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Some new methods in kinematical theory

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In the course of my recent experimental work on the vibrations of bowed strings,* it occurred to me that the ordinary Fourier analysis which has been used by previous investigators† is not a convenient or suitable method of considering the kinematics of this class of vibrations. In fact, the nature of the case is such that the Fourier analysis obscures instead of elucidating the physical processes set up in the string by the action of the bow. I have therefore developed a new method of treatment which has the great advantage of enabling the subject to be considered entirely from first principles *i.e.*, without any appeal to experiment except for confirming the fully-worked-out predictions of theory. I am using this method in a monograph on the 'Mechanics of Bowed Strings' which is under preparation and which I intend to contribute to one of the regular periodicals for physics. The brief account of this method which I propose to give here may perhaps be of some interest to mathematicians.

The general solution of the equation of wave-propagation on an infinite string not subject to damping is

$$Y = f(x - at) + F(x + at). \tag{1}$$

It is well-known that this solution for the case of an infinite string can be used to represent the configuration at any instant of a vibrating string of finite length by arranging the form of the *displacement* waves in such manner that the motion is periodic and satisfies the terminal condition $y = 0$ at the two ends of the string.

Similarly, the solution obtained by differentiating (1) with respect to time, *viz*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -af'(x - at) + aF'(x + at) \tag{2}$$

can be applied to represent the *velocity-diagram* of a finite string at any instant during its vibration, if the periodicity of the motion and the terminal conditions of velocity are secured. It is obvious that solution (2), as it stands, represents

* *Bulletin No. XI of the Indian Association for the Cultivation of Science*, pp. 43–52, May 1914. Also, '*Nature*', page 622, August 1914, and Science Abstracts, February 1915.

† Helmholtz, 'Sensations of Tone' English Translation by Ellis, Chapter V and Appendix VI.

Krigar-Menzel and Raps, Über Saitenschwingungen, *Sitzungberichte* of the Berlin Academy, 1891.

A. Stephenson, "On the Maintenance of Periodic Motion by Solid Friction," *Philosophical Magazine*, January 1911.

the velocity-waves that travel on an infinite string without change of form in the positive and negative directions respectively. In the case of a finite string of length l, the reflexions that take place at the two ends have to be taken into account, and we may write

$$\frac{dy}{dt} = \theta(x - at) + \phi(x + at). \tag{3}$$

The two functions $\theta\ (x - at)$ and $\phi\ (x + at)$ represent the velocity-waves which must be imagined as extending to infinity in both directions, and as being perfectly periodic with wavelength equal to twice the length of the string. To satisfy the terminal conditions $dy/dt = 0$, we must assume that the positive wave from $x = 0$ up to $x = l$ in its initial position is an inverted and reflected image of the negative wave from $x = l$ up to $x = 2l$, and vice-versa.

The next step in the argument is to consider the changes of velocity that take place at individual points on the string. Obviously the form of the positive and negative velocity waves must be such that by their movement and superposition they reproduce the changes of velocity at any given point on the string.

If now, some point on the string (say the point $x = x_b$) has the characteristic property of always moving with a succession of constant velocities during each period of vibration, the velocity passing in a discontinuous manner from each value to the next, we must have, at that point, the condition $d^2y/dt^2 = 0$ always satisfied, except at certain instants in each period of vibration when it becomes $\pm$ infinity. Differentiating (3) with respect to time, we have

