practicable. A rough numerical estimate of the effect of the finite width of the bow on the critical pressure necessary to maintain any given type of vibration seems however within the range of possibility, if the necessary physical data are known or can be assumed. This complicated work must necessarily be reserved for a future occasion.

The effect of the yielding at the bridge: General discussion

The yielding at the bridge and the consequent communication of energy from the vibrating string to the sound-box, and thence to the air, are essential features in the investigation of the mechanics of bowed strings. For simplicity, we have so far assumed that this yielding is very small, and that the bow maintains a strictly periodic vibration. When these assumptions are made, no special difficulty arises in discussing the mode of action of the bow; and it is found that the vibrations of the string are practically in the normal modes, and have the same frequency as that of the string with rigidly fixed ends. Cases may however arise in which either or both of the assumptions may not be justifiable *a priori*, and we have thus to enter into an examination of (*a*) the conditions, if any, under which a periodic vibration cannot be maintained by the bow, and (*b*) the possibility (or otherwise) of a periodic motion under the action of the bow, with a frequency differing from that of the string with rigidly fixed ends. For a full understanding of these problems, we have to enter into a discussion of the mechanics of the string and bridge much more detailed than that given in section II. The discussion may conveniently be divided into four parts: (1) the free vibrations of the system if there be no dissipation of energy; (2) the free vibrations as modified by the dissipation of energy; (3) the vibrations forced by a periodic force of arbitrary period; and (4) the vibrations forced by the bow under various conditions. The discussion is also valuable as enabling us to find methods for experimentally verifying the theory given in the preceding pages.

Alteration of free periods

Taking first the free vibrations, we may, to avoid undue complications, consider only the motion at the bridge transverse to the string. This neglect of the longitudinal motion of the bridge would not seriously invalidate our conclusions, as the reaction of this component of the motion of the bridge would only tend to alter the tension of the string periodically, and would not directly tend to set up a transverse motion such as that with which we are now concerned. Further, it is sufficient if we consider the motion in the plane parallel to that of the vibrations of the string, the latter being assumed to be the same as that in which vibrations are forced by the bow in usual practice. In other words, the reaction of the bridge on the motion of the string may be sufficiently nearly represented by a system capable of movement in a single direction transverse to the string, and having one, two, three or such larger number of free periods of vibration as may be necessary to obtain a sufficiently close approximation to the truth. As a first step, we may assume that the bridge may be represented as a single mass controlled by a spring. Neglecting the dissipation of energy and using the notation of section II, the free periods of vibration of the string and of the bridge may be found from the formula

$$\tan pl = \frac{-T_0 p}{f^2 - Mm^2}, \quad \text{where} \quad \mu m^2 = T_0 p^2.$$

The equation for pl may be completely solved by a graphical method. Writing the relation in the form

$$MT_0p^2/\mu - f^2 = T_0 p \cot pl,$$

the expressions on either side may be readily plotted as graphs with respect to pl. The left-hand side gives us a parabola with its vertex on the line $pl = 0$, on the negative side of the axis of pl. The right-hand side has a series of branches with the lines $pl = \pi$, $pl = 2\pi \cdots pl = n\pi$, etc. as their asymptotes. The particular point at which the parabola cuts the axis of pl determines the free period of vibration of the bridge, if the reaction of the string be not taken into account. The curves representing $T_0 p \cot pl$ evidently cut the axis of pl at the points $pl = n\pi/2$, and if the parabola also cuts the axis at one of these points, the reaction of the string would have no effect on the free period of the bridge. In other cases, the free period of the bridge and the free periods of the string are both subject to modification. The graphs show that the natural frequencies of the string which are greater than that of the bridge, are further increased by the yielding of the latter, while those which are less are further decreased. The frequency of the bridge is increased or decreased according as it is greater or less than the nearest of the two natural frequencies of the string between which it lies. The most interesting results are obtained when the free period of the bridge, and one of the free periods of the string, nearly or actually coincide, that is, when the parabola in the graph cuts the axis of pl at or near one of the points $pl = \pi$ or 2π or 3π, etc. Taking, as an example, the case in which the bridge and the string have nearly the same frequency, the points where the parabola cuts the two branches having the common asymptote $pl = \pi$, determine the two free periods of the system as modified by the mutual action of its parts. When the parabola cuts the axis of pl at a point for which $pl < \pi$, the intersection of the parabola with the inner branch determines the modified period of the bridge, and the intersection with the outer branch determines the modified period of the string. The state of matters is reversed when the parabola cuts the axis of pl at a point for which $pl > \pi$. When the parabola passes exactly through the point $pl = \pi$, it is no longer possible to define which of the two free periods of the system is distinctively that of the string and which of the bridge.

From the preceding discussion it is clear that the yielding of the bridge modifies the free periods of the string lying near its own period to a greater extent than it does those further off, and that when the string and bridge taken separately have the same free period in any case, their mutual action results in the system having two different periods which, when the free oscillations are excited, would produce "beats" somewhat similar to those of two tuned electrical circuits coupled with each other. "Beats" would also occur when the free periods, taken separately, are nearly but not quite equal to each other; but they would then be more rapid than in the case of perfect tuning. From the principle of conservation of energy, it is evident that in every such case, the vibration of the string would be a maximum when that of the bridge is a minimum, and *vice-versa*.

Effect of thickness of string on free periods

An approximate formula for the numerical calculation of the alteration in the frequencies of the string is readily found. The unaltered frequencies are given by $m/2\pi$ where $m=(T_0/\mu)^{1/2}p$ and $pl=n\pi$, n being any positive integer. Putting $pl=n\pi+p'l$ as the result of the yielding of the bridge, $p'l$ being small, $\tan pl = \tan p'l = p'l$ (nearly). The change in frequency is thus

$$\begin{aligned} m'/2\pi &= (T_0/\mu)^{1/2}p'/2\pi \\ &= T_0 m/2\pi l(Mm^2-f^2). \end{aligned}$$

If (Mm^2-f^2) is sufficiently large and either positive or negative, the value of m corresponding to the undisturbed period may be substituted on the right-hand side without sensible error. For a given value of m, that is, for a given frequency, the alteration produced by the yielding of the bridge is proportional to T_0/l, in other words to the tension divided by the length. For a given vibrating length and frequency, the heavier or thicker a string is, the greater would be the tension necessary, and therefore also the alteration in frequency produced by the yielding at the bridge. Similarly, with a given string vibrating with a stated frequency, the ratio T_0/l may be increased by increasing the tension and the vibrating length, and the alteration of frequency produced by the yielding of the bridge also increases.

When (Mm^2-f^2) is not very large, the equation may be written in a slightly different form

$$\begin{aligned} m'[M(m-m')^2+2Mm'(m-m')-f^2] \\ = T_0 m/l. \end{aligned}$$

Since $(m-m')$ is the unaltered frequency divided by 2π, the relation is evidently a quadratic in m' which may be readily solved. When the free periods of the bridge and the string considered are the same if the two are taken separately, we have to write $M(m-m')^2 = f^2$ and $m' = \pm [T_0/2\ Ml]^{1/2}$.

The formula shows that in this case the system has *two* free periods instead of one, and that the influence of the density of the string, or of its tension and vibration-length for a given frequency in determining the effect of the yielding of the bridge, is relatively not so great as at other frequencies.

Yielding of a bridge which has more than one free period

If the bridge has two or more free periods of vibration, the effect of its yielding on the free periods of the string may be readily found by the method of generalized coordinates. If ϕ_1, ϕ_2, ϕ_3, etc. be the normal coordinates of the system of which the bridge forms a part, we may write

$$\text{Kinetic energy of the system} = \tfrac{1}{2}a_1\dot{\phi}_1^2 + \tfrac{1}{2}a_2\dot{\phi}_2^2 + \tfrac{1}{2}a_3\dot{\phi}_3^2 + \text{etc.}$$

$$\text{Transverse yielding of bridge} = r_1\phi_1 + r_2\phi_2 + r_3\phi_3 + \text{etc.}$$

From this, we find

$$\tan pl = -T_0 p\left[\frac{r_1^2/a_1}{n_1^2-m^2} + \frac{r_2^2/a_2}{n_2^2-m^2} + \frac{r_3^2/a_3}{n_3^2-m^2} + \text{etc.}\right]$$

where $n_1/2\pi$, $n_2/2\pi$, $n_3/2\pi$, etc. are the natural frequencies of the bridge. When the particular free period of the string under consideration nearly coincides with the one of the free periods of the bridge, the results obtained in respect of it from this general formula, would not differ seriously from those obtained, if only the coordinate of the bridge having nearly the same free period is retained, and the others are neglected. Generally, however, two or more of the coordinates in the motion of the bridge require to be taken into account, and the alterations in a free period of the string produced by two coordinates in the motion of the bridge may be of opposite sign, and may thus tend to cancel each other out.

Decrement of free vibrations of string

We may now pass on to consider the effect of dissipation of energy on the free vibrations of the system. For this purpose, we may, at first, confine our attention to the case in which the motion of the bridge is expressed by a single coordinate, and use the same notation as in section II. Since $y=0$ when $x=0$, we find by assuming the free oscillations of the string to be given by the expression

$$Y = F_1 \exp((ip_1+q_1)(x+at)) + F_2 \exp((ip_2+q_2)(x-at)),$$

that $p_1 = -p_2$, $q_1 = -q_2$ and $F_1 = -F_2$. We may therefore write

$$Y = F[\exp((ip+q)(x+at)) - \exp(-(ip+q)(x-at))].$$

At the point $x = l$, we have the relation

$$M\frac{d^2y}{dt^2} = -T_0\frac{dy}{dx} - f^2y - g^2\frac{dy}{dt}.$$

Substituting the assumed value of y, we find

$$[Ma^2(ip+q)^2 + g^2a(ip+q) + f^2][\exp((ip+q)l) - \exp(-(ip+q)l)]$$
$$= -T_0(ip+q)[\exp(ip+q)l + \exp(-(ip+q)l)].$$

Since ql may be assumed to be small, a simplification may be effected by writing $(\exp(ql) + \exp(-ql)) = 2$ and $(\exp(ql) - \exp(-ql)) = 2ql$. Then,

$$[Ma^2(ip+q)^2 + g^2a(ip+q) + f^2][2ql\cos pl + 2i\sin pl]$$
$$= -T_0(ip+q)(2\cos pl + 2iql\sin pl).$$

Separating the real and imaginary parts and equating, we obtain

$$\tan pl = -\frac{T_0p + Bql}{A - T_0q^2l} = \frac{Aql + T_0q}{B + T_0pql},$$

where $A = [Ma^2(q^2 - p^2) + g^2aq + f^2]$

and $B = [2Ma^2pq + g^2ap]$.

Cross-multiplying and neglecting terms of the order q^3, we find that

$$qa = -\frac{g^2a^2p^2T_0}{[(f^2 - Ma^2p^2)^2l + T_0^2p^2l] + [f^2 + Ma^2p^2]T_0}.$$

This gives us the rates of decay of the free vibrations of the system. The free periods as altered by the damping are given by the equation

$$\tan pl = -(T_0p + Bql)/(A - T_0q^2l),$$

which may be solved graphically or numerically, after substitution of the approximate value of q.

We now proceed to discuss the results obtained in the preceding para.

Using the abbreviation $f^2 = Mm_1^2$, the expression for the rate of decay of the free vibrations may be written in the form

$$qa = \frac{g^2}{p^2l^2(m_1^2/m^2 - 1)M^2/\mu l + M(m_1^2/m^2 + 1) + \mu l}.$$

To obtain the rate of decay of the free vibrations of the bridge, we put $m = m_1$ and find $qa = -g^2/(2M + \mu l)$, where μl is, of course, the mass of the string. It will also be seen that if the mass of the string be small compared with that of the bridge and associated parts of the instrument, the rates of deacy of the free vibrations of the string would, in general, be small compared with the decrement of the free vibrations of the bridge. An exception should, however, be made in respect of the free modes, if any, the frequencies of which do not differ considerably from that of the bridge. The components whose frequencies are nearest to the frequency of free vibration of the bridge decay with time more rapidly than the rest, and in the particular case in which the free vibrations of the bridge and of the string in any particular mode are of nearly equal frequency, the rate of decay of the particular component is much larger and becomes comparable with that of the bridge.

Effect of damping on free periods

The effect of damping upon the free periods (as shown by the formula for tan pl) is generally quite negligible. For, if we take the maximum value of qa, that is $-g^2/2M$, and substitute it in the formula $\tan pl = -(T_0 p + Bql)/(A - T_0 q^2 l)$, it is found that $Bql = 0$, and that $T_0 q^2 l$ is quite negligible. Except for certain terms containing q^2 appearing in A, the equation for determining the free periods is thus practically the same as if there were no dissipation of energy. The free period of the bridge (in the absence of the string) is increased by a quantity of the second order in consequence of the dissipation of energy, and the effect of this on the free periods of the string which do not lie near that of the bridge is of no consequence. Even in respect of those free periods of the string, if any, which are approximately coincident with that of the bridge, the alterations produced by the mutual action of string and bridge would be nearly the same as in the absence of any dissipation of energy. But this statement may require modification, if the alteration of free periods by the mutual action of the string and the bridge be of the same order of quantities as the alteration of the free period of the bridge by damping, as would be the case if T_0/l were less than $g^4/2M$, that is, if the string were very thin, or if the damping of the bridge were considerable.

Coupled vibrations of string and bridge

From what has been said above, it is clear that, ordinarily, if the free vibrations of the string be excited, the free vibrations of the bridge which are excited at the same time would die out more or less quickly; the beats due to the superposition of its free and forced oscillations would therefore vanish, leaving a steady vibration having the modified free period of the string which dies away rather slowly. The transient beats in the vibration of the bridge become slower and slower as the frequency of the string approaches that of the bridge, and the dissipation of the energy of the vibrating string becomes much more rapid. But the beats of the free vibrations of the bridge with those excited by the string do not vanish even when the adjustment of the frequencies of the string and bridge is most accurate. On the other hand, it is at this stage that they become most prominent in the motion of the bridge, and also appear in the vibration of the string, the modification of the free period of the string by the yielding of the bridge being then a maximum. The motion is however highly damped at this stage, and dissipates itself quickly. We cannot, therefore, as in the case of undamped vibration, expect to find the epochs of the maxima of the bridge-vibration coinciding *exactly* with those of the minima of the string-vibration, or *vice versâ*. But as the free periods of the system are not considerably altered by damping, *approximate* coincidence of the maxima of the one with the minima of the other, and *vice versâ*, may still be expected.

It is worthy of note that when m_1 is not nearly equal to m, the rate of decay of the free vibrations of the string depends not only on the frequency but also upon its length and density. For a given frequency of vibration, the heavier of two strings of the same length is more strongly damped. Similarly, for a string of a given material, the damping may be increased by altering its length and tension in such a manner as to keep the frequency of vibration constant. Analogous effects have already been noted in respect of the alterations of the free period of the string produced by the yielding of the bridge. When the frequency of vibration is gradually altered so as to approach the point at which $m \equiv m_1$, the rate at which the damping of the free vibrations increases to the maximum possible, depends largely upon the density and length of the string. But in all cases, the maximum value is the same, this being attained when the string and bridge taken separately have *exactly* the same free period. That this is so in spite of the actual free periods of the system as modified by the mutual action of its parts being different in the various cases, is explained by the fact that the increased damping due to the greater density or length of the string is at this point completely set off by the decrease in the damping produced by larger alteration of free period.

So far for the free vibrations in the case in which the motion of the bridge is expressible by a single coordinate. If the bridge has two or more free periods of vibration, the yielding at the end of the string may be expressed as a linear function of the values of the normal coordinates of the bridge-system. Similarly, the force exerted by the string on the bridge (which is of the damped harmonic type) may be resolved into its normal force-components which are also of the same type. On account of the dissipation of the energy, however, it is not generally possible in these cases to express the equations of motion in the simple form in which only a single coordinate is involved in each equation. By applying a general method analogous to that described by Lord Rayleigh (*Theory of Sound*, vol. 1, 2nd edn., Art. 104), the yielding at the bridge and thence also the values for the free periods and the logarithmic decrements of the vibrations of the string may be determined. For those cases in which the free period of any component in the vibration of the string lies at or near one of the free periods of the bridge, the values thus obtained in respect of the particular component would not differ seriously from those obtained by neglecting all the coordinates in the motion of the bridge except that which becomes specially important in the circumstances.

Simple harmonic force of arbitrary period

We may now pass on to discuss the vibrations maintained by a periodic force of arbitrary frequency applied at some specified point on the string. In section II, we have already found the formulae necessary for this purpose, on the assumption that the motion of the bridge is expressible by a single coordinate. It now remains to scrutinize the results in some detail, and to extend them to the case in which the bridge has two, three or a larger number of free periods of vibration.

One important consequence of the formulae given on page 246 is as regards the phase of the motion of the bridge. When the period of the maintained vibration is much greater than that of the bridge, $(f^2 - Mm^2)$ is positive and large in comparison with g^2m. The string and the bridge are then in the same phase of vibration. On the other hand, when the period is less than that of the bridge, $(f^2 - Mm^2)$ is negative, and the two vibrations are in opposite phases. When $f^2 = Mm^2$, the vibrations differ in phase by exactly a quarter of an oscillation. The results hold good, irrespectively of whether the frequency of the applied force is, or is not, exactly the same as the frequency of the free vibrations of the string, so long as the latter are of sufficient amplitude in comparison with the motion of the bridge.[17]

The results given in section II may be generalised for the case in which the bridge has two, three or a larger number of free periods, by using the method given in Lord Rayleigh's *Theory of Sound*, I, Art. 104, to find the forced oscillation of the bridge in terms of that of the string. The relations thus found would be precisely analogous to equations (9) and (10) on page 246 except that instead of $(f^2 - Mm^2)$ and g^2m, we would have two other constants depending in a rather complicated manner upon the masses, free periods and damping coefficients of the bridge-system, upon the position of the point on the bridge over which the string passes, and also upon the assumed period of the applied force. The formulae for the forced oscillation of the string may then be deduced precisely as in section II. The quantity $(F_2^2 + G_2^2)^{1/2}$ may be regarded as expressing the forced vibration of the string, so long as the motion of the bridge does not become comparable with the motion of the string in its amplitude.

[17] Compare with the results described in the paper on the "Small Motion at the Nodes of a Vibrating String", Bulletin No. 6 of this Association.

It is given by the relation

$$(F_2^2+G_2^2)^{1/2}=\frac{E\cos\psi\sin px_0}{pT_0[\sin^2(pl+\psi)+\delta^2\cos^2 pl\cos^2\psi]^{1/2}}$$
$$=E\sin px_0/k,$$

where k is a quantity independent of the position of the bowed point which expresses the relation between the applied force and the maintained vibration. In the denominator of the expression for $(F_2^2+G_2^2)^{1/2}$, the quantity $\sin^2(pl+\psi)$ is approximately independent of the dissipation factor, provided two or more of the free periods of the system composed of the string and bridge do not lie near each other. On the other hand, the quantity $\delta^2\cos^2 pl\cos^2\psi$ is dependent entirely on the friction. Except in special cases of the kind mentioned, the amplitude of the maintained vibration would practically be a maximum when $\sin^2(pl+\psi)$ is zero, that is, when the period of the impressed force is the same as one of the free periods of the string as modified by the yielding of the bridge. When this is the case, k is proportional to δ, and is therefore entirely dependent on the rate of dissipation of energy. On the other hand, if the period of the impressed force is the same as one of the free periods of the string with the ends rigidly fixed, $\sin pl\equiv n\pi$, and k is proportional to $[\tan^2\psi+\delta^2]^{1/2}$, and depends both on the dissipation of energy and on the alteration of free period produced by the yielding of the bridge. The magnitude and phase of the force required to maintain the vibration thus depend, in general, both on the difference between the free and forced periods, and on the rate of dissipation of energy.

