^4 + 18\,e_r^2 - 77\right]\eta^3\right\}\sin 8u$$

$$+e_r^3\left\{12\left[\left(1320\,e_r^{10} + 15224\,e_r^{8} - 74096\,e_r^{6}\right.\right.\right.$$

$$\left.+117744\,e_r^{4} - 80696\,e_r^{2} + 20504\right)C^4$$

$$+\left(1980\,e_r^{10} + 22876\,e_r^{8} + 9060\,e_r^{6} - 135676\,e_r^{4} + 142768\,e_r^{2} - 41008\right)C^2$$

$$\left.+3165\,e_r^{10} - 27932\,e_r^{8} + 312905\,e_r^{6} - 279780\,e_r^{4} - 62072\,e_r^{2} + 20504\right]$$

$$-20\left[\left(3960\,e_r^{10} + 45672\,e_r^{8} - 222288\,e_r^{6} + 353232\,e_r^{4}\right.\right.$$

$$\left.-242088\,e_r^{2} + 61512\right)C^4$$

$$+\left(2112\,e_r^{10} + 19868\,e_r^{8} + 118320\,e_r^{6} - 427716\,e_r^{4}\right.$$

$$\left.+410440\,e_r^{2} - 123024\right)C^2$$

$$\left.+615\,e_r^{10} - 51786\,e_r^{8} + 98783\,e_r^{6} - 20476\,e_r^{4} - 168352\,e_r^{2} + 61512\right]\eta$$

$$+15\left[\left(5280\,e_r^{10} + 60896\,e_r^{8} - 296384\,e_r^{6} + 470976\,e_r^{4}\right.\right.$$

$$\left.-322784\,e_r^{2} + 82016\right)C^4$$

$$+\left(-5280\,e_r^{10} - 80816\,e_r^{8} + 358816\,e_r^{6} - 618096\,e_r^{4}\right.$$

$$\left.+509408\,e_r^{2} - 164032\right)C^2$$

$$\left.+1449\,e_r^{10} + 20045\,e_r^{8} - 84378\,e_r^{6} + 165224\,e_r^{4} - 186624\,e_r^{2} + 82016\right]\eta^2$$

$$
\begin{aligned}
&+495\left(1-e_r^2\right) e_r^4\left[9\, e_r^4+118\, e_r^2-280\right] \eta^3\Big\} \sin 9u \\
&+30\, e_r^4\Big\{24\left[\left(-40\, e_r^8+160\, e_r^6-240\, e_r^4+160\, e_r^2-40\right) C^4\right. \\
&+\left(-56\, e_r^8-36\, e_r^6+320\, e_r^4-308\, e_r^2+80\right) C^2 \\
&\left.+50\, e_r^8-647\, e_r^6+534\, e_r^4+148\, e_r^2-40\right] \\
&+8\left[\left(600\, e_r^8-2400\, e_r^6+3600\, e_r^4-2400\, e_r^2+600\right) C^4\right. \\
&+\left(266\, e_r^8+1338\, e_r^6-4674\, e_r^4+4270\, e_r^2-1200\right) C^2 \\
&\left.-413\, e_r^8+777\, e_r^6+138\, e_r^4-1870\, e_r^2+600\right] \eta \\
&-\left[\left(4800\, e_r^8-19200\, e_r^6+28800\, e_r^4-19200\, e_r^2+4800\right) C^4\right. \\
&+\left(-5232\, e_r^8+20976\, e_r^6-35856\, e_r^4+29712\, e_r^2-9600\right) C^2 \\
&\left.+1093\, e_r^8-4123\, e_r^6+8622\, e_r^4-10512\, e_r^2+4800\right] \eta^2 \\
&+297\left(1-e_r^2\right) e_r^4\left[-e_r^2+2\right] \eta^3\Big\} \sin 10u \\
&+15\, e_r^5\Big\{12\left[\left(8\, e_r^8-32\, e_r^6+48\, e_r^4-32\, e_r^2+8\right) C^4\right. \\
&+\left(12\, e_r^8+12\, e_r^6-76\, e_r^4+68\, e_r^2-16\right) C^2 \\
&\left.-9\, e_r^8+140\, e_r^6-109\, e_r^4-36\, e_r^2+8\right] \\
&-4\left[\left(120\, e_r^8-480\, e_r^6+720\, e_r^4-480\, e_r^2+120\right) C^4\right. \\
&+\left(64\, e_r^8+300\, e_r^6-1032\, e_r^4+908\, e_r^2-240\right) C^2 \\
&\left.-63\, e_r^8+106\, e_r^6+97\, e_r^4-428\, e_r^2+120\right] \eta \\
&+\left[\left(480\, e_r^8-1920\, e_r^6+2880\, e_r^4-1920\, e_r^2+480\right) C^4\right. \\
&+\left(-480\, e_r^8+1968\, e_r^6-3456\, e_r^4+2928\, e_r^2-960\right) C^2 \\
&\left.+97\, e_r^8-355\, e_r^6+774\, e_r^4-1008\, e_r^2+480\right] \eta^2 \\
&+33\left(1-e_r^2\right) e_r^4\left[e_r^2-2\right] \eta^3\Big\} \sin 11u\,. \qquad (4.18\text{m})
\end{aligned}
$$

In Eqs.(4.17) and (4.18), $\delta = (m_1 - m_2)/m$ and $S = \sin i$. In the circular limit Eqs.(2), (3) and (4) of [47] *modulo the tail terms* are recovered *by* setting $e_r = 0$ in

Eqs.(4.17) and (4.18) and using

$$u = \left\{1 - 3\,\tau^{2/3} - \frac{1}{2}\,(9 - 14\,\eta)\,\tau^{4/3}\right\}\phi\,, \tag{4.19}$$

obtained by inverting Eqs.(4.5) in the circular limit. This completes the solution to the 2PN generation problem for inspiraling compact binaries moving in elliptic orbits modulo the tail terms. Though, in principle, the required equations are available [128], the explicit expressions for the tail contribution to the polarizations have not been written down for elliptic orbits. Related details of tail contributions are discussed in [128, 129] and summarized in section 4.4.

Following earlier work [126, 127, 44] we have used the 'radial eccentricity' e, to represent in Eqs.(4.16), (4.17) and (4.18) the gravitational polarizations, $h_+$ and h,. Though convenient and compact for the initial computations, at higher orders it has the disadvantage that various PN contributions do not separate cleanly when written in terms e,. This is due to the $v$ term in $H^{(1)}_{+,\times}$. This term has a 1PN correction which when re-expressed in terms of e,, cannot be cleanly separated out analytically in the tan–' expansion. However, if one uses $e_\phi$ rather than e,, one can achieve a clean split of the various PN contributions to $h_+$ and h,. The following relation connecting $e_r$ to $e_\phi$ is needed to rewrite the N, 0.5PN and 1PN contributions to $h_+$ and $h_\times$ in Eqs.(4.16), in terms of $e_\phi, u$ and $\tau$,

$$\begin{aligned} e_r \;\; &= \;\; e_\phi\left\{1 - \frac{\tau^{\frac{2}{3}}}{2}\,\eta - \frac{\tau^{\frac{4}{3}}}{768\,(1-e_\phi^2)}\left[3264 - 2112\,\eta - 360\,\eta^2\right.\right. \\ &\quad \left.\left. + (1-e_\phi^2)\left(960 - 224\,\eta + 264\,\eta^2\right)\eta\right]\right\}. \end{aligned} \tag{4.20}$$

It may be noted that the above tranformation will only change the coefficients in Eqs.(4.17) and (4.18) at 1PN, 1.5PN and 2PN orders and not their 'u-harmonic' structure.

## 4.3 Influence of the orbital parameters on the waveform

To investigate the dominant effects of eccentricity and orbital inclination on the polarization waveforms, we concentrate our attention on the leading Newtonian part of $h_+$ and $h_\times$. For convenience we list them below again,

$$\begin{aligned} h_+ &= \frac{2G\eta m}{c^2 R}\tau^{2/3}\Bigg\{\frac{1}{4(1-e_r\cos u)^3}\Bigg[-4e_r^2 - e_r\Big((3e_r^2-3)C^2-7\Big)\cos u \\ &\quad +\Big((1-e_r^2)C^2+1)(-4\cos 2u + e_r\cos 3u\Big)\Bigg]\Bigg\}, \qquad (4.21a) \\ h_\times &= \frac{2G\eta m}{c^2 \mathbf{R}}\tau^{2/3}\Bigg\{\frac{C(1-e_r^2)^{1/2}}{2(1-e_r\cos u)^3}\Bigg[5\,\mathrm{e}, \sin u - 4\sin 2u + e_r\sin 3u\Bigg]\Bigg\}. \qquad (4.21b) \end{aligned}$$

