Chapter III

SEMICLASSICAL DECAY OF THE KALUZA-KLEIN VACUUM

Our aim in this chapter is to describe an important distinctive feature of higher dimensional gravity : semiclassical instability of the ground states and the corresponding decay process. Such a process is fundamentally different from all others in four dimensional gravity where, in fact, it can never occur due to reasons to be described below.

III.(A) Vacuum Decay in Field Theory

Let us first start with a very brief qualitative description of the semiclassical decay processes that arise in ordinary field theories (without gravity). This will help in bringing forth the distinguishing features of the equivalent process in the presence of gravitation when we will describe that later.

The first description of such a process was given by Voloshin, Kobzarev and Okun (1975) and the theory was further developed mainly by Coleman (1977). Consider a self-interacting scalar field @ in four-dimensional spacetime with nonderivative interactions

$$L = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - U(\Phi). \qquad (III.1)$$

Let U possesses two relative minima, $\Phi_\pm$, only one of which, Φ_- , is an absolute minimum.

The state of the classical field theory for which $\Phi = \Phi_-$ is the unique classical state of the lowest energy and, at least in perturbation theory, corresponds to'the unique vacuum state of the quantum theory. The state of the classical field theory for which $\Phi = \Phi_+$ is a stable classical equilibrium state. It is, however, rendered unstable by quantum effects, in particular by barrier penetration. It is a false or metastable vacuum. Once in a while, an energetically favourable bubble of true vacuum will form and this will grow converting the false vacuum to a true one.

The semiclassical process of bubble nucleation can be pictured as the evolution of

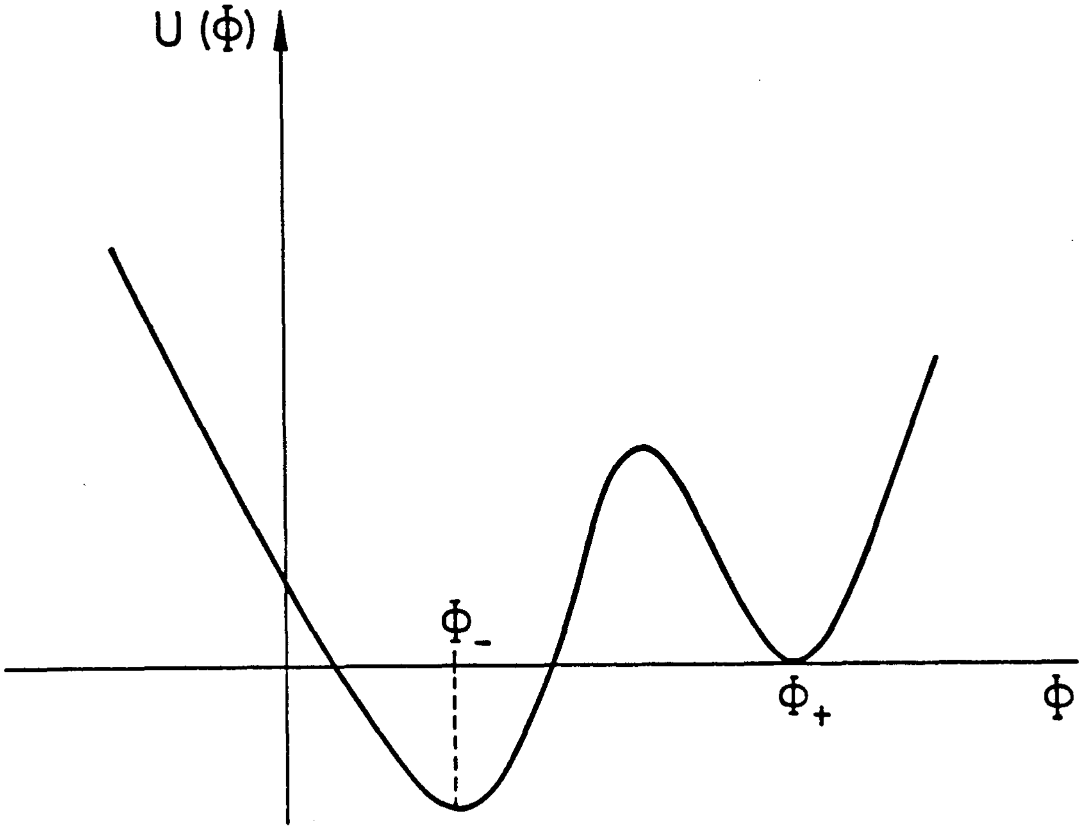

Fig.7 The assumed shape of the potential of the field Φ

the field Φ in imaginary time (t_E). To describe the process. therefore. one has to study the corresponding Euclidean field equations. The solution of this equation is called its 'bounce'. This solution approaches the false vacuum value at spacetime infinity and satisfies $\partial\Phi/\partial t_E = 0$ at $t_E = 0$. It can be shown that the field can emerge into the Lorentzian region after the tunneling process only if the eigenvalue spectrum of the small fluctuation operator (essentially the second variation of the action) possesses a negative eigenvalue. Its presence will indicate the instability of the false vacuum.

The probability of bubble nucleation per unit time per unit volume is proportional to $\exp(-I_E)$, where I_E is the Euclidean action for the bounce. The decay process is dominated by the lowest action bounce which has the important property of O(4) invariance, i.e.

$$\Phi(x, t_E) = \tilde{\Phi}(x^2 + t_E^2) \qquad (III.2)$$

To obtain a description of the classical evolution of the bubble after nucleation, one has to analytically continue the bounce solution to the Minkowskian time ($t_E \rightarrow$ it), so that

$$\Phi(x,t) = \tilde{\Phi}(x^2 - t^2) \qquad (111.3)$$

So, the O(4) invariance of the bounce solution implies that the Lorentzian evolution of the bubble is O(3,1) invariant. Equivalently, one can say that the expanding bubble looks the same to all Lorentz observers.

Let us now study, in this context, the guidelines that such a process should follow when we take gravity into account. As will be described below, the situation becomes highly nontrivial and complicated in this case.