Forced vibrations of coupled systems of nearly equal periods

The special cases in which two or more of the free periods of the system composed of the string, bridge and associated masses lie near each other require however to be separately considered, as the value of ψ cannot then be regarded as independent of the dissipation of energy. The simplest of these cases is that in which one of the free periods of the string approximately coincides with one of the free periods of the bridge, and also with the period of the applied force. As one of the coordinates in the motion of the bridge then becomes large in comparison with the others, we may neglect all the rest, and the formulae given in section II may therefore be used without serious error. Under the conditions assumed, the motion of the bridge is not negligibly small, and the motion of the string on either side of the point of application of the force cannot be quite accurately represented by the same expression; in other words, the string does not quite conform to the natural types of vibration. This is sufficiently evident from the formulae on page 246 if (f^2-Mm^2) be put nearly equal to zero. As an approximate result, however, we may still regard $(F_2^2+G_2^2)^{1/2}$ as expressing the amplitude of the maintained vibration of the string, while $(D_2^2 + E_2^2)^{1/2}$ represents that of the bridge. To find $(F_2^2 + G_2^2)^{1/2}$ for various periods of the applied force lying in the neighbourhood of the free periods of the string and the bridge, we have to consider the values of ψ and δ in such cases. Thus, $\tan\psi = \tan\theta(1 - \sin^2\phi)$ and $\delta = \tan\theta \sin\phi\cos\phi$, where $\tan\theta = T_0p/(f^2 - Mm^2)$, and $\tan\phi = g^2m/(f^2 - Mm^2)$. By substitution, $\delta = T_0p\sin^2\phi/g^2m$, and is thus always positive. Generally speaking, δ is small and approximately equal to $T_0pg^2m/(f^2 - Mm^2)^2$, being thus directly proportional to the damping coefficient g^2 of the bridge. But when f^2 approaches Mm^2 (as in the cases now being considered), δ increases rather sharply to a considerably larger maximum value T_0p/g^2m which is *inversely* proportional to the damping coefficient of the bridge. So long as $(f^2 - Mm^2)$ is not small, ψ is practically equal to θ, being positive when $f^2 > Mm^2$, and negative when $f^2 < Mm^2$. But when f^2 approaches the value Mm^2, ψ becomes numerically less than θ and, in fact, vanishes when $f^2 = Mm^2$, whereas θ increases to either of the values $\pm\pi/2$ at this point. The free period of the string is given by the relation $pl = n\pi$, if its ends be supposed rigidly fixed, and by the relation $pl + \theta = n\pi$, if the yielding of the bridge be taken into account. Thus, if the string and bridge (taken separately) have the same free period, and the applied force is also of the same period, $f^2 = Mm^2$, and $pl = n\pi$. $\mathrm{Sin}^2(pl + \psi)$ is then zero, and $G_2 = 0$. The amplitude of vibration of the string $(F_2^2 + G_2^2)^{1/2}$ is then equal to $\pm F_2$ and may be written as $E\sin px_0/pT_0\delta$ which is equal to $Eg^2m\sin px_0/p^2T_0^2$.

In other words, the force required to maintain a vibration of a given amplitude is then *inversely* proportional to the damping coefficient g^2 of the bridge. This apparently paradoxical result is easily explained[18]. For, the motion of the bridge is then $E_2\cos mt$, where $E_2 = E\sin px_0/pT_0$, and the rate of dissipation of energy which is $\frac{1}{2}g^2m^2E_2^2$ may be written as $\frac{1}{2}F_2^2T_0^2p^2/g^2$, and is thus also inversely proportional to the damping coefficient g^2. By writing the rate of dissipation of energy in the form $\frac{1}{2}F_2Em\sin px_0$, it is seen that the force required to maintain the string in a vibration of a given amplitude is directly proportional to the rate of dissipation of energy. It is also proportional to the corresponding motion at the bridge, as may be seen by writing the expression for the applied force in the form $\frac{1}{2}F_2E_2T_0pm$.

If the free periods of the string and the bridge (taken separately) and the period of the applied force are not all exactly equal to one another, but only approximately equal, the expression for the amplitude of the forced vibration of the string cannot reduce to such simple forms as those given in the preceding para. We may however trace the manner in which the amplitude of the maintained vibration varies with the period of the applied force, assuming *either* that (1) the latter is the same as the free period of the string with rigidly fixed ends,

[18]Compare with the observations of Lord Rayleigh on the reaction of a dependent system, *Theory of Sound*, **I**, Art. 117.

but differs from the period of the bridge (taken by itself), *or* (2) that the free periods of the string and of the bridge (taken separately) are equal, but differ from the period of the applied force. In the first set of cases, the expression $[\sin^2(pl+\psi) + \delta^2\cos^2\psi\cos^2 pl]^{1/2}/\cos\psi$ to which the amplitude of maintained vibration is inversely proportional, reduces to $[\tan^2\psi+\delta^2]^{1/2}$, pl being equal to π. As the period of the applied force approaches the free period of the bridge, δ^2 increases continuously and becomes a maximum when $f^2 = Mm^2$. On the other hand, the value of $\tan^2\psi$ first increases, and after attaining a maximum when f^2 is $>$ or $<$ than Mm^2, becomes zero when $f^2 = Mm^2$. But $(\tan^2\psi+\delta^2)^{1/2}$ has only one *maximum* which is attained when $f^2 = Mm^2$. For, $\tan\psi$ may be written as $T_0 p\sin\phi\cos\phi/g^2m$, and $(\tan^2\psi+\delta^2)$ is therefore equal to $(T_0 p\sin\phi/g^2m)^2$ which is a maximum when $\phi=\pi/2$. From this, it is seen that in the first set of cases, practically the *maximum* force is required to maintain a vibration of a given amplitude, when its period is the same as the free period of the bridge. In the second set of cases, it is necessary to trace the values of the expression $[\sin^2(pl+\psi)/\cos^2\psi+\delta^2\cos^2 pl]^{1/2}$. This may be written in the form $[\sin^2 pl + \sin^2 pl\tan\psi + \cos^2 pl(\tan^2\psi+\delta^2)]^{1/2}$. $\text{Sin}^2 pl$ is a minimum when $pl=\pi$ and $f^2 = Mm^2$, and increases continuously when f^2 is $>$ or $< Mm^2$. We have already seen that $\cos^2 pl(\tan^2\psi+\delta^2)$ is a maximum when $f^2 = Mm^2$. Sin $2pl\tan\psi$ is zero when $f^2 = Mm^2$, and is very small when f^2 is appreciably greater or less than Mm^2, but intermediately, it has finite *negative* values. To find whether the sum of these expressions is a minimum or a maximum when $pl=\pi$ and $f^2 = Mm^2$, we may differentiate $[\sin^2(pl+\psi)/\cos^2\psi+\delta^2\cos^2 pl]$ twice with respect to p, and then put $pl=\pi$ and $\psi=0$. The result found is that the expression is then a maximum or a minimum according as

$$4\left(\frac{T_0 M}{g^4 l}\right)^2 + 4\frac{T_0 M}{g^4 l}(1+\mu l/4M) > \text{or} < 1.$$

If we neglect $\mu l/4M$ which is one-fourth of the ratio of the mass of the string to the mass of the bridge, etc. this condition may be simply written as

$$T_0/l > \text{or} < \frac{g^4}{2M}(\sqrt{2}-1),$$

or to a closer approximation, in the form

$$T_0/l > \textit{or} < (\sqrt{2}-1)\frac{g^4}{2M}(1-\mu l/4\sqrt{2}M).$$

As the term on the right-hand side is proportional to the square of the damping coefficient of the bridge, we should expect in practice to find it smaller than the one on the left, unless, of course, the string be very thin or the damping of the bridge be unusually large. The force required to maintain the vibration would thus, *in general*, be a *maximum* when its period is the same as the free period of the string and the bridge, and when this is the case, the force required would be a *minimum* for two other periods, one of which lies on each side of the aforesaid maximum. These two periods for which the *minimum* maintaining force is required would, except in extreme cases, be more or less the same as the two free periods of the coupled system composed of the string and bridge. When, however, the string is very thin, or the damping of the bridge is unusually large, the two minima do not exist, the force required being a *minimum* and not a *maximum* at the period given by $f^2 = Mm^2$.

When the free periods of the string and of the bridge (taken separately) are nearly but not quite equal to one another, and are assigned specific values, and only the period of the applied force is regarded as a variable, it is easy to see from the discussion in the preceding para that, subject to similar restrictions regarding the relative magnitudes of the quantities, involved, the force required to maintain the string in vibration of given amplitude would, likewise, be a minimum for two periods, f^2 being greater than Mm^2 in one case and less than Mm^2 in the other. The two minima of the force necessary would however be unequal. These two periods would correspond more or less closely to the two free periods of the system, and from the discussion of the free vibrations already given, it is clear that the difference of these two periods would be a minimum when the free periods of the string and bridge taken separately are equal, and would not therefore change very rapidly when they are made slightly different.

Effect of motion of the bridge on periodic vibrations forced by the bow

When the applied force, instead of being simple in character, comprises a series of harmonics, and its period does not differ considerably from the free period of the string with rigidly fixed ends, the forced vibration may still, *as an approximation*, be expressed in terms of the amplitudes B_n and phases e_n of the normal types of vibration.

The force required to maintain the vibration may be written in the form

$$\sum_{n=1}^{n=\infty} \frac{k_n B_n \sin\left(\dfrac{2n\pi t}{T} + e_n + e'_n\right)}{\sin \dfrac{n\pi x_0}{l}}$$

where k_n, e'_n are quantities which are approximately independent of the position of the bowed point, and are determined by the differences between the free and forced periods of the components in the motion, and the logarithmic decrements of the respective free modes. If the difference of periods is zero or negligible in respect of any component or components, the corresponding constants k_n are merely the logarithmic decrements of the free vibrations multiplied by the factor $\pi n^2 T_0/l$, and should thus be readily capable of experimental determination. If the difference of periods is not negligible, the experimental determination of k_n is not such a simple matter, as it would be necessary to determine the logarithmic decrement of the free vibrations, and also to make an accurate comparison of the period of the applied force and the period of the component free vibration as modified by the yielding of the bridge. [The free periods may, of course, be calculated theoretically if the constants of the string and the actual components of the yielding of the bridge are known]. In the particular cases in which the difference of the free and forced periods or the logarithmic decrement of any component is large, the corresponding constant k_n becomes large. This might, for instance, happen on account of a particular component being nearly or exactly in resonance with one of the free modes of vibration of the bridge, in which case, as we have already seen, the component would be thrown out of harmonic relation with the rest, or would be highly damped.

The preceding theory is evidently applicable in considering the motion under the action of the bow, if we assume that this motion is a periodic vibration. It is clear that the bow should be capable of forcing a strictly regular vibration even if the yielding of the bridge be not negligible, provided, of course, the pressure and speed of the bow are in suitable relation to the position of the bowed point and other factors involved. For, the expression given above for the force requisite to maintain such a motion is exactly of the same form as that given on page 298, the only difference (to a first approximation) made by the alteration of free periods due to such yielding being in the actual numerical values of the series of constants k_n and e'_n. In other words, assuming that the motion is periodic, the theory is substantially on the same lines as that already discussed. Various special points however come up for consideration. For instance, what is the period of the vibration forced by the bow when the yielding of the bridge is not negligible? In attempting to find an answer to this question, we should recollect that the series of "constants" k_n and e'_n are not independent of the period of the forced vibration. A complete theory seems at first sight wholly impracticable in the circumstances. It is obvious, however, that in most cases of practical interest, where the yielding of the bridge is sufficiently large to enter into the theory, the alterations of the free periods of the various components in the motion produced by it, would not all be in the same direction. We may therefore infer that when the pressure of the bow is just sufficient to elicit a steady vibration including the usual retinue of upper partials, the period would not ordinarily differ to an appreciably extent from that of the free vibrations of the string with rigidly fixed ends. [The amplitude of vibration is, of course, assumed to be small, or at any rate of the same order of quantities in both cases]. Even when the yielding of the bridge results in a large alteration of the free period of some fairly important harmonic, e.g. of the fundamental or the octave, the same result might be expected. It might be shown that if in such a case, the pressure be sufficient only to excite the said component with the frequency as altered by the yielding of the bridge, and not with the normal frequency, a steady vibration would not be possible. For, when the bow is laid on, the bridge has *initially* no motion, and one of two things would happen. Either the component in question would altogether fail to be excited, or as we shall see later on, it would tend to be excited in a cyclic manner, that is, with alternately increasing and decreasing amplitude. Only the pressure sufficient to excite all the components in their normal frequencies would be capable of maintaining a regular periodic vibration of the usual kind.

Cases in which the bow maintains a periodic vibration with altered frequency

The preceding discussion makes it clear that in practically all cases in which the pressure of the bow is *just* sufficient to maintain a steady vibration, the period would not differ appreciably from that of the free vibration of the string with rigidly fixed ends. But when the pressure is much in excess of the minimum necessary, we might expect different results. For, as already remarked on page 328, when we have a very large maintaining force at our disposal, we are no longer restricted to a consideration of those types of motion which require the smallest forces to elicit. In order that a motion of a period different from the normal be possible, at least two factors are necessary—(*a*) we should have sufficient maintaining forces at our disposal, and (*b*) there should be a factor tending to alter the frequencies of all the components of free vibration in the same direction, that is, to increase all of them or to decrease all. By sufficiently increasing the pressure of the bow, factor (*a*) may be secured. We have already seen that the yielding of the bridge would not, in general, enable us to secure the result stated in (*b*). *The sliding friction of the bow might however lower the frequencies of all the component vibrations*, in much the same way as in the dynamics of a particle, the viscous resistance is found to result in an increase of the period of vibration. In other words, it is quite possible that when the pressure of the bow is sufficiently large, it might tend to lower the frequency of vibration in addition to maintaining the motion[19]. As, however, the friction exerted by the bow is not entirely determinate, we cannot ordinarily expect a regular vibration of the string under such circumstances. But under special conditions, as for instance in the region of frequencies within which the resonance of the instrument is considerable, we may expect to meet with cases in which a high bowing pressure elicits a *regular* vibration with a frequency appreciably or even considerably less than the normal. The greater the pressure of the bow, the larger would be the possible lowering of

[19] With reference to the possibility of a maintained vibration in such cases having a period different from that of free vibration, see Lord Rayleigh's *Theory of Sound*, I, Arts. 64 and 68(*a*).

frequency. For the reasons explained on pages 328 and 329, the phenomenon would be most prominent when the bow is applied at points exactly coinciding with important nodes, such as $l/3$ or $l/4$ or $l/5$, and would be inappreciable if the bow be applied near an end of the string, or at points close to, but not coincident with, important nodes.

Various other important questions arise in regard to the theory of the periodic vibrations of a uniform string forced by the bow. Has the construction of the instrument on which the string is mounted any effect in determining the possibility or otherwise of any stated form of vibration? What is the effect on the motion of the string of muting the bridge or otherwise loading the mobile parts of the instrument? How do the thickness, length and tension of the string influence the character of its motion under the action of the bow? What is the relation between bowing pressure and velocity? What happens when the amplitude of the maintained vibration is too large for the theory of "small" oscillations to be strictly applicable? These are extremely important questions for which it is essential to find an answer in order to arrive at a proper understanding of our subject. We now proceed to consider the various points raised, one after another, in considerable detail.

Influence of the instrument on the form of vibration of bowed strings

Upon the construction of the instrument depends the motion of the points of attachment of the string forced by its vibrations, and therefore also the reaction of the instrument on the vibrations of the string. This influence may be expressed mathematically in terms of the masses, free periods and damping coefficients of the bridge, and the associated mobile parts of the instrument, and in terms of the tension, length and free periods of the string, and the position of the point on the bridge over which it passes; we have already discussed the formulae necessary for this purpose in some detail. It now remains to consider in what manner, if any, the motion forced by the bow depends on the actual numerical values of the various quantities involved. From the discussion given in sections III, V, X, and pages 290 to 297, it would appear that the sequence of the phenomena obtained by bowing at any given point on the string with varying degrees of pressure or velocity, is to a large extent capable of being found from kinematical considerations depending upon the position of the bowed point, its contiguity or coincidence with some important node-point and the like, taken in conjunction with broad dynamical principles common to all cases likely to arise in practice. But the subsequent mathematical treatment (pages 297 to 327) showed that the actual magnitudes of the constants k_1, k_2, k_3 etc., e'_1, e'_2, e'_3 etc. expressing the relation between the maintaining forces and the maintained motion, are of no small importance. In order to get a connected idea of the theory, we considered on pages 302 to 314 a special class of cases in which all the constants e'_n were put equal to $\pi/2$, and the constants k_n were put equal to nk; this supposition enabled us to effect an analytical summation of the expression for the maintaining force in all such cases. We shall now see in what manner, if any, the conclusions thus arrived at have to be modified when the constants k_n and e'_n have values determined by the actual constants of the instrument on which the string is mounted.

The general equation giving k_n in terms of the constants of the instrument is

$$k_n = 2n\pi T_0[\sin^2(2n\pi l/aT + \psi_n) + \delta_n^2 \cos^2 2n\pi l/aT \cdot \cos^2 \psi_n]^{1/2}/aT \cos \psi_n.$$

If the period of the motion is assumed to be the same as that of the string with rigidly fixed ends, this reduces to the form $k_n = n\pi T_0/l \cdot [\tan^2 \psi_n + \delta_n^2]^{1/2}$, where ψ_n usually depends only on the difference between the free and forced periods of the nth component and δ_n depends on the dissipation of energy. The magnitudes of ψ_n and δ_n for any component in the motion depend mainly on those normal modes of vibration of the instrument which have frequencies nearest to it. As any actual instrument would have numerous free periods of its own forming an ascending scale of frequencies, it is not a violent supposition in trying to take a general view of the phenomena to assume that $[\tan^2 \psi_n + \delta_n^2]^{1/2}$ is of the same order of quantities for all values of n. We would then have $k_n = nf(n)k$, where $f(n)$ varies in an irregular (not progressive) manner as n is increased, and k is a constant. If we put $f(n)$ equal to unity, and make the further supposition that e'_n is equal to $\pi/2$ for all values of n, we have precisely the same conditions as those assumed in calculating the form of the frictional force curves on pages 302 to 314. The general value of e'_n is

$$\pi/2 - \tan^{-1}[-\sin(2n\pi l/aT \cdot + \psi_n)/\delta_n \cos \psi_n \cos 2n\pi l/aT].$$

When the period of forced vibration is the same as that of the string with rigidly fixed ends, this reduces to the form

$$e'_n \sim \pi/2 - \tan^{-1}(-\tan \psi_n/\delta_n).$$

From this, it is seen that when the dissipation of energy is the principal factor and not the difference, if any, between the free and forced periods, e'_n is practically equal to $\pi/2$.

