26. GRAVITATIONAL RADIATION FROM INSPIRALLING COMPACT BINARIES

MOTION, GENERATION and RADIATION REACTION

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In 1978 when I met Ajit Kembhavi he was excited about someone called Vishveshwara who had come back from USA to the Raman Research Institute in Bangalore and was the head of a relativity group consisting of D. M. Chitre and N. D. Haridass. Ajit had worked on a project with him on compact objects and Sanjeev Dhurandhar was soon to also join him as a post doc. I was then a student with Arvind Kumar at the Bombay university working on problems of quantum field theories (QFT) in curved spacetimes(CST). The first relativity meeting we went to was the Einstein Centenary symposium at Physical Research Laboratory, Ahmedabad. Though I have many wonderful memories of the symposium the most memorable one was Vishu's lecture entitled 'Black Holes for Bedtime' [1]. To me it was a magical experience; an exotic cocktail of science, art, humour and caricature. Equations of the kind I was struggling with in my thesis were not necessarily abstract and unspeakable. They could as well be translated in the best literary tradition.

In the Einstein meeting all the well-known speakers like Vishu seemed to be inaccesible stars and hence I was elated when he accepted me as a postdoc at RRI in 1980. QFT in CST was then past its peak and Vishu encouraged me to explore problems in classical general relativity with possible astrophysical implications. Over the years we worked on problems related to existence and stability of ultracompact objects, accretion in presense of magnetic fields, exact solutions, separability properties of the Dirac equation, black holes in higher dimensions and Gauss Bonnet theories, and Frenet Serret methods for black holes and gyrosopic precession.

It has always been a pleasure working with Vishu. There is no pressure, no generation gap, a natural possibility to grow and contribute your best, an easy personal rapport, a refreshing sense of humour, an unassuming erudition and most importantly a warm and wonderful human being.

Over the years we have tread many trails together: research, teaching, editing, reading, movies, schools, organisation, science education, friendship, discussions, dreams, beer! There is a trail we want to intensely explore: To write a book together. I hope that we can do it in the coming years.

1. Introduction

The Binary pulsars 1913+16 and 1534+12 establish the reality of gravitational radiation [2]. They also provide proof of the validity of Einstein's general relativity in the strong field regime [3]. More importantly they are prototypes of inspiralling compact binaries which are strong sources of gravitational waves for ground based laser interferometric detectors like LIGO and VIRGO

With an orbital period of eight hours, the frequency of gravitational waves from the binary pulsar 1913+16 today is very low: (10^{-4} Hz). However in about three hundred million years the two stars will inspiral and the gravitational waves will sweep upward in frequency to about 10 Hz. In the following fifteen minutes before the neutron stars collide and coalesce the frequency will rise to about 1000 Hz with increasing amplitude producing a characteristic chirp waveform containing about 16000 cycles. Though the gravitational wave signal is extremely weak and buried deep in the detector noise the large number of precisely predictable cycles in the detector bandwidth brings the characterstic signal strength to the realm of the measurable. This enables one to use the technique of matched filtering initially for detection and later for estimation of parameters of the inspiralling binary [4]. The information content in these events is of excellent quality. If they are detected with a suitably high signal to noise ratio they should allow one to do astronomy. For instance, it could (i) provide precise measurements of the masses of the objects, possibly of their spins and probably, in the case of neutron stars, of their radii; (ii) allow one to measure cosmological distance directly and provide a cleaner determination of H_0 and q_0; (iii) test nonlinear structure of radiative gravitation; (iv) perform new tests of the existence of a scalar component to gravitation (v) probe black hole physics [5]. Estimates of the rate of such coalescence events are about a few per year upto 200 Mpc. Advanced LIGO which would look upto cosmological distances [5] would get to numbers of hundreds per year.

The phenomenal success of the high-precision radio wave observation of the binary pulsar makes crucial use of an accurate relativistic 'Pulsar timing formula' [6, 7].

$$\phi_n^{\mathrm{PSR}} = F[t_n; p_i] \,, \tag{1}$$

linking the rotational phase of the spinning pulsar (stroboscopically observed when $\phi_n^{\rm PSR} = 2\pi n$ with $n \in \mathcal{N}$) to the time of arrival t_n on Earth of an electromagnetic pulse, and to some parameters p_i. Similarly precise gravitational-wave observation of inspiralling compact binaries would require an equivalent accurate 'Phasing formula' i.e. an accurate mathematical model of the continuous evolution of the gravitational wave phase

$$\phi^{\rm GW} = 2\Phi = F[t; p_i]\,, \tag{2}$$

involving a set of parameters $\{p_i\}$ carrying information about the emitting binary system (such as the two masses m_1 and m_2). Since the equations of motion is not yet available at such higher orders the conventional approach heuristically relies on a standard energy-balance argument. From this it follows that the time evolution of the orbital phase Φ is determined by two functions: an energy function $E(v)$, and a flux function $F(v)$. The argument v is defined by $v = (\pi\, m\, f^{\rm GW})^{1/3}$, which can be rewritten in terms of the instantaneous *orbital* angular frequency Ω, $v \equiv (m\Omega)^{1/3} \equiv x^{1/2}$ ($m \equiv m_1 + m_2$ denotes the total mass of the binary). The (dimensionless) energy function E is defined by

$$E_{\rm tot} = m(1+E) \tag{3}$$

where $E_{\rm tot}$ denotes the total relativistic energy (Bondi mass) of the binary system. The flux function $F(v)$ denotes the gravitational luminosity of the system. The three quantities v, E and F are invariantly defined (as global quantities in the instantaneous center of mass frame), so that the two functions $E(v)$, $F(v)$ are coordinate-independent constructs. Denoting the symmetric mass ratio by $\eta \equiv m_1 m_2/(m_1+m_2)^2$, the energy balance equation $dE_{\rm tot}/dt = -F$ gives the following parametric representation of the phasing formula Eq. (2) (for the orbital phase)

$$t(v) = t_c + m\int_v^{v_{\rm lso}} dv\,\frac{E'(v)}{F(v)}\,, \tag{4}$$

$$\Phi(v) = \Phi_c + \int_v^{v_{\rm lso}} dv v^3\,\frac{E'(v)}{F(v)}\,, \tag{5}$$

where t_c and Φ_c are integration constants and $v_{\rm lso}$ is the velocity corresponding to the 'last stable orbit'. Note that $E'(v) < 0$, $F(v) > 0$ so that both t and Φ increase with v.

The accurate mathematical modelling of gravitational wave signals from inspiralling compact binaries requires solutions to two different but related problems referred to respectively as the "wave generation problem" and the "radiation reaction problem" [8], adequate for treating compact objects. The wave generation problem deals with the computation of the gravitational waveforms generated by the binary when the orbital phase and frequency of the binary take some given values ϕ and ω taking into account propagation and nonlinear effects. This problem involves computing the (tensorial) amplitude of each harmonic of the wave corresponding to frequencies which are multiples of the orbital frequency, with the predominant harmonic being at twice the orbital frequency. The radiation reaction problem consists of determining the evolution of the orbital phase $\phi(t)$ itself as a function of time, from which one deduces the orbital frequency $\omega(t) = d\phi(t)/dt$. The actual time variation of $\phi(t)$ is nonlinear because the orbit evolves under the effects of gravitational radiation reaction forces. In principle it should be determined from the knowledge of the radiation reaction forces acting locally on the orbit. However these forces are at present not known with sufficient accuracy, so in practice the phase evolution is determined by equating post-Newtonian energy flux in the waves or energy loss (averaged over one orbit) and the decrease of the correspondingly accurate binding energy of the binary. In order not to suffer a very severe reduction in signal-to-noise ratio, one will have to monitor the phase evolution with an accuracy of one tenth of a cycle over the tens of thousands of cycles during the entire passage through the frequency bandwidth of the detector. Consequently, the radiation-reaction part of the problem which determines the time evolution of the phase of the gravitational wave signal needs more crucial attention for successful detection. Determination of the 'Energy function' on the other hand requires the solution to the problem of 'motion'. The problems of motion, generation and radiation reaction are the constitutive elements of this analysis and the algebraic complexity introduced by nonlinearity makes it mandatory to make as many independent checks and counter checks as is possible.