III.(B) Positive Energy Theorem and Higher Dimensional Gravity

We recall here that in all reasonable classical field theories, the global energy can be easily expressed as the integral of the local energy density, T_{00}. Since T_{00} is always positive and definite, that naturally ensures the stability of the ground state. However, in gravity, the situation is not so straightforward. In fact, a well-defined concept of local energy density is totally absent in this case. Attempt has been made to realize this by a definition of energy momentum pseudotensor. But first of all, it is not a true tensor and also not positive definite. However, progress along this line has been able to provide a satisfactory definition of the total energy for a gravitating system. The system should, however, be quasi-Minkowskian in nature, so that the metric can be expressed as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$., where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ vanishes at infinity. Then the total energy can be calculated to be in the form of a surface integral :

$$E = \frac{1}{16\pi G} \int \left\{ \frac{\partial h_{ij}}{\partial x^j} - \frac{\partial h_{jj}}{\partial x^i} \right\} dS^i \qquad (III.4)$$

The integral is taken over a large surface S'. This surface integral is popularly known to be ADM(ArnoWitt, Deser, Misner) mass [for details, see sec.7.6, Weinberg, 1972].

It was proved first by Schoen and Yau(1979) using classical methods and then by Witten(1981a) using spinor algebra that this total energy E, in the absence of matter field is always either zero (only for flat Minkowski space) or positive. When matter is present, the statement of the positive energy theorem remains unchanged provided the matter contribution is positive everywhere. We may now summarize all the finer points in the above discussion in a compact statement of the theorem :

Positive Energy Theorem : The total energy of any solution of the four dimensional Einstein equations for which T_{00} of the matter field is either positive or zero at each point in spacetime and in each local Lorentz frame and which asymptotically approaches the flat Minkowski spacetime at infinitely large distances should always be positive or zero, and zero only for flat Minkowski space.

This theorem, therefore, attributes a uniqueness to the gravitational ground state in four dimensions thereby ensuring its semiclassical stability. Unfortunately, the proof of this, theorem can not be fully generalised to higher dimensional spacetimes. It is comparatively easier to realise this from the proof forwarded by Witten(1981a). The important steps towards the proof are being described below.

Witten's Proof : Witten's proof crucially depends on the possibility of defining spinors uniquely on an asymptotically Euclidean initial value hypersurface in a gravitating system as shown in Fig.8. Witten begins with the observation that *no* nonzero spinor ϵ that satisfies the Dirac equation $\not{D}\epsilon = \gamma^i D_i \epsilon = 0$ on some initial value hypersurface [the index i denotes spatial coordinates of the hypersurface, γ_i are curved space Dirac Gamma metrices] can, as well, vanish at infinity. He showed that, in the case when matter is present and the Dirac equation is valid, one may write

$$(i \not{D})^2 \epsilon = -\bar{D}^i D_i \epsilon + 4\pi G(T_{00} + T_{0j}\gamma^0\gamma^j)\epsilon = 0. \qquad (III.5)$$

Multiplying by ϵ^* and integrating over the three surface,

$$\int d^3x\sqrt{g}(\bar{D}_i\epsilon^* D^i\epsilon) + 4\pi G\int d^3x\sqrt{g}\epsilon^*[T_{00} + T_{0j}\gamma^0\gamma^j]\epsilon = 0 \qquad (III.6)$$

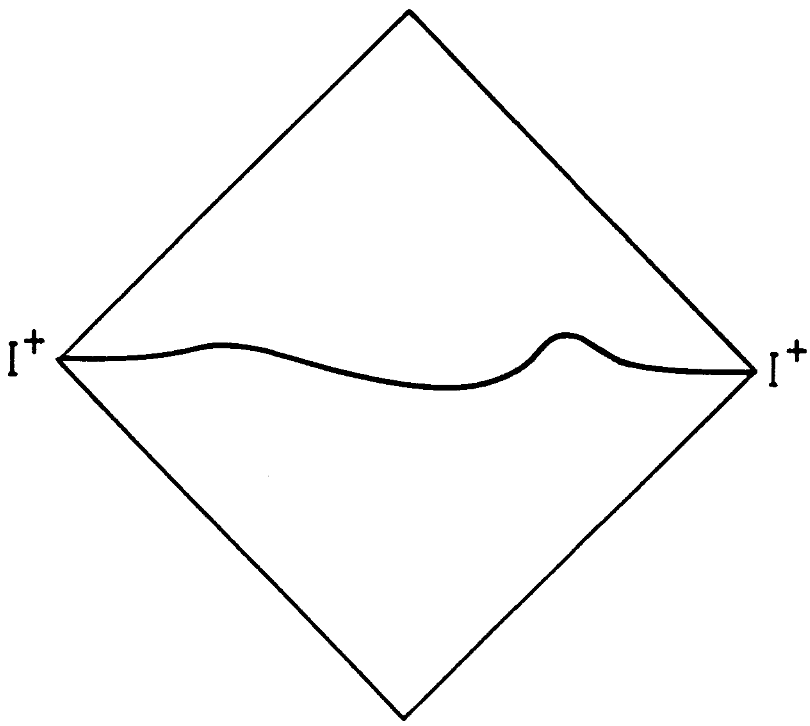

Fig.8 Asymptotically Euclidean *initial* value *hypersurface*

The surface term vanishes if $\epsilon \to 0$ at infinity. Now, by dominant energy condition which ensures that in any orthonormal basis T_{00} dominates the other components of the energy-momentum tensor, the second term should always be positive (semi-definite). Thus the equation is valid only if the second term is zero (no matter field present) as well as $D_i\epsilon = 0$ which means ϵ =constant. However, if ϵ vanishes at infinity, ϵ should be zero throughout and the above-stated observation follows.

Now, to study spinors that satisfy Dirac's equation but do not vanish at infinity? Witten writes $\epsilon = \epsilon_1 + \epsilon_2$ where ϵ_1 has the asymptotic behaviour at large r of the form $\epsilon_1 = \epsilon_0 + \mathcal{O}(1/r)$; ϵ_0 being a constant and ϵ_2 vanishes at spatial infinity at a rate faster than $\mathcal{O}(1/r)$. He proves that there always exists such an ϵ satisfying Dirac's equation. Then repeating the entire sequence of the previous analysis done for vanishing ϵ, Witten obtains almost the same result but with an extra surface term that vanished before :

$$\int d^3x \bar{D}_i \epsilon^* D^i \epsilon + 4\pi G \int d^3x\, \epsilon^* (T_{00} + \gamma^0 \gamma^j T_{0j}) \epsilon = \int dS^i \epsilon^* D_i \epsilon. \qquad (III.7)$$

dS^i is the area element in a large surface at infinity bounding the three dimensional initial value hypersurface. This surface term can be expressed in terms of the arbitrary spinor ϵ_0 and the linearized (or asymptotic) form of the spin connection Γ_k. The explicit calculation of S identifies it to be proportional to the ADM mass. Since the L.H.S. of the above equation can never be negative, the global energy should, therefore, always be positive semi-definite.

