Random-phase reservoir and a quantum resistor: The Lloyd model

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We introduce phase disorder in a one-dimensional quantum resistor through the formal device of "fake channels" distributed uniformly over its length such that the outcoupled wave amplitude is reinjected back into the system, but with a phase which is random. The associated scattering problem is treated via invariant embedding in the continuum limit, and the resulting transport equation is found to correspond exactly to the Lloyd model. The latter has been a subject of much interest in recent years. This conversion of the random phase into the random Cauchy potential is a notable feature of our work. It is further argued that our phase-randomizing reservoir, as distinct from the well-known phase-breaking reservoirs, induces no decoherence, but essentially destroys all interference effects other than the coherent backscattering.

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The Lloyd model[1–8] is known to be one of the very widely used models of disorder for quantum-electronic systems. Indeed, very recently, it has been the subject of detailed analysis for electronic transport in a quantum resistor, providing deeper insights into the scaling ideas of localization in a one-dimensional (1D) system.[7,8] In the Lloyd model for a tight-binding disordered system, the site-energies are taken to be distributed identically, independently, and randomly with a Cauchy probability distribution. The latter is a fat-tailed distribution with infinite variance. Its simple two-pole structure in the complex site-energy plane makes for an exact analytical treatment. In this work, we show that the Cauchy site-energy disorder (i.e., the random site-diagonal potential) can be formally viewed as arising from a certain process of phase randomization. The latter is introduced through the formal device of "fake or side channels" distributed uniformly along the length of the 1D resistor, wherein the outcoupled wave amplitude is reinjected back into the system, but with the proviso that its phase is shifted randomly over 2π. Such a phase disorder or "dephasing"—without causing decoherence—has been invoked recently[9,10] in the context of mesoscopic conductors in calculating the full-counting statistics. Our objective here, however, is different; namely, to study how such a random-phase distribution leads to a "potential" disorder giving the Lloyd model. This phase randomization is formally incorporated through an invariant embedding treatment as known in the context of quantum transport in disordered conductors,[11–14] where the object of interest is an emergent quantity such as the reflection and/or transmission coefficient or, equivalently, the resistance and/or conductance. The evolution equation so derived for the emergent quantity (the reflection amplitude in our case) in sample length is found to correspond exactly to the continuum limit of the Lloyd model. This emergence of the Lloyd model with a Cauchy-potential disorder arising from the phase randomization through our phase reservoir is a striking result. It is further argued that our phase-randomizing reservoir, unlike the well-known phase-breaking (decohering) reservoirs,[15–17] cannot eliminate the coherent backscattering. The phase randomization considered here by us involves effectively parallel addition of quantum resistors (as introduced originally in Ref. 18) via the scattering matrices providing outcoupling to the side channels. Of course, strictly speaking, being "quenched" in nature, it can cause no reservoir-induced decoherence.

Let us first introduce our phase-randomizing reservoir with its fake channels. In its simplest form, it is modeled here by the three-port scatterer with an energy-independent and symmetric S matrix[15]

$$S=\begin{pmatrix} \frac{1}{2}(\sqrt{1-2\epsilon}-1) & \frac{1}{2}(\sqrt{1-2\epsilon}+1) & \sqrt{\epsilon} \\ \frac{1}{2}(\sqrt{1-2\epsilon}+1) & \frac{1}{2}(\sqrt{1-2\epsilon}-1) & \sqrt{\epsilon} \\ \sqrt{\epsilon} & \sqrt{\epsilon} & -\sqrt{1-2\epsilon} \end{pmatrix} \quad (1)$$

connecting the outgoing amplitudes (o_1,o_2,o_3) with the incoming amplitudes (i_1,i_2,i_3) as shown in Fig. 1. Here, ϵ is the outcoupling to the transverse fake channel labeled 3 with $0\leq\epsilon\leq\frac{1}{2}$. Channels 1 and 2 are the transport channels (leads) through which the device is to be inserted into the 1D quantum conductor. Our random-phase reservoir differs essentially from the well-known decoherence-inducing reservoirs[15,16] in that the amplitude outcoupled into the fake channel is here reinjected (rescattered) back into the system, but now with a phase shift ϕ which is assumed random over 2π.