None of the above problems can be solved exactly. They are treated by a combination of approximation methods like Post-Newtonian approximation, Post-Minkowskian approximation and Perturbations about a Curved Background. Since the details of the last approach are reviewed in Sasaki's article in this volume we only discuss the first two here and list the main features of these two schemes in the following section:

2. Post-Newtonian versus Post-Minkowskian Approximation

The two main approximation schemes that feature in the analytical studies of inspiralling binaries are:

1. Post-Newtonian Approximation (PNA): It is based on the assumption of an everywhere weak gravitational field together with that of slow motions. It is an expansion in $\beta \sim v/c \sim L/\lambda \sim (L/c)/P$ where v, L, λ and P are the characteristic velocity, size, wavelength and period of the

system respectively. It uses newtonian concepts like absolute space with an Euclidean metric and absolute time. It uses newtonian techniques and in this viewpoint Einstein theory provides small numerical corrections to Newtonian theory. The equations in this scheme are a hierarchy of Poisson equations which are solved by instantaneous potentials. In the PNA one looks for solutions of the field equations which are formal expansions in $1/c$ and hence it is also called the slow motion expansion. At higher orders PNA lead to to *divergent* integrals since they are based on instantaneous potentials – such potentials lead to divergent integrals at some order since they correspond to expansion in powers of r/c which grow like positive powers of r causing *infrared divergent* integrals. However, this may be avoided by assuming its validity in the near-zone only. This implies that the source is well-within the near-zone where retardation effects are small and also that time derivatives of the field are smaller than space-derivatives. More seriously, the PNA breaks down at 4PN since hereditary effects arise at this level and instantaneous potentials can no longer suffice to approximate the situation. A word about nomenclature: Each power of $(v/c)^2$ corresponds to one post-Newtonian order (1PN) and thus 4PN above refers to corrections including $(v/c)^8$. All the corrections(orders) are relative to the lowest order. This, 1PN in the equations of motion refers to terms including $1/c^2$ while 1PN accurate energy flux refers to terms including $1/c^{12}$.
2. Post-Minkowskian Approximation (PMA): It is based on the assumption of the weakness of the gravitational field and hence an expansion in $\gamma_i = GM/c^2R$ where M is the characteristic mass and R the characteristic size of the compact object. It makes crucial use of the conceptual framework of Minkowski geometry and its causality properties. The equations in this scheme reduce to a hierarchy of wave equations on Minkowski background which are solved by retarded potentials. In the PMA one looks for solutions to Einstein's equations which are formal expansions in powers of G. It is also called nonlinearity, weak field, or fast motion approximation. The basic complication is in the nonlinear iteration. The PMA shows no signs of internal inconsistency. It applies all over the weak field zone. In vacuum the PM expansion is reliable i.e., any solution of the perturbation equations comes from the Taylor expansion, when $G \to 0$, of a family of exact solutions [9].

3. Problem of Motion

It may be worth mentioning that unlike linear EM, non-linear GR has the feature that its field equations contain the equations of motion. In their original form the Einstein equations(EE) $G_{\mu\nu} = 8\pi G T_{\mu\nu}/c^4$ are mathematically not very convenient since they do not form a partial differential system of well defined type. To alleviate this difficulty one 'relaxes' the field equations by use of special coordinate conditions. One of the most common choice is the harmonic (De Donder or Lorentz) coordinate condition

$$\mathcal{G}^{\mu\nu}_{,\nu} = 0,$$

where the 'Gothic' contravariant metric is defined by

$$\mathcal{G}^{\mu\nu} = \sqrt{g} g^{\mu\nu} \quad ; \quad g = -\det(g_{\mu\nu}).$$

The EE then become

$$\mathcal{G}^{\alpha\beta}\mathcal{G}^{\mu\nu}_{,\alpha\beta} + Q^{\mu\nu}(\mathcal{G}, \partial\mathcal{G}) = \frac{16\pi G}{c^4} g T^{\mu\nu}.$$

where $Q^{\mu\nu}$ is a complicated quadratic form in the first derivatives of $\mathcal{G}$. In terms of deviation from the flat metric $h^{\mu\nu} = \mathcal{G}^{\mu\nu} - \eta^{\mu\nu}$ the above equations become

$$\partial_\nu h^{\mu\nu} = 0,$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}(h),$$

where $\Lambda^{\mu\nu}$ includes all the nonlinearities of the field equations and is at least quadratic in h and its first and second derivatives: $\Lambda^{\mu\nu} = N^{\mu\nu}(h,h) + O(h^3)$. The harmonic coordinates are well adapted to Lorentz-covariant formulations. Here all the ten metric coefficients satisfy hyperbolic equations.

Within the PMA one then assumes a formal asymptotic expansion

$$\mathcal{G}^{\alpha\beta} \equiv \sqrt{g}\; g^{\alpha\beta} = \eta^{\alpha\beta} \;+\; \gamma_i h_1^{\alpha\beta} \;+ \gamma_i^2 h_2^{\alpha\beta} + \cdots \gamma_i^n \; h_n^{\alpha\beta} + \cdots \tag{6}$$

One is thus led to a formal hierarchy of inhomogeneous wave equations for the $h_n^{\mu\nu}$ of the form:

$$\Box_f \; h_n^{\alpha\beta} \;=\; T^{\alpha\beta} + N_n^{\alpha\beta} \;(h_1, h_2 .... h_{n-1}) \equiv S_n^{\alpha\beta}. \tag{7}$$

The equations must be supplemented by a prescription to pick the physically correct solution. The Fock conditions consisting of appropriate fall-off together with the 'no incoming-radiation' condition at past infinity implies that the solution is determined by the flat spacetime retarded Green function as:

$$h_n^{\mu\nu} = \int d^4x G^{(f)}_{\rm ret}(x - x') S_n^{\mu\nu}(x'). \tag{8}$$

The N-body problem as in newtonian gravity is decomposed into an external problem and an internal problem. The former refers to the problem of defining and determining the motion of the center of mass and the latter to motion of each body around the center of mass. The effacement of internal structure in the external problem and effacement of external structure on the internal problem involves subtle issues in the problem of motion and we cannot do better than refer the reader to the beautiful review by Damour [10].

The topic of EOM for compact binary systems received careful scrutiny in the years following the discovery of the binary pulsar. There have been three different approaches to the complete kinematical description of a two body system upto the level where radiation damping first occurs (2.5PN). Damour's method explicitly discusses the external motion of two condensed bodies without ambiguities. The method employs the best techniques to treat various subproblems. (a) A PMA to obtain the gravitational field outside the bodies incorporating a natural 'no incoming-radiation condition' whose validity is not restricted to only the near-zone. (b) A matched asymptotic expansion scheme to prove effacement and uniquely determine the gravitational field exterior to the condensed bodies. (c) An Einstein Infeld Hoffmann Kerr(EIHK) type approach to compute equations of orbital motion from knowledge of the external field only. The n^{th} approximate EOM is obtained from the integrability condition on the $(n+1)^{th}$ approximated vacuum field equations. (d) Use of Riesz's analytic continuation technique to evaluate surface integrals. The final EOM at 2.5PN level are expressed only in terms of instantaneous positions, velocities and spins in a given harmonic coordinate system and given explicitly in Ref.[10]. The two mass parameters in these formulas are the Schwarzschild masses of the two condensed bodies.