Unfortunately. this proof cannot be fully generalized to spacetimes with more than four dimensions. Witten's proof applies in any number of dimensions provided the topology of the initial value hypersurface is such that one can consistently define a spinor field on it.

We can now see why the theorem cannot be extended to higher dimensional spacetime with nontrivial topology of initial value hypersurface. We observed that in a spacetime with zero energy, any nonzero spinor must satisfy $\int dS^i \epsilon^* D_i \epsilon = 0$ and, therefore, must be covariantly constant at spatial infinity.

The five dimensional Kaluza-Klein ground state is assumed to be a product of the four dimensionsal Minkowski space and a compactified dimension, $M^4 \times S^1$ and thus represents a multiply connected spacetime. The initial value hypersurface has a topology $\boldsymbol{R}^3 \times S^1$. The presence of the extra compactified dimension introduces a constraint that the phase gained by the spinor on returning to its original value after parallel transport around S^1 must be zero, so that there do not exist inequivalent ways to define spinors.

But if we can find an alternative spacetime with the following properties : (i) with zero energy, (ii) approaches flat $M^4 \times S^1$ spacetime at infinity, (iii) has initial value hypersurface of different topology that, however, approaches $\boldsymbol{R}^3 \times S^1$ at infinity; then it is possible that the different topology of the hypersurface of this second spacetime would induce a non-zero phase in ϵ defined on $M^4 \times S^1$ flat spacetime. So, a consistent spinor field that will be covariantly constant at spatial infinity can never be constructed and we will not be able to apply Positive energy theorem in such cases. It is quite possible that in this case the ground state may decay into another spacetime of same or lower energy. That is just the way by which Witten proved the semiclassical instability of $M^4 \times S^1$ by finding an alternative spacetime (known as Witten Bubble spacetime) that possesses all the three properties above.

But before getting into that, let us sum up here the general technical procedure that is to be followed up to study the semiclassical instability of the ground state of any higher dimensional theory of gravitation. Since the ground state corresponds to zero energy, as discussed above, it can decay into another spacetime of zero energy only. So, the existence of the alternative spacetime will essentially disprove the uniqueness of the vacuum. The procedure for obtaining the alternative solution is as follows :

(1) Try to get a solution of the Euclidean field equations such that it approaches the analytically continued Euclidean version of the assumed ground state at infinity. This solution is the instanton-like 'bounce' solution which interpolates between the assumed vacuum and whatever spacetime it decays into.

(2) Search for the negative action modes in the functional determinant obtained for small fluctuation around this 'bounce' solution. If such modes exist, the gaussian integral around this solution will contribute an imaginary part to the energy of the vacuum state, thus representing the instability [for details, see Gross, Perry, Yaffe,1982].

(3) To obtain the spacetime into which the assumed ground state decays, analytically continue the 'bounce' solution back to the Minkowski space. If it remains to be real-valued metric there, then that will represent the alternative spacetime.

In the next section we are going to discuss how Witten found the bubble solution.

III.(C) The Witten Bubble Solution

The Euclideanised version of the Kaluza-Klein ground state has topology $R^4 \times S^1$ which has an asymptotically (in fact, everywhere) $S^3 \times S^1$ boundary.

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 + dt^2 + d\chi^2 \\ &= dr^2 + r^2 d\bar{\Omega}_3^2 + d\chi^2 \end{aligned} \qquad (III.8)$$

where χ represents the compactified dimension. To search for the bounce solution, Witten realised that the five dimensional Euclidean Schwarzschild solution

$$ds^2 = \left(1 - \frac{2GM}{r^2}\right)dt_E^2 + \left(1 - \frac{2GM}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \qquad (III.9)$$

also has an $S^3 \times S^1$ boundary with an asymptotically flat metric. One may write it in a somewhat different way as

$$ds^2 = \left(1 - \frac{\alpha}{r^2}\right)^{-1} dr^2 + \left(1 - \frac{\alpha}{r^2}\right)d\chi^2 + r^2 d\Omega_3^2, \qquad (III.10)$$

so that it still remains to be a solution of the Euclideanised field equations. The quantity a should not be interpreted to be equal to $2GM$ here. Rather it is to be considered as a parameter. Now, studying the behaviour of this metric at $r = \sqrt{\alpha}$, he found that χ has to be periodic with a period $2\pi\sqrt{\alpha}$ so that the r–χ subspace remains nonsingular. Thus, $a = R_0^2$ where R_0 is the radius of the fifth dimension, a completely free parameter of the theory.

Since the boundary conditions of Eqs. 111.8 and 10 are now same, (111.10) can, therefore, represent the Kaluza-Klein instanton. Also, Euclidean Schwarzschild solution has one transverse traceless negative mode for small oscillations. So, the one loop determinant is imaginary representing the decay of flat space.

The instanton has a discrete $\mathcal{Z}_2$ time symmetry. Thus, it also has a surface $(t_E = 0)$ on which the time derivative of the metric vanishes, or stated more geometrically, the extrinsic curvature vanishes. So, after the analytical continuation $(t_E \rightarrow \pi/2 + i\tau)$ the Minkowski solution is obtained as

$$ds^2 = -r^2 d\tau^2 + \left[1 - \frac{R_0^2}{r^2}\right]^{-1} dr^2 + r^2 \cosh^2\tau (d\theta^2 + \sin^2\theta d\phi^2) + \left[1 - \frac{R_0^2}{r^2}\right] d\chi^2 \quad (III.11)$$

where r has the range $R_0 \leq$ r $< \infty$.

The topology of the $\tau = 0$ surface is $R^2 \times S^2$, although in its geometry it is asymptotic to flat metric on $R^3 \times S^1$. As described in the previous section, this solution may thus induce a nonzero phase in a spinor (upon parallel transport) defined on $R^3 \times S^1$ in Kaluza-Klein vacuum and become a cause for the semiclassical instability.

The instanton represents a tunneling from flat $R^3 \times S^1$ to $R^2 \times S^2$ and thus involves a topology change. This is one of the rarest examples of the topology changing processes in gravitation [see Strominger, 1989].

The spacetime (111.11) is known as Witten Bubble. Its evolution properties are being described in the next section.

