Multipartite entanglement configurations: Combinatorial offshoots into (hyper)graph theory and their ramifications

Sudebkumar Prasant Pal*, Somesh Kumar† and R. Srikanth**

*Department of Computer Science and Engineering, and Centre for Theoretical Studies, Indian Institute of Technology, Kharagpur - 721302, India
†Department of Mathematics, and Centre for Theoretical Studies, Indian Institute of Technology, Kharagpur- 721302, India
**Raman Research Institute, Sadashiva Nagar, Bangalore- 560080, Karnataka, India

Abstract. Computations in a distributed environment comprising a network of spatially separated nodes may require the exchange of classical and quantum information. The amount of classical communication may be reduced in such computations by using multipartite entanglement. Following the combinatorial approach developed in [25, 27], we study *entanglement configurations* over a set of nodes, where each entanglement configuration is a collection of multipartite entanglement (CAT or GHZ) states shared within different combinations of subsets of nodes. The main problem is to determine whether LOCC transformations can generate an entanglement configuration B from another entanglement configuration A, written as $B <_{LOCC} A$. We characterize the resulting partial order introduced on *unitarily equivalent classes* of entanglement configurations due to LOCC transformations. This study includes the communication complexity of generating higher cardinality multipartite CAT states from smaller sized CAT state configurations. We also study classes of *incomparable* entanglement configurations where no pair (A,B) of configurations satisfies $A <_{LOCC} B$. This leads us to investigate certain combinatorial properties of hypergraphs and hypertrees following initial results in [25, 27]. We study the unique reconstruction of vertex labelled r-uniform hypertrees on n vertices, where $r \leq n$ is a constant, and each hyperedge has the same number r, of vertices. We conclude by discussing several problems and open questions in the context of entanglement configurations.

Keywords: entanglement characterization and manipulation; combinatorics; hypergraphs; entanglement configurations
PACS: 03.67.Mn, 03.65.Ud

INTRODUCTION

Quantum entanglement has been used as a resource in quantum information processing [7, 20]. Entanglement is also useful in the computation of functions of several variables in a distributed environment where spatially separated parties are provided with input values for the different variables; the communication complexity in such computations can be reduced substantially for certain functions by exploiting multipartite entanglement (see [1, 9, 2, 21]). Entanglement properties of bipartite states are reported in [8, 12]. Bipartite states possess the elegant mathematical property known as the *Schmidt decomposition* [20]. The Schmidt coefficients characterize all non-local properties. Even though no such structure is known for multipartite systems, there are some approaches using certain generalizations of Schmidt decomposition [5, 14, 22] and group theoretic

CP864, *Quantum Computing*, edited by D. Goswami

or algebraic methods [15, 16, 17]. Methods for comparing, quantifying or qualifying entanglement for bipartite systems and/or pure states, include entanglement of formation [4], entanglement cost [4, 28], distillable entanglement [4, 23], relative entropy of entanglement [11], negativity [29], concurrence [30] and entanglement witnesses [13].

Exchange of information is possible by both classical and quantum communication. Entanglement assisted quantum communication as in *superdense coding* can convey upto two bits of information for every qubit communicated. Certain operations like *entanglement teleportation* or creation of multipartite entanglement states can however be done also by using only classical communication, with the aid of only preshared bipartite entanglements between pairs (or subsets) of parties (see [25, 26, 32, 6]). The study of multipartite entanglement transformations under *local operations and classical communication (LOCC)* is therefore an important and interesting stream of research, partly due to the advantage of not having to use channels for qubit communication, and more importantly, because apriori bipartite entanglement sharing patterns between parties or nodes can suffice in creating and using multipartite states between parties (see [25, 26]). Necessary multi-qubit entangled states can certainly be created at a single node and suitably shared with other nodes by means of qubit communication. Indeed, Yao's model [31] formalizes the framework for quantum communication based computation and its complexity in terms of qubits communicated. Our focus is on transformations between multipartite entangled states using LOCC; in situations where LOCC cannot support such transformations, we propose a mixture of LOCC and requisite quantum communication.