Though, as we have seen, the assumptions that $e'_n = \pi/2$ and $k_n = nk$ serve as a broad basis for discussion, the values of k_n and e'_n in individual cases, especially for the first eleven or twelve harmonics which are of importance, may indeed stand to each other in very different relations. As an illustration of this, we may consider a few examples. The first is that in which the gravest mode of free vibration of the instrument is far higher in frequency than the gravest mode of vibration of the string, and has, say, exactly ten times its frequency. So far as the first ten harmonics are concerned, we may altogether neglect the influence of the modes of free vibration of the instrument higher in frequency than the gravest. The values of $\tan \psi_n$ for the first nine harmonics increase in proportion to the ratio $n/(100 - n^2)$, and for the tenth is zero. The value of δ_n similarly increases with n in the ratio $n^2/(100 - n^2)^2$ if $n < 10$, and when $n = 10$, becomes very large. Since $k_n = n\pi T_0/l[\tan^2 \psi_n + \delta_n^2]^{1/2}$, it is seen that is would increase at a vastly greater rate than in proportion to n, as we move up from $n = 1$ to $n = 10$. In the curves for the maintaining force calculated on this basis for any assumed type of motion and position of the bowed point, the first three or four harmonic components would be far less pronounced in comparison with the rest than in the curves found when k_n is simply proportional to n. One obvious consequence of this would be that the critical pressure required for eliciting types of vibration in which the fundamental or octave is dominant, would be smaller than usual, and for those in which the higher harmonics are dominant, would be increased; so far from making it difficult to excite vibrations of the latter kind, this might, for some positions of the bowed point, actually make the production of such types of motion easier by increasing the difference between the critical pressures of alternative modes possible in any given case. An exception should however be made in respect of the extreme case of a type in which the tenth harmonic is dominant, as owing to the resonance of the instrument, this would require an unusually large force to elicit, and might thus actually be incapable of being realised in experiment.

As another example, we shall consider the case in which the gravest period of free vibration of the instrument is graver than the fundamental period of the string, and the other periods are assumed to be so small that their influence on the first eleven or twelve harmonics is negligible. As an extreme case, we may take the period of the instrument to be infinitely long, that is, regard the bridge as having inertia subject to damping but without spring. The values of $\tan \psi_n$ and δ_n would then vary as $1/n$ and $1/n^2$ respectively. The value of k_n would therefore actually decrease as n is increased. Since the amplitudes of the harmonics in the vibration are of the order $1/n^2$, the magnitude of the higher components in the maintaining force would be negligibly small, the first three or four terms of the series being pre-eminent. The prominence of the first component would be even more marked, if instead of taking the free period of the instrument as infinitely long, we take it as only a little greater than, or actually equal to, the fundamental period of the string. For example, let the period of the instrument be $\frac{11}{10}$ times the fundamental period of the string. The values of ψ_n and δ_n would then vary as $n/(100 - 121n^2)$ and $n^2/(100 - 121n^2)^2$ respectively. It is obvious that k_1 would then be much greater than k_2, k_3, etc. This preponderance of the first component in the maintaining force would have an important influence on the possibility of certain types of motion under the action of the bow. For example, we may consider the case of a string bowed near one end. If the pressure of the bow be sufficient, we should get the first type of vibration in which the principal component is dominant, the motion at the bowed point being of the simple two-step zig-zag form. When the pressure is reduced below the limit necessary for this type of vibration, the general theory leads us to expect that the motion would alter to the second type of vibration in which the motion at the bowed point is of the four-step zig-zag form (see pages 292 and 293), or to one of the transitional forms intermediate between the first and the second types (see pages 324 to 327). But in the present case, the fact that k_1 is large leads to a difficulty. In order that a four-step zig-zag type of motion be possible at the bowed point, it is necessary that the frictional force should be equal, or nearly equal in the two stages at which slipping occurs. This, in its turn, is only possible when the *second* component (not the *first*) is prominent in the maintaining force. (This remark will be clearly understood on a reference to the last three pairs of curves in figure 21 on page 313). We see that under the conditions assumed, the only steady modes of vibration ordinarily possible would be those in which the first harmonic is either dominant or entirely absent. This remark is of importance, as will appear later when we consider the theory of cyclic vibrations.

Other examples may also be furnished of the manner in which the values of the individual constants k_1, k_2, etc. may affect the sequence of the phenomena obtained by bowing at any given point on the string with varying pressures or velocities. For instance if k_1 is very large, it would require a considerable pressure to elicit the first type of vibration in which the fundamental is dominant, even when the bow is applied at a point far removed from the end of the string, e.g. between $l/2$ and $l/3$. We may then find, as an exception to the general rule stated on pages 327 and 328, that this is not the type which requires the smallest pressure to elicit within the range referred to. Similarly, a large k_2 would result in the octave being very prominent in the curves for the maintaining force, and we might find the sequences indicated for the range $l/2$ to $l/3$ in figure 15 on page 296 entirely altered in such a case, especially in regard to modes of vibration involving complicated motions at the bowed point.

Relation of bowing pressure to the tension, linear density and length of string

From a musical point of view, the thickness of the string is a factor of great importance. Leaving aside the complications due to possible non-uniformity or imperfect flexibility of the string (which may be considered separately), the thickness of the string is principally of importance as it determines the linear density and therefore also the tension necessary for any given frequency of vibration. For purposes of comparison of the bowing pressures necessary, we may assume the length of the string and the frequency of vibration to be the same in all cases. For a given velocity of bowing, the amplitude of vibration would also be constant. The nth harmonic component of the maintaining force is proportional to k_n, that is $n\pi T_0/l[\tan^2\psi_n + \delta_n^2]^{1/2}$. Since T_0 is the tension of the string, and since $\tan\psi_n$ and δ_n are also proportional to T_0, the maintaining forces necessary are proportional to the square of the tension, that is, also to the square of the linear density of the string. This becomes intelligible when we remember that the rate of emission of energy from the instrument is also proportional to the square of the tension or of the linear density. The use of heavy strings is thus indispensable for obtaining a powerful tone, and if we assume that the frictional coefficients are the same, the minimum bowing pressure required for any given type of vibration is also proportional to the square of the linear density. With strings of the same material, the linear density is proportional to the square of the diameter, and the bowing pressure necessary increases as the fourth power of the diameter. It is worthy of notice that the maximum possible area of contact between the bow and the string increases at a much smaller rate than the bowing pressure necessary, as the thickness of the string is increased. Further, as a thin string requires a much smaller bowing pressure than a thick one, a greater variety of vibration-forms would, in general, be possible with the former, within a given range of bowing pressures, than with the latter.

Comparison may also be made of the bowing pressures necessary when the tension and length of the string are simultaneously varied in such a manner as to retain the frequency of vibration unaltered. It is easily seen that under the conditions assumed, the amplitude of vibration remains constant, provided the velocity of the bow and the relative position of the point at which it is applied, are the same in all cases. The frequency being constant, the tension necessary varies as the square of the length of the string, and the bowing pressure necessary is also proportional to the square of the length of the string.

The manner in which the bowing pressure necessary for the maintenance of the motion varies when the frequency of vibration is altered by changing the tension or length of the string, cannot however be expressed in such a simple manner as in the preceding cases. Taking, for instance, the case in which the frequency is altered by varying the tension of a string of constant length, the expression for the bowing pressure is seen to consist of two factors, one of which increases continuously with the tension, and the other fluctuates between a series of maxima and minima. As the frequency increases, the amplitude of vibration for a given velocity of bowing falls off, being inversely proportional to the square root of the tension; but the values of constants k_1, k_2, etc. contain a factor proportional to the square of the tension. The bowing pressure therefore contains a factor proportional to $T_0^{3/2}$ which increases continuously as T_0 is increased. The other factor depends on the relation between the frequencies of the impressed vibration and the natural frequencies of resonance of the instrument. For example, if the instrument has only one free period of vibration, one of the factors which determines the bowing pressure has a series of maxima corresponding to the cases in which the first, second, third or higher harmonic has the same frequency as the free vibration of the instrument. The bowing pressure necessary, if plotted as a function of the frequency of vibration, should then show a series of maxima and minima, the highest value being reached when the frequency of the principal component in the forced vibration of the string is the same as that of the free vibration of the instrument. If the instrument has more than one free period of vibration, the variation of bowing pressure with frequency would be determined by the relation of the impressed frequencies of vibration to the frequencies of all the free periods of vibration of the instrument. In general, we should expect the pressure-frequency curve to show a series of maxima and minima, the most important maxima corresponding to the frequencies of resonance of the instrument to the dominant harmonic in the vibration. Superposed on these maxima and minima, we should expect to find a tendency of the bowing pressure necessary to increase continuously as the tension is increased. Similar results should be obtained when the frequency of vibration is altered by decreasing the length of the string, both the tension and the ratio in which the bowed point divides the string being retained constant.

Effect of muting on the minimum bowing pressure

The next question to be dealt with is the effect of muting or otherwise loading the mobile parts of the instrument upon the periodic vibrations elicited by the bow. The addition of mass represented by the mute results, of course, in an alteration of the constants of the instrument relevant to our present investigation. The free periods of the instrument are all increased. It is obvious that this would profoundly influence the character of the forced vibration of the body of the instrument and therefore also its reaction on the string. We have also to consider the alteration in the modes of vibration of the various parts of the instrument produced by the load. The point at which the mute or other load is added tends to become a node, or point of rest, so far as the modes of higher frequencies are concerned; on the other hand, the point loaded would tend to become a point of maximum vibration for the gravest mode or modes. The contiguity or otherwise of the load and of the point on the bridge over which the string passes, is thus a matter of some importance. Considering the bridge, belly and the air enclosed in the instrument as a single dynamical system, we may use the Lagrangian equations of motion to investigate the alteration in its forced vibrations produced by the load.

Let ϕ_1, ϕ_2, ϕ_3, etc. be the normal coordinates of the system, and Φ_1, Φ_2, etc. be the normal force-components corresponding to a particular harmonic in the force exerted on the bridge by the vibrating string. As the result of the addition of the load, ϕ_1, ϕ_2, ètc. are no longer the normal coordinàtes, and we may write the kinetic and potential energies of the system in the form[20]

$$T = \tfrac{1}{2}a_1\dot{\phi}_1^2 + \tfrac{1}{2}a_2\dot{\phi}_2^2 + \ldots\ldots + \tfrac{1}{2}a(f_1\dot{\phi}_1 + f_2\dot{\phi}_2 + \ldots)^2$$
$$V = \tfrac{1}{2}c_1\phi_1^2 + \tfrac{1}{2}c_2\phi_2^2 + \ldots\ldots .$$

[20] *Cf.* Routh's *Advanced Rigid Dynamics*, Sixth Edition, Secs. 76 to 78, and Lord Rayleigh's *Theory of Sound*, Second Edition, Art. 92(*a*), where the effects of addition of inertia or constraints on *the free*

Footnote (contd. on p. 352)

The Lagrangian equations for the forced vibration are

$$(a_1\lambda^2 + c_1)\phi_1 + f_1\alpha\lambda^2(f_1\phi_1 + f_2\phi_2 + \ldots) = \Phi_1$$
$$(a_2\lambda^2 + c_2)\phi_2 + f_2\alpha\lambda^2(f_1\phi_1 + f_2\phi_2 + \ldots) = \Phi_2$$
$$\text{etc.} \qquad \text{etc.}$$

Multiplying the first equation by $f_1/(a_1\lambda^2 + c_1)$, the second equation by $f_2/(a_2\lambda^2 + c_2)$ and so on, and then adding the results, we find

$$(f_1\phi_1 + f_2\phi_2 + \text{etc.}) = \frac{\sum f\Phi/(a\lambda^2 + c)}{1 + \alpha\lambda^2\sum f^2/(a\lambda^2 + c)}.$$

Since $\frac{1}{2}\alpha(f_1\phi_1 + f_2\phi_2 + \text{etc.})^2$ is the kinetic energy of the load, α being its mass, its forced vibration is given by $(f_1\phi_1 + f_2\phi_2 + \text{etc.})$. Writing $\lambda^2 = -m^2$ and dividing out by the product $a_1a_2a_3$, etc., the preceding expression reduces to

$$\frac{\Phi_1 f_1/a_1\cdot(n_2^2 - m^2)(n_3^2 - m^2), \text{ etc.} + \text{similar terms}}{(n_1^2 - m^2)(n_2^2 - m^2), \text{ etc.} - am^2[f_1^2/a_1(n_2^2 - m^2)(n_3^2 - m^2), \text{ etc.} + \ldots]}$$

where $n_1/2\pi$, $n_2/2\pi$, $n_3/2\pi$, etc. are the natural frequencies of the system in ascending order of magnitude before the addition of the load. The forced vibration for any given frequency of the impressed force is thus inversely proportional to the magnitude of the expression $(n_1^2 - m^2)(n_2^2 - m^2)$, etc. $-\alpha m^2[f_1^2/a_1\cdot(n_2^2 - m^2)(n_3^2 - m^2)$, etc. $+ \ldots]$ which when put equal to zero gives also the equation for the free periods of the system as altered by the load. We may readily trace the effect of increasing the mass α of the mute from zero to any desired value, upon the value of this expression (say Δ), and therefore also upon the amplitude of the forced vibration. Taking first a case in which the frequency of the impressed force is smaller than the smallest natural frequency of the system $(m < n_1)$, it is seen that both terms in Δ are positive, and their difference must

vibrations of a system are discussed. Strictly speaking, since the kinetic energy of a particle of mass a is $\frac{1}{2}a(x^2 + y^2 + z^2)$, the addition to the kinetic energy of the system should be represented by three terms thus:

$$T = \tfrac{1}{2}a_1\dot{\phi}_1^2 + \tfrac{1}{2}a_2\dot{\phi}_2^2 + \ldots + \tfrac{1}{2}a(f_1\dot{\phi}_1 + f_2\dot{\phi}_2 + \ldots)^2$$
$$+ \tfrac{1}{2}a(g_1\dot{\phi}_1 + g_2\dot{\phi}_2 + \ldots)^2 + \tfrac{1}{2}a(h_1\dot{\phi}_1 + h_2\dot{\phi}_2 + \ldots)^2.$$

However, for the sake of simplicity in interpreting the results, we assume (following Routh) that the addition to the kinetic energy consists of only one term of this type. This would be quite correct if the displacement of the particle due to variations of the coordinates ϕ_1, ϕ_2, etc. were all in the same straight line, and in any case may be regarded as sufficient for our present purpose. Lord Rayleigh has discussed the general theory for the free vibrations when the addition to the kinetic energy consists of any number of terms less than the number of degrees of freedom of the system. *It may be remarked here that the effect of the sound-post of the violin could probably be regarded as equivalent to that of a geometrical constraint imposed on the vibrations of the sides of the sound-box, and if this is the case, the effect could be discussed mathematically in a simple manner.*

therefore gradually diminish as α is increased, till for a particular value of α, it vanishes altogether. With subsequent increases of α, Δ changes sign, its numerical value becoming larger and larger. The effect of the load is thus, at first, to *increase* the forced vibration, till a stage is reached at which the latter becomes very large. This occurs when the frequency of the impressed force is the same as the lowest natural frequency of the system as altered by the load.[21] With a further increase in the load, the forced vibration diminishes, finally vanishing when the load is very large.

We may next consider a case in which the frequency of the impressed force lies between two of the natural frequencies of the system. The result of the load is to decrease the natural frequencies of the system, and it is a well known result (which is easily deduced from the equation for free periods) that, whatever be the magnitude of the load, the altered frequencies form a series *separating* the original frequencies of the system. Thus, if

$$N_1/2\pi, N_2/2\pi, N_3/2\pi, \text{etc.}$$

be the natural frequencies of the system as altered by the load (assumed to be very great), the quantities

$$N_1, n_1, N_2, n_2, N_3, n_3, \text{etc.},$$

form an ascending scale of magnitudes. (N_1 is practically zero if the load be very large). The sequence of changes in the forced vibration due to a gradual increase in the load is sufficiently illustrated by considering a case in which $n_1 < m < n_2$. This has to be sub-divided into two categories (*a*) [$n_1 < m < N_2$] and (*b*) [$N_2 < m < n_2$]. In the category (*a*), the determinant Δ is negative throughout, and increases numerically as the mass α of the load is increased. The effect of the load is thus to *diminish* the forced vibration throughout. In the category (*b*), Δ is negative when α is zero, and decreases numerically as α is increased, becoming zero for a particular value of the latter. For larger values of α, Δ is positive. The effect of the load in the second category of cases is thus to *increase* the forced vibration up to a certain stage at which it becomes very large, subsequent additions of load *decreasing* the forced vibration, till it finally vanishes in the limit when the load is large. The difference between the two categories is that in the former, the frequency of the impressed force does not, for any load, coincide with a natural frequency of the system.

So far, we have only considered the forced vibration of the system at the particular point at which the load is fixed. For our present purpose, it is necessary to know also the effect of the load on the forced vibration of the other parts of the

[21] Of course, in such a case, the equations would have to be modified so as to take the dissipation of energy into account.

system, particularly the point at which the string passes over the bridge. [This, of course, need not be the same point as that at which the load is fixed]. The motion of the system is fully defined by the coordinates ϕ_1, ϕ_2, etc., though these are not the normal coordinates when the load is attached to the system. The values of ϕ_1, ϕ_2 etc. may be found by eliminating $(f_1\phi_1 + f_2\phi_2 + ..)$ from the Lagrangian equations of motion.[22]

We obtain as the result,

$$\begin{aligned}\Delta\phi_1 = {} & \Phi_1/a_1\cdot(n_2^2 - m^2)(n_3^2 - m^2), \text{ etc.,}\\ & + \alpha m^2/a_1\cdot[f_2/a_2(f_1\Phi_2 - \Phi_1 f)(n_3^2 - m^2)(n_4^2 - m^2), \text{ etc.,}\\ & + f_3/a_3(f_1\Phi_3 - \Phi_1 f_3)(n_2^2 - m^2)(n_4^2 - m^2), \text{ etc.,}\\ & + \ldots.],\end{aligned}$$

and analogous expressions for $\Delta\phi_2$, etc. Writing the transverse yielding of the string at the bridge in the form $(r_1\phi_1 + r_2\phi_2 + \text{etc.})$, we find $\Phi_1 = r_1 E\cos mt$, $\Phi_2 = r_2 E\cos mt$, and so on, where $E\cos mt$ is the periodic transverse force exerted by the vibrating string. If the point at which the load is fixed coincides with that over which the string passes, $f_1/r_1 = f_2/r_2 = f_3/r_3 = \ldots.$, and we have

[22]In the particular case in which the impressed force acts on the system at the point at which the load is fixed the effect of the latter on the forced vibration may be readily found from elementary considerations without the use of the Lagrangian equations, provided it is assumed that this point is capable of movement only in a single direction. (It might, for instance, be supposed that the load is fixed on a string, bar, membrane or plate, capable only of transverse vibration at the point of application of the force). If the forced vibration at this point is $kE\cos mt$ when the component of the impressed force resolved in the same direction is $E\cos mt$ and there is no load, we may write the forced vibration as $k' E\cos mt$ when a load of mass α is attached to it. The alteration in the forced vibration is simply due to the reaction of the load, and we find at once that $k' = k/(1 - k\alpha m^2)$. If k be negative, that is, if the forced vibration in the absence of any load at the driving point be in a phase opposite to that of the impressed force, the effect of the load is *to decrease the forced vibration in the ratio* $1:(1 - k\alpha m^2)$ *which is the same at every point of the system.* On the other hand, if k be positive, that is, if the vibration and the force at the driving point be in the same phase when there is no load, the effect of the load is, in the first instance, *to increase the forced vibration of every part of the system* in the ratio $1:(1 - k\alpha m^2)$. When the load is such that $k\alpha m^2 = 1$, the forced vibration is very large, showing that at this stage the frequency of the impressed force coincides with a natural frequency of the system as altered by the load. Subsequent additions of load result in a continuous decrease of the forced vibration at every point of the system which finally vanishes for a very large load. These results are exactly the same as those deduced from the general Lagrangian equations, and clearly show from elementary dynamical considerations that the addition of a load at a given point of a system cannot lower the natural frequencies in such manner that more than one of them falls from above to below any assigned frequency. This is practically Routh's theorem regarding the separation of the roots of Lagrange's determinant as altered by the addition of inertia to any part of a dynamical system, and affords a clearer view of the fundamental nature of this theorem.