III.(D) Evolution of The Witten Bubble

Let us first introduce here 'spherical Rindler' coordinates to describe the four dimensional Minkowskian subspace of the Kaluza-Klein ground state. A spherical array of uniformly accelerated observers uses such type of 'hyperbolic' coordinates. These are related to the Minkowskian coordinates in the following way :

$$\begin{aligned} t &= r \sinh \tau, \\ x^1 &= r \cosh \tau \cos \phi \sin \theta, \\ x^2 &= r \cosh \tau \sin \phi \sin 8, \\ x^3 &= r \cosh \tau \cos \theta. \end{aligned} \qquad (III.12)$$

Here, we are using these coordinates to make the vacuum metric comparable with the metric of the Witten bubble. The five dimensional Kaluza-Klein vacuum metric can now be written as

$$ds^2 = -r^2 d\tau^2 + dr^2 + r^2 \cosh^2 \tau (d\theta^2 + \sin^2 \theta d\phi^2) + d\chi^2. \qquad (III.13)$$

This metric is valid for r < 0. For $\tau > 0$, the decay state of the Kaluza-Klein vacuum has to be described by the Witten Bubble metric (111.11).

As a result of this decay, a microscopic hole of radius R_0 will be spontaneously formed in space. Like the bubble wall in conventional vacuum decay, this hole will start expanding to infinity with a uniform acceleration (Fig.9) and, therefore, will approach the velocity of light. The evolution of the Witten bubble is also $O(3,1)$ invariant and looks to be same to all lorentzian observers. But we should emphasize here that the range of r ($R_0 \leq r < \infty$) actually represents the fact that, unlike conventional decay where the inside of a bubble corresponds to the true vacuum, the Witten bubble has no interior at all. The Physical spacetime corresponds to the bubble wall and its exterior only.

Also, we cannot call the bubble surface to be a 'horizon' because, unlike 'horizons' of black hole or Rindler system, the hyperbola corresponding to the wall represents the termination of the spacetime itself and no information can be received from or sent to any other region. As it has already been discussed in the previous section, the 'troublesome part' of the metric or the r $--\chi$ subspace reduces to a planar surface at $r = R_0$. This implies that the manifold is 'smooth' or geodesically complete there. The $r--\chi$ subspace asymptotically approaches the line element of a cylinder.

The bubble surface is a 2-sphere of area $4\pi R_0^2 \cosh^2 \tau$. So, at any particular instant t, its radius is $r(t) = \sqrt{R_0^2 + t^2}$. For very large r, the metric (111.11) asymptotically approaches the M^4 x S^1 spacetime described by Eq.III.13.

These interesting properties of the Witten bubble have also been verified by the study of both time-like and null geodesics by Brill and Matlin (1989). In the next chapter, we shall study scalar waves in the Witten Bubble background. This will reveal some more interesting features of the evolution of such a spacetime.

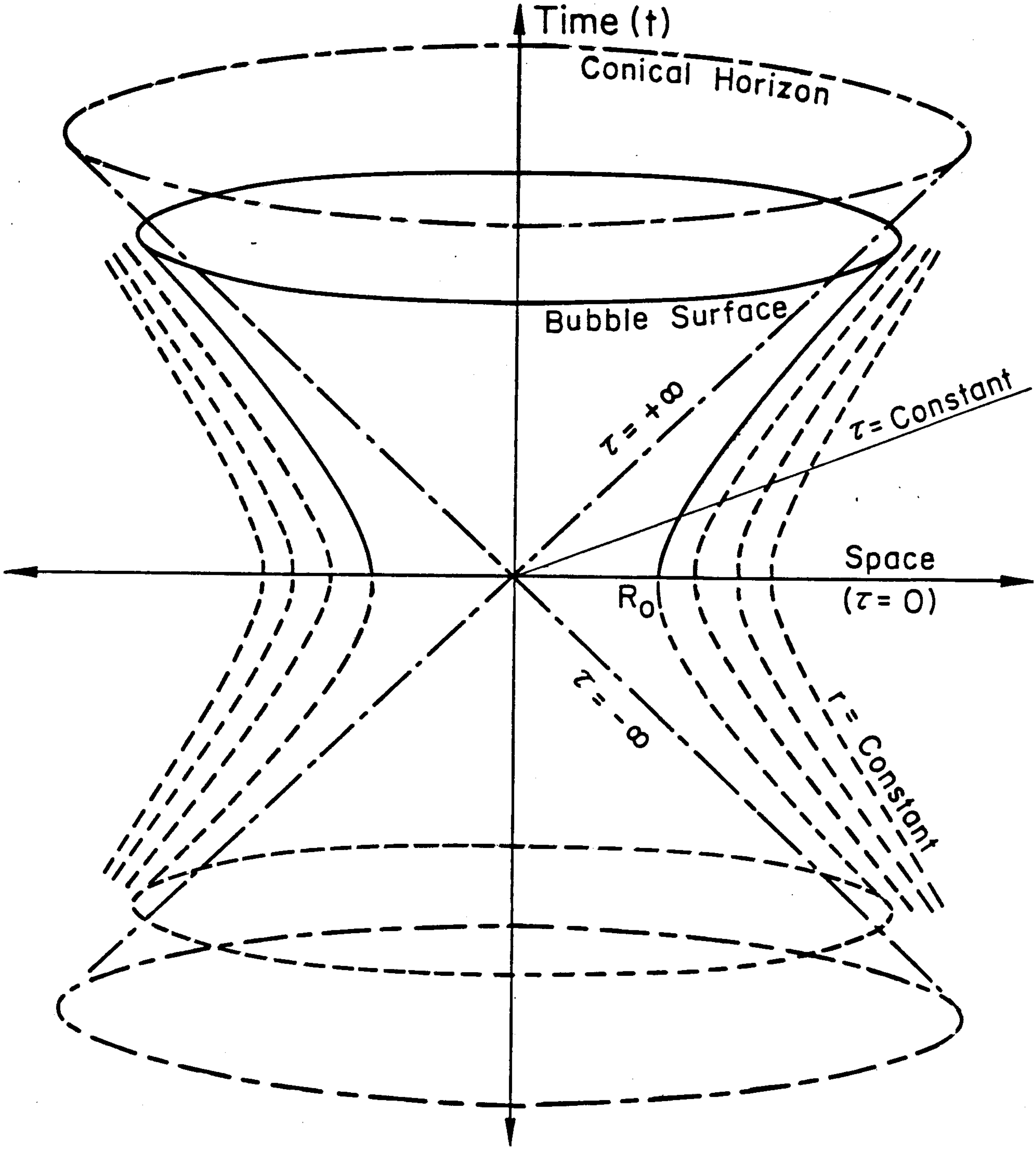

Fig.9 *Evolution of Bubble in a 2+1 dimensional Minkowski subspace. Radial lines represent constant r lines, whereas any hyperbola corresponds to a constant value of r. The bubble is formed at r = 0.*

Chapter IV

SCALAR WAVES IN THE WITTEN BUBBLE BACKGROUND

Different classical properties of the Witten Bubble spacetime introduced in the last chapter have been studied in detail by Matlin(1991). He also investigated different semi-classical phenomena, e.g. particle production, back reaction problem etc., involved in the process of the formation and evolution of the bubble.