The conservative part of the EOM upto 2PN (excluding the secular 2.5PN terms) are not deducible from an conventional Lagrangian (function of positions and velocities) in harmonic coordinates, but only from a generalised Lagrangian (depending on accelerations). This is consistent with the result in classical field theory that in Lorentz-covariant field theories there exists no (ordinary) Lagrangian description at $O(c^{-4})$ [11]. This Lagrangian is invariant under the Poincare group and thus allows one to construct ten Noetherian quantities that would be conserved during the motion. These include the 'Energy', 'Angular Momentum', 'Center of Mass' and thus a solution to the problem of 'motion' provides the first element $E(v)$ that enters into the phasing formula. For inspiralling compact binaries the 2.5PN accurate binding energy reads

$$E(v) = -\frac{c^2}{2} m\eta x \left[1 - \frac{1}{12}(9+\eta)\,x - \frac{1}{8}\left(27 - 19\eta + \frac{\eta^2}{3}\right) x^2 + \mathcal{O}(x^3)\right], \tag{9}$$

where m, η, v, and x are defined as before. The EOM to 2PN accuracy in this case is given by:

$$\mathbf{a} = -\frac{Gm}{r^3}\left[1-(3-\eta)\gamma+\left(6+\frac{41}{4}\eta+\eta^2\right)\gamma^2\right]\mathbf{x} \tag{10}$$

where $\gamma = Gm/c^2r$. The EOM for the general case is given in [10] and crucially used in the following studies of generation [12, 13] and radiation reaction [14].

Schafer's [15] approach on the other hand is based on the Hamiltonian approach to the interaction of spinless point particles with the gravitational wave field. The Hamiltonian formulation is best done in the Arnowitt-Deser-Misner (ADM) coordinates in which two metric coefficients satisfy hyperbolic equations (evolution) while the remaining eight are of elliptic type(constraints). It uses a different gauge that allows an elegant separation of conservative and damping effects. One recovers the damping force acting on the Hamiltonian subsystem of instantaneously interacting particles coming from its interaction with the dynamical degrees of freedom of the gravitational field. In this approach point masses are used as sources and regularisation uses Hadamard's 'partie finie' based on Laurent's series expansion regularisation.

The last approach due to Grischuk and Kopejkin [16] on the other hand is based on (a) PNA scheme (b) assumption that bodies are non-rotating 'spherically-symmetric' fluid balls. The symmetry is in the coordinate sense. The EOM of the center of mass of each body are obtained by integration of the local PN EOM. These are explicitly calculated retaining all higher derivatives that appear. One then reduces the higher derivatives by EOM and obtains the final results. Formally collecting the various relativistic corrections into a 'effective mass', one can have a PN proof of effacement of internal structure and provide a plausibility argument for validity of 'weak field formulas' for compact objects.

The fact that three independent methods: PM + EIHK, PN + Perfect fluid and PN + delta functions give formally identical equations of motion at 2PN order is a strong confirmation of the validity of the numerical coefficients in the EOM. This work provides the basis for the timing formula mentioned earlier. The damping terms can be considered as perturbation to a Lagrangian system which is multiperiodic – a radial period and a angular period corresponding to periastron precession – and leads to the observed secular acceleration effect in the binary pulsar. No balance argument is involved at any stage.

The work on 3PN generation crucially requires the EOM at 3PN accuracy and the situation is now under investigation at the 3PN level. The equations of motion are the geodesic equations in the space-time generated by the two particles. The gravitational field is computed using the standard post-Newtonian theory from the stress-energy tensor appropriate for point particles (i.e. involving delta functions). Because the metric coefficients are to be evaluated at the location of the particles, the geodesic equations must be computed using a process of regularization of the infinite self-field of point particles. Work is in progress to obtain the 3PN contributions by different techniques. These include the MPM method supplemented by Hadamard 'partie-finie' [17], the Epstein Wagoner Will Wiseman method [18] as also the Hamiltonian formalism [19]. As mentioned above upto 2.5PN three distinct computational techniques led to a unique EOM. Though, preliminary investigations have even raised questions about whether this sort of uniqueness will persist at 3PN [20], the work of Blanchet, Faye and Ponsot [17] indicate that unique results will obtain.

4. The Multipolar Post-Minkowskian Formalism

The use of multipole expansion methods in combination with the PMA scheme offers one of the most powerful techniques in gravitational radiation studies [21]. After decomposing the gravitational field generated by an isolated source in the exterior in terms of a formal post-Minkowskian series one decomposes each coefficient of the PM expansion in terms of its multipole moments. The use of a multipolar decomposition for the field simplifies the resolution of the field equations. Since the gravitational field is a second rank tensor, symmetric trace-free tensors are a very convenient representation to implement multipole decomposition rather than tensor spherical harmonics. They lead to a more transparent definition of multipole moments. Thus we have,

$$h_n^{\alpha\beta} = \sum_{\ell\geq 0} h_{nL}^{\alpha\beta}\, \hat{n}^L\,(\theta,\phi), \tag{11}$$

$$\hat{n}^L \equiv n^{<i_1 i_2 \cdots i_l>} \equiv STF_{i_1 i_2 \cdots i_l} n^{i_1 i_2 \cdots i_l}. \tag{12}$$

Choosing harmonic coordinates such that $\partial_\beta\, h_n^{\alpha\beta} = 0$ we look for MPM metrics which are stationary in the past and Minkowskian at spatial infinity then. This allows for a well-defined iteration at any order incorporating Fock's no incoming radiation condition. In combination with a PNA in the inner region this leads to a powerful method to deal with not only generation of gravitational waves by relativistic sources but also reaction of gravitational waves on the source.

The source-free scalar wave equation has as its most elementary retarded solution $\phi = F(t-r/c)/r$, where F is an arbitrary function of retarded time. From this one can construct the most general retarded solution by repeated spatial differentiation: $\phi_{i_1 i_2 \cdots i_l} \equiv \phi_L = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}\phi \equiv \partial_L\phi$. Similarly in the tensor case relevant here the most general solution for linearised gravity can always be written – modulo an infinitesimal gauge transformation that preserves the harmonic gauge condition – in terms of two arbitrary functions M_L and S_L which can be chosen to be symmetric and trace-free(STF) with respect to all the l indices L.

To proceed further we note that the source term for h_2 that involves h_1 are in the form of multipole expansions. Consequently one cannot use the usual retarded integral to solve the wave equation since the multipole expansion is valid only in D_e. To deal with this problem one considers a fictitious source constructed by multiplying the actual source by with r^B where B is a complex number. The retarded integral of this source admits after analytic continuation a Laurent expansion near $B \to 0$ whose finite part (i.e. coefficient of zeroth power of B in the expansion) is the solution we are looking for. This solution is also in the form of a multipole expansion; however it does not by itself satisfy the harmonic gauge condition. Thus we supplement our solution by a solution of the homogeneous wave equation to satisfy the harmonic gauge condition. This treatment can be extended to higher orders and thus we have

$$h_n^{\mu\nu} = p_n^{\mu\nu} + q_n^{\mu\nu}, \tag{13}$$

$$p_n^{\mu\nu} = FP_{B=0}\Box_R^{-1}(r^B \Lambda_n^{\mu\nu}), \tag{14}$$

$$\partial_\nu q_n^{\mu\nu} = -\partial_\nu p_n^{\mu\nu}. \tag{15}$$

The above solution is proved to be the most general and thus the general radiative field outside an isolated system depends on only two sets of time-dependent multipole moments. In the exterior near zone the field admits a PN expansion in terms of functions of the form $(\ln c)^p/c^k$ where $p, k \in N$.