$(f_1\Phi_2 - \Phi_1 f_2) = (f_1\Phi_3 - \Phi_1 f_3) = \ldots = 0$. From this it follows

$$\Delta\phi_1 \doteq \Phi_1/a_1 \cdot (n_2^2 - m^2)(n_3^2 - m^2), \text{ etc.,}$$

$$\Delta\phi_2 = \Phi_2/a_2 \cdot (n_1^2 - m^2)(n_3^2 - m^2), \text{ etc.,}$$

and so on.

In other words, each coordinate of the system is determined exclusively by the corresponding generalized component of force, and as the load is altered, *the motion of every point on the system is affected in the same way,* increasing or decreasing in inverse proportion with the determinant Δ, and vanishing when the load is increased indefinitely. A similar result would be obtained even if the point of attachment of the load is not the same as that over which the string passes, provided the ratios of the displacements at the two points due to variations of the coordinates are all identical, that is if $f_1/r_1 = f_2/r_2 = \ldots\ldots$. In general, however, if the two points are not coincident, these quantities would not be identical, and an addition of load would not alter the forced vibration at different points of the system in the same proportion. The forced vibration would not also vanish when the load is increased indefinitely; for α (the mass of the load) appears in the numerator as well as in the denominator of the expression for each of the coordinates. In such cases, the forced vibration as altered by the load may be considered as consisting of two parts. The first part alters with increasing load in inverse proportion to the value of Δ, (the determinant for free periods), and vanishes when the load is indefinitely increased. The second part alters with increasing load in the ratio α/Δ, and its changes would thus be somewhat similar to that of the first part, except that when α is very large, it does not vanish but reaches a finite limiting value.

It is instructive to trace the effect of successive small changes in the load on the frequencies and modes of free vibration of the system and upon its forced oscillations. To a first approximation, the alteration in the free period of a normal mode produced by a small load is unaffected by the change of type which accompanies it,[23] and may be readily calculated from the modified equations of motion by putting all the other coordinates equal to zero. For instance, with the preceding notation, the frequencies as altered by a small load α are given by the equations

$$m_1^2 = c_1/(a_1 + \alpha f_1^2), \quad m_2^2 = c_2/(a_2 + \alpha f_2^2) \quad \text{and so on.}$$

The alteration of free period is in each case determined by the increase in the kinetic energy of the system due to the addition of load. *This furnishes us with a simple quantitative method for exploring the motion of the system,* and determining the ratio of the amplitudes of its vibration at different points in each of the normal

[23] Lord Rayleigh's *Theory of Sound*, I, Art. 88.

modes. With larger loads, the alteration in the types of free vibration have an influence on the free periods which cannot be neglected. This is obvious at once on comparing the approximate values of the frequencies found as above (all of which become infinitely small when α is increased indefinitely), with those given by the equation $\Delta = 0$, the roots of which, according to the well-known theorem due to Routh, all remain finite (except the smallest), when the load is increased indefinitely. It is possible to find the normal coordinates of the loaded system by the general method of transformation. (see, for example, Routh's *Advanced Rigid Dynamics*, section 56). But a more instructive method is that used by Lord Rayleigh (*Theory of Sound*, I, Art. 90). The normal vibrations of the system in the absence of any load correspond to a variation of but one of the coordinates ϕ_1, ϕ_2, ϕ_3, etc. at a time. The alteration of any one of these types produced by the addition of a small load may be expressed by superposing small synchronous variations of the other coordinates upon that of the principal coordinates involved. For example, the principal vibration given by the coordinate ϕ_1 as modified by a small load α would involve also small variations in the coordinates ϕ_2, ϕ_3, etc. the magnitudes of which are given by the ratios

$$\phi_2/\phi_1 = n_1^2/(n_2^2 - n_1^2) \times \alpha f_1 f_2/a_2$$

$$\phi_3/\phi_1 = n_1^2/(n_3^2 - n_1^2) \times \alpha f_1 f_3/a_3 \quad \text{and so on.}$$

The modification of the natural types produced by the load may become very important in some cases. For instance, if the vibration corresponding to the coordinate ϕ_2 involves large displacements at the point of attachment of the load, and its frequency does not differ much from that of the vibration ϕ_1, the latter as modified by the load would involve also large synchronous variations of ϕ_2. When the load is large enough to depress the pitch of the normal vibration in ϕ_2 sufficiently, the two principal vibrations may even completely interchange their characters. Examples of this will be referred to later. In fact, when the load is sufficiently great, the gravest modes of vibration are those involving large displacements of the point at which it is attached, and similarly the higher modes are those which involve little or no displacement of this point. This is a feature which is of importance in reference to our present investigation.

It has been stated above that the effect of the load on the forced vibration is in some cases to increase it, and in other cases to decrease it, and that in the particular case in which the load is very large and it is fixed to the system at the point of application of the force, the forced vibration vanishes. It may now be shown that, in general, the effect of a *moderate* load is to *increase* the forced vibrations of graver pitch, and to *diminish* those of higher pitch. This is a consequence of the fact that a dynamical system such as that with which we are now concerned has a number of free periods which form a series, and that a moderate load fixed at any given point is sufficient to make the pitch of the higher modes of vibration approximate to their limiting values, or at any rate pass the first stage in which the effect of the load is to *increase* the forced vibration. This may be proved in the following manner. The kinetic energy of the system in the absence of any load may be written in the form

$$\tfrac{1}{2}a_1\dot{\phi}_1^2 + \tfrac{1}{2}a_2\dot{\phi}_2^2 + \tfrac{1}{2}a_3\dot{\phi}_3^2 + \text{etc.}$$

For any actual system (such as a violin) composed of bars and elastic plates, the quantities a_1, a_2, a_3, etc. are all of the same order of magnitudes. From this fact and the approximate formula given previously, it follows that a given *small* load would lower the frequency of one of the higher modes by roughly about the same *interval* as it would in the case of one of the graver modes. The *absolute* lowering of the frequency would thus be much greater in the former case. By Routh's theorem, the maximum permissible lowering is equal to the difference of the frequencies of the given mode and that of the one next below it. If, as is generally the case with the higher modes, this difference is not large, a small load would be sufficient to make the frequency approximate to its limiting value, or at any rate pass the stage up to which the forced vibration is *increased* in amplitude.

From what has been said above, it is clear that the effect of a mute is to *increase* the yielding of the bridge and the dissipation of energy for vibrations of low pitch, and to *decrease* them for vibrations of high pitch. When the string is excited by bowing, the effect of the harmonic components of the force exerted by it on the bridge, would be variously influenced by the mute, according to the frequency of each. If the pitch of the fundamental vibration of the string is sufficiently high, the mute would decrease the forced vibration of the instrument, and the bowing pressure necessary for exciting the string would be reduced. On the other hand, if the pitch of the fundamental is sufficiently low, the effect of the principal components of force would be increased by the mute, and the bowing pressure necessary would be increased. The constants k_1, k_2, k_3, etc. expressing the relation between the periodic forces exerted by the bow and the vibration of the string maintained by it, would be profoundly modified by the action of the mute, and the possible influence of this on the modes of vibration of the string excited by the bow may be discussed as in the preceding pages.

Relation between bowing pressure and speed

From equations (22) and (25) given on pages 298 and 299 respectively, it is easy to see the relation between the amplitude of vibration of the string (as determined by the speed of the bow) and the minimum bowing pressure necessary for cliciting any given type of vibration. When the speed of the bow increases, the amplitudes of the harmonic components of the vibration in the given mode increase in proportion. From (22) and (25), we find accordingly that

$$P \nless S \cdot v_B/(\mu - \mu_A)$$

where P is the pressure required to excite the given type of vibration, S is a constant depending upon the type and the point of application of the bow, v_B is the velocity of the bow, and $(\mu - \mu_A)$ is the difference between the statical and dynamical coefficients of friction. If $(\mu - \mu_A)$ be independent both of the pressure and the speed of bowing, the minimum bowing pressure necessary is proportional to the speed of the bow. If, however, $(\mu - \mu_A)$ depends on the velocity of slip, this result does not hold good, and in the particular case in which the difference between the statical and dynamical coefficients of friction is proportional to the velocity of slip at which the latter is measured, the minimum bowing pressure is actually independent of the velocity of the bow. In practice, it may be expected, the truth lies between the two extremes. As remarked earlier, the property of an efficient generator such as rosined horse-hair is that the friction diminishes with increasing velocity, at first rapidly, and later more slowly. This may be represented graphically as in figure 28(a), the friction reaching a limiting value for large relative velocities. If we assume that the frictional force is proportional to the pressure, in other words that the frictional coefficient depends only on the relative velocity, the graphical relation of the type shown in the figure may be approximated to by an analytical expression of the form

$$(\mu - \mu_A) = \mu_0(1 - (\exp(-v/v_1)))$$

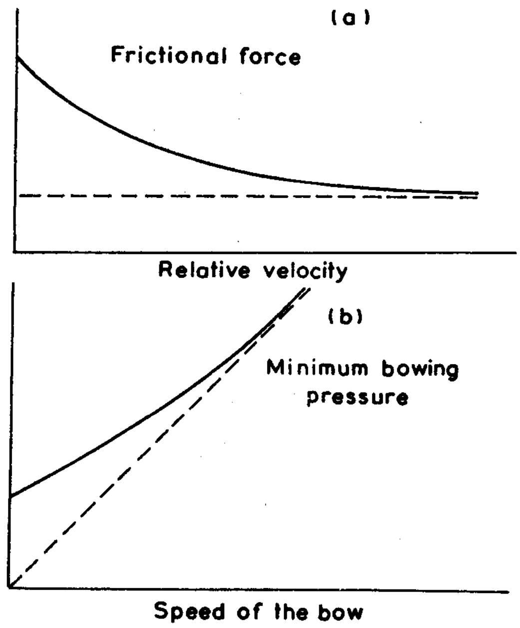


Figure 28

where μ_0 and v_1 are constants, and v is the relative velocity. For any given type of vibration, the relative velocity during the stages of slipping at the bowed point is proportional to the speed of the bow. Writing $v = rv_B$, where r is a number, the minimum bowing pressure is given by the formula

$$P = S/\mu_0 \cdot \frac{v_B}{1 - \exp(-v_B \cdot r/v_1)}.$$

This is shown graphically in figure 28(b). For large bowing speeds, the bowing pressure necessary practically varies as the speed; but for smaller speeds, the pressure required is greater than in proportion to the speed of the bow or the amplitude of the maintained vibration. For very small speeds, the pressure required would, as seen from the graph, converge upon a finite limiting value. This limit is easily shown to be given by

$$P = S/\mu_0 \cdot v_1/r$$

and is very small if the 'decay' of the friction from the statical to the limiting dynamical value be sufficiently rapid.

$$\mathrm{d}P/\mathrm{d}v_B = S/\mu_0 \cdot \frac{1 - \exp(-v_B \cdot rv_1)(1 - v_B \cdot r/v_1)}{(1 - \exp(-v_B \cdot r/v_1))^2}.$$

From this, it follows that

$$Lt\left(\frac{\mathrm{d}P}{\mathrm{d}v_B}\right) = S/2\mu_0 \quad \text{and}$$

$$v_B = 0$$

$$Lt\left(\frac{\mathrm{d}P}{\mathrm{d}v_B}\right) = S/\mu_0.$$

$$v_B = \infty.$$

The preceding discussion is based on the assumption that the speed of the bow is not so large that the resulting vibration exceeds the limit for which the theory of "small" oscillations is applicable. We now proceed to consider the effect of increasing the amplitude of vibration beyond this limit.

Effect of large amplitudes

The question arises whether a finite amplitude of vibration has any effect on the frequency of a bowed string, and in what manner, if any, the finiteness of the motion influences the mode of vibration of the string and of the instrument. This may be considered in the following way. Assuming the motion under the action of the bow to be periodic and of the form

$$y = \sum B_n \sin\frac{n\pi x}{l}\sin\left(\frac{2n\pi t}{T} + e_n\right),$$

the finiteness of the coefficients B_1, B_2, etc. necessitates our considering certain terms of the second order in the forces acting upon each element of the string and upon its point of attachment to the bridge. The length of the string at any instant from $x = 0$ to $x = l$, is given by

$$\int_0^l [1 + (\mathrm{d}y/\mathrm{d}x)^2]^{1/2}\,\mathrm{d}x = l + \int_0^l \tfrac{1}{2}(\mathrm{d}y/\mathrm{d}x)^2\,\mathrm{d}x, \quad \text{(appx.)}$$

$$= l + \pi^2/8l \cdot \sum n^2 B_n^2 - \pi^2/8l \sum n^2 B_n^2 \cos\left(\frac{4n\pi t}{T} + 2e_n\right).$$

If the ends of the string have no longitudinal motion, the alterations in the length due to the finite amplitude of vibration would result in corresponding alterations of the tension. These may be of considerable importance if (as is generally the case) the modulus of extension of the string be a large quantity in relation to its tension when at rest. It is seen that there is a non-periodic increase of tension proportional to $\sum n^2 B_n^2$, and there are also periodic changes of tension. The influence of these alterations of tension upon the forced vibrations is readily found by analogy with the case of a simple vibrator executing forced oscillations of large amplitude. The equation of motion of a symmetrical vibrator under periodic forcing is

$$\ddot{u} + k\dot{u} + (n_1^2 + \beta u^2)u = E\cos mt.$$

If we write the solution of this in the form

$$u = A_1 \sin mt + B_1 \cos mt + A_3 \sin 3mt + B_3 \cos 3mt + \text{etc.}$$

it is found on substitution that to a first approximation

$$(n_1^2 - m^2 + F)A_1 - km\,B_1 = 0$$
$$(n_1^2 - m^2 + F)B_1 + km\,A_1 = E$$

where

$$F = \frac{3\beta}{4}(A_1^2 + B_1^2)$$

and is thus proportional to the square of the amplitude of vibration. It is evident from this that the effect of the term βu^2 in the expression for the restitutional coefficient is equivalent to an alteration in the natural frequency of the vibrator; instead of n_1^2 we have to write simply

$$n_1^2 + \frac{3\beta}{4}(A_1^2 + B_1^2).$$

It should be remarked that the restitutional coefficient $(n_1^2 + \beta u^2)$ may be expanded and written in the form

$$n_1^2 + \frac{\beta}{2}\cdot(A_1^2 + B_1^2) - \frac{\beta}{2}\cdot(A_1^2 + B_1^2)\cos 2(pt + \varepsilon)$$

where

$$= \tan^{-1} A_1/B_1.$$

It follows that the effect of the *periodic* variation in spring of double frequency

$$\left[-\frac{\beta}{2}(A_1^2 + B_1^2)\cos 2(pt + \varepsilon)\right]$$

is equivalent to a *permanent* increase in spring of half the amount

$$\left[\frac{\beta}{4}(A_1^2 + B_1^2)\right],$$

in respect of the principal oscillation. Returning now to the case of the bowed string, it is obvious that the non-periodic increase of tension which is

$$\frac{\text{Young's modulus}}{\text{Tension}} \times \frac{\pi^2}{8l^2}\sum n^2 B_n^2$$

raises the natural frequencies of all the component vibrations and does not disturb their harmonic relation. The periodic part of the variation of tension is

$$-\frac{\text{Young's modulus}}{\text{Tension}} \times \frac{\pi^2}{8l^2}\sum n^2 B_n^2 \cos\left(\frac{4n\pi t}{T} + 2\varepsilon_n\right).$$

Each harmonic component of this periodic variation is equivalent to a further increase, namely,

$$\frac{\text{Young's modulus}}{\text{Tension}} \times \frac{\pi^2}{8l^2}\cdot\frac{n^2 B_n^2}{2}$$

in the tension of the string, but this increase, unlike the first, is effective only in respect of the particular (n^{th}) harmonic component of the vibration having half the frequency of the variation of spring, and not to all alike. This is readily seen on substituting the variable tension in the normal equation for free vibrations. Since, in the case of the bowed string, the amplitudes B_n vary generally as $1/n^2$, $n^2 B_n^2/2$ also varies as $1/n^2$, and the harmonic relation of the natural frequencies is thus disturbed. The frequencies of all the natural vibrations are raised, but those of the higher components to a lesser extent in proportion than in the case of the graver components. In practice, this would be probably set off to some extent by the effect of stiffness which raises the pitch of the higher components to a relatively greater extent. The general result however remains that the effect of a finite amplitude of vibration is to raise the pitch of the bowed string, and also, to some extent, to throw the *natural* frequencies of the higher components out of the harmonic relation and thus to *increase* the forces required to be exerted by the bow for the maintenance of the motion beyond what would otherwise be necessary.

That the vibration of a bowed string, *if of large amplitude,* cannot remain as a free oscillation in the absence of the bow can be shown directly from geometrical considerations. Taking, for instance, the simplest type of vibration (figure 1 on page 261 in which the configuration of the string at any instant consists of two straight lines, the point of intersection of which travels on two parabolic arcs, it is obvious that the length of the string and therefore also the tension would be a maximum when the two straight lines intersect at the vortex of either parabola, and would be a minimum when the string passes through its position of equilibrium. The velocity of travel of the wave along the string cannot, in the circumstances, be uniform. This is inconsistent with the assumed type of free vibration.

The periodic variations of tension in the string produced by a large amplitude may have an effect on the forced vibrations of the instrument. This will now be considered.