In order to introduce the random-phase reservoirs uniformly over the length of the 1D quantum resistor, we now use the method of invariant embedding and solve the scattering problem for the emergent quantity (amplitude reflection

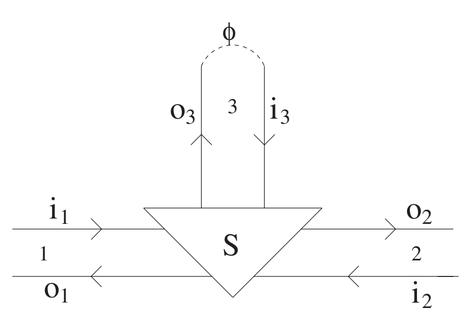

FIG. 1. A schematic showing the random-phase reservoir with the "fake channel" 3. Outcoupled amplitude is reinjected with random-phase shift ϕ.

FIG. 2. A schematic description of the "invariant embedding" method for a 1D conductor with random-phase reservoirs distributed uniformly along the length.

coefficient in the present case). Following the general philosophy of invariant embedding for a scattering problem, we now embed the scattering sample of length L in a supersample of length $L+\Delta L$, and then study the change ΔS of the total S matrix as ΔL tends to zero (Fig. 2). Here, ΔL contains the elementary random-phase reservoir with the outcoupling ϵ of order ΔL, i.e., $\epsilon/\Delta L \to$ finite as $\Delta L \to 0$. Thus, the parameter $\epsilon/\Delta L$ measures the strength per unit length with which the phase is randomized. The corresponding change ΔS in the S matrix is then given by

$$\Delta S = \begin{pmatrix} -\epsilon/2 & 1-\epsilon/2 & \sqrt{\epsilon} \\ 1-\epsilon/2 & -\epsilon/2 & \sqrt{\epsilon} \\ \sqrt{\epsilon} & \sqrt{\epsilon} & -(1-\epsilon) \end{pmatrix}. \tag{2}$$

In writing ΔS above, we have made use of the fact that ϵ is small, of order ΔL in Eq. (1). Next we calculate the incremental transmission (ΔT) and the reflection (ΔR) amplitudes in terms of the matrix elements (in obvious notation) $t_{13} = t_{23} = \sqrt{\epsilon}$, $t_{12} = 1-\epsilon/2$, $r_{33} = -(1-\epsilon)$, and $r_{11} = r_{22} = -\epsilon/2$ from the ΔS above. Taking into account the multiple scatterings involving reinjection from the fake channel, we obtain

$$\begin{aligned} \Delta T &= t_{12} + t_{13}e^{i\phi}t_{32} + t_{13}e^{i\phi}r_{33}e^{i\phi}t_{32} + \ldots \\ &= t_{12} + \frac{t_{13}e^{i\phi}t_{32}}{1 - r_{33}e^{i\phi}} = 1 - \frac{\epsilon}{2} + \frac{\epsilon e^{i\phi}}{1 + (1-\epsilon)e^{i\phi}} \end{aligned} \tag{3}$$

and

$$\Delta R = r_{11} + \frac{t_{13}^2 e^{i\phi}}{1 - r_{33}e^{i\phi}} = \frac{(e^{i\phi} - 1)\epsilon/2}{1 + (1-\epsilon)e^{i\phi}}. \tag{4}$$

Now, consider a plane wave incident on the right-hand side of the supersample of length $L+\Delta L$. Summing over all processes of direct and multiple reflections and transmissions from the right-hand side of the sample of length L and with the phase reservoir inserted in the interval $[L, L+\Delta L/2]$, we have

$$R(L+\Delta L) = \Delta R + \frac{\Delta T^2 e^{2ik\Delta L} R(L)}{1 - \Delta R R(L) e^{2ik\Delta L}}, \tag{5}$$

where k is the wave vector magnitude for the incident electron wave. Expanding the right-hand side of Eq. (5) using the values of ΔT and ΔR from Eqs. (3) and (4), and keeping terms to order ΔL, we obtain

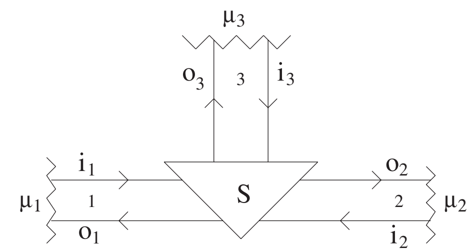

FIG. 3. A schematic of the single-channel phase-breaking reservoir.