5. Parametric Representation of Motion

At the newtonian level the motion of a binary system can be expressed in the parametric form:

$$\frac{2\pi}{P}t = u - e\sin u, \tag{16}$$

$$r = a(1 - e\cos u), \tag{17}$$

$$\phi = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan(\frac{1}{2}u)\right). \tag{18}$$

In the above P is the orbital period, e the ecentricity and a the semimajor axis of the orbit. All these parameters are functions of the the energy E and angular momentum J of the orbit. The parameter u is called the eccentric anomaly.

Damour and Deruelle [6] devised a very useful parametrisation structurally similar to the above and valid at 1PN level. This quasi-Keplerian representation uses, instead of an eccentricity e, three different eccentricities e_r, e_ϕ and e_t. The above construction has been generalised to the 2PN order by Damour, Schafer and Wex [22, 23]. This generalised quasi-Keplerian representation introduces in addition another parameter v and in this case we have:

$$
\begin{aligned}
2\pi\frac{t-t_0}{P} &= u - e_t \sin u + \frac{f_p}{c^4}\sin v + \frac{g_p}{c^4}(v-u), && (19)\\
r &= a(1-e_r\cos u), && (20)\\
2\pi\frac{\phi-\phi_0}{\Phi} &= v + \frac{f_\phi}{c^4}\sin(2v) + \frac{g_\phi}{c^4}\sin(3v), && (21)\\
v &= 2\arctan\left(\sqrt{\frac{1+e_\phi}{1-e_\phi}}\tan(\frac{1}{2}u)\right). && (22)
\end{aligned}
$$

All the parameters P, Φ, a, e_t, e_r, e_ϕ, f_p, g_p, f_ϕ, and g_ϕ are functions of the 2PN conserved E and J characterising the orbit. The above representation is one of the inputs to derive the timing formula to 2PN accuracy [24, 25]. It has also been used to discuss the evolution of the orbital elements to 2PN accuracy under 2PN radiation reaction [13] and obtain a useful form for the 2PN accurate polarisations in the case of quasi-elliptic orbits [26].

6. The Wave Generation Formalism

Einstein's far field quadrupole equation is the solution of the generation problem to the lowest order (hence referred to as Newtonian order) but applies only to objects held together by non-gravitational forces. Fock and Landau-Lifshiftz provided two very different methods to generalise to weakly self-gravitating systems. The above two approaches are the starting points for the two methods available today to calculate gravitational wave generation to higher orders: The Blanchet-Damour-Iyer(BDI) [27] approach and the Epstein-Wagoner-Thorne-Will-Wiseman (EWTWW) [28, 29, 30] approach. The relativistic corrections are called post-Newtonian corrections according to the PN order of the EOM of the source needed to reduce all accelerations with consistent accuracy. Thus the 1PN formalism retains all terms in the radiation field and reaction computed consistently using 1PN EOM. An interesting and different approach to the quadrupole formula is due to Haridass and Soni [31] who obtained the one graviton transition operator and thus the classical energy loss formula for gravitational radiation from Feynman graphs of helicity ± 2 theories of gravitation.

6.1. BDI APPROACH

Blanchet, Damour and Iyer build on a Fock type derivation using the double-expansion method of Bonnor [32]. This approach makes a clean separation of the near-zone and the wave zone effects. It is mathematically well defined, algorithmic and provides corrections to the quadrupolar formalism in the form of compact support integrals or more generally well defined analytically continued integrals. The BDI scheme has a modular structure: the final results are obtained by combining an 'external zone module' with a 'radiative zone module' and a 'near zone module'. For dealing with strongly self-gravitating material sources like neutron stars or black holes one needs to use a 'compact body module' together with an 'equation of motion module'. It correctly takes into account all the nonlinear effects.

It should be noted that, in generation problems, as one goes to higher orders of approximation two independent complications arise. Though algebraically involved in principle the first is simpler: contributions from higher multipoles. The second complication is not only algebraically tedious but technically more involved: contributions from higher nonlinearities e.g for 2PN generation cubic nonlinearities need to be handled.

The general approach to solve the generation problem may be broken up into the following steps:

1. Integrate the Einstein field equations in the vacuum exterior region D_e by means of a Multipolar Post-Minkowskian series. The exterior solution is parametrised by moments M_L and S_L called the algorithmic moments. The mass monopole and dipole moments as well as the current dipole moment are necessarily constant to satisfy the harmonic gauge condition.

2. In the far wave zone rewrite the solution in suitable coordinates to find the observable moments of the radiative field that a detector would measure. This involves going over from the harmonic coordinates to the radiative or Bondi coordinates to correct for the logarithmic deviation of the true light cones from the flat line cones in the wave zone D_w. In these coordinates we have,

$$h_{ij}^{\mathrm{TT}}(\mathbf{X},T) = \frac{4G}{c^2R}\mathcal{P}_{ijkm}(\mathbf{N})\sum_{\ell=2}^{\infty}\frac{1}{c^\ell \ell!}\Big\{N_{L-2}U_{kmL-2}(T-R/c) - \frac{2\ell}{(\ell+1)c}N_{aL-2}\varepsilon_{ab(k}V_{m)bL-2}(T-R/c)\Big\} + \mathcal{O}\left(\frac{1}{R^2}\right) \quad (23)$$

where U_L and V_L are the 'mass' and 'current' type radiative moments. In terms of these radiative moments the total power or luminosity is given by,

$$\mathcal{L} = \sum_{\ell=2}^{+\infty}\frac{G}{c^{2\ell+1}}\Big\{\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!}U_L^{(1)}U_L^{(1)}$$

$$+\frac{4\ell(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!c^2}V_L^{(1)}V_L^{(1)}\Big\} \,. \qquad (24)$$

The observable or radiative moments U_L and V_L are related to the algorithmic moments and we have:

$$U_L(T_R) = M_L^{(l)}(T_R) + \sum_{n\geq 2}\frac{G^{n-1}}{c^{3(n-1)+2k}}X_{nL}(T_R) \qquad (25)$$

$$\varepsilon_{ai_li_{l-1}}V_{aL-2}(T_R) = \varepsilon_{ai_li_{l-1}}S_{aL-2}^{(l-1)}(T_R) + \sum_{n\geq 2}\frac{G^{n-1}}{c^{3(n-1)+2k}}Y_{nL}(T_R) \qquad (26)$$

where k is a positive integer representing the number of contractions between indices on the moments in that particular term and functions X_{nL} and Y_{nL} represent some nonlinear and in general nonlocal functionals of n moments M_L and S_L whose general form reads:

$$X_{nL}(T),\ Y_{nL}(T) = \sum\int_{-\infty}^{T}dv_1\cdots\int_{-\infty}^{T}dv_n\mathcal{X}(T,v_1,\cdots,v_n)M_{L_1}^{(a_1)}(v_1)\cdots S_{L_n}^{(a_n)}(v_n). \qquad (27)$$