We continue investigations in this paper into characterizations of various patterns of multipartite states between spatially separated parties, following the paradigm developed in [25, 26, 27]. We study *entanglement configurations* over a set S of all n nodes, where each entanglement configuration is a collection of multipartite entanglement (CAT or GHZ) states shared within different combinations of subsets of nodes in S. Such an entanglement configuration H may have several (i) EPR pairs (ii) GHZ triples and (iii) m-CAT states. In general, any subset $A \subseteq S$ of cardinality m, may have nodes of A sharing an m-CAT, $1 \leq m \leq n$. The main problem is to determine whether LOCC transformations can generate an entanglement configuration H' from another entanglement configuration H, written as $H' <_{LOCC} H$. We characterize the resulting partial order introduced on classes of entanglement configurations. We study the correlation between the amount of classical communication required and the increase in overall entropy during creation of m-CAT states from CAT states that entangle less than m parties. We also study classes of *incomparable* entanglement configurations where no pair (A,B) of configurations satisfies $A <_{LOCC} B$. This leads us to investigate certain combinatorial properties of hypergraphs and hypertrees. We study the unique reconstruction of vertex labelled hypertrees on n vertices for the special case where each hyperedge has the same number $r \leq n$, of vertices. Here r is a constant and the reconstruction is considered given pairs of vertices $\{v,u\}$ such that v and u belong to the same hyperedge. Such hypergraphs are called r-uniform hypertrees.

Since LOCC can at the best increase only classical correlations, it is considered to be very important in quantifying entanglement; it is desirable that a good measure of entanglement should not increase under LOCC. A necessary and sufficient condition for the possibility of such transformations in the case of bipartite states was given by

Nielsen [19]. An immediate consequence of his result was the existence of *incomparable* states (the states that can not be obtained by LOCC from one another). Bennett et al. [5] formalized the notions of reducibility, equivalence and incomparability to multi-partite states and gave a sufficient condition for incomparability based on *partial* entropic criteria. We study certain LOCC transformations between entanglement configurations with respect to entropy changes in the whole system and the amount of classical communication in those transformations. For mutually *incomparable* entanglement configurations, we further extend the studies done in [25, 27]. We believe that systematic exploration of entanglement configurations through this combinatorial approach can simplify the study of entanglement in future networks of quantum computing nodes. Such analysis may be used to interpret entanglement topologically. We intend to extend our studies to entanglement configurations of non-maximal and mixed multipartite states.

The paper is organized as follows. First we state the definitions and notation for multipartite entanglement configurations. In the next section we develop the partial order of comparable entanglement configurations. In the subsequent section we study the correlation between the amount of classical communication required and the increase in overall entropy during creation of m-party CAT states from CAT states that entangle less than m parties. These transformations are viewed as moving down in the partial order of configurations. Finally, we discuss the *bicolored merging* technique introduced in [26, 27] for showing incomparability of entanglement configurations based on the principle of *monotonicity*. We conclude by discussing several future research directions.

MULTIPARTITE ENTANGLEMENT CONFIGURATIONS

In this section we state definitions and notations about the combinatorics of multipartite entanglement, mostly following [25, 26, 27]. We define an *EPR graph* $G(V,E)$ to be a graph whose vertices are elements of V and whose edges are elements $\{u,v\}$ of E where u and v are in V. This graph represents shared entanglement in the form of EPR pairs between vertex pairs from the set E. If $\{u,v\} \in E$, where $u,v \in V$, then parties u and v share an EPR pair. A *spanning tree* is a graph which connects all vertices without forming cycles. There is a unique path between any two vertices in a spanning tree. There may be more than one path between a pair of vertices in an arbitrary graph. If there is a path for every pair of vertices then the graph is called a *connected graph*. In that sense, a spanning tree is a minimally connected graph with exactly $n-1$ edges, where n is the number of vertices of the graph. An EPR graph $G=(V,E)$ is called a *EPR spanning tree* if the undirected graph $G=(V,E)$ is a spanning tree. We use the notion of a *connected component* of a graph: a subset $A \subseteq V$ of vertices forms a *connected component* of $G=(V,E)$ if it is a maximal set with a path between every pair of its elements (pairs of parties in A).