The nature of the time-like and null geodesics in this spacetime has been studied by Brill and Matlin (1989). As in the case of the geodesics, the study of the behaviour of scalar waves also probes the geometry of the spacetime. The scattering phenomenon throws light on the nature of the bubble as well as its effect on the surrounding spacetime. Further, it provides us valuable information about the bound states and the stability of the spacetime. Also, the investigation of scalar waves in an exact solution such as the Witten metric offers insight into the propagation of waves in strong gravitational fields.

In this chapter, we describe in detail all these topics related to scalar waves in the Witten bubble [Bhawal and Vishveshwara, 1990]. Associated with this chapter are Appendices B and C – the former contains alternative scalar wave solutions in the Witten bubble metric and the latter is a discussion on some coordinate transformation that we use in this chapter.

IV.(A) The Klein-Gordon Equation

The Klein-Gordon equation for a massive scalar field is given by

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\mu\nu}\Phi_{,\nu})_{,\mu} - M^2\Phi = 0 \qquad (IV.1)$$

where M is the mass of the field.

The metric (111.11) is independent of the fifth coordinate χ and, therefore, there is a Killing symmetry in the fifth dimension. The solution of the Klein-Gordon equation is found to be

$$\Phi = \mathcal{R}(r) H^{\ell m}_{i\omega}(\tau, \theta, \phi)\, e^{i m_1 \chi} \tag{IV.2}$$

$$= \mathcal{R}(r)\, T^{\ell}_{i\omega}(\tau)\, Y^{m}_{\ell}(\theta, \phi)\, e^{i m_1 \chi}, \tag{IV.3}$$

where $Y_l^m(\theta\phi)$ are the spherical harmonics. Other functions appearing in this solution will be discussed in the following subsections.

Scalar waves for which the fifth dimensional component vanishes ($m_1 = 0$) represent the propagation of ordinary scalar waves in such a spacetime. The case of nonzero m_1 cannot readily be interpreted in terms of realistic scalar waves [see Bailin and Love, 1987]. Since the standard studies in five dimensional Kaluza-Klein theory take the compactified radius of the extra dimension to be of the order of the Planck length, $L_{\rm Pl}$, a mode characterised by the quantum number m_1 then corresponds to a wavelength along the fifth dimension of the order of $2\pi L_{\rm Pl}/m_1$. The corresponding momentum or the relativistic kinetic energy then turns out to be of order $m_1.M_{\rm Pl}$ (in units $c = {\rm G} = \hbar = 1$), where $M_{\rm Pl}$ is the Planck mass ($\simeq 10^{19}$ GeV). Such highly energetic waves do not represent realistic ones encountered in our observed physical environment.

So, in this work, we consider $m_1 = 0$ throughout. We shall see that this will greatly facilitate the solution of the radial equation in such a spacetime.

(a) τ-Equation

The wave-field represented by the solution (IV.2) does not oscillate in a simple harmonic way, but in a more complicated way given by the hyperbolic harmonics $H^{\ell m}_{i\omega}(\tau, 8, \mathrm{d})$. These hyperbolic harmonics are characterised by the generalised frequency w, which labels the representation of the Lorentz group SO(3,1). These are actually the eigenfunctions of the D'Alembertian on the unit time-like hyperboloid :

$$[-\frac{1}{\cosh^2\tau}\frac{\partial}{\partial\tau}\cosh^2\tau\frac{\partial}{\partial\tau} + \frac{1}{\cosh^2\tau}\nabla^2_{\theta,\phi}] H^{\ell m}_{i\omega} = (\omega^2 + 1) H^{\ell m}_{i\omega}. \tag{IV.4}$$

These form a complete orthonormal set with respect to the Lorentz-invariant volume on the time-like hyperboloid :

$$\int_0^\infty \int_0^{2\pi} \int_0^\pi H_{i\omega}^{\ell m}(\tau,\theta,\phi)\, H_{i\omega'}^{*\ell' m'}(\tau,\theta,\phi)\, \cosh^2\tau \sin\theta\, d\theta d\phi d\tau = \delta(\omega-\omega')\delta_{\ell\ell'}\delta_{mm'}. \tag{IV.5}$$

A detailed construction of these hyperbolic harmonics has been discussed by Gerlach(1983) in the appendix of his paper. Some inadvertant errors seem to have crept into his constructions, probably stemming from the original sources used. In this work, we use explicit expressions for the solutions, thereby avoiding any possible ambiguities.

$$\text{Also,} \quad H_{i\omega}^{\ell m}(\tau,\theta,\phi) = T_{i\omega}^{\ell}(\tau) Y_\ell^m(\theta,\phi) \tag{IV.6}$$

where $T_{i\omega}^{\ell}(\tau)$ satisfies the equation

$$[-\frac{1}{\cosh^2\tau}\frac{d}{d\tau}\cosh^2\tau\frac{d}{d\tau} - \frac{\ell(\ell+1)}{\cosh^2\tau}]T_{i\omega}^{\ell}(\tau) = (\omega^2+1)T_{i\omega}^{\ell}(\tau) \tag{IV.7}$$

Introducing the function

$$u_{i\omega}^{\ell}(\tau) = \cosh\tau T_{i\omega}^{\ell}(\tau), \tag{IV.8}$$

we can write Eq.(IV.7) in the Schrodinger-form

$$[-\frac{d^2}{d\tau^2} - \frac{\ell(\ell+1)}{\cosh^2\tau}]u_{i\omega}^{\ell}(\tau) = \omega^2 u_{i\omega}^{\ell}(\tau)$$

In this section, we are confining our discussion only to the lowest mode ($\ell = 0$). The higher mode solutions ($\ell = 0$) will be discussed in section IV.(C).

The lowest mode solution is given by

$$u_{i\omega}^{0}(\tau) = \frac{e^{i\omega\tau}}{\sqrt{4\pi}} \tag{IV.10}$$

where $1/\sqrt{4\pi}$ factor has been taken to normalise the function.

$$\int_0^\infty u_{i\omega}^{0}(\tau)u_{i\omega'}^{0}(\tau)d\tau = \delta(\omega-\omega'). \tag{IV.11}$$

$$\text{Therefore,} \quad H^{0m}_{i\omega}(\tau,\theta,\phi) = \frac{1}{\sqrt{4\pi}}\frac{e^{i\omega\tau}}{\cosh\tau}e^{im\phi} \qquad (IV.12)$$

An alternative solution for τ-equation (IV.7) is discussed in Appendix A.