$$\frac{d^2y}{dt^2} = -a\theta'(x - at) + a\phi'(x + at). \tag{4}$$

Since, at the point x_b, d^2y/dt^2 is generally zero, we must have

$$\theta'(x_b - at) = \phi'\ (x_b + at). \tag{5}$$

If the velocity-waves $\theta\ (x - at)$ and $\phi\ (x + at)$ are represented graphically, equation (5) may be given a geometrical significance. If any two points are taken, one on the positive wave and one on the negative wave, the distances of which from the point x_b measured along the string are equal but in opposite directions, we should find the slopes of the waves at the two points to be equal. As already mentioned, the form of the velocity-waves must satisfy certain other conditions, viz., that they are periodic with wavelength $2l$, and that initially the form of the positive wave from $x = 0$ up to $x = l$ is an inverted and reflected image of the negative wave from $x = l$ up to $x = 2l$, and *vice-versa*. It is a simple geometrical problem to find the form of the positive and negative waves which would simultaneously satisfy these three conditions. By inspection, the following remarkably simple and significant solution is obtained: if the point x_b divides the string in an *irrational* ratio, the only possible form of the velocity-waves is that in which the slope is everywhere the same, *i.e.* they are representable by a number of straight lines that are all parallel to one another, a discontinuity intervening wherever one straight line leaves off and the next begins.

The next step in the argument is to show that the *modus operandi* of the bow requires that d^2y/dt^2 should be generally zero at the bowed point except at certain instants in each period of vibration when it becomes $\pm$ infinity. This can be proved from dynamical considerations, for which I must refer the reader to my forthcoming monograph, and it is also shown there that the velocity at the bowed point must alternate between two and only two constant values. The preceding theory is thus applicable when the velocity-waves travel on the string without any appreciable change of form, and the discussion shows the form of the velocity-waves to be representable by a number of parallel straight lines with intervening discontinuities, when the bow is applied at a point dividing the string in a rational ratio, as well as in the cases in which it divides the string in an irrational ratio. The condition that the velocity at the bowed point alternates between two and only two constant values is then used to find the form of the velocity-waves. When the bowed point divides the string in an irrational ratio, the discontinuities in the velocity-waves are all numerically equal in magnitude to one another, and to the arithmetical sum of the two speeds possible at the bowed point. The types of vibration may then be classified according to the *number* of the discontinuities (one, two, three or more) per wave-length, in each wave. In the detailed discussion of these cases, the conditions under which they are excited and the kinematical relations involved therein are investigated, and the modifications that occur when the bow is applied at a *rational* point are also worked out. The general procedure adopted is very simple. If there are n equal discontinuities in each velocity-wave, the lines in the velocity-graph of the string being parallel to one another must evidently all pass through the nodes of the nth harmonic or the ends of the string (there being situated at equal intervals along the string). The position of the intervening discontinuities is, in general, arbitrary. From this we get at once, the general kinematical relation

$$w = \frac{nx_n}{l} \qquad (6)$$

where w is the ratio of the time during which the bowed point moves with the larger of the two speeds, to the total period of a complete vibration, n is an integer, and x_n is the distance of the bowed point from the *nearest* node of the nth harmonic. This relation holds good both for 'rational' and 'irrational' points of application of the bow. When $n = 1$, we have the well-known kinematical relation $w = x/l$ discovered by Helmholtz. The relation has been verified by me for the other cases in which $n = 2$, 3, or 4 etc.

It is impossible here to enter into further detail of the various developments of the theory outlined above, and I must therefore content myself with summarising briefly the main results of the research which are as follows:

(*a*) The theory gives a completely satisfactory account of the observed types of vibration, proceeding from first principles and including the so-called 'complicated' types of motion.

(*b*) It predicts the general kinematical relation (6) given above, of which only the first case ($n = 1$) was known previously through the work of Helmholtz. The general relation has since been verified by me experimentally.

(*c*) It predicts the effect to be observed by varying the pressure or velocity of bowing at any given point, rational or irrational, and has an important bearing on the musical applications of the subject.

(*d*) The actual form of the vibration-curve for any given point on the string for any one of the possible or actual types of vibration can be readily traced by a very simple graphical process which dispenses entirely with the tedious methods of harmonic analysis, and the curve thus traced from purely theoretical principles, can be compared directly with that observed in experiment. A large number of examples will be given in my complete monograph.

(*e*) Many of the conclusions arrived at by Krigar-Menzel and Raps as the result of their work require to be largely modified.

(*f*) The whole treatment gives a far more vivid idea of the kinematics of bowed strings than can possibly be conveyed by the Fourier analysis.