Effect of the longitudinal motion of the bridge

The periodic forces exerted by the vibrating string on the bridge at the point over which it passes may be resolved into two sets of components, one set transverse to the string and the other set parallel to the string. The magnitude of these forces and their generalized components may readily be calculated. T_0 being the stationary value of the tension of the string, the tension at any given instant may be written in the form $T_0 + \delta T_0$. If the string at the bridge makes an angle θ with its position of equilibrium, the force exerted by it may be resolved into two components $(T_0 + \delta T_0)\cos\theta$ and $(T_0 + \delta T_0)\sin\theta$ respectively parallel and transverse to the string. The transverse force may to a sufficient approximation be written simply as $T_0\theta$ and is thus of the first order of small quantities. The longitudinal component is

$$(T_0 + \delta T_0)(1 - \theta^2/2)$$

and may be written as

$$T_0 + \delta T_0 - T_0\theta^2/2.$$

If Y be the modulus of extension of the string, l_0 its unstrained length, and Δl_0 the extension necessary to give it a tension T_0, the length of the string in the position of equilibrium is

$$l_0 + \Delta l_0 = l, \quad \text{(say)}.$$

If $l + \delta l$ be the actual length of the string at the given epoch during the vibration, we may write

$$\text{Longitudinal force} = Y(\Delta l_0/l_0 + \delta l/l_0 - \theta^2 \Delta l_0/2l_0).$$

Of the three terms within the brackets, only the second and the third include periodic quantities. The deflection of the string from its position of equilibrium produces an increase of length, and in the preceding sub-section, δl has already been calculated on the assumption that the bridge has no motion parallel to the string. It is of the order $\theta^2 l_0$, and $\delta l/l_0$ is thus of the order θ^2. The third term $\theta^2 \Delta l_0/2l_0$ is therefore of a smaller order of quantities than even the second term which is itself proportional to the square of the amplitude of vibration of the string. It is thus seen that the only part of the periodic longitudinal force which is sensible is that due to the fluctuation of the effective length of the string when in vibration. If η be the longitudinal motion of the bridge, we may write

$$\delta l = \eta + \pi^2/8l \cdot \sum_n n^2 B_n^2 \left(1 - \cos \overline{\frac{4n\pi t}{T} + 2e_n}\right)$$

and the longitudinal force $= Y \cdot \delta l/l$.

In finding the generalized components of force corresponding to the normal modes of vibration of the instrument, and the transverse periodic components, of the tension, we are only concerned with the motion of the bridge transverse to the string. Similarly in finding the generalized components of the longitudinal forces, we are only concerned with the longitudinal motion of the bridge. If the principal normal modes of vibration of the body of the instrument do not involve any longitudinal motion of the bridge, the effect of the longitudinal components of force may be ignored. In any case, these components are of the second order of smallness in magnitude and can be ignored if the amplitude of vibration of the string be sufficiently small. In special circumstances, however, they may attain some importance. The periodic part of the longitudinal force depending on the vibration of the string is of double its frequency, and if one of the frequencies of resonance of the instrument is of suitable value, the string may succeed in forcing an appreciable vibration of double its own frequency. As remarked in the preceding sub-section, the periodic variations of tension in the string vibrating with finite amplitude may also appreciably influence the frequency of its excitation by the bow and the magnitude of the force required to maintain the motion.

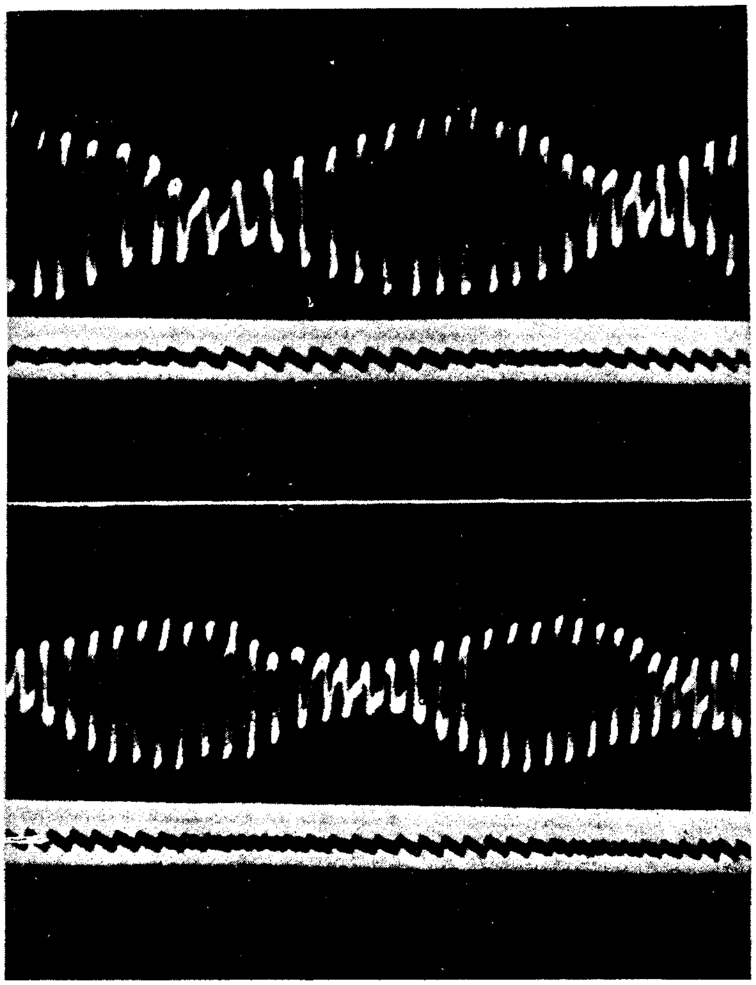


Plate I. Simultaneous vibration-curves of belly and G-strings of violoncello at the "wolf-note" pitch showing cyclical changes.

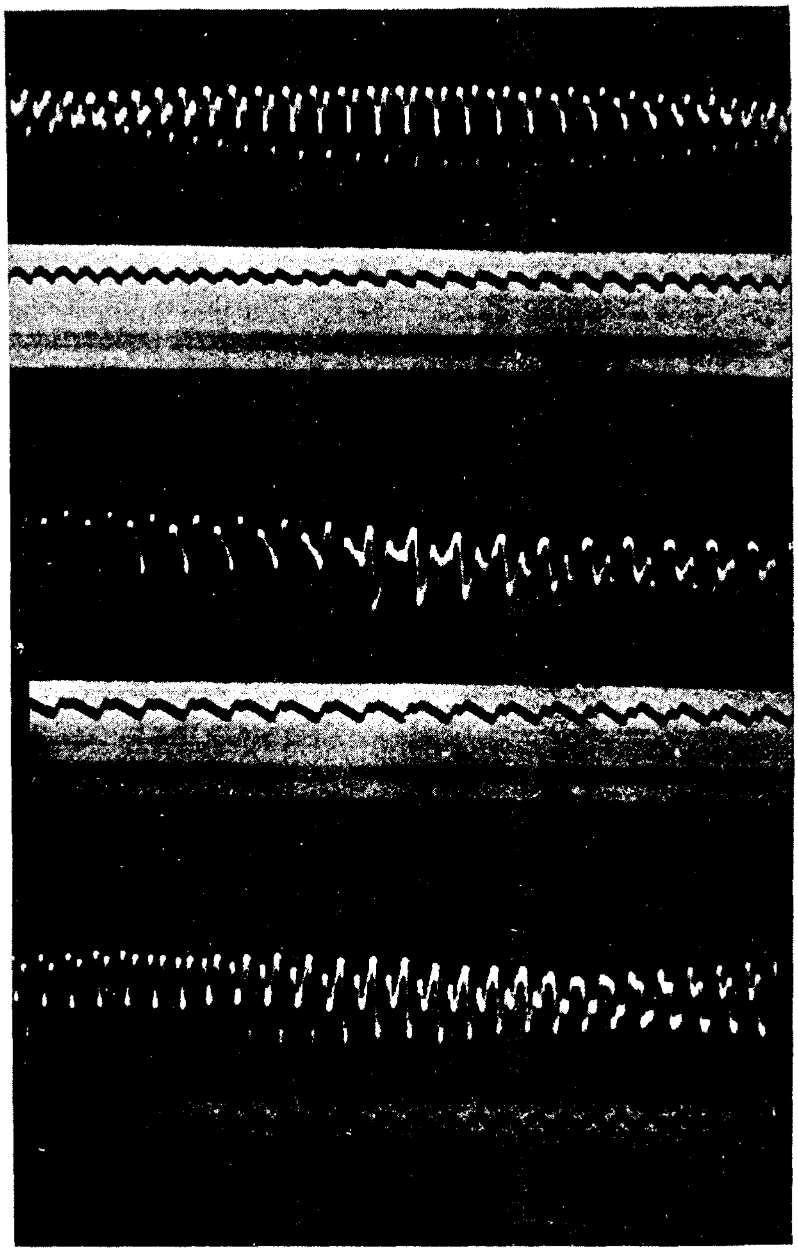


Plate II. Simultaneous vibration-curves of belly and G-string of violoncello at half the "wolf-note" pitch showing the resonance of the octave and cyclical changes.

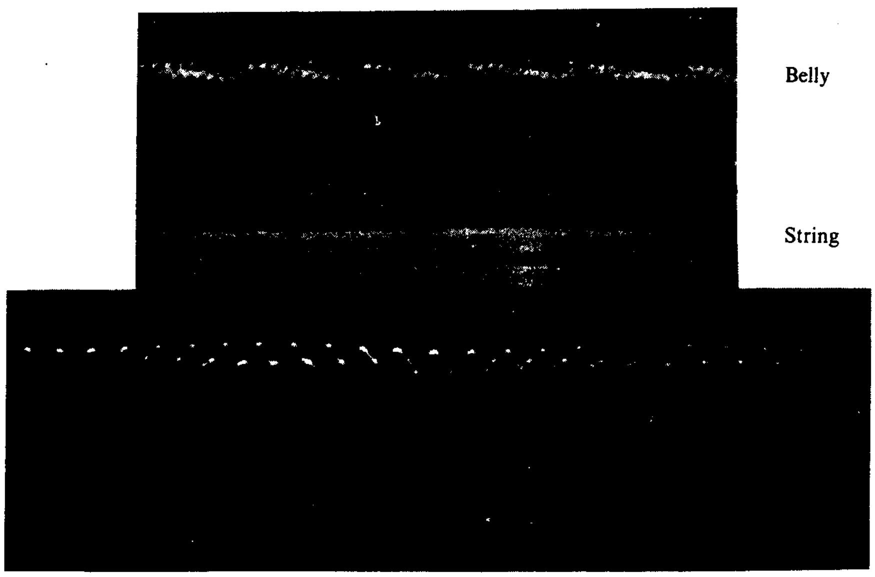


Plate III. Simultaneous vibration-curves of belly and G-string of violoncello at 264 vibrations per second, showing cyclical changes.

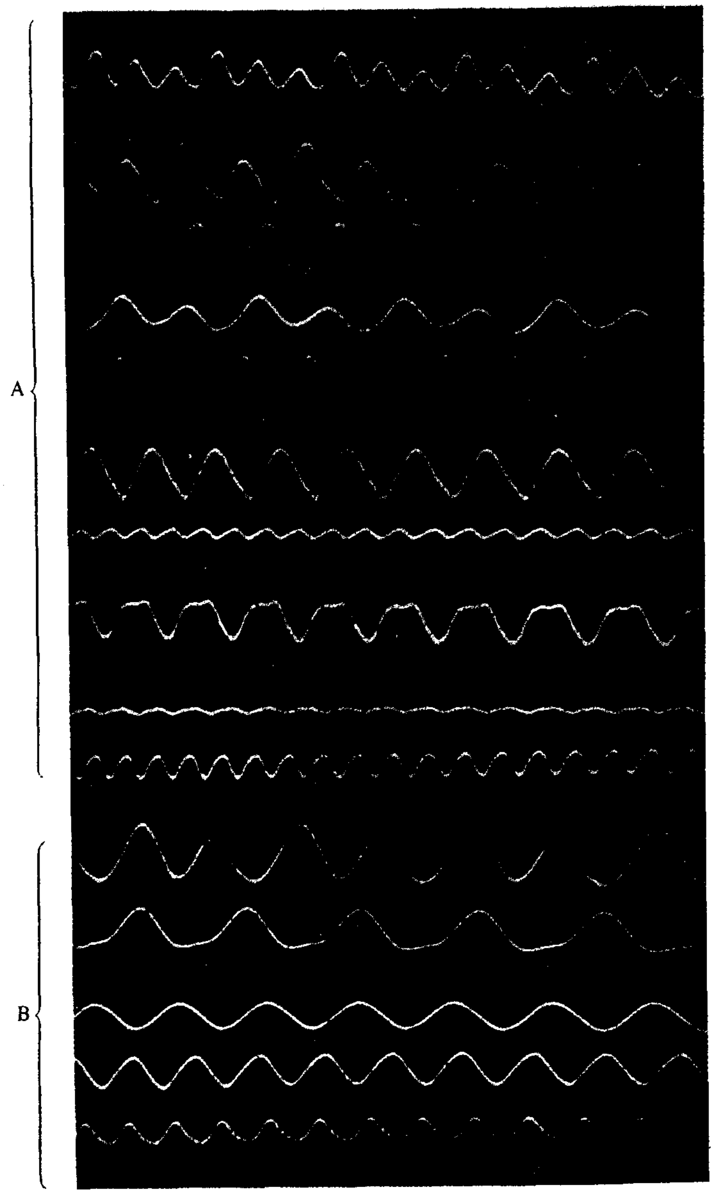
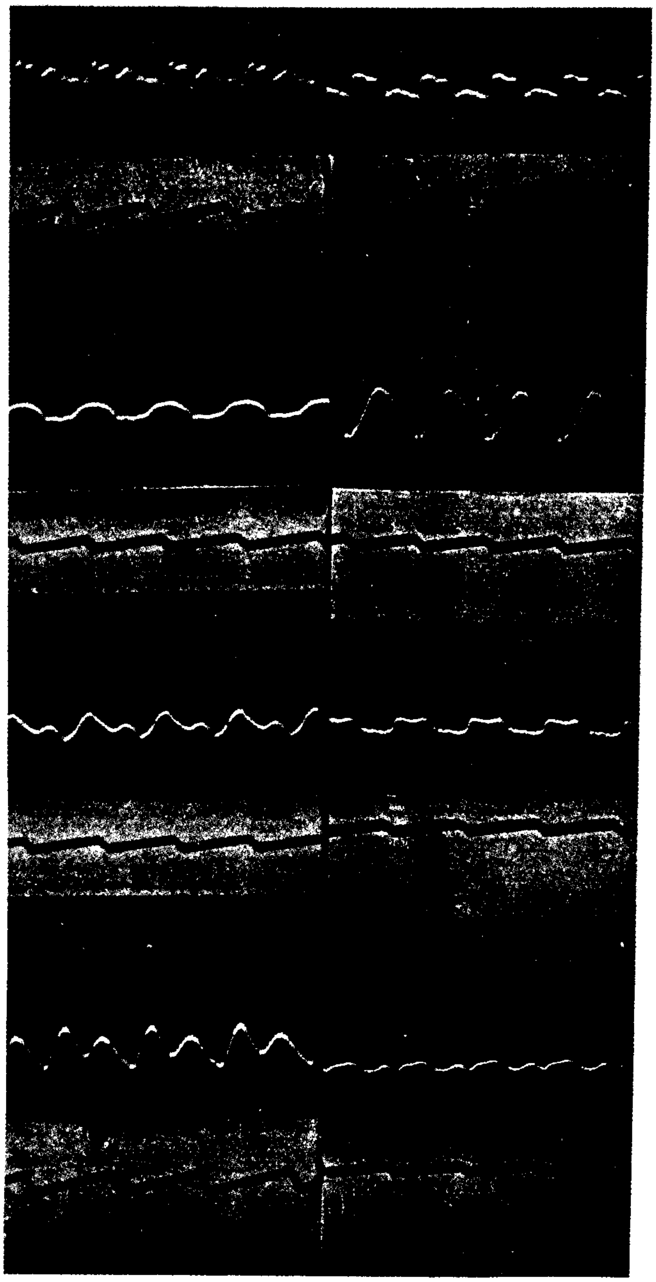


Plate IV. Resonance-curves of belly of violoncello. **A.** Without any load on the bridge: "Wolf-note" frequency 176 vibrations per second. **B.** With a load of 40·4 grammes fixed on top of bridge. "Wolf-note" frequency 137 vibrations per second.

G-String Bowed. D-String Bowed.



Without load.

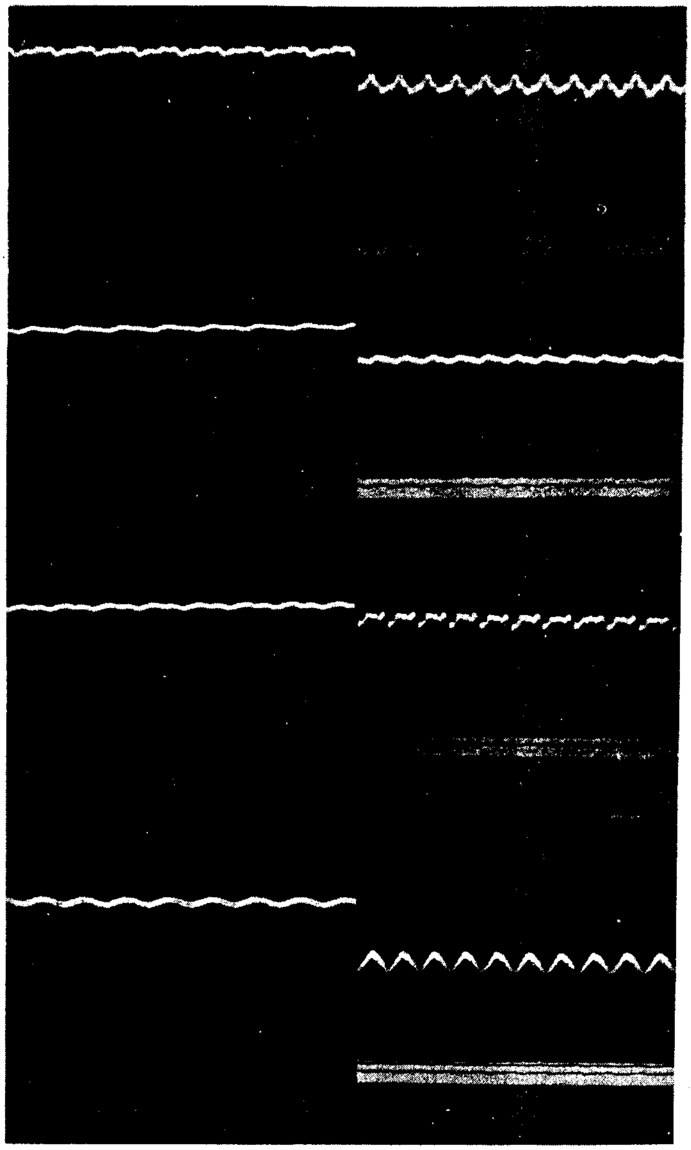
With load of 17 grammes fixed on top of bridge.

Ditto; fixed on left foot of bridge.

Ditto; fixed on right foot of bridge.

Plate V. Simultaneous vibration-curves illustrating effect of loading the violin bridge on its horizontal motion transverse to the strings (observed at the G-string corner).

A-String Bowed. E-String Bowed.