$$\frac{dR}{dl} = 2iR(l) + \frac{i}{2}\eta \tan\frac{\phi(l)}{2}[1 + R(l)]^2, \tag{6}$$

where we have introduced dimensionless length $l = kL$, and $\eta = \epsilon/k\Delta L$ as $\Delta L \to 0$, with the initial condition $R(l) = 0$ for $l = 0$. Here, the random phase $\phi(l)$ is distributed uniformly over $0-2\pi$. Transforming $\eta\tan(\phi(l)/2) = V(l)$, we find the distribution $P_l(V)$ of $V(l)$

$$P_l(V) = \frac{1}{\pi}\frac{\eta}{V^2(l) + \eta^2}, \tag{7}$$

which is the Cauchy probability distribution. Finally, with the above transformation from the random phase to the random potential (Cauchy), we obtain

$$\frac{dR}{dl} = 2iR(l) + \frac{i}{2}V(l)[1 + R(l)]^2. \tag{8}$$

This invariant embedding equation for evolution in l has the form of a Langevin equation for the complex reflection amplitude R with a Cauchy noise potential $V(l)$. It corresponds to the underlying quantum-mechanical Hamiltonian for a 1D disordered continuum with a potential $V(x)$, $0 \le x \le l$. The corresponding tight-binding Hamiltonian will have the site (Cauchy) potential $V(n)$ with $0 \le n \le N$ and $N = l/ka$, where a is the lattice constant. Thus, the phase randomization is mapped onto the Cauchy random potential $V(n)$ for a tight-binding Hamiltonian—the Lloyd model.

Having, thus, discussed the provenance of the Cauchy potential disorder (and, therefore, the Lloyd model) in terms of our random-phase reservoir, it will be in order now to compare the latter with the phase-breaking reservoirs, giving the reservoir-induced decoherence, as due originally to Büttiker.[15,16] For an isolated single-channel phase-breaking reservoir, the S matrix is as given in Eq. (1) and the corresponding schematic as in Fig. 3. It shows explicitly the connections to the three terminals with three chemical potentials: μ_1, μ_2 for the longitudinal (or transport channels), and μ_3 for the "potentiometric" (transverse) channel; the latter being determined from the condition of zero net current. This can be readily shown to give the two-probe conductance (G_{12}) between terminals 1 and 2,

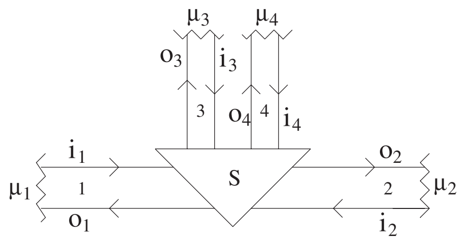

FIG. 4. A schematic of the phase-breaking reservoir with two uncoupled transverse channels 3 and 4.

$$G_{12} = \frac{e^2}{\pi\hbar}\left\{\left[\frac{1}{2}(\sqrt{1-2\epsilon}+1)\right]^2 + \frac{\epsilon}{2}\right\}. \qquad (9)$$

In our corresponding random-phase reservoir with a single fake channel, we have the same three-terminal S matrix except for the reinjection at the fake channel 3 with a random phase ϕ. For a given value of the phase ϕ, the two-terminal conductance G_{12}^{ϕ} can be readily shown to be

$$G_{12}^{\phi} = \frac{e^2}{\pi\hbar}\left| t_{12} + \frac{t_{13}e^{i\phi}t_{32}}{1-r_{33}e^{i\phi}}\right|^2, \qquad (10)$$

with the coeffiecients $t_{12}=\frac{1}{2}(\sqrt{1-2\epsilon}+1)$, $t_{13}=t_{32}=\sqrt{\epsilon}$, and $r_{33}=-\sqrt{1-2\epsilon}$. Averaging now G_{12}^{ϕ} over ϕ, we find

$$\langle G_{12}^{\phi}\rangle_{\phi} \equiv \frac{1}{2\pi}\int_0^{2\pi} G_{12}^{\phi} d\phi = G_{12}, \qquad (11)$$

i.e., both the reservoirs give identical results for the two-probe conductance between terminals 1 and 2.