In order to compare with existing results for the spectral analysis of Newtonian part, of $h_+$ and $h_\times$ [132, 133], we require the following expansion of the eccentric anomaly '$u$' in terms of the mean anomaly $\mathrm{M} = \mathrm{n}(\mathrm{t} - t_0)$ to the Newtonian order, available in the standard textbooks of celestial mechanics [148]

$$u = M + \sum_{p=1}^{\infty}\left(\frac{2}{p}\right) J_p(p\,e_r)\,\sin pM\,, \qquad (4.22)$$

where $J_p(p\,e_r)$ is the Bessel function of the first kind of order p. Further, the trigonometric functions of the eccentric anomaly 'u' appearing in Eqs.(4.21) can also be expanded in terms of a Fourier-Bessel series of the mean anomaly $\mathrm{M} = \mathrm{n}(\mathrm{t} - t_0)$ using standard relations available in the literature [148]. We display them below

$$\begin{aligned} \frac{1}{(1-e_r\cos\mathrm{u})} &= 1 + 2\sum J_p(p\,e_r)\cos p\,M \qquad (4.23a) \\ \cos qu &= \sum\left(\frac{q}{\mathrm{p}}\right)\Big(J_{p-q}(p\,e_r) - J_{p+q}(p\,e_r)\Big)\cos pM \qquad (4.23b) \\ \sin qu &= \sum\left(\frac{q}{\mathrm{p}}\right)\Big(aJ_{p-q}(p\,e_r) + J_{p+q}(p\,e_r)\Big)\sin pM\,, \qquad (4.23c) \end{aligned}$$

where $\mathrm{p},\mathrm{q} \geq 1$ and all sums are from $\mathrm{p} = 1$ to $\mathrm{p} = \infty$. As is well known [149], these expressions are generally convergent for $e_r < 0.66$ only. To compute the power spectra for $h_\times$ and $h_+$, we keep the first 40 terms in Eqs.(4.23) and also Taylor expand

$J_p(p\, e_r)$ to $O(e_r^{41})$. Since these number of terms exhibit reasonable convergence we have not gone to hundred terms as in [132]. Using these expressions we compute $|(h_\times)_p|^2$ and $|(h_+)_p|^2$, the strength of the harmonic p of the fundamental orbital frequency for the 'plus' and 'cross' polarization at the Newtonian order. The results obtained for the first ten harmonics for $e_r = 0.1, 0.22, 0.4, 0.6$ and 0.9 in the case of 'cross' polarization are presented in Table 4.1. We observe that for small and medium eccentricities ($e_r = 0.1 \ldots 0.4$) the second harmonic has the maximum amplitude. Moreover, for $e_r = 0.22$ the third harmonic is 30% of the the second one. We also find that for $e_r = 0.6$ the maximum amplitude harmonic is the fourth one and there is appreciable power in all the first ten harmonics, all these are in agreement with [132, 133]. Though we also observe that the first harmonic is dominant for $e_r = 0.9$ as noted by [132], we have little confidence in the values presented in the last column of Table 4.1, due to poor convergence of Eqs.(4.23) for e, > 0.66. Note that the element in the $9^{th}$ row, $6^{th}$ column of the Table 4.1 is negative. This is an indication that the the number of terms retained in our computation is not sufficient to achieve the limit of the poorly convergent infinite series involving Bessel functions. The behaviour for $|(h_+)_p|^2$ is similar and we do not list it here.

Table 4.1: The power spectrum $|(h_\times)_p|^2$ scaled by $(\frac{Gm\eta}{c^2 R}\,\tau^{(\frac{2}{3})})^2$, corresponding to different values of p and eccentricity $e_r$

| Harmonic, p | $e_r = 0.1$ | $e_r = 0.22$ | $e_r = 0.4$ | $e_r = 0.6$ | $e_r = 0.9$ |
|---|---|---|---|---|---|
| 1 | $\sim 10^{-2}$ | 0.1022 | 0.2892 | 0.4686 | 0.2752 |
| 2 | 3.8037 | 3.1152 | 1.6262 | 0.3271 | $\sim 10^{-2}$ |
| 3 | 0.1931 | 0.7746 | 1.4184 | 0.8681 | $\sim 10^{-3}$ |
| 4 | $\sim 10^{-3}$ | 0.1171 | 0.6944 | 0.9572 | $\sim 10^{-3}$ |
| 5 | $\sim 10^{-4}$ | $\sim 10^{-2}$ | 0.2779 | 0.8139 | $\sim 10^{-2}$ |
| 6 | $\sim 10^{-6}$ | $\sim 10^{-3}$ | 0.1006 | 0.6127 | $\sim 10^{-2}$ |
| 7 | $\sim 10^{-8}$ | $\sim 10^{-4}$ | $\sim 10^{-2}$ | 0.4299 | $\sim 10^{-2}$ |
| 8 | $\sim 10^{-9}$ | $\sim 10^{-5}$ | $\sim 10^{-2}$ | 0.2884 | 0.4494 |
| 9 | $\sim 10^{-11}$ | $\sim 10^{-6}$ | $\sim 10^{-3}$ | 0.1875 | -1.02554 |
| 10 | $\sim 10^{-13}$ | $\sim 10^{-7}$ | $\sim 10^{-3}$ | 0.1192 | 39.0874 |

Table 4.2: The spectrum of $|(h_\times)_n|^2$, where 'harmonics' are in terms of eccentric anomaly $u$ corresponding to different values of eccentricity e,. Here too $h_\times$ is scaled by $\frac{Gm\eta}{c^2 R}\tau^{(\frac{2}{3})}$

| Harmonic, n | $\|h_\times\|^2$,$e_r = 0.1$ | $\|h_\times\|^2$,$e_r = 0.4$ | $\|h_\times\|^2$, $e_r = 0.9$ |
|---|---|---|---|
| 1 | $\sim 10^{-3}$ | $\sim 10^{-2}$ | 0.7622 |
| 2 | 4.0201 | 4.3561 | 7.7536 |
| 3 | $\sim 10^{-2}$ | 1.1449 | 13.5365 |
| 4 | $\sim 10^{-4}$ | 0.159379 | 14.8223 |
| 5 | $\sim 10^{-6}$ | $\sim 10^{-2}$ | 12.8813 |
| 6 | $\sim 10^{-8}$ | $\sim 10^{-3}$ | 9.7166 |
| 7 | $\sim 10^{-11}$ | $\sim 10^{-4}$ | 6.6532 |
| 8 | $\sim 10^{-13}$ | $\sim 10^{-6}$ | 4.2439 |
| 9 | $\sim 10^{-15}$ | $\sim 10^{-7}$ | 2.5639 |
| 10 | $\sim 10^{-18}$ | $\sim 10^{-8}$ | 1.4836 |
| 20 | $\sim 10^{-43}$ | $\sim 10^{-20}$ | $\sim 10^{-3}$ |
| 30 | $\sim 10^{-68}$ | $\sim 10^{-33}$ | $\sim 10^{-8}$ |
| 40 | $\sim 10^{-93}$ | $\sim 10^{-46}$ | $\sim 10^{-14}$ |
| 50 | $\sim 10^{-119}$ | $\sim 10^{-60}$ | $\sim 10^{-25}$ |

In the circular limit $u = \phi$, the waveforms are relatively simple, and multiples of $\phi$ correspond to higher harmonics of the dominant gravitational wave frequency. The situation is more involved in the elliptic case discussed here due to the presence of the factor $(1- e_r \cos u)^3$ in the denominator of Eqs.(4.21). To obtain another simple characterization of the 'harmonic' content in the Newtonian part h,, using the eccentric anomaly u, we Taylor expand $(1-$ e, $\cos u)^{-3}$ around cos u $= 0$ to high accuracy by keeping the first 100 terms. From the the resultant expression for $h_\times$, we compute $|(h_\times)_p|^2$, where p $= 1, \ldots 100$. The results are summarized in Table 4.2. It is clear from the Table 4.2 that for small and medium eccentricities (e, $= 0.1$ and e, $= 0.4$), the second 'u-harmonic' contribution to $|h_\times|^2$ is dominant and $|(h_\times)_p|^2$ is negligible beyond p $= 10$. However for very high values of e,, (e, $= 0.9$) the higher 'u-harmonics' contribute substantially to $|h_\times|^2$. In fact for e, $= 0.9$ the 'harmonic' contributing most is the fifth one and moreover, $|(h_\times)_p|^2$ is not negligible until p = 20. Similar results hold for $|(h_+)_p|^2$. This qualitative observation regarding the dominant 'u-harmonic' for very high values of eccentricities, is different from a similar discussion in [133] and the last column of Table 4.1. However, there may be more reliability on the discussions based on the 'u-harmonics' for high values of e, since the Fourier-Bessel expansion of the true or eccentric anomaly in terms of the mean anomaly and Eqs.(4.23) are not in general applicable for $e_r > 0.66$, while no such restriction applies when we Taylor expand $(1 - e_r \cos u)^{-3}$.

It is also evident from Eqs.(4.21) that the orbital inclination i changes the magnitudes of $|h_\times|^2$ and $|h_+|^2$ appreciably. In Figures (4.1), (4.2), (4.3) and (4.4) we have plotted $h_\times$ and $h_+$ scaled by $\frac{G m \eta}{c^2 R} \tau^{(\frac{2}{3})}$, for various $e_r$'s and $i$'s when eccentric anomaly $u$ goes from 0 to $2\pi$, corresponding to one complete orbit.