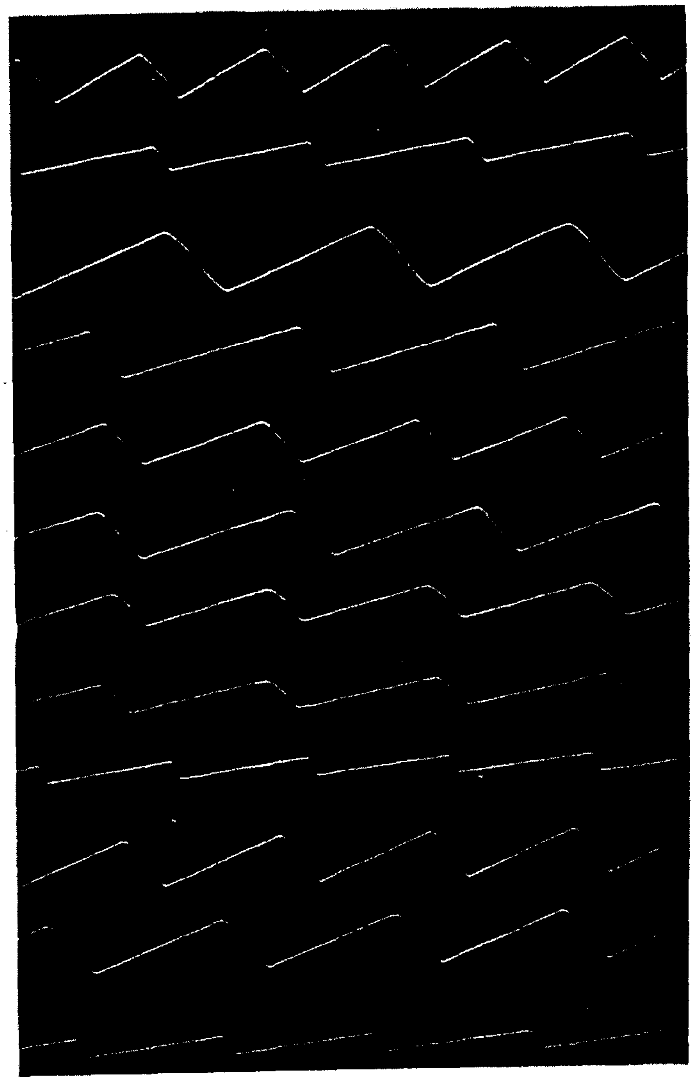
Without load.

With load of 17 grammes fixed on top of bridge.

Ditto; fixed on left foot of bridge.

Ditto; fixed on right foot of bridge.

Plate VI. Simultaneous vibration-curves illustrating effect of loading the bridge on its horizontal motion transverse to the strings, (observed at the G-string corner).



Bowed and observed at L/3·26

Low pressure.

Higher pressure.

L/3·52

Low pressure.

Higher pressure.

L/4·23 with successive increments of pressure.

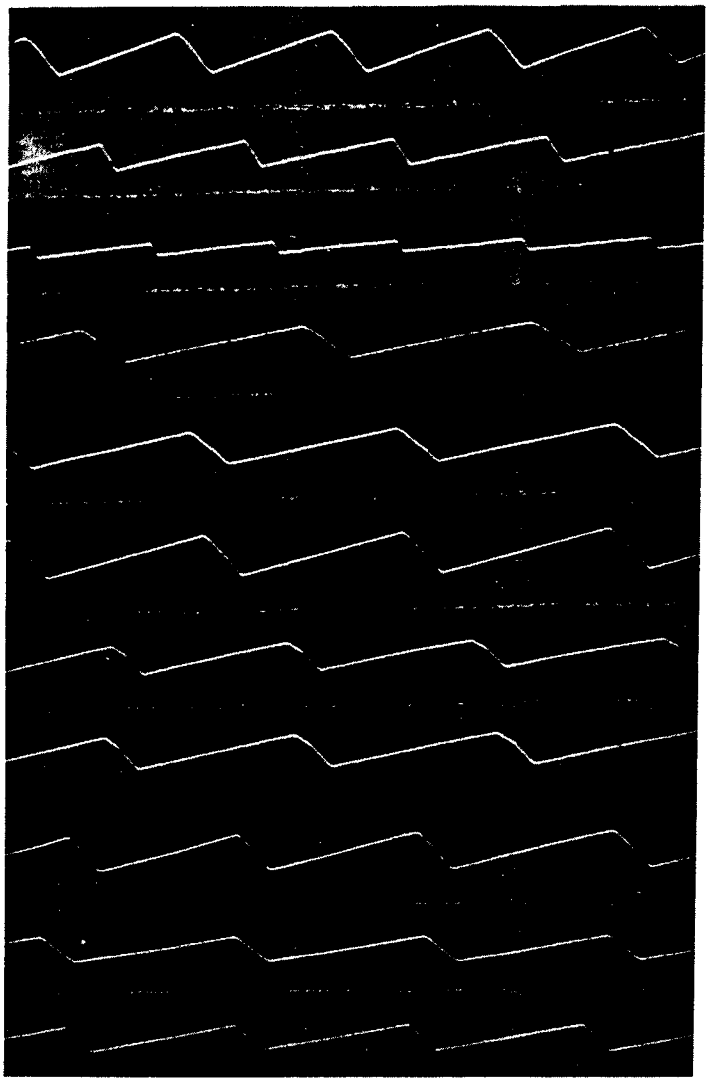
L/3·73

Low pressure.

Ditto.

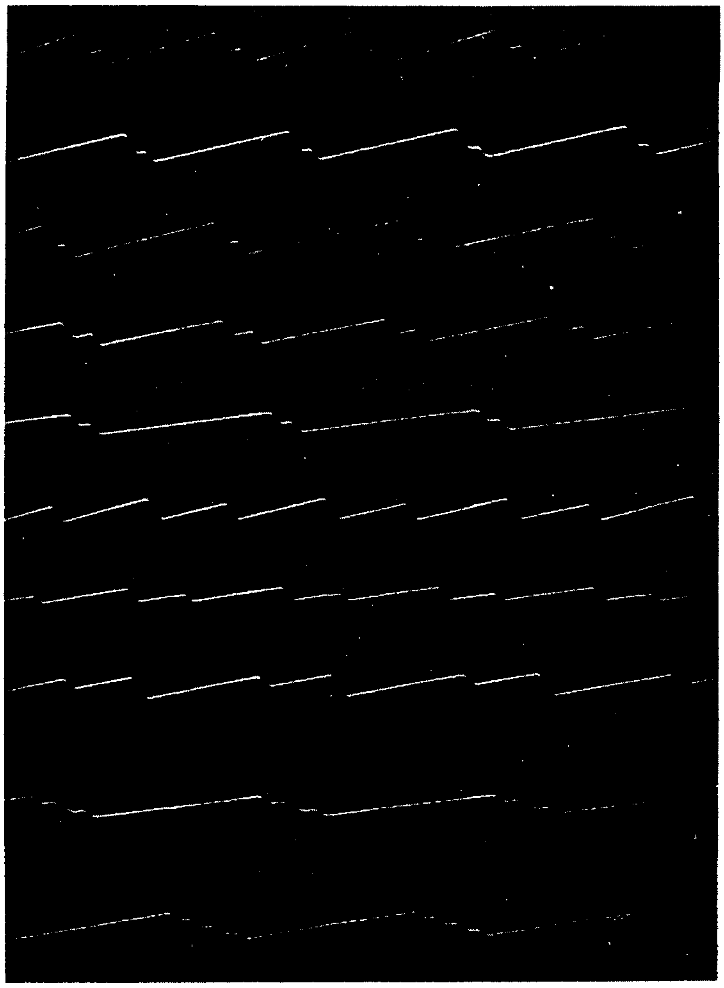
High Pressure.

Plate VII. Modified two-step zig-zag vibration-curves at the bowed point, showing the practical constancy of the velocity of ascent with the bow, and the variability of the velocity of descent against the bow.



Bowed and observed at L/4·50
Low pressure.
High pressure.
L/4·73
High pressure.
Low pressure
L/5·24
with two different pressures.
L/5·50
with three different pressures.
L/5·74
L/6.46

Plate VIII. Modified two-step zig-zag vibration-curves at the bowed point, showing the practical constancy of the velocity of ascent with the bow, and the variability of the velocity of descent against the bow.



Bowed and observed at L/3·26

L/4·23 with three different pressures.

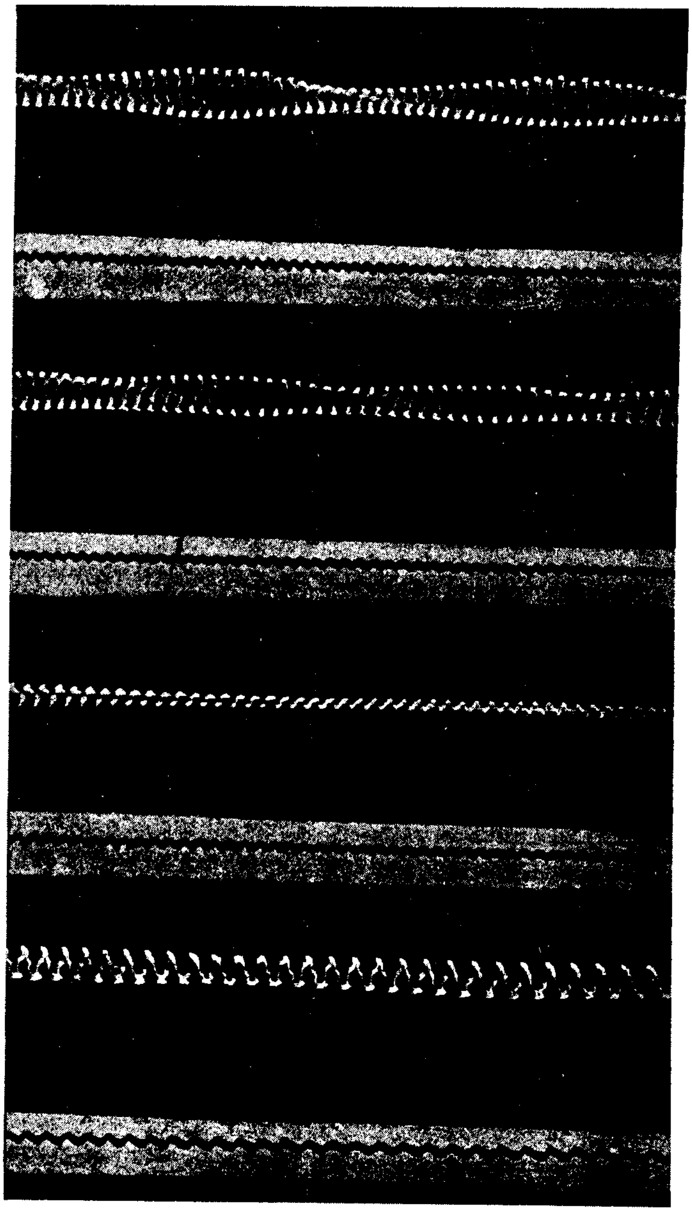
L/4·15

L/15 with three different pressures.

L/3·08

L/3·08

Plate IX. Modified four-, six-, and eight-step zig-zag vibration-curves at the bowed point, showing the practical constancy of the velocity of ascent with the bow, and the variability of the velocity of descent against the bow.



Bridge.

D-String bowed, frequency 360 vibrations per sec.

Bridge.

D-String bowed, frequency 360 vibrations per sec.

Bridge.

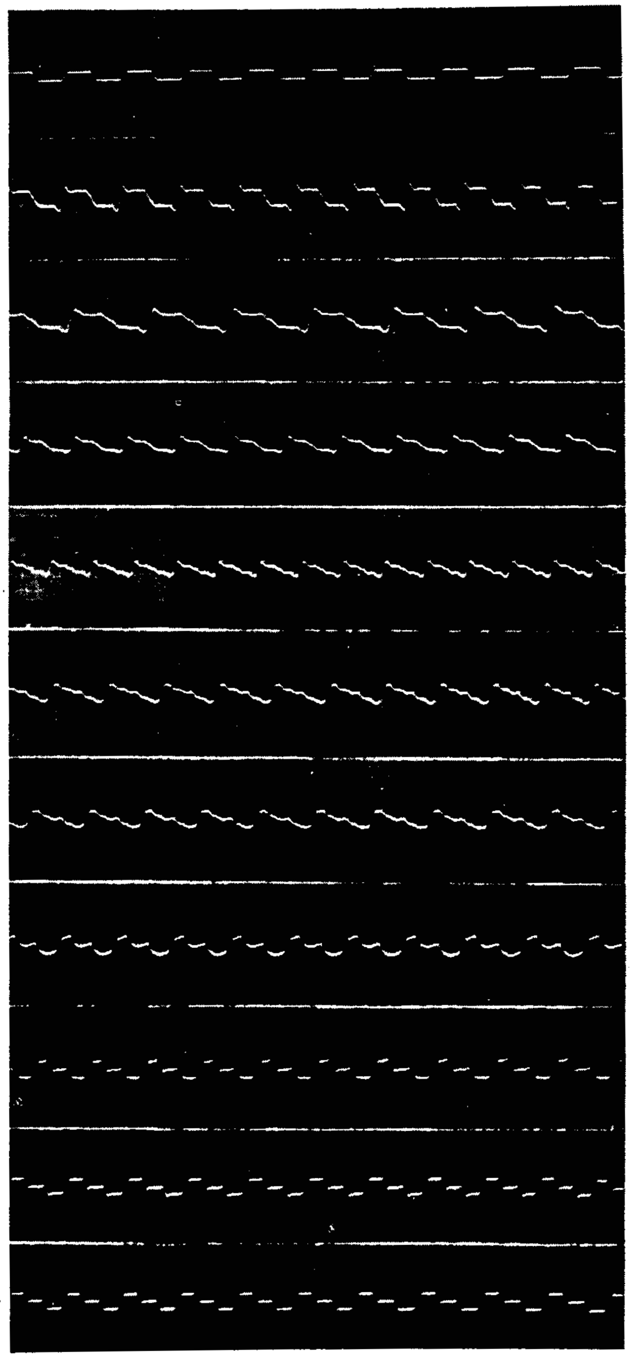
D-String bowed, frequency 240 vibrations per sec.

Bridge.

D-String bowed, frequency 240 vibrations per sec.

Plate X. Simultaneous vibration-curves of the bridge and D-string of a violoncello, showing cyclical changes at frequencies other than that of the "wolf-note".

Observed at L/10.



Bowed at

L/2·00

L/2·13

L/2·26

L/2·32

L/2·39

L/2·43

L/2·50 or 2 L/5

L/2·65

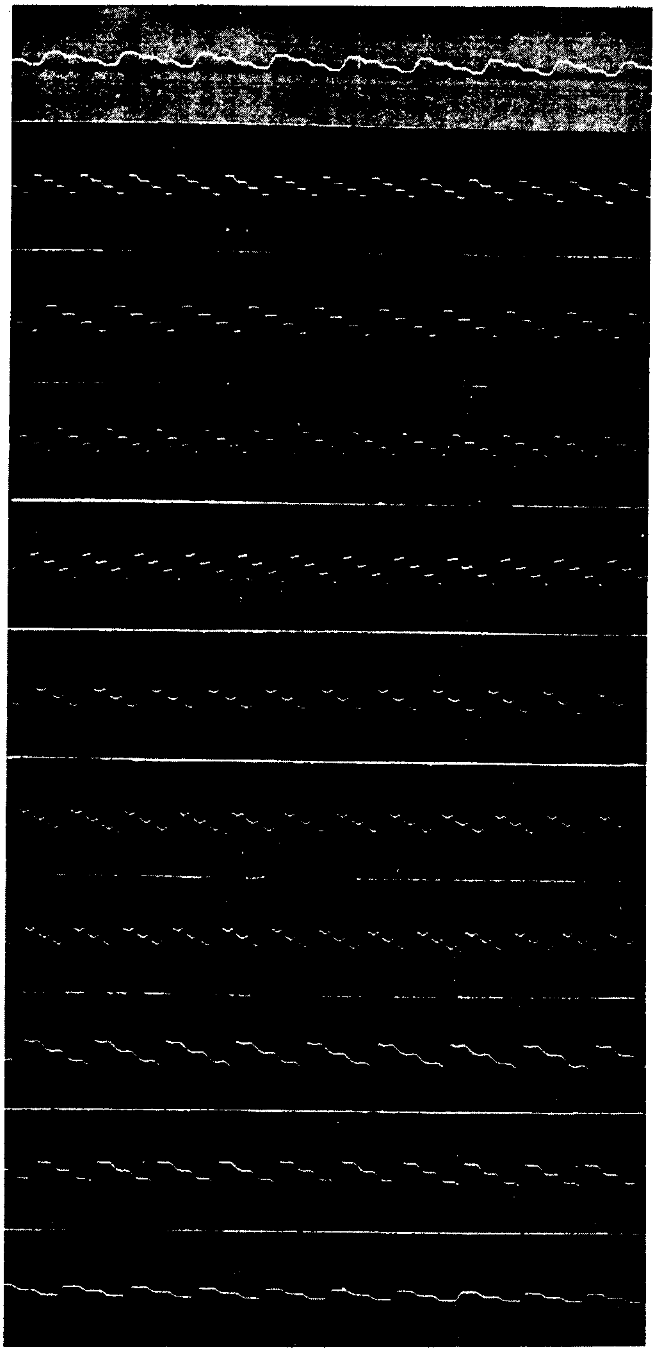
L/2·74

L/2·85

L/2·91

Plate XI. The first type of vibration: its rational and transitional modifications.

Observed at L/10



Bowed at

L/4·00

L/3·85

L/3·71

L/3·59

L/3·5 or 2 L/7

L/3·43

L/3·37

L/3·33 or 3 L/10

L/3·20

L/3·12

L/3·00

Plate XII. The first type of vibration: its rational and transitional modifications.