(b) Radial equation

The radial function $\mathcal{R}(r)$ satisfies the following equation :

$$\begin{aligned} 0 =& \frac{1}{r^3}\left[1-\frac{R^2}{r^2}\right]\left[r^3(1-\frac{R_0^2}{r^2})\mathcal{R}_{,r}\right]_{,r} - M^2\left[1-\frac{R_0^2}{r^2}\right]\mathcal{R} \\ & - m_1^2\mathcal{R} + \frac{\omega^2+1}{r^2}\left[1-\frac{R_0^2}{r^2}\right]\mathcal{R} \end{aligned} \qquad (IV.13)$$

As discussed in the beginning of this section, we shall take $m_1 = 0$. Then for massless $(M = 0)$ scalar waves, Eq.(IV.13) turns out to be

$$\frac{1}{r}[r(r^2 - R_0^2)\mathcal{R}_{,r}]_{,r} + (\omega^2+1)\mathcal{R} = 0 \qquad (IV.14)$$

Now, we make a change of variable such that

$$\frac{dr}{dx} = \sqrt{r^2 - R_0^2} \qquad (IV.15)$$

$$\text{or,} \qquad \text{x} = \cosh^{-1}(\frac{r}{R_0}) \qquad (IV.16)$$

As $r \to R_0$, x $\to 0$. As $r \to +\infty$, x goes to both $\pm\infty$. Here. we are choosing the limit to be x $\to \infty$. Use of such a coordinate transformation has a natural significance which we shall discuss in Appendix C.

After this transformation, Eq.(IV.14) becomes

$$\mathcal{R}_{,x,x} + f(r)\mathcal{R}_{,x} + (\omega^2+1)\mathcal{R} = 0 \qquad (IV.17)$$

$$\text{where} \qquad f(r) = \frac{r}{\sqrt{r^2-R_0^2}} + \frac{\sqrt{r^2-R_0^2}}{r} \qquad (IV.18)$$

The radial equation is still not free of first derivative terms. Now, if we define

$$\mathcal{R}(r) = \frac{\Psi(r)}{\sqrt{r\sqrt{r^2-R_0^2}}} \qquad (IV.19)$$

then it brings Eq.(IV.17) to the form of a Schrodinger equation :

$$\Psi_{,x,x} + \left[\omega^2 + \frac{1}{\sinh^2 2x}\right]\Psi = 0 \qquad (IV.20)$$

with an effective potential

$$V_{\rm eff} = -\frac{1}{\sinh^2 2x} \qquad (IV.21)$$

This Schrodinger equation has a very simple effective potential whose behaviour can be readily visualised. Qualitative features of the wave behaviour can also be easily discussed. To solve this equation, let us introduce a new variable

$$y = -\sinh^2 2x \qquad (IV.22)$$

so that Eq.(IV.20) becomes

$$y(1-y)\Psi_{,y,y} + (\frac{1}{2} - y)\Psi_{,y} + \left(\frac{1}{16y} - \frac{\omega^2}{16}\right)\Psi = 0. \qquad (IV.23)$$

Then we define a new function

$$W = y^{-1/4}\Psi \qquad (IV.24)$$

which now satisfies the following equation

$$y(1-y)W_{,y,y} + [1 - \frac{3}{2}y]W_{,y} - \frac{\omega^2+1}{16}W = 0. \qquad (IV.25)$$

This is in the form of a hypergeometric equation. Therefore. the analytical solutions of this equation near y = 0 can be found in terms of hypergeometric series. These are

$$W_1 = F(a,b;1;y) \qquad (IV.26)$$

$$\text{and} \quad W_2 = \ln y F(a,b;1;y) + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(n!)^2} y^n S(n) \qquad (IV.27)$$

for | y |< 1, where

$$a = \frac{1}{4}(1 + iw),$$

$$b = \frac{1}{4}(1 - iw),$$

$$F(a,b;1;y) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(n!)^2} y^n,$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

$$S(n) = \psi(a+n) - \psi(a) + \psi(b+n) - \psi(b) - 2\psi(n+1) + 2\psi(1),$$

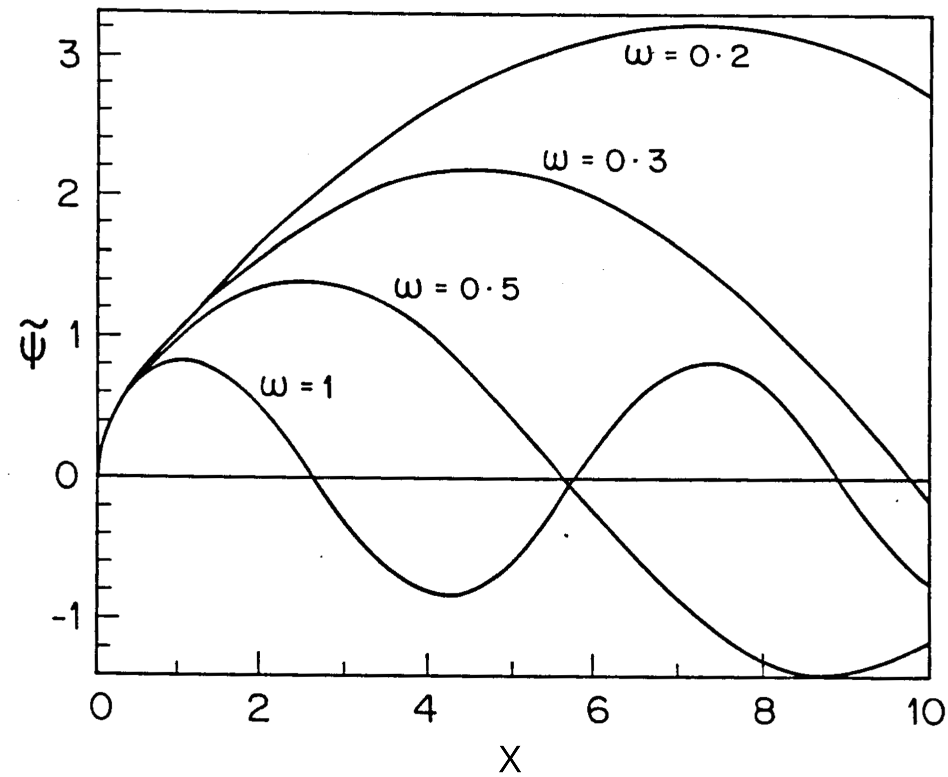

Fig.10 The solution $\Psi_1(x)$ for different frequencies w

where ψ is the logarithmic derivative of the Gamma function.