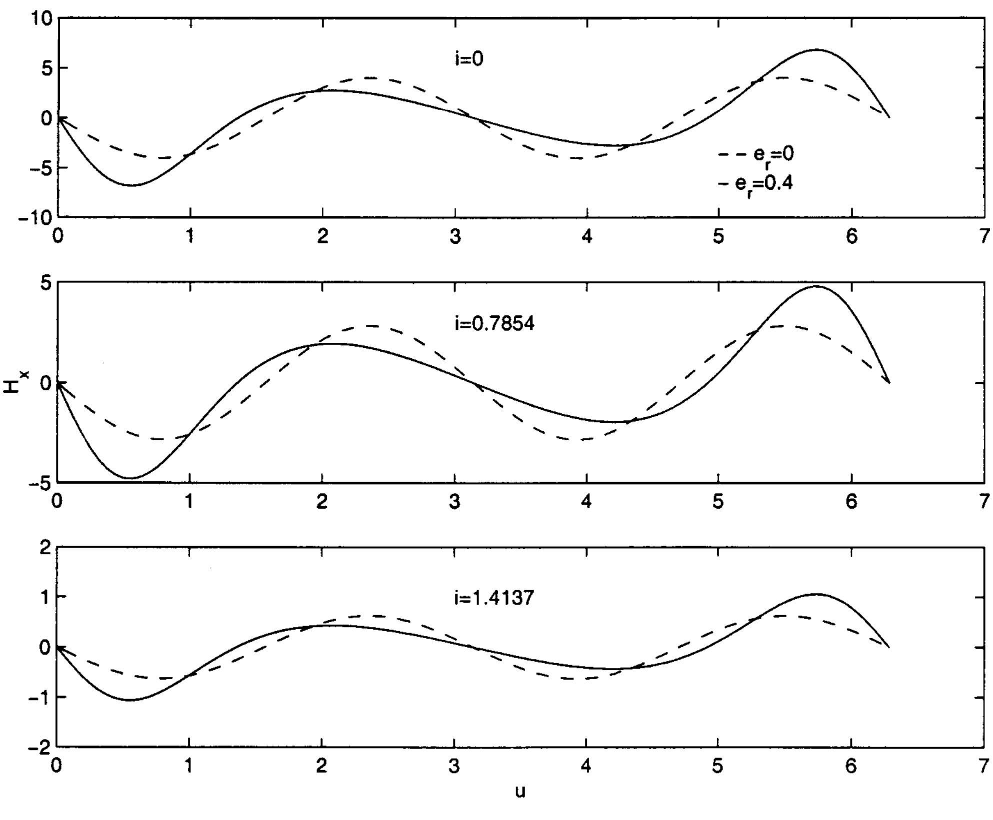


Figure 4.1: The effect of the inclination angle i on the Newtonian part of $h_\times$ when $e_r$ takes values 0 and 0.4. Note that $h_\times$is scaled by $\frac{G m \eta}{c^2 R} \tau^{(\frac{2}{3})}$.

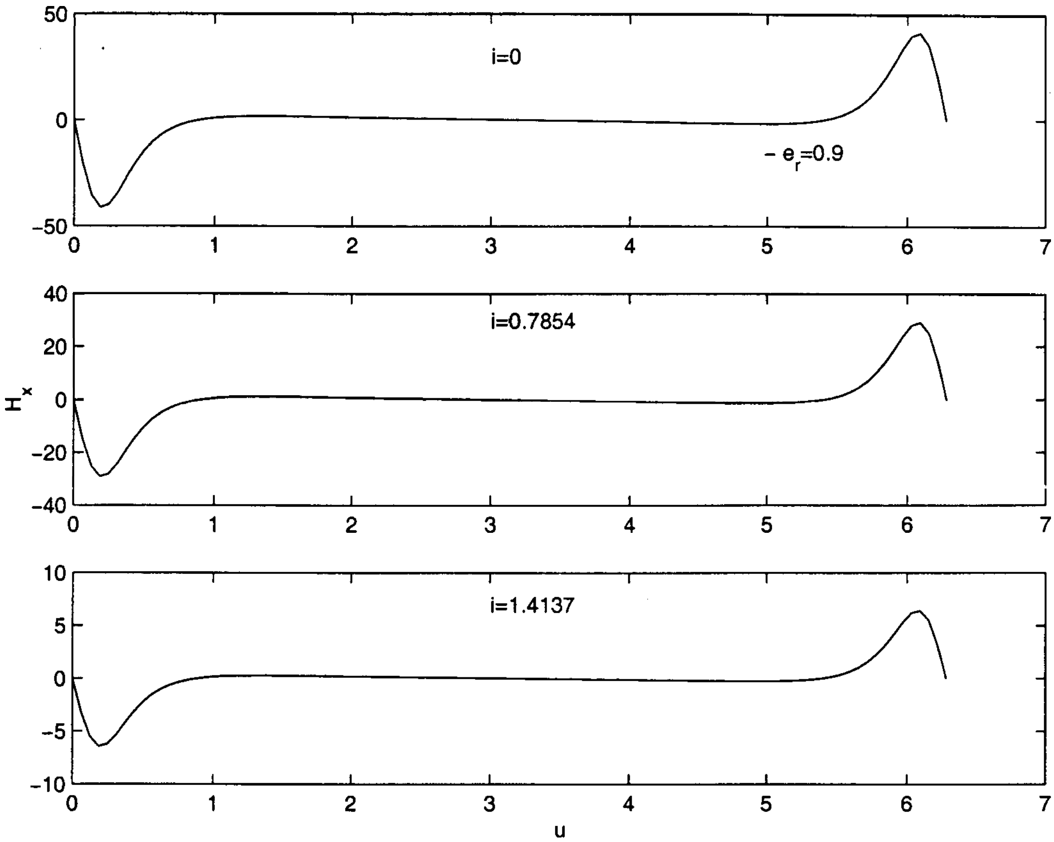


Figure 4.2: The effect of the inclination angle $i$ on the Newtonian part of $h_\times$ when $e_r = 0.9$. Here also we scale $h_\times$ by $\frac{G m \eta}{c^2 R}\,\tau^{(\frac{2}{3})}$.

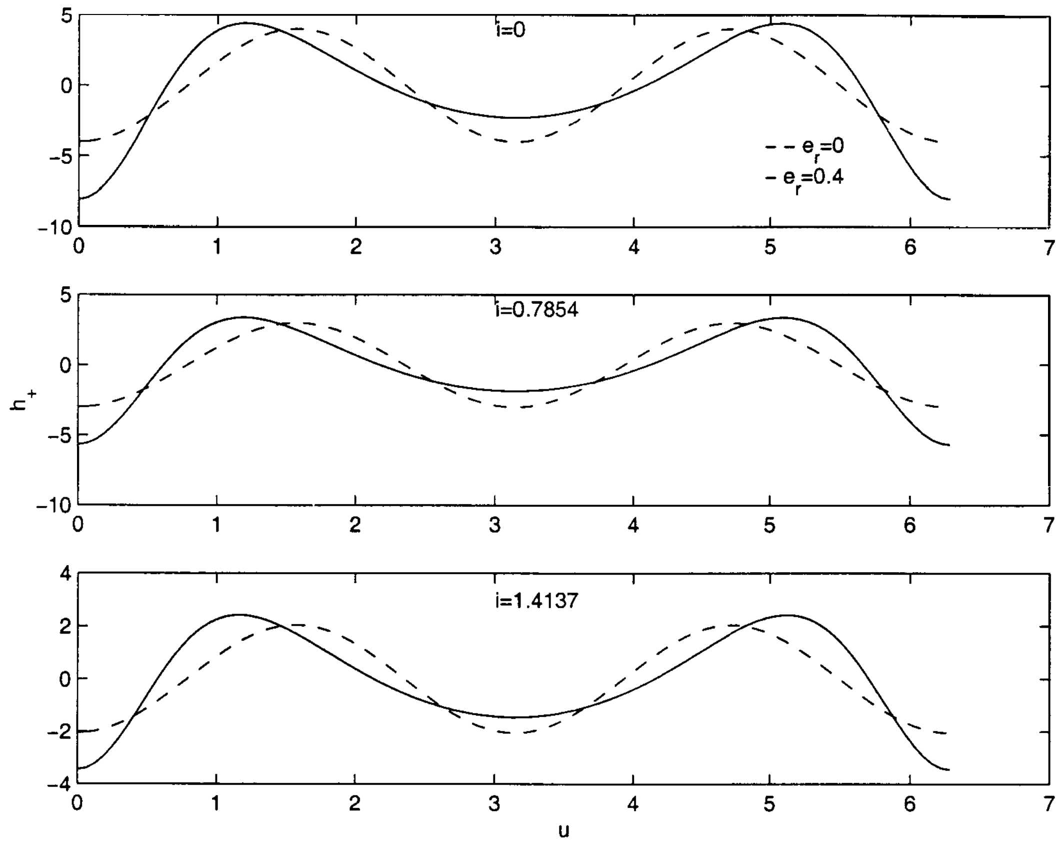


Figure 4.3: The effect of the inclination angle i on the Newtonian part of $h_+$ scaled by $\frac{Gm\eta}{c^2 R}\tau^{(\frac{2}{3})}$. Here $e_r$ takes values 0 and 0.4.