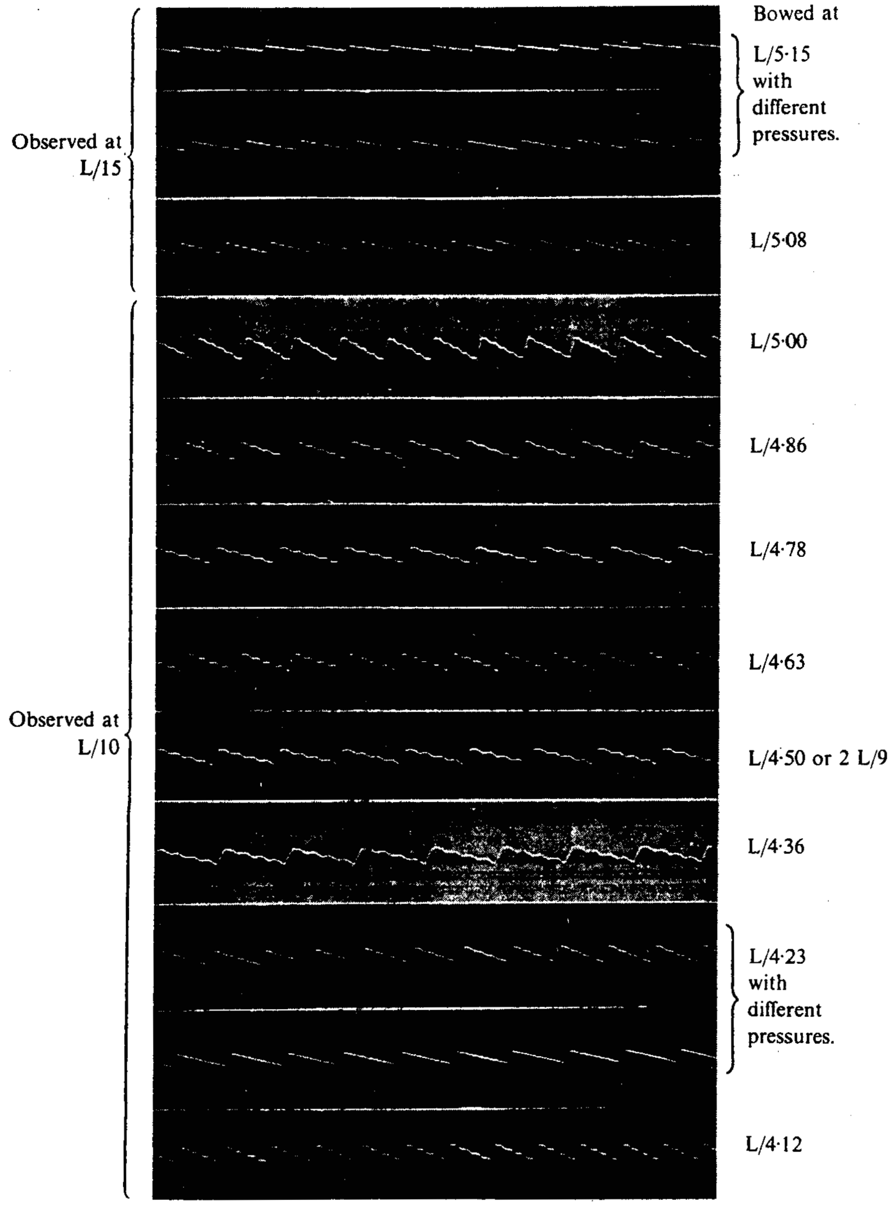
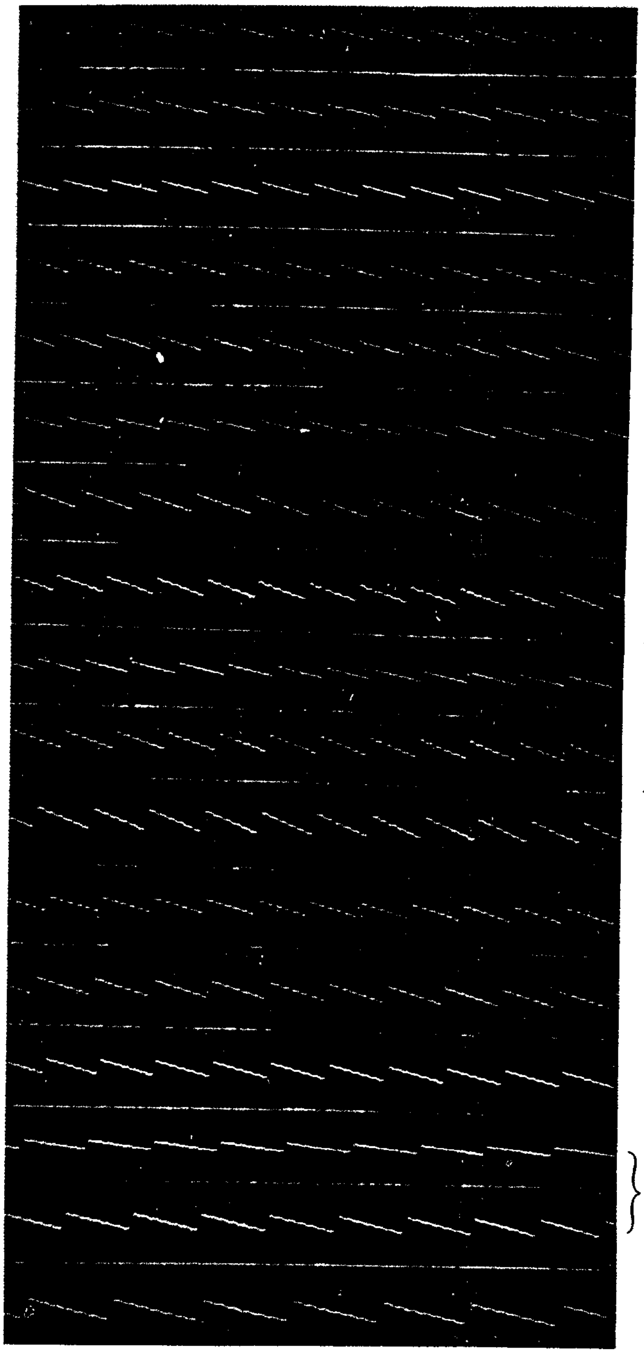


Plate XIII. The first type of vibration: its rational and transitional modifications.

Observed at L/15.



Bowed at L/5·22

L/5·35 with different pressures.

L/5·49 with different pressures.

L/5·61

L/5·81 with different pressures.

L/6·00

L/6·25

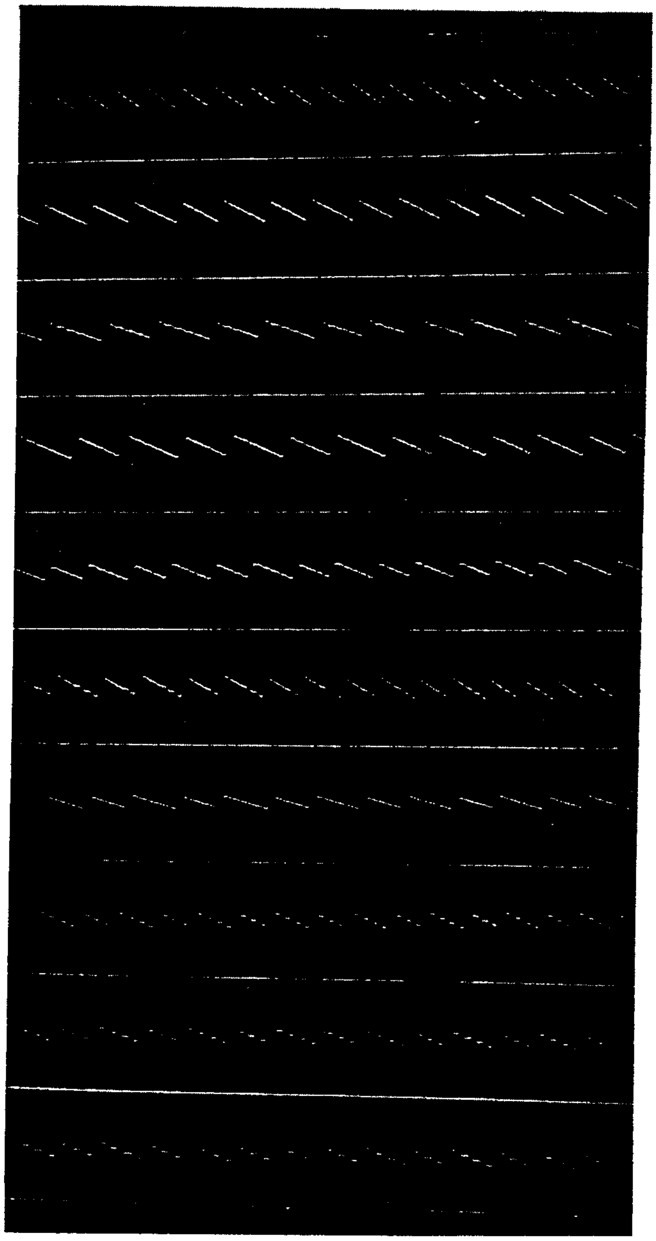
L/6·50 with different pressures.

L/6·75

L/7·00

Plate XIV. The first type of vibration: its rational and transitional modifications.

Observed at L/15



Bowed at

L/2·11

L/2·16 or 6 L/13

L/2·19

L/2·20 or 5 L/11

L/2·23

L/2·24

L/2·25 or 4 L/9

L/2·29

L/2·30

L/2·35 or 3 L/7 nearly.

Plate XV. The second type of vibration: its rational and transitional modifications.

Observed at L/15

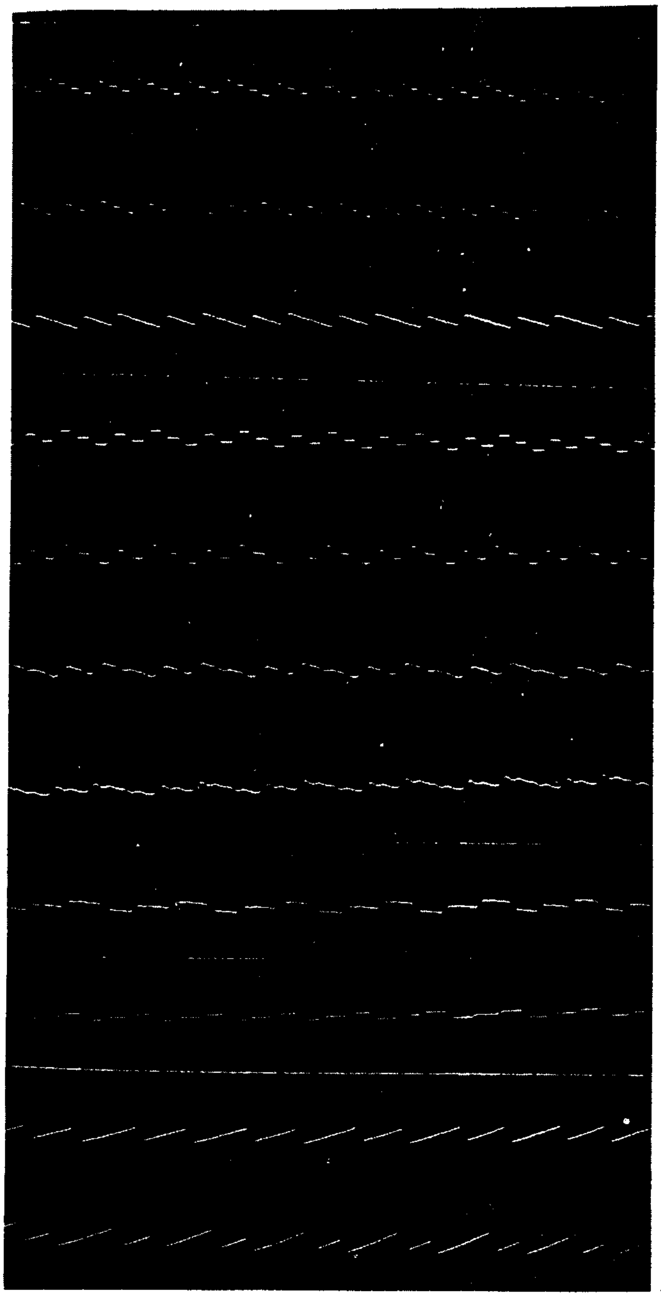
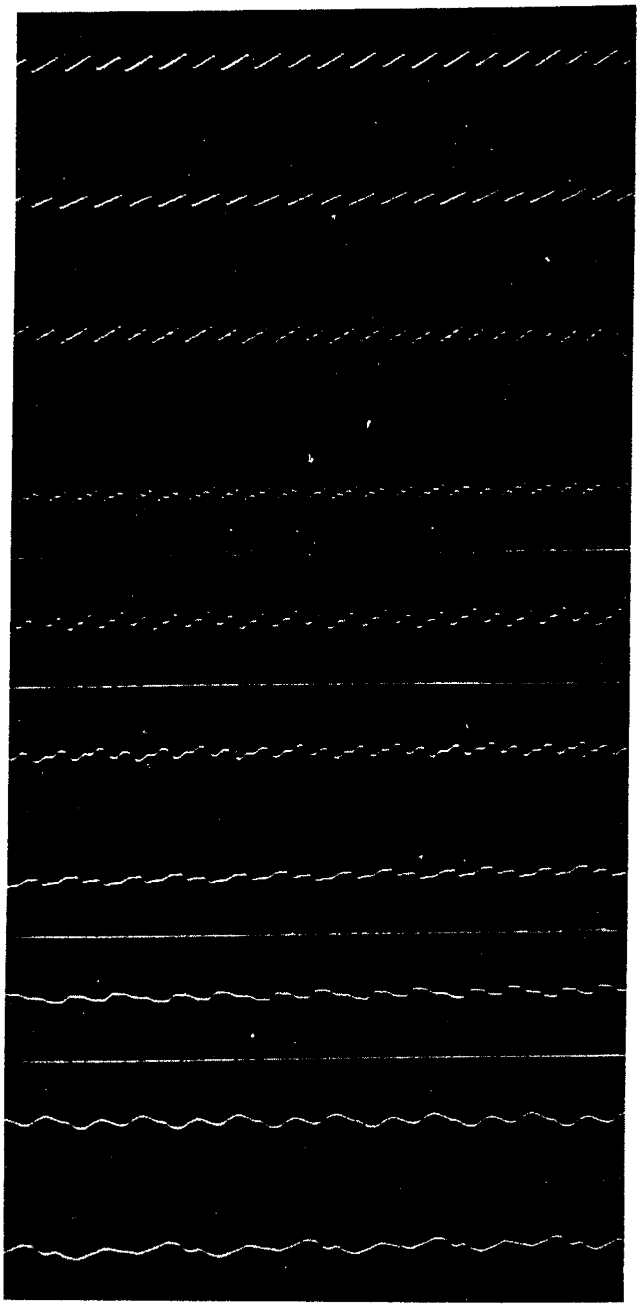


Plate XVI. The second type of vibration: its rational and transitional modifications.

Observed at L/15.



Bowed at

L/2·86

L/2·81

L/2·76 or 4 L/11

L/2·72

L/2·68 or 3 L/8

L/2·64

L/2·58

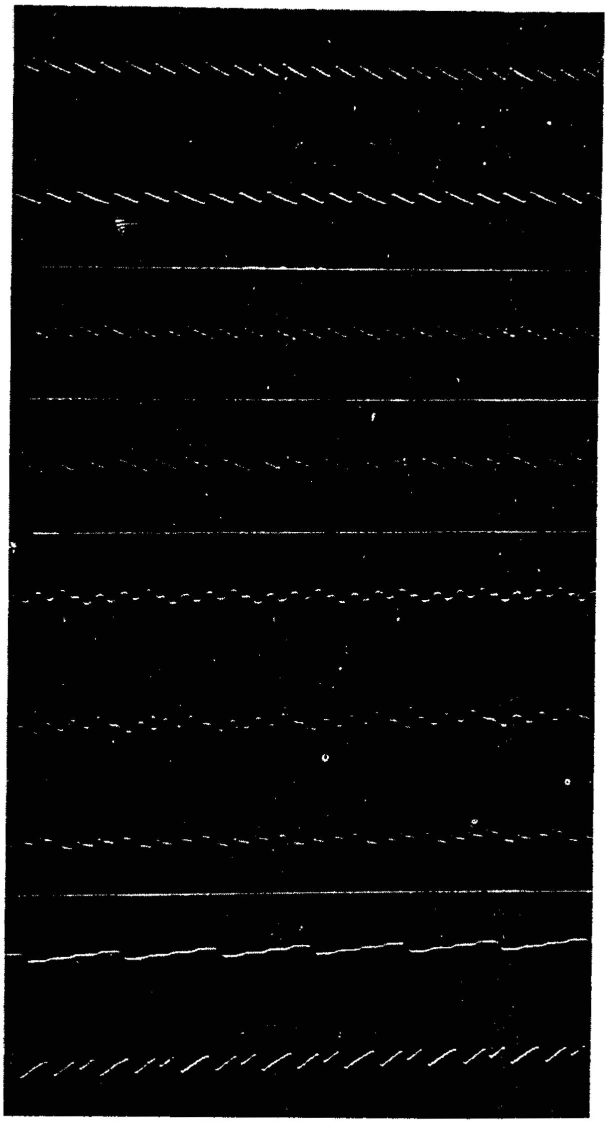
L/2·51 or 2 L/5 nearly.

L/2·40

L/2·35 or 3 L/7 nearly.

Plate XVII. The third (negative) type of vibration: its rational and transitional modifications.

Observed at L/15.



Bowed at

L/3·15

L/3·26

L/3·33 or 3 L/10

L/3·44

L/3·54 or 2 L/7 nearly.

L/3·65

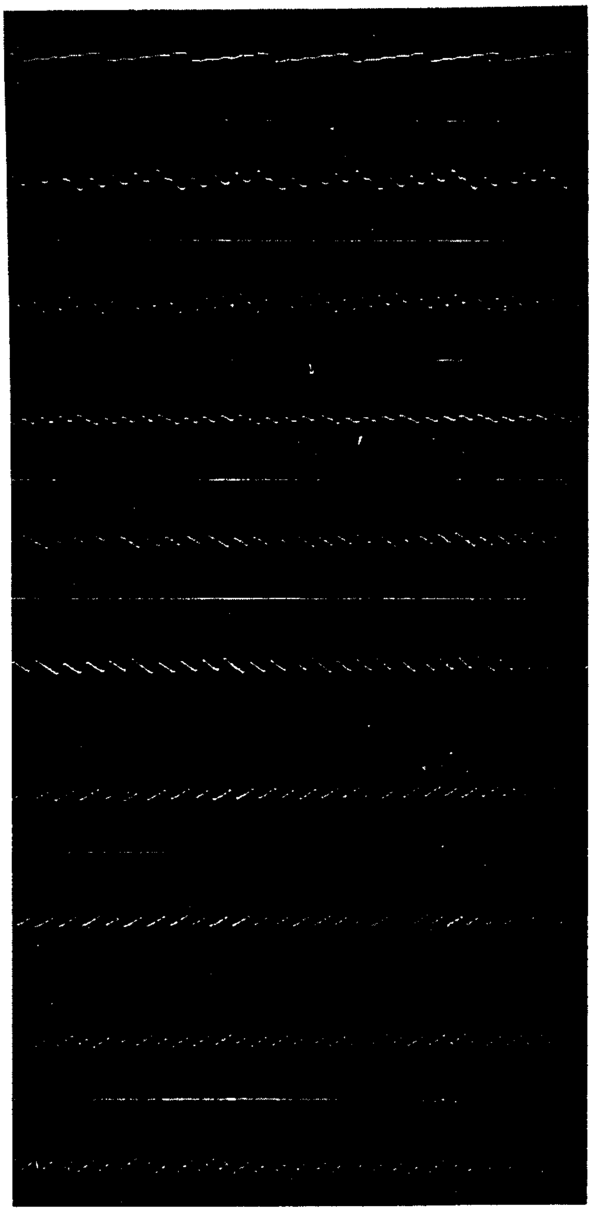
L/3·79

L/4·00

L/20

Plate XVIII. The third (positive) type of vibration: its rational and transitional modifications.

Observed at L/15.



L/5·00

L/4·69

L/4·53

L/4·48 or 2 L/9 nearly

L/4·41

L/4·27

L/3·79

L/3·74

L/3·70

L/3·62

Plate XIX. The fourth type of vibration: its rational and transitional modifications.