Notions about graphs can be generalized for multipartite entanglement using *hypergraphs*. A pair of vertices defines an edge in a graph. A *hyperedge* is defined by a subset with 2 or more elements. Let S be a set of n parties represented as vertices, and $F=\{E_1,E_2,\cdots,E_m\}$, where $E_i \subseteq S; i=1,2,\cdots,m$ and E_i is such that its elements (parties) are in the *maximally entangled* $|E_i|$-CAT state. Then, the hypergraph (set system) $H=(S,F)$ is called an *entanglement configuration* of the n parties. From the combinatorial viewpoint, a simple and interesting connection can be made between multipartite entanglement and hyperedges: an m-CAT state corresponds to a hyperedge of size m. In particular, an EPR state corresponds to a simple edge connecting only two vertices. EPR graphs and EPR spanning trees as defined above are special cases of entanglement configurations where all hyperedges are pairs of vertices (parties) sharing EPR pairs. In the rest of the paper we use *EC hypergraphs* and *EC hypertrees* to mean *entanglement configurations* that are hypergraphs or hypertrees, respectively. Since each vertex in a graph or a hypergraph represents a single party in a multiparty environment, we use the equivalent terms *vertex* and *party* throughout the paper.

We define *connectedness* for hypergraphs as follows. A sequence of j hyperedges E_1, E_2, ..., E_j in a hypergraph $H=(S,F)$ is called a *hyperpath* (path) from a vertex $a \in S$ to a vertex $b \in S$ if E_i and E_{i+1} have a common vertex v_i in S, for all $1 \leq i \leq j-1$, $a \in E_1$, and $b \in E_j$, where the vertices v_i are distinct. If there is a hyperpath between every pair of vertices of S in the hypergraph H, we say that H is connected. We use the notion of a *connected component* of a hypergraph: a subset $A \subseteq S$ of parties forms a *connected component* of $H=(S,F)$ if it is the largest set with a hyperpath between every pair of its elements (pairs of parties in A).

Analogous to an EPR spanning tree we state the definition of a *hypertree entanglement configuration* as follows. An EC hypergraph $H=(S,F)$ is a *hypertree entanglement configuration* if it contains no cycles. In other words, no pair of vertices from S has two distinct hyperpaths connecting them. An r-uniform hypertree is an EC hypertree where there are exactly r vertices in every hyperedge. Here r is a fixed integer greater than 1.

In ordinary graphs, a vertex belonging to a single edge is called a *pendant* vertex. This concept is extended to the case of hypergraphs. A vertex of a hypergraph $H=(S,F)$ belonging to exactly one hyperedge from the set F is called a *pendant* vertex in H.

PARTIAL ORDERS OF COMPARABLE ENTANGLEMENT CONFIGURATIONS

Let A and B be entanglement configurations (EC hypergraphs) on n nodes such that $A <_{LOCC} B$. Such EC hypergraphs A and B are called *comparable*. We interpret each hyperedge E_i in an EC hypergraph as one instance of an $|E_i|$-CAT state (an m-partite GHZ state $\frac{1}{2}(|0^m\rangle + |1^m\rangle)$ or its m-partite locally unitarily equivalent state), shared between the nodes of E_i. If neither $A <_{LOCC} B$ nor $B <_{LOCC} A$, then we say that A and B are *incomparable*. The partial order defined on EC hypergraphs by the LOCC relation $<_{LOCC}$ is interesting.