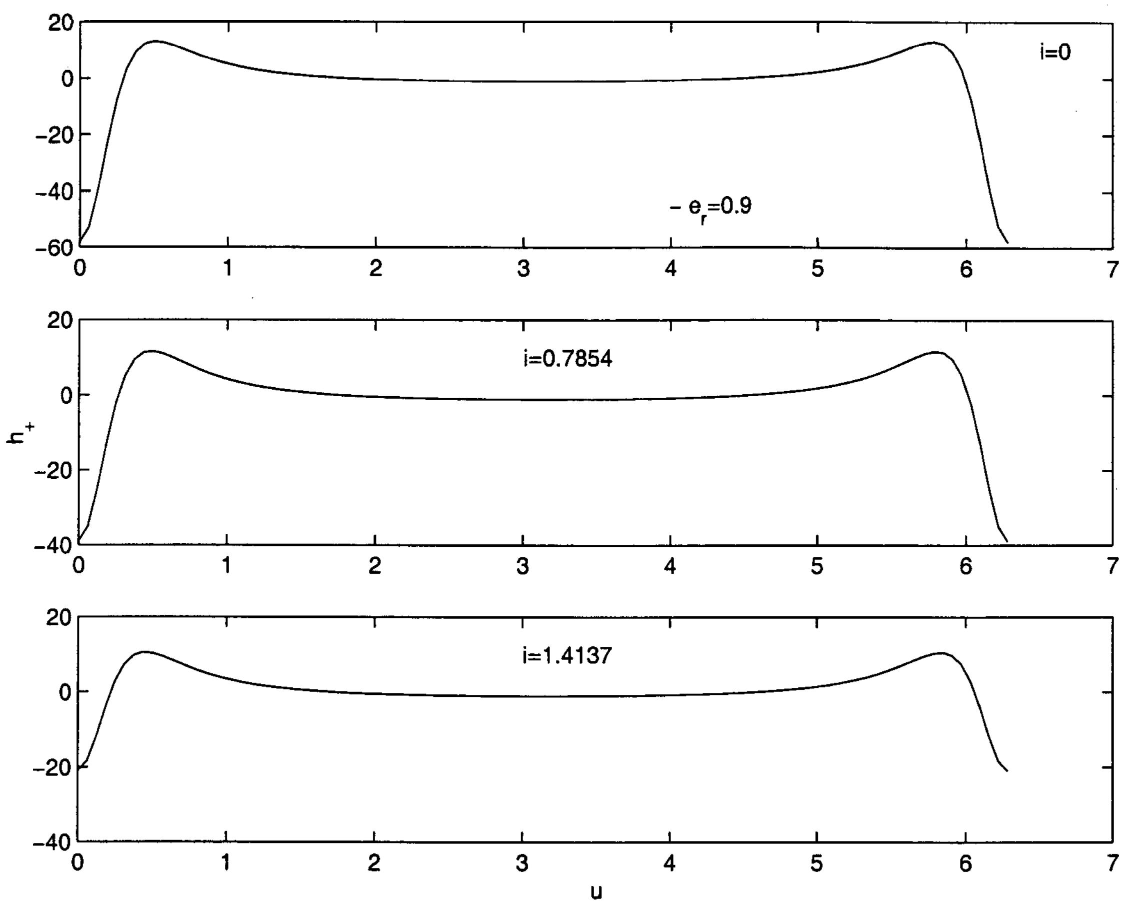


Figure 4.4: The effect of the inclination angle i on the Newtonian part of $h_+$, scaled by $\frac{Gm\eta}{c^2R}\tau^{(\frac{2}{3})}$ for e, $= 0.9$.

For $e_r = 0.9$, $|h_+|^2$ is reduced by a factor of 4 whereas $|h_\times|^2$ goes down by a factor $> 45$ when i is varied from 0 to $0.45\pi$. It is clear from above plots that for small and medium eccentricities, reduction in $|h_\times|^2$ and $|h_+|^2$ is small compared to higher $e_r$'s, when i is varied from 0 to $\pi/2$. This is consistent with [133].

We also compute the square of the ratio between $h_+$ and $h_\times$, to see if we can use it to obtain an estimate of the orbital inclination i.

$$\left(\frac{h_\times}{h_+}\right)^2 = \left\{\frac{-2C\,(1-e_r^2)^{1/2}\sin u\Big(e_r - 2\cos u + e_r\cos u^2\Big)}{\Big((1+(1-e_r^2)\,C^2)\ \{-e_r\cos u^3 + 2\cos u^2 - 1\} + e_r\,(e_r - \cos u)\Big)}\right\}^2. \tag{4.24}$$

In Figs. (4.5) and (4.6) we plot Eq.(4.24) for various eccentricities and eccentric anomalies when i is varied from 0 to $\pi/2$.

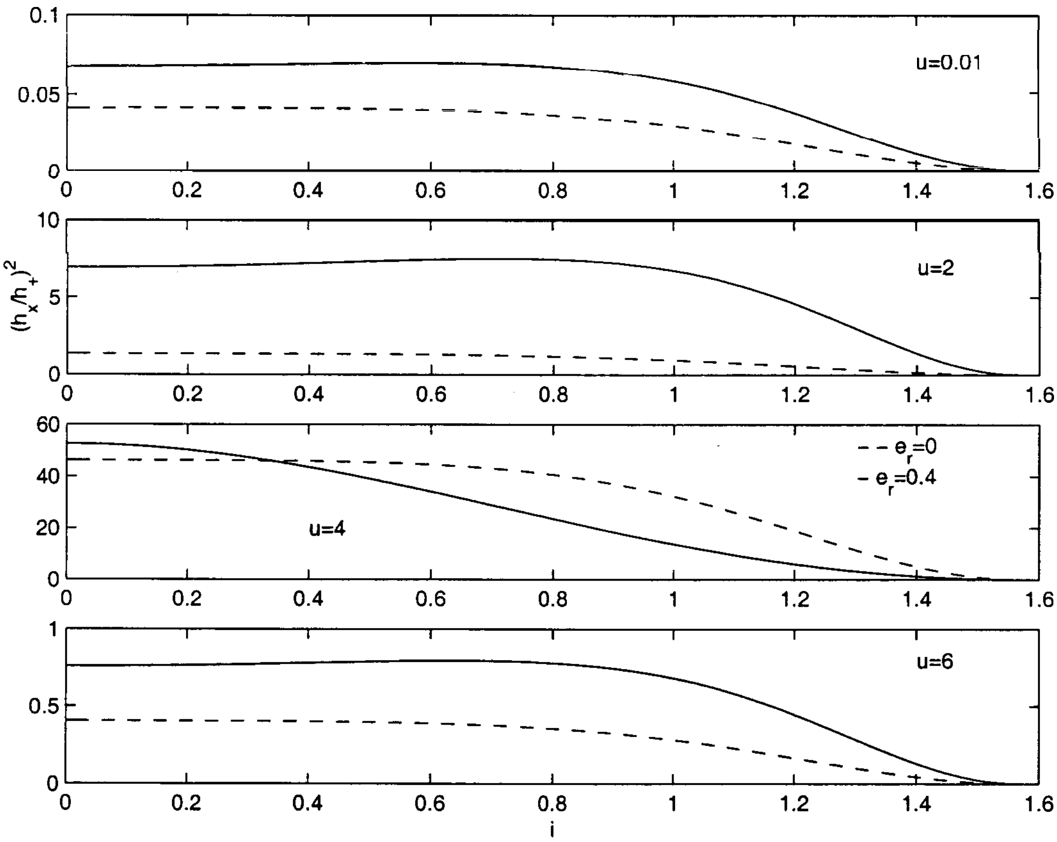


Figure 4.5: Plots of $(\frac{h_\times}{h_+})^2$. Here $i$(x-axis) is varied from 0 to $\pi/2$ and $e_r$ takes values 0 and 0.4.

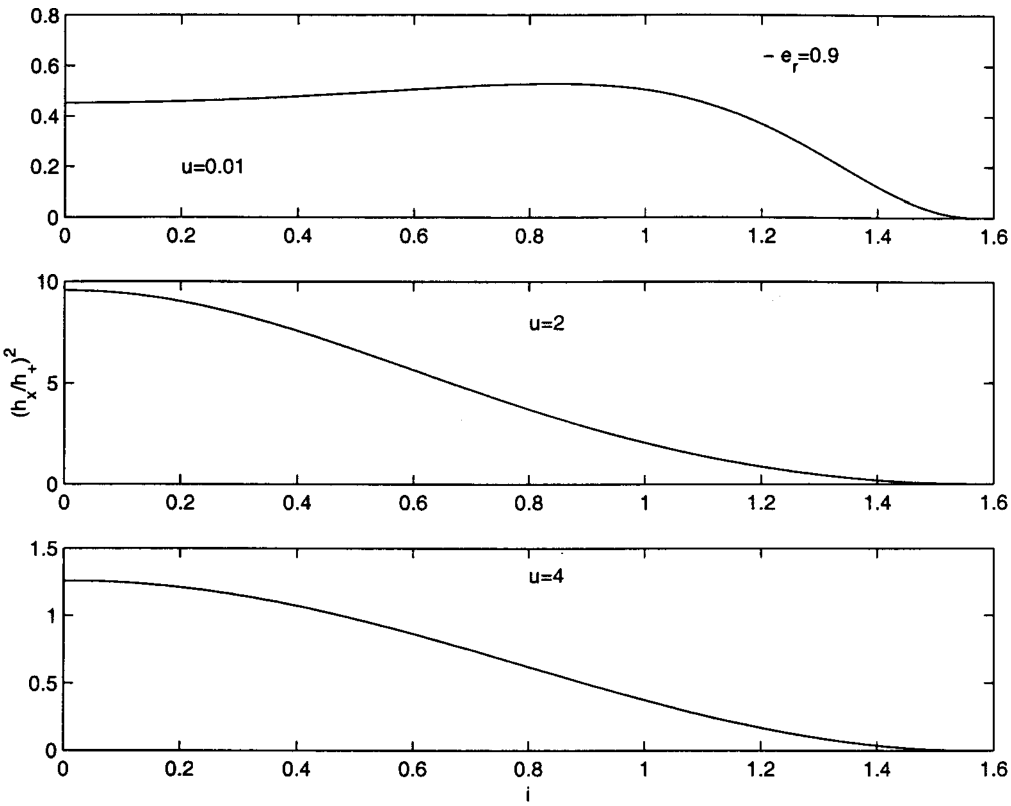


Figure 4.6: Plots of $(\frac{h_\times}{h_+})^2$ when i (x-axis) is varied from 0 to $\pi/2$, for $e_r = 0.9$

We observe that for u = 2 the ratio can be used as a good indicator for the orbital inclination for very small to very high eccentricities.