Using Eq.(IV.24), we can now get the solutions of the Schrodinger equation (IV.20) to be

$$\Psi_1 = y^{1/4} F(a, b; 1; y) \qquad (IV.28)$$

$$\Psi_2 = y^{1/4} \left[\ln y F(a, b; 1; y) + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} y^n S(n) \right] \qquad (IV.29)$$

Both these solutions go to zero, as $y \to 0$.

.We have also solved the Eq.(IV.20) numerically and plotted the solutions in figure 10 for different frequencies w.

We observe that starting from $x = 0$, the solution rises very rapidly to a maximum value and then starts oscillating like a cosine wave. As w increases, the influence of the spacetime on the waves reduces and the solution starts oscillating very close to the bubble wall.

However, if we look at the corresponding solutions for R-equation by using Eq.(IV.19)

$$\mathcal{R}_1 = \frac{1+i}{R_0} F(a,b;1;y) \qquad (IV.30)$$

$$\mathcal{R}_2 = \frac{1+i}{R_0}\left[\ln y\, F(a,b;1;y) + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} y^n S(n)\right] \qquad (IV.31)$$

We observe that at r $= R_0$,

$$\mathcal{R}_1 = \frac{1+i}{R_0} \; . \qquad \text{whereas} \quad \mathcal{R}_2 \to -\infty \qquad (IV.32)$$

Therefore, as far as the Schrodinger equation is concerned, the second solution behaves properly in that coordinate system. But when we consider the actual radial equation, the corresponding solution blows up at $r = R_0$. So, we are discarding the second solution throughout our further calculations.

From Eq.(IV.20), we can readily obtain the asymptotic behaviour of its solution as $x \to \infty$

$$\Psi(x \to +\infty) = Ae^{-i\omega x} + Be^{+i\omega x} \qquad (IV.33)$$

where A and B are arbitrary constants. Then, using Eq.(IV.19), we get

$$\mathcal{R}(r \to +\infty) = \frac{Ae^{-i\omega x} + Be^{+i\omega x}}{r}, \qquad (IV.34)$$

since as $r \to +\infty$, $\sqrt{r\sqrt{r^2 - R_0^2}} \to r$.

We shall now apply these considerations to the wave scattering by the bubble.

IV.(B) Scattering and Bound States

The total scalar wave solution in its lowest mode can now be written in its asymptotic limit to be

$$\Phi(m_1, \ell=0, r \to +\infty) = \frac{e^{im\phi}}{\sqrt{4\pi}} \frac{e^{i\omega\tau}}{\mathrm{r}\cosh\tau}(Ae^{-i\omega x} + Be^{+i\omega x}) \qquad (IV.35)$$

The factor $(\mathrm{r}\cosh\tau)$ in the denominator ensures that the total flux of energy passing through a unit solid angle $d\Omega$ does not depend on r or τ for very large r.

What is the relation between A and B ? The answer follows immediately, if we just consider the behaviour of the differential equation (IV.17). The hypergeometric series of Eq.(IV.30) is always real, since a and b are complex conjugates of each other. Now, if we use initial condition (IV.32) in Eq.(IV.17) and study the evolution of $\mathcal{R}$, we shall see that the real and imaginary parts of this equation will evolve independently. However, at any point, both these parts will be equal. Considering this fact and matching the solution with (IV.34) in the asymptotic limit, one can show by a very simple calculation that this is a case which corresponds to $\mid A \mid = \mid B \mid$.

The actual expressions for 'A' and 'B' can be obtained by analytically extending the solution (IV.28) to infinitely large negative values of the argument

$$y = -\sinh^2 2x \to -2^{-2}e^{4x}. \qquad (IV.36)$$

Then the solution is

$$\begin{aligned}\Psi(x \to +\infty) &= 2^{-1/2}e^{x}\left[\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(1-a)}2^{2a}e^{-4ax} + \frac{\Gamma(a-b)}{\Gamma(a)\Gamma(1-b)}2^{2b}e^{-4bx}\right] \\ &= \frac{\Gamma(-i\frac{\omega}{2})e^{i\frac{\omega}{2}\ln 2}}{\Gamma(\frac{1}{4}-i\frac{\omega}{4})\Gamma(\frac{3}{4}-i\frac{\omega}{4})}e^{-i\omega x} + \frac{\Gamma(+i\frac{\omega}{2})e^{-i\frac{\omega}{2}\ln 2}}{\Gamma(\frac{1}{4}+i\frac{\omega}{4})\Gamma(\frac{3}{4}+i\frac{\omega}{4})}e^{+i\omega x}.\end{aligned} \qquad (IV.37)$$

Matching with Eq.(IV.33), we get expressions for A and B. Since $\Gamma(\bar{z}) = \bar{\Gamma}(z)$, we can easily see that A and B are complex conjugates of each other and, therefore, $\mid A \mid = \mid B \mid$.

From the foregoing discussion, we see that only one of the two independent solutions is acceptable. This solution is well-behaved at infinity and consists of incoming and reflected wave components with equal amplitudes. Further, this solution goes to zero at $r = R_0$. Since the other solution is not well-behaved at $r = R_0$, there is no scope for superposition of the two solutions, thereby obtaining other boundary conditions, e.g., standing waves that do not go to zero at $r = R_0$.

On the other hand, the boundary conditions that have naturally arisen fit in well with the notion of a bubble surface enclosing a region r $< R_0$ that does not correspond to points in physical space. One expects the incoming wave to be totally reflected from the bubble surface. This phenomenon is, in fact, happening here. We may also note here that by a similar argument, one can rule out quasinormal modes of the bubble, since waves purely incoming at $r = R_0$ and purely outgoing at $r \to \infty$ cannot be obtained. This indicates that the bubble surface acts as a perfectly reflecting rigid barrier.

To investigate the bound states of this problem, we have to consider imaginary frequencies. Let us replace $i\omega \to$ w,. Then, for T-equation (IV.9), a discrete set of square-integrable wave functions can be obtained as bound states. These have been constructed in detail by Gerlach(1983).

To obtain bound states in the radial equation(IV.20), we see that the parameters a and b in solution (IV.28) have now become real. Then, in the asymptotic expressions (IV.37), the first term behaves as $e^{-\omega_n x}$ and the second as $e^{+\omega_n x}$. Bound states are possible, only if the coefficient of $e^{+\omega_n x}$ in the second term vanishes. However, all Γ functions in this coefficient have a positive real argument. Therefore, no Γ function in the denominator can ever blow up and make the factor vanish. Consequently, no bound state is possible.