We start with a fundamental and well known result. Let EC hypergraphs $H1$, $H2$ and $H3$ on three vertices A, B and C, be defined by hyperedge sets $F_1 = \{\{A,B\}\}$, $F_2 = \{\{A,B,C\}\}$, and $F_3 = \{\{A,B\},\{A,C\}\}$, respectively. Then, we have $H1 <_{LOCC} H2$ and $H2 <_{LOCC} H3$. In other words, if A shares an EPR pair with each of B and C, then a GHZ state can be prepared between the three parties by LOCC. Also, the GHZ can be transformed into a single EPR pair (without loss of generality) between A and B. It is not difficult to argue using the principle of *monotonicity* as defined in [12] that these transformations cannot be reversed by LOCC. We distinguish bewteen these two LOCC transformations: when $H3$ is transformed to $H2$, we go from maximally entangled states of two parties to a maximally entangled state of three parties, whereas, $H2$ transforms to $H1$ by reducing the number of entangled parties. The common feature is that in either case some classical communication is required, which is determined by local measurement(s). Before we state our observations about the partial order, we state some notation. Let S be a set of n parties and $H=(S,F)$ be an EC hypergraph with a single connected component spanning the vertex subset $A \subseteq S$. We have the following lemma.

Lemma 1 *Let $Reach(H)=\{H'|H'=(S,F')$ is an EC hypergraph such that $H' <_{LOCC} H\}$. Then, $(Reach(H), <_{LOCC})$ is a lattice.*

Proof: (Sketch) The *join (least upper bound or lub)* and *meet (greatest lower bound or glb)* are well defined for any pair $(H1,H2)$ of EC hypergraphs in $Reach(H)$. The join(meet) is the EC hypergraph $HM(HJ)$ such that $H1 <_{LOCC} HM(HJ <_{LOCC} H1)$, and $H2 <_{LOCC} HM(HJ <_{LOCC} H2)$, and for every other $HM'(HJ')$ satisfying $H1 <_{LOCC} HM'(HJ' <_{LOCC} H1)$, and $H2 <_{LOCC} HM'(HJ' <_{LOCC} H2)$, we have $HM <_{LOCC} HM'$ and $HJ' <_{LOCC} HJ$. Such an $HM(HJ)$ exists for each pair $(H1,H2)$ of EC hypergraphs in $Reach(H)$. In particular $H' <_{LOCC} H$ and $HE <_{LOCC} H'$, for every EC hypergraph H' in $Reach(H)$, where HE is the empty EC hypergraph.

□

It is easy to see that the above result can be extended to EC hypergraphs with multiple connected components. It will be interesting to characterize EC hypergraphs H with the property that $Reach(H)$ is also a *distributed lattice* and/or a *complemented lattice* and finally whether it is a *boolean algebra*. We observe $Reach(HCE)$, for EC hypergraph $HCE=(S,FCE)$, is not a distrubutive lattice where $S=\{A,B,C\}$ and FCE has hyperedges $\{A,B\}$, $\{B,C\}$ and $\{C,A\}$. Here, $Reach(HCE)$ has HC hypergraphs (in addition to HCE) with hyperegde sets as follows: (i) $cat=\{\{A,B,C\}\}$ $(3-CAT)$ (ii) $a=\{\{A,B\}\}$ $(AB-EPR)$ (iii) $b=\{\{B,C\}\}$ $(BC-EPR)$ (iv) $c=\{\{C,A\}\}$ $(CA-EPR)$ (v) $d=\{\{A,B\},\{B,C\}\}$ $(AB-BC-EPRs)$ (vi) $e=\{\{B,C\},\{C,A\}\}$ $(BC-CA-EPRs)$ (vii) $f=\{\{C,A\},\{A,B\}\}$ $(CA-AB-EPRs)$ (viii) $null=\{\}$ (NULL). For distributivity we require that $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$, where $a\vee(\wedge)b$ is the lub(glb) of a and b. The left hand side is $a\vee(b\wedge c)$ = $a\vee$ NULL =a. The right hand side is $(a\vee b)\wedge(a\vee c)=cat\wedge cat=cat$. So, $Reach(HCE)$ is not distributive. It turns out that this lattice is also not complemented.

Open question 1 *Characterize EC hypergraphs $H=(S,F)$ whose lattice $Reach(H)$ is distributive and/or complemented.*

Note that the number of EC hypergraphs on a set of n parties is no more than 2^{2^n}. So, we may define the (finite) metric, the *LOCC distance* $dist^H_{LOCC}(H1,H2)$, between two EC hypergraphs $H1$ and $H2$ in the lattice $(Reach(H), <_{LOCC})$ as follows.