The different post-Newtonian contributions; Newtonian, 0.5PN, and 1PN, to $h_\times$ and $h_+$ scaled by $\frac{Gm\eta}{c^2 R}$ for a binary with following parameters f $= 0.01$ Ha, i $= 0.45\pi, m_1 = 10M_\odot, m_2 = 1.4M_\odot$ are plotted over an orbit for various values of $e_r$ in Figures (4.7), (4.8), (4.9) and (4.10). To compare the variations with the Newtonian order, we scale 0.5 PN corrections by a factor of $10^9$ and 1PN corrections by $10^{19}$.

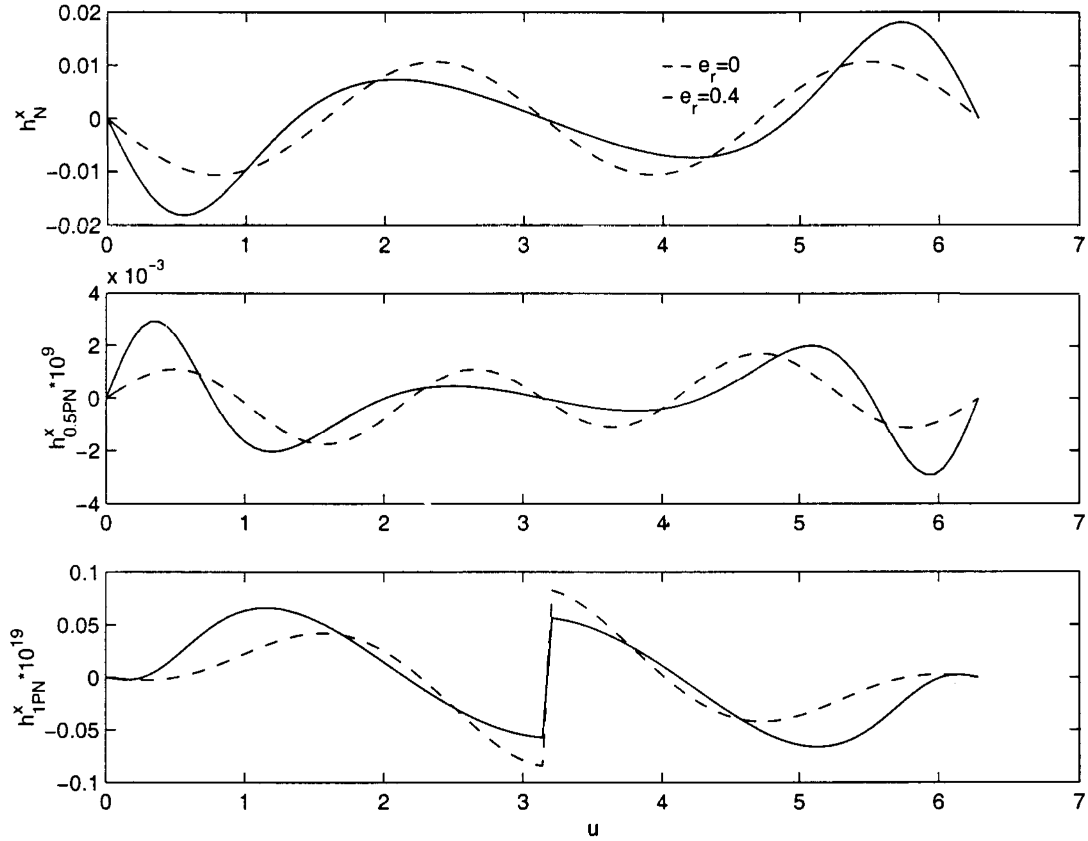


Figure 4.7: Plots of N, 0.5PN and 1PN contributions to $h_\times$ scaled by $\frac{Gm\eta}{c^2 R}$ for f $= 0.01$Hz, $i = \pi/4$, m $= 11.4$, for an orbital period, when $e_r$ takes the values 0 and 0.4. The 0.5PN and 1PN contributions are scaled by $10^9$ and $10^{19}$ respectively for comparison with the N contribution.

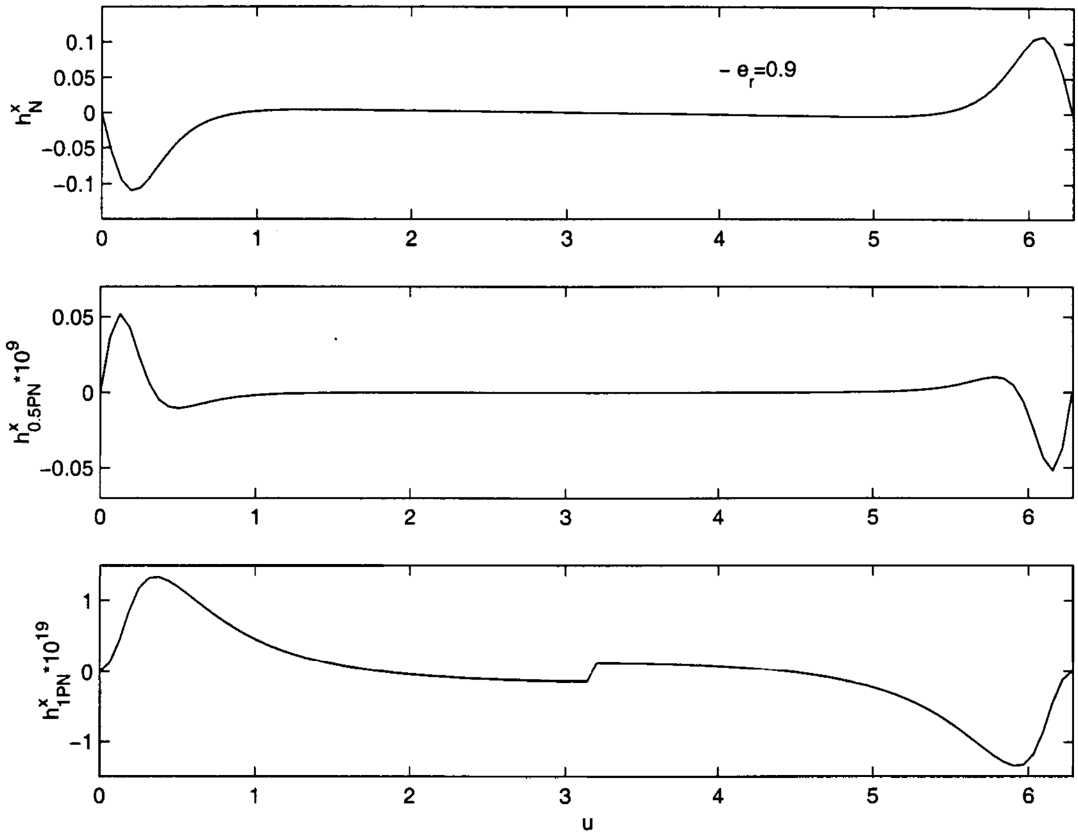


Figure 4.8: Plots of N, 0.5PN and 1PN contributions to $h_\times$ scaled by $\frac{Gm\eta}{c^2 R}$ for f $= 0.01$Hz, $i = \pi/4$, m $= 11.4$ when u is varied from $0$ to $2\pi$ for $e_r = 0.9$. The 0.5PN and 1PN contributions are scaled by $10^9$ and $10^{19}$ respectively for comparison with the N contribution.