The kernel $\mathcal{X}$ made from Kronecker and Levi- Civita symbols, has a complicated index structure and depends on variables having dimensions of time. The structure of X_{nL} and Y_{nL} embody the fact that the gravitational field in higher approximations depends on the 'history' of the source and that propagation of radiation is not only along light cones but also inside them.
3. Finally one needs to relate the field in D_e to the inner field in the source. To this end one does two things: Re-expand the external post-Minkowskian field in a post-Newtonian expansion. Integrate the *non-vacuum* field equations in the near zone D_i by means of a post-newtonian expansion using as source variables $\sigma = (T^{00}+T^{ss})/c^2$, $\sigma_i = T^{0i}/c$ and $\sigma_{ij} = T^{ij}$. This choice simplifies the 1PN solution and hence the subsequent iterations. Starting with the source terms at the lowest order one solves for the gravitational field $h^{\mu\nu}$. This solution for $h^{\mu\nu}$ is then used in the relevant nonlinear terms to generate a more accurate source term at the next order. This in turn determines a h to higher accuracy. To 2PN accuracy the solutions are determined in terms of potentials V, V_i and W_{ij} which are retarded integrals associated with sources σ, σ_i and $\sigma_{ij}+(1/4\pi G)(\partial_iV\partial_jV-(1/2)\delta_{ij}\partial_kV\partial_kV)$ respectively.
4. One finally matches the two solutions in the exterior near zone $D_i\cap D_e$ to relate the algorithmic moments to the source properties. This is most

conveniently done by relating the 'exterior potentials' in terms of which the exterior solution is expressed to multipole expansions of the corresponding 'inner potentials'. This leads one to a result that the linear piece of the external potential $h_1^{\mu\nu}$ is – modulo a linear gauge transformation – given by a multipole expansion of the form

$$Gh_1^{\mu\nu}[M_L, S_L] = -\frac{4G}{c^4}\sum_{l\geq 0}\frac{(-)^l}{l}\partial_L\left[\frac{1}{r}\mathcal{F}_L^{\mu\nu}\left(t-\frac{r}{c}\right)\right] + \mathcal{O}(\varepsilon^7)$$

where the reducible moment is given by

$$\mathcal{F}_L^{\mu\nu} = FP_{B=0}\int d^3x|\mathbf{x}|^B\hat{x}_L\int_{-1}^{1}dz\delta_l(z)\left(\overline{|g|}\overline{T}^{\mu\nu} + \frac{c^4}{16\pi G}\overline{\Lambda}^{\mu\nu}(V,W)\right)$$

In the above equation the bar above any symbol is a reminder that the corresponding symbol is post-Newtonian expanded to the appropriate accuracy and not retained in its retarded form. In the final step the 'reducible' moment needs to be decomposed into its 'irreducible' parts and this technical problem is the same as discussed in the multipole analysis of linearised gravity [33]. For instance, one finally obtains for the mass-algorithmic moment to 2PN accuracy:

$$\begin{aligned}
M_L(t) &= \mathrm{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\Big\{\hat{x}_L\Big[\sigma + \frac{4}{c^4}(\sigma_{ii}U - \sigma P_{ii})\Big] \\
&+ \frac{|\mathbf{x}|^2\hat{x}_L}{2c^2(2\ell+3)}\partial_t^2\sigma - \frac{4(2\ell+1)\hat{x}_{iL}}{c^2(\ell+1)(2\ell+3)}\partial_t\left[\left(1+\frac{4U}{c^2}\right)\sigma_i\right. \\
&+ \left.\frac{1}{\pi G c^2}\left(\partial_k U[\partial_i U_k - \partial_k U_i] + \frac{3}{4}\partial_t U\partial_i U\right)\right] \\
&+ \frac{|\mathbf{x}|^4\hat{x}_L}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\sigma - \frac{2(2\ell+1)|\mathbf{x}|^2\hat{x}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)}\partial_t^3\sigma_i \\
&+ \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\hat{x}_{ijL}\partial_t^2\left[\sigma_{ij} + \frac{1}{4\pi G}\partial_i U\partial_j U\right] \\
&+ \frac{1}{\pi G c^4}\hat{x}_L\Big[-P_{ij}\partial_{ij}^2 U - 2U_i\partial_t\partial_i U + 2\partial_i U_j\partial_j U_i \\
&\qquad -\frac{3}{2}(\partial_t U)^2 - U\partial_t^2 U\Big]\Big\} + \mathcal{O}(\varepsilon^5)\ . \qquad (28)
\end{aligned}$$

After elimination of the mathematical intermediaries appearing in the formalism like the algorithmic moments, the basic structure of the final results of the BDI formalism is the following: The observable 'radiative moments' U_L and V_L giving the angular dependence of the asymptotic gravitational wave amplitude $h_{ij}^{TT}(T,R,\theta,\phi)$ are given in terms of the source-related potentials as a series of terms of increasing nonlinearity.

6.1.1. *A Sampler of Nonlinear effects*

The nonlinear nature of general relativity leads to interesting physical phenomena which we catalog next. The first such effect is the interaction between the time dependent mass quadrupole and the static mass monopole. This represents the back-scatter of linear waves by the space-time curvature generated by the mass energy and referred to as tails. Thus gravitational radiation not only propagates on the light cone but also inside it. Tails appear in the radiation field at 1.5PN order and contribute to the far zone flux. They appear also in the radiation reaction forces at 1.5PN order so that the 'balance equations' are correctly satisfied. In the formal structure of the theory the appearance of tails imply a dependence on the past history of the source and hence a non-locality in time. As mentioned earlier this signals the breakdown of the PNA at 4PN. It has important observational consequences in the dynamics of coalescing binaries.

The second such effect is the interaction of the quadrupole moment with itself [34]. This includes a non-local contribution which causes a permanent change in the wave amplitude before and after the burst as first pointed out by Christodoulu [35]. The physical interpretation of this effect as the re-radiation of gravitational waves by the stress-energy tensor of the linear waves was clarified by Thorne [36]. It appears at 2.5PN in the radiation field. The memory effect does not contribute to the energy loss and hence has poor observable consequences. In addition to this non-local effect the quadrupole-quadrupole interaction includes many instantaneous terms; these unlike the non-local term are transients.

The last such investigated effect involves the cubic interaction between the time varying mass quadrupole and two static mass monoples [37]. In addition to a second order scattering of the linear waves, it also includes the scattering of the tail of waves from the static mass monopole M. The latter called 'tails of tails' is of order 3PN in the radiation field and, though small, is still important for the detection of inspiralling compact binaries.

6.2. THE EPSTEIN-WAGONER FORMALISM

The Epstein and Wagoner (EW) [28] approach, also starts by rewriting the Einstein equations in a "relaxed" form. As in electromagnetism one can write down a *single* formal solution valid everywhere in spacetime based on the flat-spacetime retarded Green function. The retarded integral equation for $h^{\alpha\beta}$, can then be iterated in a slow-motion ($v/c < 1$), weak-field ($||h^{\alpha\beta}|| < 1$) approximation as shown by Thorne [29]. Unlike in the electromagnetic case, however, the non-linear field contributions make the integrand of this retarded integral non-compact. The EW formalism leads to integrals that are not well defined, or worse, are divergent. Though at the first few PN orders different arguments were given to ignore these issues they provide no justification that the divergences do not become fatal at higher orders. Consequently, the EW formalism did not appear to be a reliable route to discuss higher PN approximations. Recently, Will and Wiseman have critically examined the EW formalism and provided a solution to the problem of its divergences. The resolution involves taking literally the statement that the solution is a *retarded* integral, *i.e.* an integral over the *entire* past null cone of the field point. Unlike in the original EW treatment, only the part of the integral that extends over the intersection between the past null cone and the material source and the near zone is approximated by a slow-motion expansion involving spatial integrals of moments of the 'source'. Undefined and divergent integrals result from PN expansions if these spatial integrals are extended to infinity. Will and Wiseman restrict these integrals to the boundary of the near zone $\mathcal{R}$ chosen roughly to be a wavelength of the gravitational radiation. The slow motion expanded form is not used to evaluate the integral over the rest of the past null cone exterior to the near zone ("radiation zone"). A coordinate transformation is used to convert the integral into a convenient form for easy evaluation and it is manifestly convergent for reasonable past behavior of the source. All integrations are explicitly finite and convergent and all contributions from the near-zone spatial integrals that grow with $\mathcal{R}$ (and that would have diverged had $\mathcal{R} \to \infty$) are actually *cancelled* by corresponding terms from the radiation-zone integrals. The procedure, as expected, has no dependence on the artificially chosen boundary radius $\mathcal{R}$ of the near-zone. The new EW method proposed by Will and Wiseman can thus be carried to higher orders in a straightforward, albeit very tedious manner and the result is a manifestly finite, well-defined procedure for calculating gravitational radiation to high PN orders. Moreover, part of the tail terms at 3/2PN and 2PN order serve to guarantee that the outgoing radiation propagates along true null directions of the asymptotic curved spacetime, despite the use of flat spacetime wave equations in the solution.