1. If $H1$=$H2$ then $dist^H_{LOCC}(H1,H2)=0$.
2. If (i) $H1$ and $H2$ are distinct, (ii) $H2 <_{LOCC} H1$ (without loss of generality), and (iii) there is no EC hypergraph $H' \in Reach(H)$ distinct from $H1$ and $H2$ such that $H2 <_{LOCC} H'$ and $H' <_{LOCC} H1$, then $dist^H_{LOCC}(H1,H2)=1$.
3. If $H1$, $H2$ and H' are distinct and (i) $H1 <_{LOCC} H'$, and (ii) $H' <_{LOCC} H2$, then $dist^H_{LOCC}(H1,H2)=min_{H'\in Reach(H)}\{dist^H_{LOCC}(H1,H')+dist^H_{LOCC}(H',H2)\}$.

The amount of classical communication required along the path with distance $dist^{H}_{LOCC}(H1,H2)$ in the lattice $(Reach(H), <_{LOCC})$, is the minimum required communication for transforming $H1$ to $H2$. Here the transformation is restricted to transit through EC hypergraphs in $Reach(H)$, where H can be transformed by LOCC to $H1$ as well as $H2$. As already defined in this paper, hyperedges in any EC hypergraph represent only maximally entangled CAT states.

An important parameter in any partial order is the maximum number of elements that are mutually incomparable. We call this parameter the *width* of the partial order. In the case of $Reach(H)$, we would pose the following problem for its width denoted by $width(H)$.

Open question 2 *For an EC hypergraph H, determine $width(H)$, the maximum number of EC hypergraphs in $Reach(H)$ that are mutually incomparable.*

Seemingly, $width(H)$ may be viewed as an upper bound on the total number of parameters required to represent the number of equivalence classes of all quantum states that can be obtained by local unitary transformations from EC hypergraphs in $Reach(H)$.

ENTROPY CHANGE AND COMMUNICATION FOR CREATING AN *m*-CAT STATE

In this section we consider the basic LOCC operation of creating an m-CAT state from CAT states shared by less than m parties. Consider the transformation of two EPRs between parties A and B, and B and C, respectively, into a GHZ state between A, B and C. We present a protocol from [25, 26] for this transformation, expressed in the form of the circuit in Figure 3. This protocol involves a total of five qubits, three of which are from A. We explain the working of this circuit in some detail in order to analyse entropy change and communication complexity. The von Neumann entropy of the system of five qubits is initially zero. Finally, after the measurements on qubits $a2$ and $a3$ in A, the entropy of the system rises to 2. This is due to the probability with which each of these qubits gets set to either the $|0\rangle$ state or the $|1\rangle$ state, with equal probability for each state. The details follow.

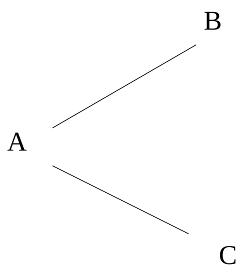

FIGURE 1. *A* shares an EPR pair with each of *B* and *C*.

Without loss of generality let us assume that the sharing arrangement is as in Figure 1. A shares an EPR pair with B and another EPR pair with C but B and C do not share an EPR pair. This means that we have the states $(|0_{a1}0_{b}\rangle + |1_{a1}1_{b}\rangle)/\sqrt{2}$ and

$(|0_{a2}0_c\rangle + |1_{a2}1_c\rangle)/\sqrt{2}$ where $a1$ and $a2$ denote the first and second qubits with A and b and c denote qubits with B and C, respectively. Our aim is to prepare $(|0_{a1}0_b0_c\rangle +$

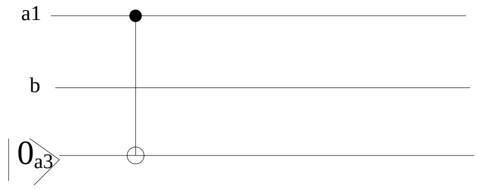

FIGURE 2. Entangling qubits $a3$ with the EPR pair between A and B.