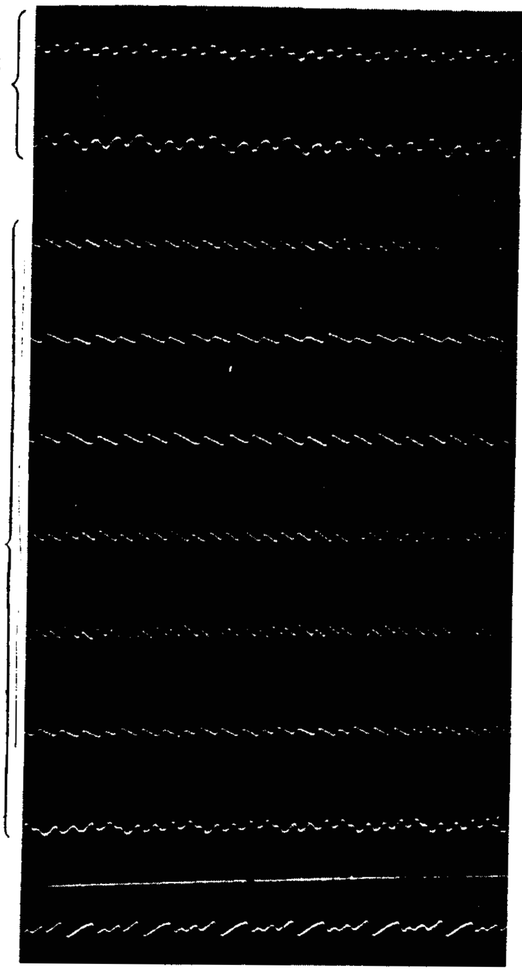
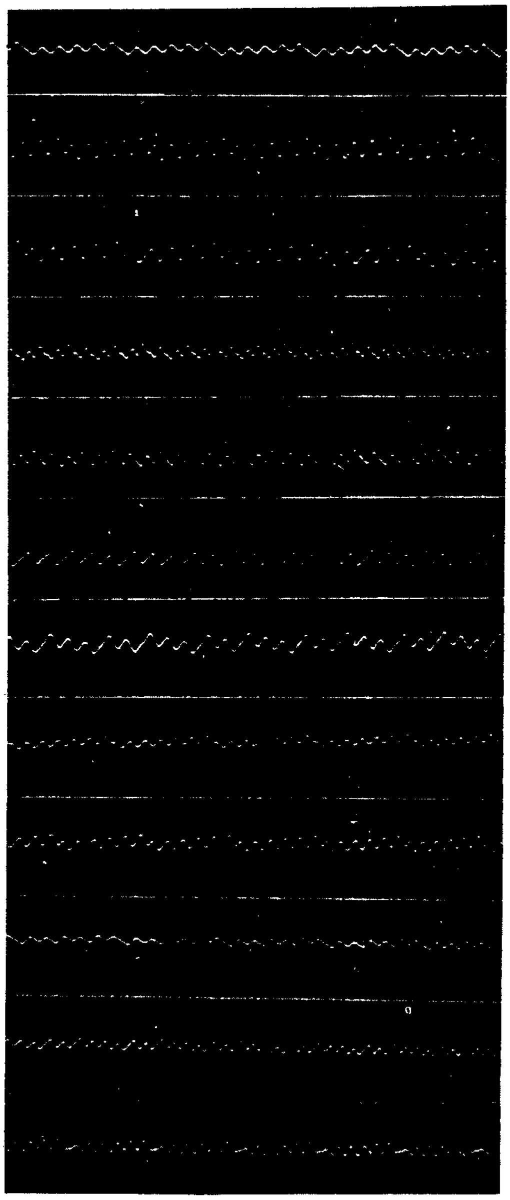


Plate XX. The fourth type of vibration: its rational and transitional modifications.

Bowed at



L/5·47
L/4·77
L/4·66
2L/5·19
2L/5·14
2L/4·90
2L/4·82
L/6·55 or nearly 2L/13
L/5·68
2L/6·30
2L/5·72
3L/6·30

The fifth type and its modifications.

nearly 5L/13

nearly 5L/12

The sixth type of vibration with approximately two-step zig-zag motion at bowed point.

Ditto with a four-step zig-zag motion at bowed point.

Ditto with complicated motion at bowed point.

Plate XXI

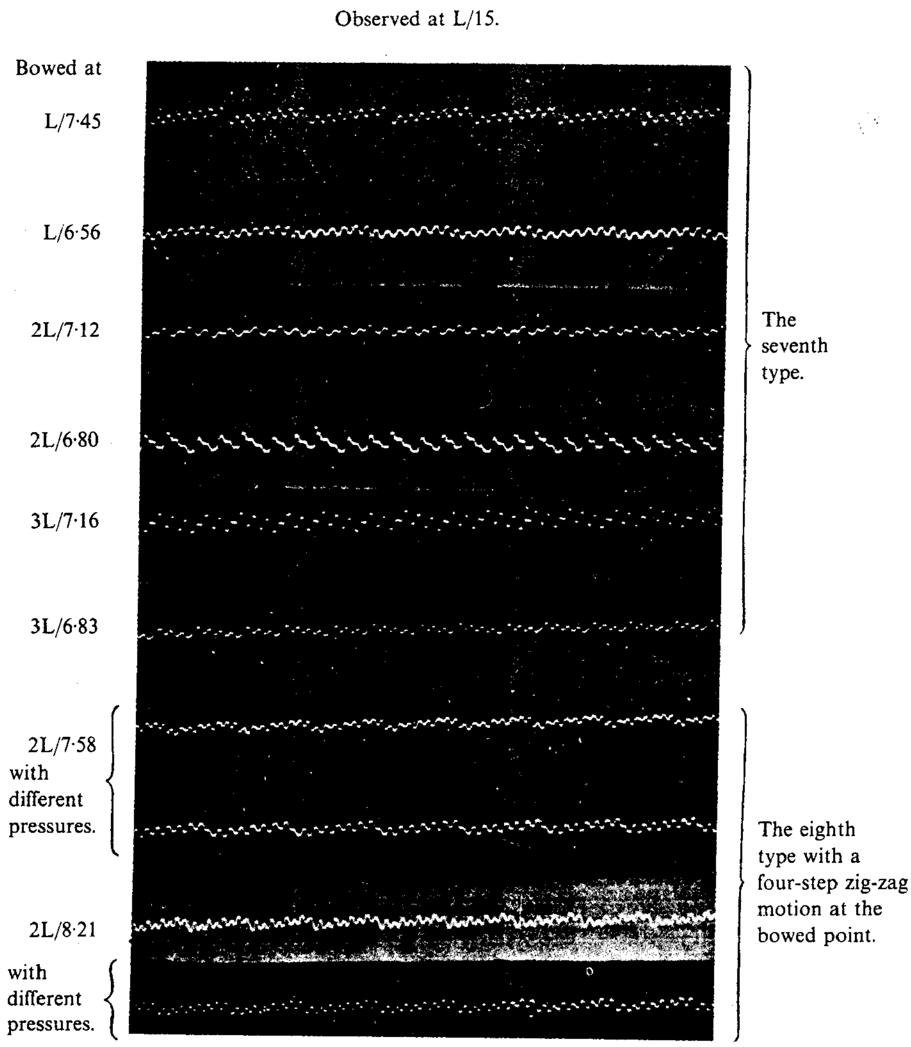
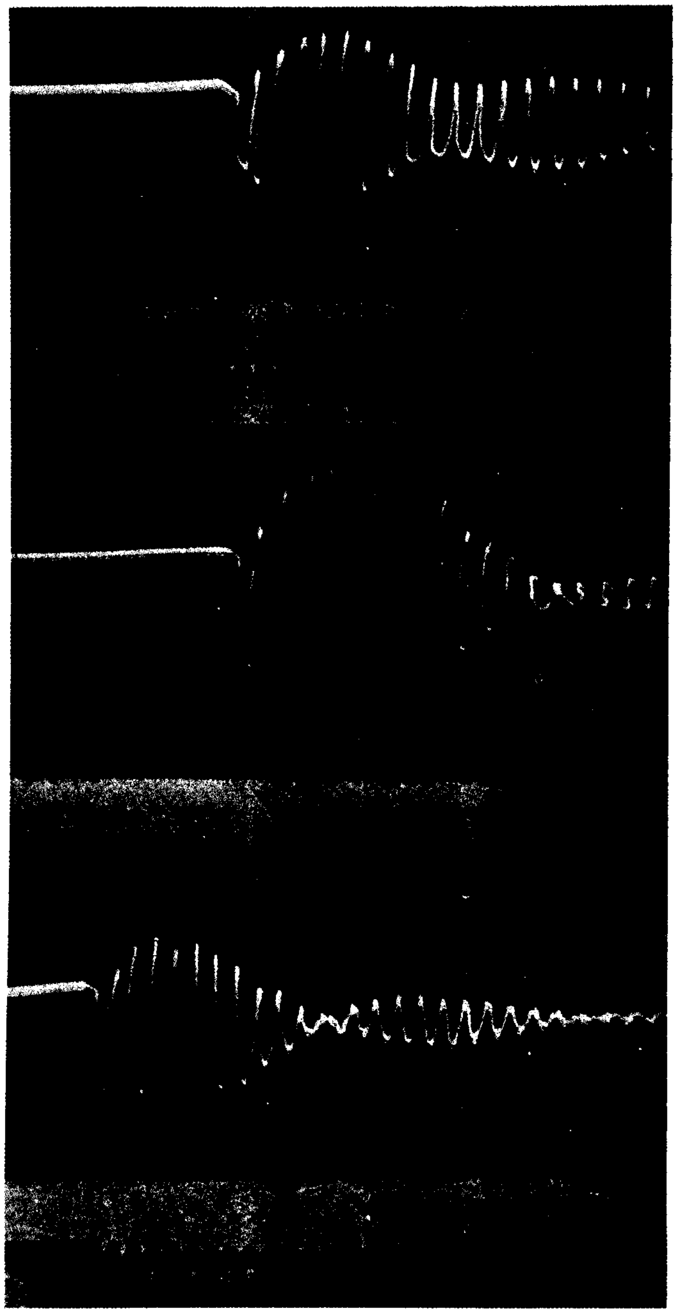


Plate XXII. The seventh and eighth types: their rational and transitional modifications.

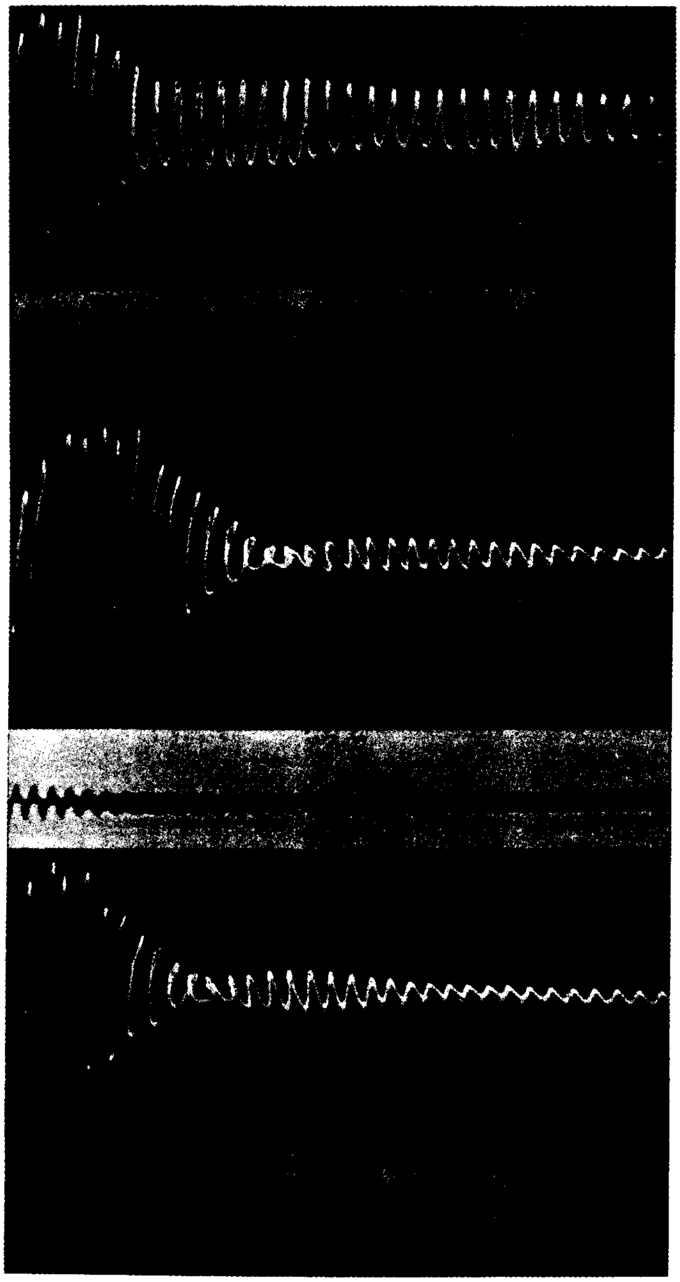


Slightly below 'Wolf-note' pitch.

At the 'Wolf-note' pitch, showing alteration of free periods.

Slightly above 'Wolf-note' itch.

Plate XXIII. Simultaneous vibration-curves of the bridge and G-string of a 'cello (plucked).

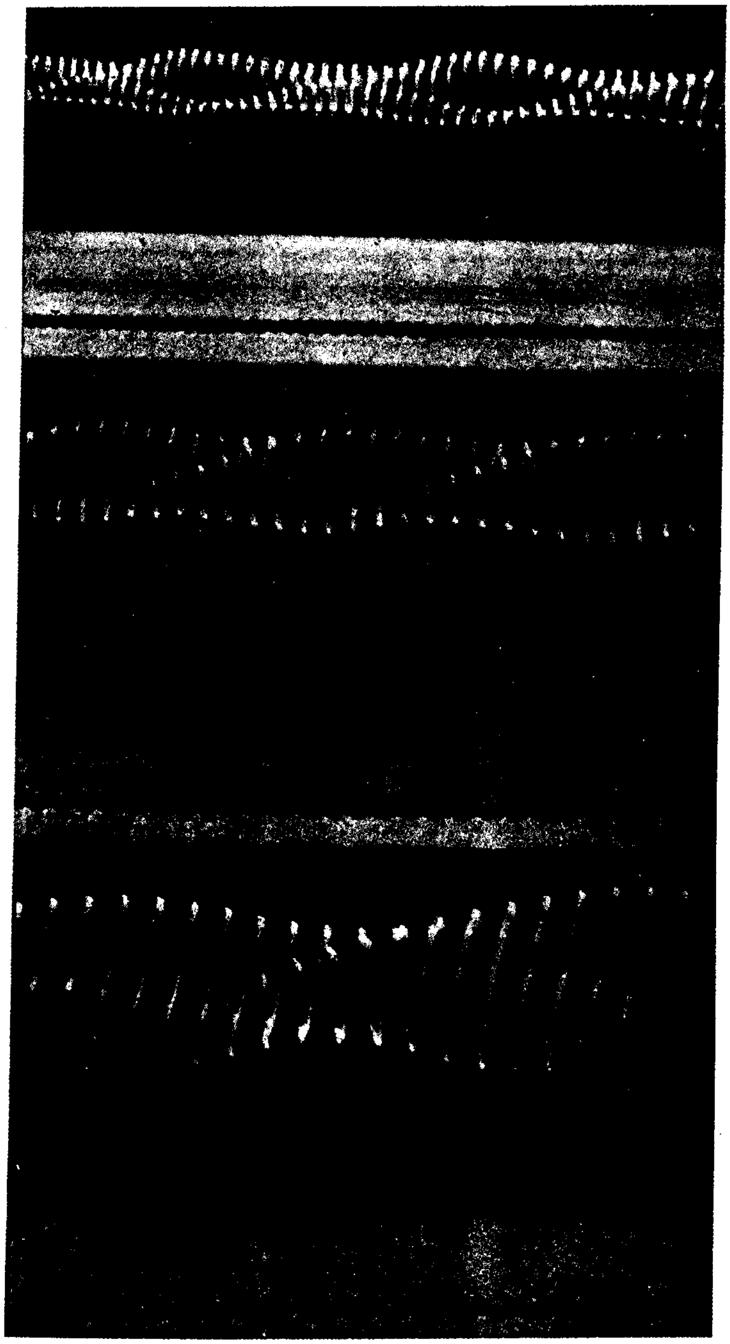


Slightly below 'Wolf-note' pitch.

At the 'Wolf-note' pitch, showing the rapid dissipation of energy.

Ditto, and the beats in the motion of the bridge.

Plate XXIV. The simultaneous vibration-curves of the bridge and G-string of a 'cello (plucked).

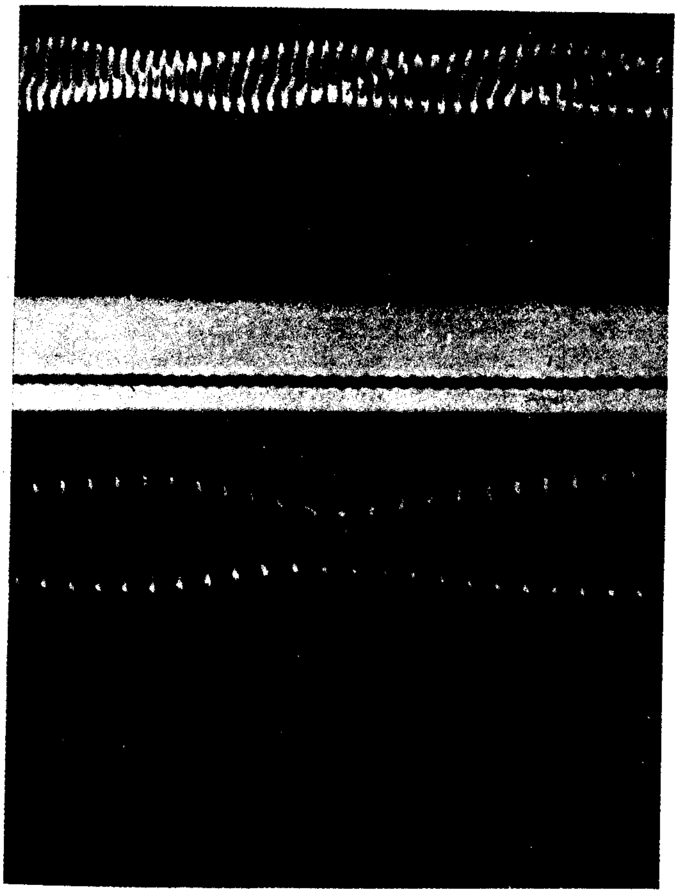


Frequency
148 vibrations
per sec.
Very small
lag.

156 vibrations
per sec.
Lag one-sixth
of a cycle.

162 vibrations
per sec.
Lag one-fifth
of a cycle.

Plate XXV. Cyclical vibrations of G-string and bridge of 'cello near 'wolf-note' pitch, showing the differences in lag.



Frequency
157 vibrations
per second.
Bowed near
one end.

Frequency
165 vibrations
per second
Bowed at distance of
two-fifths
of length from
one end.

Plate XXVI. Cyclical vibrations of G-string and bridge of 'cello near the 'wolf-note' pitch.