6.3. SUMMARY OF RESULTS

The end result of the computations of the previous subsection are expressions for the radiative mass and current multipole moments characterising the source distribution. Once they are on hand one can proceed to compute the associated gravitational waveform. From the waveform, the far zone energy flux may be computed by time differentiation (this is why one needs the EOM) and integration over all directions. The energy flux can also be computed directly from the moments and this provides a simple check on the algebraic correctness of the long computations. The angular momentum flux can also be computed for non-circular orbits.

For nonspinning compact objects (mass monopoles) Wagoner and Will [38] and later Blanchet and Schafer [39] obtained the 1PN accurate energy flux and discussed the evolution of orbital period. The corresponding angular momentum flux and evolution of other orbital elements were studied by Junker and Schafer [40]. Wiseman discussed the linear momentum flux and the recoil effect in binaries [41]. Blanchet and Schafer discussed the tail effect in energy [42] while Rieth and Schafer extended it to angular momentum [43]. At 2PN the cubic nonlinearity needs to be handled and this was provided by Blanchet, Damour and Iyer [44] in the case of circular orbits. This was independently computed by Will and Wiseman [12] using their improved Epstein Wagoner formalism. They also provided the waveform and energy flux for general (non- circular) orbits. A summary of these results is presented in [45] and the associated 2PN accurate gravitational polarisations is available in ref.[46]. Recently, Gopakumar and Iyer [13] using the BDI approach obtained the waveform, energy flux and associated angular momentum flux and proved the equivalence to the Will Wiseman results. They also used the generalised quasi-Keplerian representation of Damour, Schafer and Wex to compute the evolution of the orbital elements to 2PN accuracy.

The extension of these results to 3PN accuracy is an algebraically heavy and conceptually involved exercise. The multipolar post-Minkowskian approach has been extended to compute the 3PN accurate mass quadrupole (source) moment, 2PN current quadrupole moment and 2PN mass octupole moment of a system of two point masses moving on a circular orbit [47, 48]. From the moments the total energy flux has been computed to 3.5PN order. The regularization of the equations of motion should be consistent with the computation of the multipole moments. It is shown that the arbitrary constants associated with the Hadamard partie finie drop out from the final result (providing one of the sensitive tests of the computation). The Hadamard regularization, based on the Hadamard partie finie, thus seem to provide a good method in this context. Furthermore there is agreement with the known test particle limit. Hopefully in the near future the EW formalism [18, 19] should provide a check on these results.

The extension of the above results to spinning bodies (current dipole) has also been given by Kidder, Will and Wiseman [49, 51] and Owen, Tagoshi and Ohashi [52]. The effects of rotationally induced and tidally induced quadrupole and higher moments on orbital evolution and gravitational wave generation [53, 54] have been also investigated. These are found to be negligible except in the final coalescence stage for neutron star binaries.

As a sample we quote below the 2PN accurate mass quadrupole for circular orbits [45, 46]:

$$\begin{aligned} I_{ij} &= \eta m STF_{ij}\left\{x_{ij}\left[1-\frac{\gamma}{42}(1+39\eta)-\frac{\gamma^2}{1512}(461+18395\eta+241\eta^2)\right]\right. \\ &\left.+\frac{r^2}{c^2}v_{ij}\left[\frac{11}{21}(1-3\eta)+\frac{\gamma}{378}(1607-1681\eta+229\eta^2)\right]\right\}. \end{aligned} \tag{29}$$

The general expression for non-circular orbits may be found in [13]. The corresponding 2PN accurate energy flux is given by:

$$\begin{aligned} \mathcal{L} &= \frac{32c^5}{5G}\eta^2x^5\left\{1-\left(\frac{1247}{336}+\frac{35}{12}\eta\right)x+4\pi x^{3/2}\right. \\ &+\left(-\frac{44711}{9072}+\frac{9271}{504}\eta+\frac{65}{18}\eta^2\right)x^2 \\ &\left.-\left(\frac{8191}{672}+\frac{535}{24}\eta\right)\pi x^{5/2}+\mathcal{O}(x^3)\right\}, \end{aligned} \tag{30}$$

with γ and x as defined earlier. The solution to the generation problem thus provides the second input for phasing once we make the assumption of energy balance. In terms of the adimensional time variable :

$$\Theta = \frac{c^3\eta}{5GM}(t_c - t)\ , \tag{31}$$

where t_c denotes the instant of coalescence, the orbital phase is given by

$$\begin{aligned} \phi(t) &= \phi_0 - \frac{1}{\eta}\left\{\Theta^{5/8}+\left(\frac{3715}{8064}+\frac{55}{96}\eta\right)\Theta^{3/8}-\frac{3\pi}{4}\Theta^{1/4}\right. \\ &+\left(\frac{9275495}{14450688}+\frac{284875}{258048}\eta+\frac{1855}{2048}\eta^2\right)\Theta^{1/8} \\ &\left.-\left(\frac{38645}{172032}+\frac{15}{2048}\eta\right)\pi\ln\Theta+\mathcal{O}(\Theta^{-1/8})\right\}, \end{aligned} \tag{32}$$

where ϕ_0 is a constant phase determined by initial conditions.

In the next section we discuss the possible checks we can make to verify this.

7. Radiation Reaction Problem

As in electromagnetism, radiation reaction forces arise in gravitation from the use of retarded potentials satisfying time asymmetric boundary conditions like no-incoming boundary condition at past null infinity. As in earlier cases the problem is more complicated because of the nonlinearity of general relativity.

The approach to gravitational radiation damping has been based on the balance methods, the reaction potential or a full iteration of Einstein's equation. The first computation in general relativity was by Einstein [55] who derived the loss in energy of a spinning rod by a far-zone energy flux computation. The same was derived by Eddington [56] by a direct near-zone radiation damping approach. He also pointed out that the physical mechanism causing damping was the effect discussed by Laplace [57], that if gravity was not propagated instantaneously, reactive forces could result. An useful development was the introduction of the radiation reaction potential by Burke [58] and Thorne [59] using the method of matched asymptotic expansions. In this approach, one derives the equation of motion by constructing an outgoing wave solution of Einstein's equation in some convenient gauge and then matching it to the near-zone solution. Restricting attention only to lowest order Newtonian terms and terms sensitive to the outgoing (ingoing) boundary conditions and neglecting all other terms, one obtains the required result. The first complete direct calculation à la Lorentz of the gravitational radiation reaction force was by Chandrasekhar and Esposito [60]. Chandrasekhar and collaborators [61, 62] developed a systematic post-Newtonian expansion for extended perfect fluid systems and put together correctly the necessary elements like the Landau-Lifshiftz pseudotensor, the retarded potentials and the near-zone expansion. These works established the balance equations to Newtonian order, albeit for weakly self-gravitating fluid systems. The revival of interest in these issues following the discovery of the binary pulsar and the applicability of these very equations to binary systems of compact objects follows from the works of Damour [63, 64] and Damour and Deruelle [6] discussed earlier.