$|1_{a1}1_b1_c\rangle)/\sqrt{2}$. We need three steps to do so.
Step 1: A prepares a third qubit in the state $|0\rangle$. We denote this state as $|0_{a3}\rangle$ where the subscript $a3$ indicates that this is the third qubit of A.
Step 2: A prepares the state $(|0_{a1}0_b0_{a3}\rangle + |1_{a1}1_b1_{a3}\rangle)/\sqrt{2}$ using the the circuit in Figure 2.
Step 3: A sends her third qubit to C with the help of the EPR channel $(|0_{a2}0_c\rangle + |1_{a2}1_c\rangle)/\sqrt{2}$.

A straightforward way to execute Step 3 is through standard teleportation, where only one of the parties B and C is *dynamically involved*. By a party being *dynamically involved* we mean that the party is involved in applying local operations for the completion of the (teleportation) transformation, with final creation of the GHZ state. In our teleportation

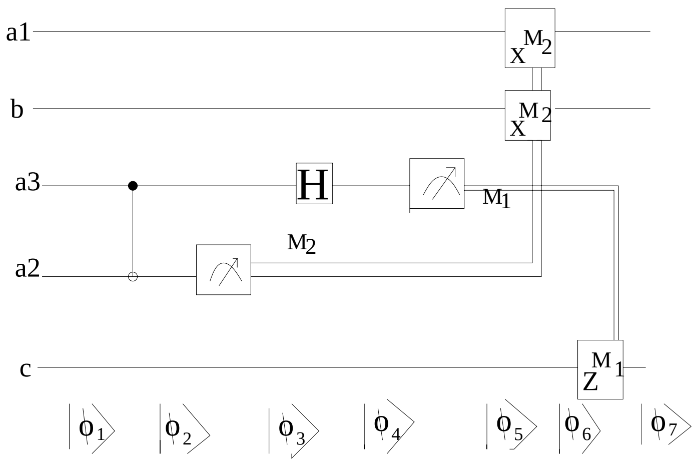

FIGURE 3. Circuit for creating a GHZ state from two EPR pairs with dynamic involment of both the B and C.

circuit as shown in Figure 3, B and C act dynamically. The circuit works as follows. A has three qubits and can do any operation she wishes to be performed on them. Initially the five qubits are jointly in the state $|\phi_1\rangle$. A first applies a controlled NOT gate on her second qubit controlling it from her third qubit changing $|\phi_1\rangle$ to $|\phi_2\rangle$. Then she measures her second qubit yielding measurement result M_2 and bringing the joint state to $|\phi_3\rangle$. She then applies a Hadamard gate on her third qubit and the joint state becomes $|\phi_4\rangle$. A measurement on the third qubit is then done by her yielding the result M_1 and bringing the joint state to $|\phi_5\rangle$. She then applies a NOT (Pauli's X operator) on her first qubit, if M_2 is 1. Now she sends the measurement results M_2 to B and M_1 to C. B applies an X gate on his qubit if he gets 1 and C applies a Z gate (Pauli's Z operator) if he gets 1. The order in which B and C apply their operations does not matter. The final state is $|\phi_7\rangle$. The circuit indeed produces the GHZ state between A, B and C as can be seen from the detailed mathematical explanation given in [26]. It may be noted that the protocol requires two cbits of communication.

Even though the roles of B and C are symmetrical, there is a condition on what operations should be performed when each receives a single cbit from A. B performs an X operation and C performs a Z operation, as required. We set the cyclic ordering $A \to B \to C \to A$. If A shares EPR pairs with the other two, then it is the first one in the ordering. The second one, B, must perform an X operation when he gets a single cbit from A. The third one, C, must perform a Z operation on his qubit when he gets a single cbit from A. If B is the one sharing EPR pairs with the other two then C must apply an X and A must apply a Z.

Methods for creating a GHZ state from Bell pairs have been presented earlier by Zukowski et al. [33] and Zeilinger et al. [32]. The first of these uses three Bell pairs for this purpose. The later, however, uses only two Bell pairs. The motivation for developing our protocol is the dynamic involvement of both B and C which was lacking in these earlier methods. Dynamic involvement of multiple parties might me desirable in multiparty interactive quantum protocols and multiparty cryptograhy (such as in secret sharing). Dynamic involvement may result in fairness. By fairness we mean that every party has similar and symmetric participation in the protocol in creating the final output. Implementation of such protocols with fairness may have interesting applications.