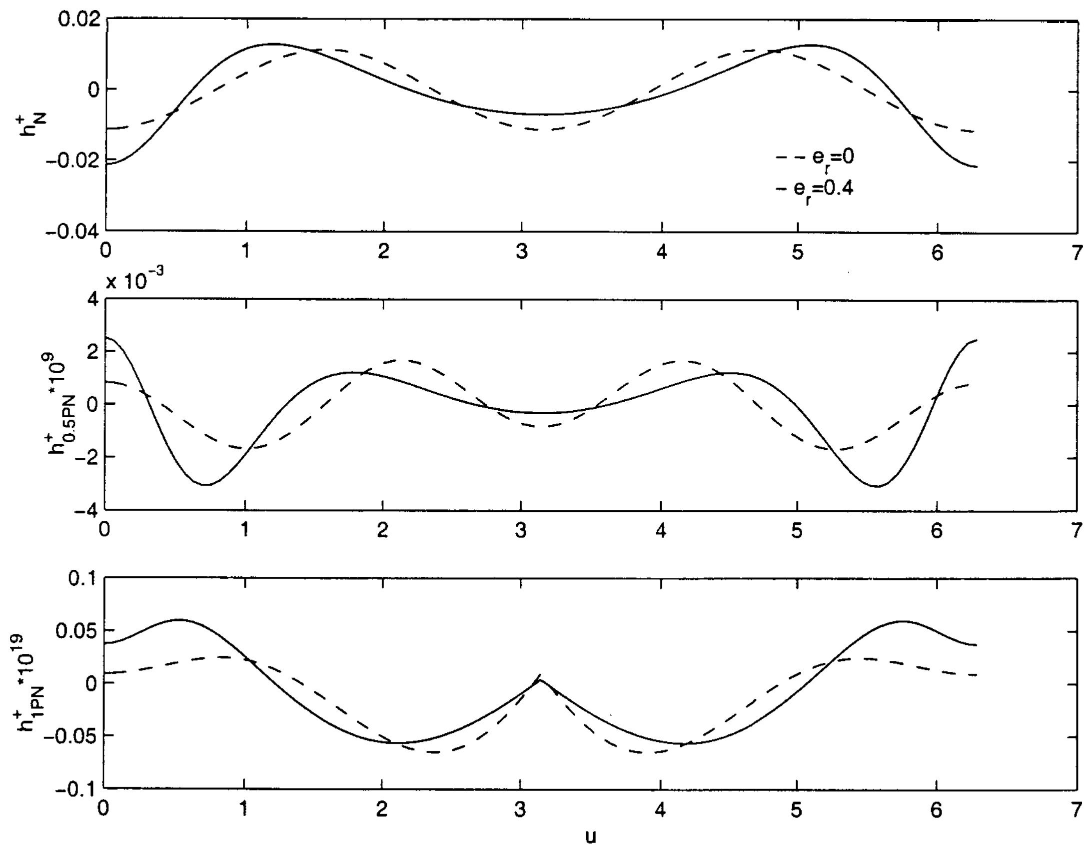


Figure 4.9: Plots of N, 0.5PN and 1PN contributions to $h_+$ scaled by $\frac{Gm\eta}{c^2 R}$ for $\mathrm{f} = 0.01$ Hz, $\mathrm{i} = \pi/4$, $\mathrm{m} = 11.4$, for an orbital period, when $e_r$ takes the values 0 and 0.4. The 0.5PN and 1PN contributions are scaled by $10^9$ and $10^{19}$ respectively for comparison with the N contribution.

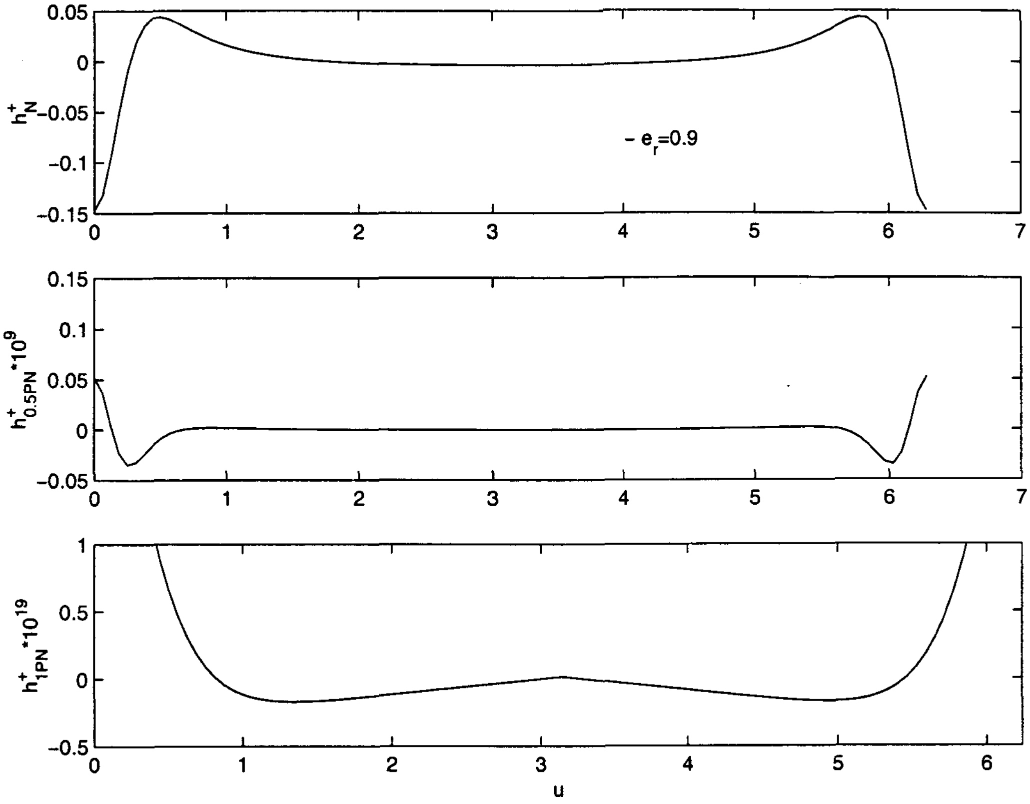


Figure 4.10: Plots of N, 0.5PN and 1PN contributions to $h_+$ scaled by $\frac{Gm\eta}{c^2R}$ for f $= 0.01\text{Hz}, i = \pi/4$, m $= 11.4$, for an orbital period, when e, $= 0.9$. The 0.5PN and 1PN contributions are scaled by $10^9$ and $10^{19}$ respectively for comparison with the N contribution.

Here we have not plotted the 1.5PN and the 2PN contributions to $h_+$ and $h_\times$ for the following reasons. The 1.5PN terms are not structurally different from the 1PN terms but only $\sim 10^9$ times smaller than that. For the 2PN terms, as mentioned earlier when one employs e,, the 2PN corrections from the v terms in $H^{(1)}_{+,\times}$ do not analytically seperate out cleanly. Hence these orders are not plotted in this chapter. A comment is in order regarding the cusp and discontinuity in the above Figures at the 1PN order. These features are due to the v terms present in the 1PN contributions to $h_+$ and $h_\times$ generated by the Taylor expansion of $\cos\phi$'s and $\sin\phi$'s at Newtonian order to 1PN accuracy and directly involve the periastron constant k as seen from Eqs.(4.5).

The explicit effect of periastron precession is explored in the set of Figures where the waveforms are compared with and without the inclusion of the periastron precession.

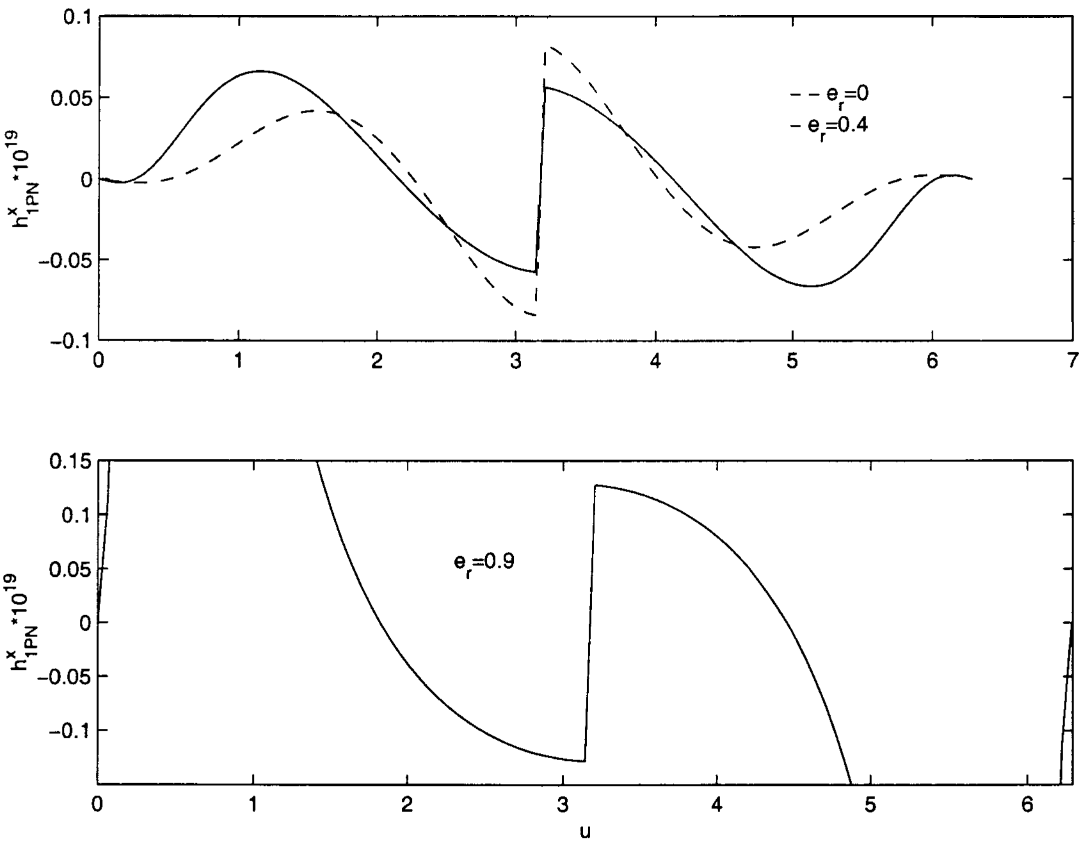


Figure 4.11: The modulation due to periastron precession at the 1PN order, for the 'cross' polarization. We concentrate on the same binary as in Fig (4.9).