Many other approaches to radiation reaction problems have emerged in the last five years. E.g., given the formulas for the far-zone energy and angular momentum fluxes to a particular PN accuracy, to what extent can one infer the radiation reaction acceleration in the (local) EOM? Given the algebraic complexity of various computations and subtle evaluations of various small coefficients, it is worthwhile to check the obvious consistency requirement on the far-zone fluxes. To this end, Iyer and Will (IW) [65, 66] proposed a refinement of the text-book [67] treatment of the energy balance method used to discuss radiation damping. This generalization uses both energy and angular momentum balance to deduce the radiation reaction force for a binary system made of nonspinning structureless particles moving on general orbits. Starting from the 1PN conserved dynamics of the two-body system, and the radiated energy and angular momentum in the gravitational waves, and taking into account the arbitrariness of the 'balance' upto total time derivatives, they determined the 2.5PN and 3.5PN terms in the equations of motion of the binary system. The part not fixed by the balance equations was identified with the freedom still residing in the choice of the coordinate system at that order. The explicit gauge transformations they correspond to has also been constructed. Blanchet [68], on the other hand, obtained the post-Newtonian corrections to the radiation reaction force from first principles using a combination of post-Minkowskian, multipolar and post-Newtonian schemes together with techniques of analytic continuation and asymptotic matching. By looking at "antisymmetric" waves – a solution of the d'Alembertian equation composed of retarded wave minus advanced wave, regular all over the source, including the origin – and matching, one obtains a radiation reaction tensor potential that generalizes the Burke-Thorne reaction potential, in terms of explicit integrals over matter fields in the source. The *validity* of the balance equations upto 1.5PN is also proved. By specializing this potential to two-body sytems, Iyer and Will [66] checked that this solution indeed corresponds to a unique and consistent choice of coordinate system. This provides a delicate and non-trivial check on the validity of the 1PN reaction potentials and the overall consistency of the direct methods based on iteration of the near-field equations and indirect methods based on energy and angular momentum balance. It should be noted that the 'balance method' by itself cannot fix the particular expression for the reactive force in a given coordinate system. In order to solve a practical problem (in which we erect a particular coordinate system), the method is in principle insufficient by itself, but it provides an extremely powerful check of other methods based on first principles. Gopakumar, Iyer and Iyer [14] have applied the refined balance method to obtain the 2PN radiation reaction – 4.5PN terms in the equation of motion. Different facets of the IW choice like the functional form of the reactive acceleration have been systematically and critically explored and a better understanding of the origin of redundant equations is provided by studying variants obtained by modifying the functional forms of the ambiguities in energy and angular momentum. These reactive solutions are general enough to treat as particular cases any reactive acceleration obtained from first principles in the future. The radiative 3.5PN ADM hamiltonian has been obtained by Jaranowski and Schafer [69].

Work on radiation reaction in the test particle case has focussed on understanding the evolution of Carter constant in Kerr geometry. Ryan [70] has investigated the effect of gravitational radiation reaction, first on circular, and later even for non-equatorial orbits around a spinning black hole. Kennefick and Ori [71] developed a computational scheme in which the radiation reaction force is determined by the 'physical retarded' radiation field rather than the radiative field. This allows them to determine the evolution of all associated constants of motion. Capon and Schutz[72] have looked at a 'local expression' for radiation reaction by evaluating its self field as an integral over the particle world line. Recently Mino, Sasaki and Tanaka [73] have derived the leading order correction to the equation of motion of a particle which presumably describes the effect of gravitational radiation reaction by two methods: One approach is analogous to the DeWitt and Brehme [74] method in the case of electromagnetic radiation, where the conservation law of the total (matter + e.m. field) stress-energy tensor is integrated across a tube surrounding the particle world-line, giving the equations of motion including radiation reaction. The other method uses on the other hand asymptotic matching. Quinn and Wald [75] have discussed an axiomatic approach to gravitational radiation reaction and their results are consistent with those of Ref. [73]. Gergely, Perjes and Vasuth [76] have included the spin effects on gravitational radiation reaction using the BDI approach and their results are in accordance with those of Refs. [51, 70]

8. Concluding remarks

Far from being an esoteric and abstruse theory driven by aesthetic considerations, we are in a situation where experiments are driving the theory of general relativity. It is interesting that in the macroscopic world the computations of small higher order corrections so reminiscent of Lamb shift corrections in quantum electrodynamics are inexpendable. We are on the threshold of opening another window to this marvellous universe and with the inauguration of the new gravitational wave astronomy more than ever before general relativity will have found its true home.

Vishu began his career working on Gravitational Radiation[77] and should be happy twice over: Glad he was not so ahead of his time that he gained a place of notoriety in the Guinness book of records [78]! Glad once again that the path he almost tread at the beginning of his research career is probably the most explored today not only the world over but also in India! Maybe he will join the black hole hunt with gravitational waves.. Life after all begins at sixty!!

References

1. C. V. Vishveshwara in *Gravitation, Quanta and the Universe*, Eds. A. R. P. Prasanna, J. V. Narlikar and C. V. Vishveshwara, Wiley Eastern, New Delhi (1980).
2. J. H. Taylor, A. Wolszczan, T. Damour, and J. M. Weisberg, Nature, **355**, 132 (1992).
3. T. Damour, *Gravitational Radiation*, Eds. N. Deruelle and T. Piran (North Holland, Amsterdam, 1983), p.59 and references therein.
4. C. Cutler, T. A. Apostolatos, L. Bildsten, L. S. Finn, É. E. Flanagan, D. Kennefick, D. M. Marković, A. Ori, E. Poisson, G. J. Sussman, and K. S. Thorne, Phys. Rev.