Nevertheless, we should point out here that if one performs the following integration

$$\int_0^\infty \sqrt{E - V_{\rm eff}}\, dx$$

for E = 0 in this case, one obtains

$$\int_0^\infty \frac{1}{\sinh 2x}\, dx = \frac{1}{2} \ln \tanh x \Big|_0^{+\infty} = +\infty.$$

Following Merzbacher (1970), this means the existence of an infinite number of bound states. However, our explicit calculation has shown that there is no bound state at all. This apparent contradiction is due to our discard of solution (IV.29), though it was behaving well throughout the range of variable x in Schrodinger equation (IV.20). Had we considered both solutions, we would have obtained an infinite number of bound states. But those are not realistic as far as our problem is concerned.

Now, since the same w appears in both radial and r equations, the nonexistence of bound states also confirms that modes exponentially growing with τ do not exist. This shows the mode stability of the bubble spacetime against scalar perturbations. Further. since the scattering modes form a complete set, the bubble spacetime is stable with respect to any arbitrary scalar perturbations.

IV.(C) Higher Mode ($\ell > 0$) Solutions

As we have seen in section IV.(A), the lowest mode ($\ell = 0$) solution given in Eq.(IV.10) is 6-function normalised. Now, to study higher mode solution. following Gerlach(1983). we can introduce the raising and lowering operators by factorization method in Eq.(IV.9).

$$L^{\ell}_{\pm} = \ell \tanh \tau \mp d\tau \qquad (IV.38)$$

Then one can write $u^{\ell}_{i\omega}(\tau)$ as an eigenfunction of $L^{\ell}_{+}L^{\ell}_{-}$, with the eigenvalue $(\omega^2 + \ell^2)$.
Now the general eigenfunction can be written in its normalised form to be

$$u^{\ell}_{i\omega}(\tau) = \left[(\omega^2+\ell^2)\cdots(\omega^2+1^2)\right]^{-1/2} L^{\ell}_{+}\cdots L^{1}_{+}\frac{e^{i\omega\tau}}{\sqrt{4\pi}} \qquad (IV.39)$$

$$\text{Also,} \quad u^{\ell+1}_{i\omega}(\tau) = \left[\omega^2+(\ell+1)^2\right]^{-1/2} L^{\ell+1}_{+} u^{\ell}_{i\omega}(\tau). \qquad (IV.40)$$

For $\ell = 1$, we obtain from Eq.(IV.39),

$$\begin{aligned} u^{1}_{i\omega}(\tau) =& (\omega^2+1)^{-1/2}(\tanh\tau - \frac{d}{d\tau})\frac{e^{i\omega\tau}}{\sqrt{4\pi}} \\ =& \mathcal{A}_1(\tau)e^{-i\Theta_1(r)}\frac{e^{i\omega\tau}}{\sqrt{4\pi}} \end{aligned} \qquad (IV.41)$$

$$\text{where} \qquad \mathcal{A}_1(\tau) = \left(\frac{w^2 + \tanh^2 r}{\omega^2 + 1}\right)^{1/2} \qquad (IV.42)$$

$$\text{and} \qquad \Theta_1(r) = \arctan \frac{w}{\tanh r} \qquad (IV.43)$$

As $\tau \to \infty$, $\mathcal{A}_1(\tau) \to 1$, $\Theta_1(r) \to \tan^{-1} w$. One can continue this process and see that any higher mode solution can always be written in the form

$$u^{\ell}_{i\omega}(\tau) = \mathcal{A}_\ell(\tau) e^{-i\Theta_\ell(\tau)} \frac{e^{i\omega\tau}}{\sqrt{4\pi}}. \qquad (IV.44)$$

Therefore, any higher mode solution is nothing but a phase-modulated wave function of the lowest mode solution. However, this is a transient case of phase modulation, since, in every case, Θ_ℓ very rapidly approaches a constant value, as τ increases. The amplitude of this modulating wave is also time-dependent. But as r increases, $\mathcal{A}_\ell(\tau)$ also approaches the value 1 very rapidly. So, for a sufficiently large value of r, any higher mode solution will look like

$$u^{\ell}_{i\omega}(\tau) = \frac{e^{i(\omega\tau - \Theta_c)}}{\sqrt{4\pi}} \qquad (IV.45)$$

where Θ_c is a constant.

For the sake of completeness, we are writing below the explicit expressions for $\mathcal{A}_\ell$ and Θ_ℓ for a few other higher mode solutions.

$$< \ell = 2 >$$

$$\mathcal{A}_2^2 = \frac{w^4 + (3\tanh^2 r + 2)w^2 + (9\tanh^4 \tau - 6\tanh^2 \tau + 1)}{\omega^4 + 5w^2 + 4}$$

$$\tan\Theta_2 = \frac{3\omega \tanh r}{3\tanh^2 r - (1 + \omega^2)}.$$

$$< \ell = 3 >$$

$$\mathcal{A}_3^2 \ :$$

$$\text{Numerator} = \omega^6 + \omega^4(8 + 6\tanh^2 r) + w^2(45\tanh^4 \tau - 12\tanh^2 \tau + 16)$$
$$+ (225\tanh^6 r - 270\tanh^4 r + 81\tanh^2 r).$$

$$\text{Denominator} = \omega^6 + 14\omega^4 + 49w^2 + 36.$$

$$\tan\Theta_\ell = \frac{15\omega\tanh^2 \tau - 4\omega - w^3}{15\tanh^3 r - (6w^2 + 9)\tanh^2 \tau}$$

...etc.

IV.(D) Concluding Remarks

In the previous sections, we have developed the mathematical formalism for and studied the behaviour of scalar field in the Witten Bubble spacetime. We have written the eigenfunctions of the temporal equation as hyperbolic harmonics which manifest wave-behaviour in all of its modes. By choosing the null coordinate system, we could transform the radial equation into a very simple Schrodinger form. Studying the scattering problem, we have observed that our results are consistent with the concept of bubble as a perfectly reflecting wall. At large enough distance, we could get both incoming and outgoing waves with the same amplitude, thus giving the value of the reflection to be unity. On the other hand, near the bubble, the wave behaviour gets distorted. The higher the frequency, the lower is the distortion produced by the spacetime. A high frequency wave starts manifesting its wave behaviour very near to the bubble wall.

A study of bound states confirms the stability of the spacetime against arbitrary scalar perturbations. For a complete stability analysis of such a spacetime, the study of electromagnetic and gravitational perturbations is also necessary. This study may be able to project a clearer concept of some inherent aspects of the Witten bubble and lead to further studies related to such a spacetime.