Now we turn to comparing the phase-breaking reservoirs with two transverse channels and our corresponding random-phase reservoir also with two fake channels, as shown in Figs. 4 and 5. The corresponding four-terminal S matrix is[16]

$$S = \begin{pmatrix} 0 & \sqrt{1-\epsilon} & \sqrt{\epsilon} & 0 \\ \sqrt{1-\epsilon} & 0 & 0 & \sqrt{\epsilon} \\ \sqrt{\epsilon} & 0 & 0 & -\sqrt{1-\epsilon} \\ 0 & \sqrt{\epsilon} & -\sqrt{1-\epsilon} & 0 \end{pmatrix} \qquad (12)$$

with $0 \le \epsilon \le 1$. Note the reinjections shown in dashes with random phases ϕ_1 and ϕ_2 at the fake channels 3 and 4 (Fig. 5). It is to be noted that in Fig. 4, the potentiometric condition for zero net current is being imposed here for the two transverse channels 3 and 4 separately. With this, it can now be readily shown how that the two-probe conductances are again equal:

$$G_{12} = \langle G_{12}^{\phi_1,\phi_2}\rangle_{\phi_1,\phi_2} = \frac{e^2}{\pi\hbar}\frac{2(1-\epsilon)}{2-\epsilon}. \qquad (13)$$

Now, however, for the case of the two-channel phase-breaking reservoirs with the potentiometric condition of zero net current imposed summarily[16] on the two coupled transverse or side channels 3 and 4, the conductances turn out to be different. Some thought will convince that this is so because the phase-breaking reservoir and the random-phase reservoir differ essentially inasmuch as the former induces decoherence (can destroy all interference effects) while the latter cannot eliminate the coherent backscattering (CBS). Indeed, for the case of coupled transverse channels, one can easily trace the CBS alternatives. We may say that our random-phase reservoir leads to a purification of interference effects to coherent backscattering.

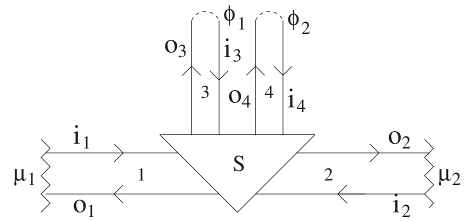

FIG. 5. A schematic of the random-phase reservoir with two uncoupled fake channels.

Now some comments and clarifying remarks on the use of the reservoirs, in general, and the physical realization of the random-phase reservoir, in particular, as used here by us, seem to be in order. In the original Landauer-Buttiker scattering approach[19–21] to quantum transport through a conductor, dissipation and associated decoherence are viewed as taking place in the reservoirs at the two ends of the sample. Physically, however, the dissipation takes place in the sample throughout its length. This latter feature has been modeled,[15,16,22,23] admittedly phenomenologically, through the formal device of reservoirs distributed along the sample length and connected to it through the appropriately chosen S matrices, whereby the outcoupled amplitude is absorbed and reemitted into the sample where it adds incoherently to the coherent transport amplitude. This constitutes the now well-known reservoir-induced decoherence. Now, we can also have a random-phase reservoir where outcoupled amplitude is reinjected back into the conductor with a phase shift distributed randomly over 2π as in the work presented here. We emphasize that this is a quenched phase disorder that causes no decoherence or phase breaking. The invariant embedding, in fact, allows us to introduce both the decoherence[12,24] and the phase randomization over the conductor through a proper choice of ΔS's appearing in Eq. (2), and calculate the emergent quantities like reflection and/or transmission coefficients. The random-phase reservoir is physically equivalent to the phase disorder as considered by others.[18,25] A literally physical realization of the random-phase reservoir would be through the chaotic cavities (with a long dwell time) terminating the side channels, wherein the random phase shifts result from the deterministic quantum chaos.[9,10] The idea underlying the use of these formal devices (reservoirs) is that the strength of the outcouplings can be used to effectively parametrize some of the physical effects of interest.

In conclusion, we have demonstrated analytically a conversion of random phases into random potentials that correspond exactly to the Lloyd model. To this end, we have introduced a formal device of random-phase reservoir with fake channels. Despite the apparent similarity to the well-known phase-breaking reservoirs, the two types are essentially different. Thus, while the phase-breaking reservoirs with absorption and reemission of electrons cause the well-known reservoir-induced decoherence (that can suppress all interference effects), our random-phase reservoirs having fake channels subtending reinjection with random phases cannot eliminate the coherent backscattering.

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