Lett. **70**, 2984 (1993).
5. K. S. Thorne in *300 Years of Gravitation*, Eds. S. W. Hawking and W. Israel, (Cambridge Univ. Press, Cambridge, 1987), p.330; *Proc. of IAU symposium 165, Compact Stars in Binaries*, Eds. J. Van Paradijs, E. Van den Heuvel and E. Kuulkers, (Kluwer Academic Publishers, 1995); *Proc. of the Snowmass 95 summer study on particle and nuclear astrophysics and cosmology*, Eds. E. W. Kolb and R. Peccei, (World Scientific, 1995)
6. T. Damour and N. Deruelle, C. R. Acad. Sci. Paris **293**, 537 (1981); **293**, 877 (1981); Phys. Lett., 87A, 1981, 81.
7. T. Damour, B. R. Iyer and B. S. Sathyaprakash, Phys. Rev. D **57**, 885 (1998).
8. T. Damour, *Gravitation in Astrophysics*, Eds. B. Carter and J. B. Hartle, (Plenum Press, New York and London, 1986), p.3
9. T. Damour and B. G. Schmidt, J. Math. Phys., **31**, 2441().
10. T. Damour, in *300 Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, London, 1987), p. 128.
11. J. Martin and J. L. Sanz, J. Math. Phys. **20**, 25 (1979).
12. C. M. Will and A. G. Wiseman, Phys. Rev. D **54**, 4813 (1996).
13. A. Gopakumar and B. R. Iyer, Phys. Rev. D **56**, 7708 (1997).
14. A. Gopakumar, B. R. Iyer and Sai Iyer, Phys. Rev. D **55**, 6030 (1997).
15. G. Schafer, Ann. Phys. (N.Y.) **B161**, 81 (1985); *Gravitational wave detection*, Ed. A. Krolak, (Banach center publications, Warzawa, 1997), p. 43.
16. L. P. Grishchuk and S. M. Kopejkin, in *Relativity in Celestial Mechanics and Astrometry*, edited by J. Kovalevsky and V. A. Brumberg (Reidel, Dordrecht, 1986), p. 19.
17. L. Blanchet, G. Faye and B. Ponsot, work in progress.
18. C. M. Will and M. E. Pati, work in progress;
19. G. Schäfer and P. Jaranowski, grqc-9712075 (1997), grqc-9802030 (1998).
20. P. Jaranowski, in *Gravitational wave detection*, Ed. A. Krolak, (Banach center publications, Warzawa, 1997), p. 55.
21. L. Blanchet and T. Damour, Phil. Trans. R. Soc. London A **320**, 379 (1986); Phys. Rev. D **37**, 1410 (1988).
22. T. Damour and G. Schäfer, Nuovo Cimento B **101**, 127 (1988).
23. G. Schäfer and N. Wex, Phys. Lett. **174 A**, 196, (1993); erratum **177**, 461.
24. N. Wex, CQG **12**, 983, (1995).
25. N. Wex and R. Rieth, in *Symposia Gaussiana*. Eds. Behera, Fritsch and Lintz (W. Gruyter and Co., Berlin, 1995).
26. A. Gopakumar and B. R. Iyer, in preparation(1998).
27. L. Blanchet, Proc. R. Soc. Lond. A **409**, 383 (1987); L. Blanchet and T. Damour, Ann. Inst. H. Poincaré (Phys. Théorique) **50**, 377 (1989); T. Damour and B. R. Iyer, Phys. Rev. D **43** 3259 (1991); Ann. Inst. H. Poincaré (Phys. Théorique) **54**, 115 (1991); CQG **11**, 1353 (1994); L. Blanchet and T. Damour, Phys. Rev. D**46**, 4304 (1992). L. Blanchet, Phys. Rev. D **51**, 2559 (1995); L. Blanchet, T. Damour, and B. R. Iyer, Phys. Rev. D **51**, 5360 (1995); L. Blanchet, Phys. Rev. D **54**, 1417 (1996).
28. R. Epstein and R. V. Wagoner, Astrophys. J. **197**, 717 (1975).
29. K. Thorne, Rev. Mod. Phys. **52**, 299 (1980).
30. C. M. Will and A. G. Wiseman, Phys. Rev. D **54**, 4813 (1996).
31. N. D. Haridass and V. Soni, J. Phys. A **15**, 473, (1982).
32. W. B. Bonnor, Philos. Trans. R. Soc. London A **251**, 233 (1959).
33. T. Damour and B. R. Iyer, Phys. Rev. D **43** 3259 (1991).
34. L. Blanchet, *Quadrupole Quadrupole gravitational waves*, CQG (1998) to appear.
35. D. Christodoulou, Phys. Rev. Lett. **67**, 1486 (1991).
36. K. Thorne, Phys. Rev. D **45**, 520 (1992).
37. L. Blanchet, *Gravitational wave tails of tails*, CQG (1998) to appear.
38. R. V. Wagoner and C. M. Will. Astrophys. J **210**, 764 (1976).

39. L. Blanchet and G. Schafer, Mon. Not. R. Astron. Soc. **239**, 845 (1989).
40. W. Junker and G. Schafer, Mon. Not. R. Astron. Soc. **254**, 146 (1992).
41. A. G. Wiseman, Phys. Rev. D **46**, 1517 (1992); Phys. Rev. D **48**, 4757 (1993).
42. L. Blanchet and G. Schäfer, CQG **10**, 2699 (1993).
43. R. Rieth and G. Schafer, CQG **14**, 2357 (1997).
44. L. Blanchet, T. Damour, and B. R. Iyer, Phys. Rev. D **51**, 5360 (1995).
45. L. Blanchet, T. Damour, B.R. Iyer, C. M. Will and A. G. Wiseman, Phys. Rev. Lett. **74**, 3515 (1995).
46. L. Blanchet, B. R. Iyer, C. M. Will and A.G. Wiseman, CQG **13**, 575, (1996).
47. L. Blanchet, B. R. Iyer and B. Joguet, paper in preparation;
48. L. Blanchet, G. Faye, B. R. Iyer, B. Joguet and B. Ponsot, work in progress;
49. L. E. Kidder, C. M. Will, and A. G. Wiseman, Phys. Rev. D **47**, R4183 (1993).
50. L. E. Kidder, Phys. Rev. D **52**, 821 (1995).
51. L. E. Kidder Phys. Rev. D **52**, 821 (1995).
52. B. Owen, H. Tagoshi and A. Ohashi, gr-qc??
53. L. E . Kidder (unpublished).
54. K. Taniguchi and M. Shibata, *Phys. Rev. D.* (submitted), Kyoto University preprint KUNS 1464, Osaka University preprint OU-TAP 68.
55. A. Einstein, Sitzber. Preuss. Akad. Wiss. (Berlin), (1918).
56. A. S. Eddington, *The mathematical theory of relativity*, (Cambridge Univ. Press, Cambridge, 1924).
57. P. S. Laplace, *Mécanique céleste*, (Courcier, Paris). Second Part: Book 10, chap. 7.
58. W. L. Burke, unpublished Ph. D. Thesis, California Institute of Technology (1969); J. Math. Phys. **12**, 401 (1971).
59. K. S. Thorne, Astrophys. J. **158**, 997 (1969).
60. S. Chandrasekhar and F. P. Esposito, Astrophys. J. **160**, 153 (1970).
61. S. Chandrasekhar, Astrophys. J. **158**, 45 (1969).
62. S. Chandrasekhar and Y. Nutku, Astrophys. J. **158**, 55 (1969).
63. For a comprehensive review and detailed references to original literature see T. Damour in *Gravitational radiation* edited by N. Deruelle and T. Piran, (North Holland, Amsterdam, 1982).
64. T. Damour, C. R. Acad. Sc. Paris, **294**, 1355 (1982).
65. B. R. Iyer and C. M. Will, Phys. Rev. Lett. **70**, 113 (1993).
66. B. R. Iyer and C.M. Will, Phys. Rev. D **52** ,6882 (1995).
67. L. D. Landau and E. M. Lifschitz, *The Classical Theory of Fields* (Pergamon, New York, 1975), 4th ed., p. 357.
68. L. Blanchet, Phys. Rev. D **47**, 4392 (1993); L. Blanchet, Phys. Rev. D **55**, 714 91997).
69. P. Jaranowski and G. Schafer, Phys. Rev. D **55**, 4712 (1997).
70. F. D. Ryan, Phys. Rev. D **52** , R3159 (1995); Phys. Rev. D **53** , 3064 (1995).
71. D. Kennefick and A. Ori, Phys. Rev. D **53**, 4319 (1996); A. Ori, Phys. Rev. D **55**, 3444 (1997).
72. R. A. Capon (1997) in *Mathematics of Gravitation: Gravitational Wave Detection*, **41**, p. 65, Ed. A. Królak, Banach Center Publications, Warsaw.
73. Y. Mino, M. Sasaki and T. Tanaka, Phys. Rev. D **55**, 6030 (1997).
74. B. S. DeWitt and R. W. Brehme, Annals of Physics, **9**, p. 220, (1960).
75. T. C. Quinn and R. M. Wald, Phys. Rev. D **56** 3381 (1997).
76. L. Gergely, Z. Perjés and M. Vasúth, Phys. Rev. D.**57**, 876, (1998); **57**, 3423 (1998).
77. C.V. Vishveshwara, *Gravitational Radiation from Binary Systems*, University of Maryland Technical Report (1965).
78. C. V. Vishveshwara *On the black hole trail...: A personal journey* Current Science, **71**, 824 (1996).