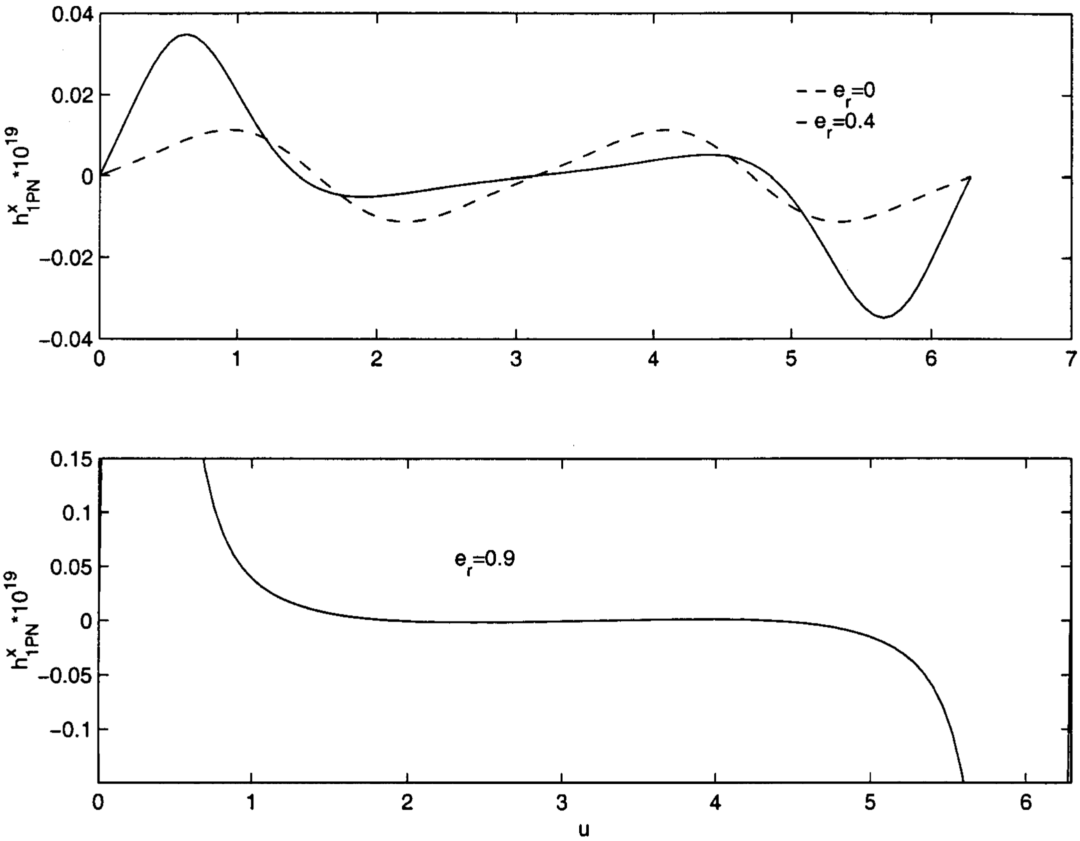


Figure 4.12: The plot of 1PN contribution to $h_\times$ when $v$ terms appearing in $H_\times^{(1)}$ are neglected. The binary parameters are as in Fig (4.9).

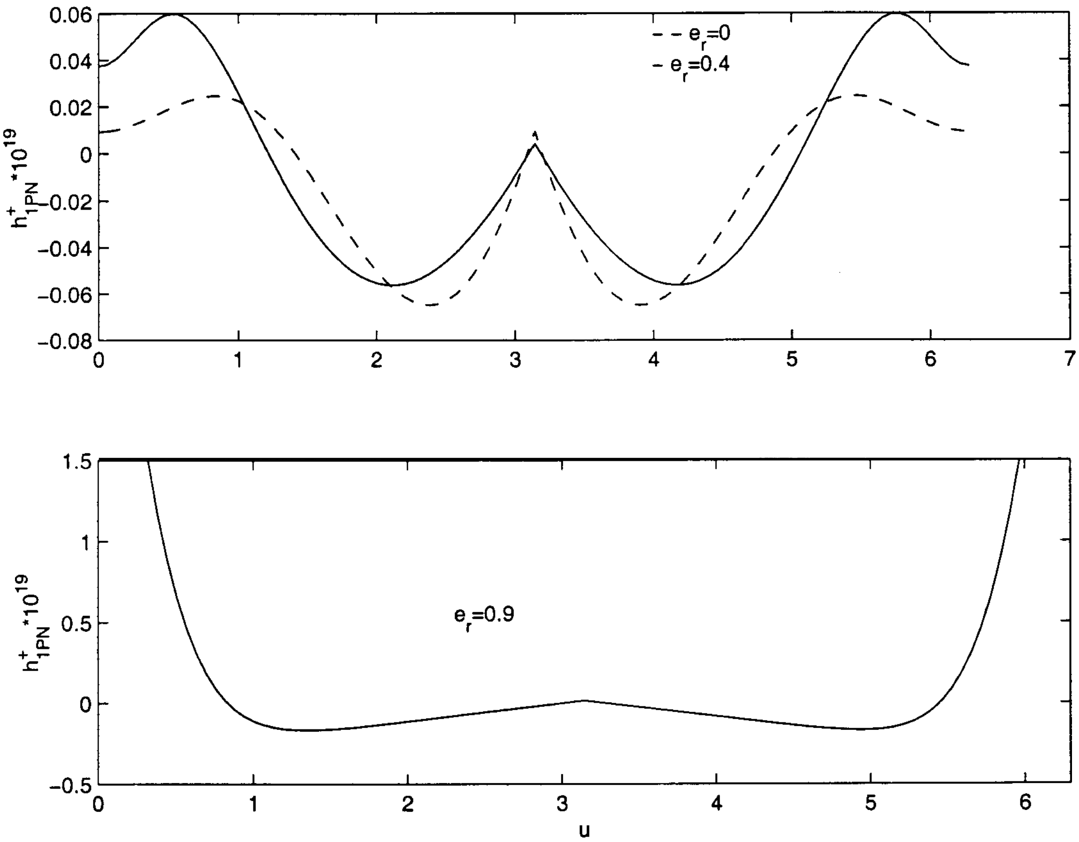


Figure 4.13: A plot for the 1PN contribution to $h_+$ similar to Fig (4.11)

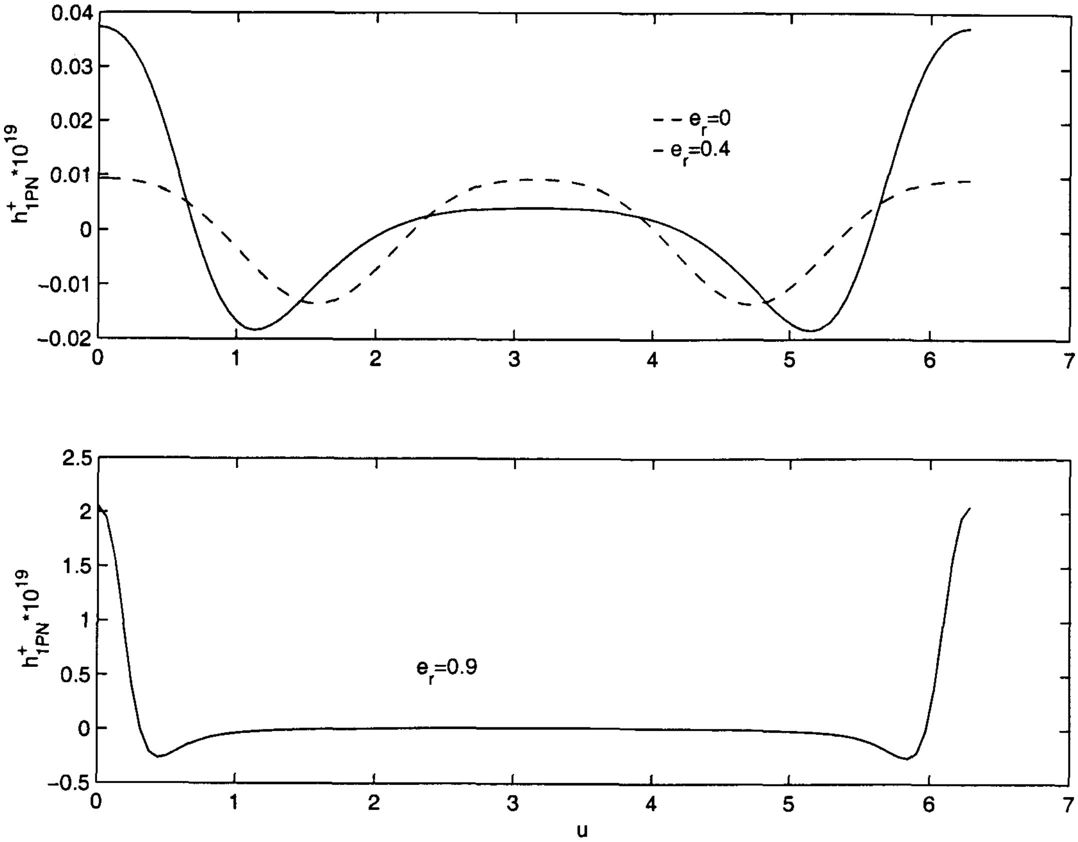


Figure **4.14:** The plot similar to Fig **(4.12)** for the 'plus' polarization.

From above Figures it is clear that periastron precession modulates the waveform.

The next Figure contains plots of the real anomaly versus the eccentric anomaly u for values of eccentricities e, $= \mathbf{0.1}$ and e, $= .9$ respectively.

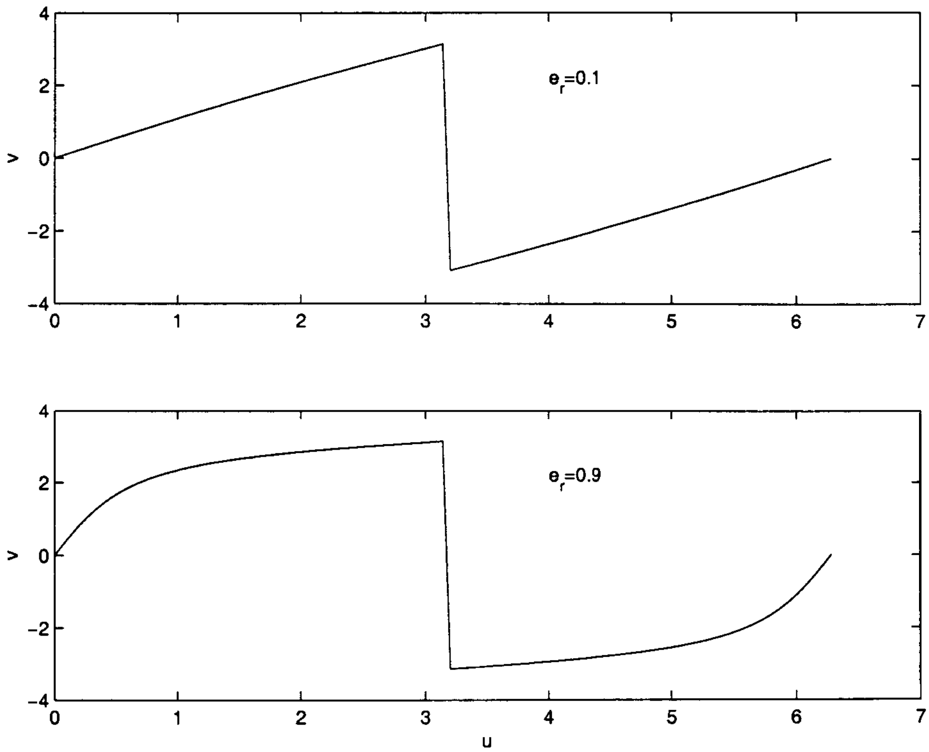


Figure 4.15: The plot of the true anomaly as a function of the eccentric anomaly. Note that discontinuity occurs at $u = \pi$ regardless of eccentricity.

Irrespective of the value of $e_r$ the real anomaly $v$ as a function of eccentric anomaly $u$ has a discontinuity at $u = \pi$. The combined effect of this discontinuity in v and the oscillatory behaviour of various harmonics present at 1PN leads to the cusp and discontinuity in the waveforms at $u = \pi$ at these orders. Such features are also present at the 1.5PN and 2PN orders. Finally, it should be noted that the values in these Figures (4.11) and 4.13) have been scaled by $10^{19}$ and hence these features may not be relevant in practice.

## 4.4 Phasing

The explicit time evolution for the 'plus' and 'cross' polarizations is obtained by computing the time dependence of $u$, w and $e_r$ and inserting these relations back into Eqs.(4.16), (4.17) and (4.18) for $h_+$ and $h_\times$. To obtain the time evolution of $u$ we first need to expand $u$ in terms of M to the 2PN order, generalizing Eq.(4.22)

[146]. The orbital elements $e_r$ and $w$ appearing in the above relation will evolve due radiation reaction. Unlike in the quasi-circular case, the solution $e_r(t)$ and $\omega(t)$ is not explicit but implicitly contained in the following coupled system of first order differential equations. The equations are obtained earlier in [44] to the required 2PN order but rewritten here in terms of the 'gauge-invariant' variable $\tau = \frac{G m \omega}{c^3}$. We have,

$$
\begin{aligned}
< \frac{dw}{dt} > \;=\; & \frac{1}{5}\frac{\eta\, c^6}{G^2 m^2}\frac{\tau^{11/3}}{(1-e_r^2)^{11/2}}\Big\{\left(96 + 292\, e_r^2 + 37\, e_r^4\right)\,(1-e_r^2)^2 \\
& + \frac{\tau^{2/3}\,(1-e_r^2)}{56}\Big\{(1036\,\eta - 13147)\; e_r^6 - (6636\,\eta + 189154)\; e_r^4 \\
& - (54096\,\eta + 193624)\; e_r^2 - 14784\,\eta - 11888 \\
& + \tau^{4/3}\Big[\frac{1}{6048}\Big(\left(34188\,\eta^2 - 1649430\,\eta + 7135065\right)\; e_r^8 \\
& - \left(1038408\,\eta^2 - 680310\,\eta - 143408034\right)\; e_r^6 \\
& \left(2190804\,\eta^2 - 151843320\,\eta - 202085400\right)\; e_r^4 \\
& + \left(7007280\,\eta^2 + 131090976\,\eta - 4673632\right)\; e_r^2 \\
& + 1903104\,\eta^2 + 4514976\,\eta - 360224\Big) \\
& + (1-e_r^2)^{3/2}\,(5-2\eta)\left(48 + 298\, e_r^2 + 79\, e_r^4\right)\Big]\Big\}, \qquad (4.25a) \\
< \frac{de_r}{dt} > \;=\; & -\frac{1}{15}\frac{\eta\, c^3}{G m}\frac{\tau^{8/3}\, e_r^2}{(1-e_r^2)^{9/2}}\Big\{2\left(304 + 121\, e_r^2\right)\,(1-e_r^2)^2 \\
& + \frac{(1-e_r^2)\,\tau^{2/3}}{84}\Big[(16940\,\eta - 168303)\; e_r^4 \\
& - (60060\,\eta + 858504)\; e_r^2 - 180320\,\eta - 196632\Big] \\
& + \tau^{4/3}\Big[\frac{1}{1008}\Big(\left(50820\,\eta^2 - 3172554\,\eta + 11204991\right)\; e_r^6 \\
& - \left(1173480\,\eta^2 - 6557598\,\eta - 91575254\right)\; e_r^4 \\
& - \left(117180\,\eta^2 - 75705732\,\eta - 25245996\right)\; e_r^2 \\
& + 3144960\,\eta^2 + 16402608\,\eta - 12161360\Big) \\
& + 5\,(1-e_r^2)^{(3/2)}\,(5-2\eta)\left(304 + 121\, e_r^2\right)\Big]\Big\}. \qquad (4.25b)
\end{aligned}
$$

The solution to the above system gives us $\omega(t)$ and $e_r(t)$, the evolution of w and $e_r$ under the effect of gravitational radiation reaction. Using this solution in the 2PN accurate expansion connecting u and $M$ one gets $u(t)$, the time evolution of the eccentric anomaly. Finally inserting $u(t)$, w$(t)$and $e_r$(t) into Eqs.(4.16), (4.17) and (4.18) one obtains $h_+(t)$ and $h_\times(t)$, the time evolution of the 'plus' and 'cross' polarizations, under gravitational radiation reaction.

The above equations are complete to 2PN accuracy modulo the tail terms. The contribution of tail terms to the flux of energy and angular momentum has been obtained in [128] and [129]. The consequent contribution to the evolution of orbital frequency and eccentricity is also discussed there. After adding on these contributions at 1.5PN the phasing equations are complete and accurate to 2PN order and should provide the starting point for a numerical solution to the phasing problem in the quasi-elliptic case.

## 4.5 Conclusions

In this chapter we have computed all the 'instantaneous' 2PN contributions to $h_+$ and $h_\times$ for two compact objects of arbitrary mass ratio moving in elliptical orbits, using 2PN corrections to $h_{ij}^{TT}$ and the generalized quasi-Keplerian representation for the 2PN motion. The expressions for $h_+$ and $h_\times$ obtained here represent gravitational radiation from an elliptical binary during that stage of inspiral when orbital parameters are essentially the same over a few orbital periods, in other words when the gravitational radiation reaction is negligible. We investigated the effect of eccentricity and orbital inclination on the amplitude of the Newtonian part of $h_+$ and $h_\times$. We observed that orbital inclination i changes the magnitudes of $|h_+|^2$ and $|h_\times|^2$ appreciably. The reduction in $|h_+|^2$ and $|h_\times|^2$ for small and medium eccentricities, is small compared to higher $e_r$'s, when $i$ is varied from 0 to $\pi/2$, which is consistent with the earlier work [133]. We compute $\left(\frac{h_\times}{h_+}\right)^2$ at the Newtonian order and conclude that this ratio for u = 2 can be used as good indicator for the orbital inclination, for very small to very high values of e,. The modulation of $h_+$ and $h_\times$ due to the precession of the periastron, which occurs at 1PN order is also explicitly shown.

As mentioned earlier, following [2] the construction of the search templates for gravitational radiation may be done in two steps. The first step deals with the construction of the 'plus' and 'cross' gravitational wave polarizations, which was performed here for compact binaries of arbitrary mass ratio, moving in elliptical orbits. The second step involves the determination of the evolution of the orbital elements (the orbital phase and parameters like eccentricity) as a function of time. The parameters describing the orbit vary in a nonlinear manner with respect to time, as the orbit evolves under the action of gravitational radiation reaction forces. In principle, the evolution of the orbital elements should be determined from the knowledge of the radiation reaction forces acting locally on the orbit. In practice, as discussed in this chapter, this is determined ***assuming*** energy and angular momentum balance and the far-zone expressions for energy and angular momentum fluxes. The complete determination of the radiation reaction terms in the equations of motion requires a full iteration of the Einstein's field equations in the near-zone. In the absence of this complete result an interesting question to pose is the following. To what extent do the expressions of energy and angular momentum fluxes and the assumption of energy and angular momentum balance constrain the equations of motion? In the next chapter we address this question using the 'refined balance procedure' proposed by Iyer and Will [33, 34] and discuss radiation reaction to 2PN order beyond the quadrupole approximation *i.e.* the 4.5PN terms in the equations of motion.