A more familiar alternative would be the use of the standard teleportation circuit as in [3, 20], where an ancilla qubit is first entangled with $a1$ (and therefore with b in B), and then teleported to B using the EPR pair $(a2, c2)$ between B and C. This does the same GHZ creation as in Figure 3; also, the entropy of this system of five qubits goes up by 2, as one can quickly verify. The difference in the case of the standard teleportation circuit is that both the cbits need to be used by party C, whereas, in Figure 3, one cbit is needed at A and the other at C. In either case, the number of cbits communicated is 2, very much the same as the increase in entropy. Both these methods can be used to convert an m-CAT state amongst m parties to an $(m+1)$-CAT state amongst $m+1$ parties where the $(m+1)$th party initially shares an EPR pair with any one of the initial m parties. In this case too, we can verify that two cbits of communication are required and the entropy goes up by two cbits. If we start with only a EPR spanning tree of n vertices, we need $n-2$ stages of entanglement teleportation until we get an n-CAT state. We generalize our observation posing an open problem as follows.

Open question 3 *Given a connected entanglement configuration (EC hypergraph) between n parties, determine the relationship between the amount of communication required in the creation of a pure n-CAT state between the parties in terms of the total rise in the quantum (von Neumann) entropy of the whole system of qubits in the n parties.*

Here, the given connected entanglement configuration may be a hypergraph, hypertree, an EPR graph. or simply an EPR spanning tree; the necessary condition as shown in [26] is that the configuration must be connected. In other words, there must be a hyperpath (or a path) in the hypergraph (graph), between every pair of vertices.

Further, note that any n-CAT state (n-GHZ state) essentially has exactly two basis states $|0^n\rangle$ and $|1^n\rangle$. Local X and Z operations done over the last $n-2$ qubits in the process of generating an n-CAT state, using the standard teleportation technique can create upto 2^{n-2} unitarily equivalent states. [Toggling using X, each of n qubits would give 2^{n-2} possibilities, of which only half are distinct for n-CAT states; using local phase change by Z would again double this to make a total of 2^{n-2} possibilities]. Choosing the right one out of them would require at least $n-2$ cbits of information to be communicated over the $n-2$ stages of the protocol, thereby accounting only partially for the total entropy change of $2(n-2)$ during the $n-2$ stages of the creation of the n-CAT state. This iterative protocol is therefore inefficient but no more than twice as costly as the one which would require just $n-2$ cbits to be communicated overall. Note also that the final n-CAT state produced has the $n-2$ second qubits of the initial EPR pairs (provided we started with a spanning EPR tree). Since the first qubits of the EPR pairs are measured causing $n-2$ units of entropy, we can partially account for the rise in entropy in the iterative $n-2$ stages of teleportations. Further, the ancillae used in the $n-2$ nodes account for $n-2$ more units of entropy. Therefore, we note that the partial entropy of the n qubits of the final n-CAT state is indeed $2(n-2)$; this is with respect to the whole system of $3n-4$ qubits ($n-2$ new ancillae and the old $2(n-1)$ qubits from all EPR pairs taken together). Now consider minimizing the amount of communication required for converting a star-shaped EPR spanning tree into an n-CAT state. Instead of performing the iterative process above as applicable to any spanning EPR tree, we may use the technique of Bose et al. [6], whereby a the central star node performs a measurement on the n-partite maximally entangled basis of its own set of $n-1$ qubits (of the $n-1$ EPR pairs it shares separately with the $n-1$ other parties), and an additional qubit of a private EPR pair. It is easy to see that only $n-1$ cbits (the measurement results in the central star node) need to be broadcast to all the parties so that the mixed state (comprising all the 2^n possible maximally entangled states) can be converted by local unitary operations in respective parties resulting in the desired pure n-CAT state. For an arbitrary EPR spanning tree as the initial entanglement configuration (of EPR pairs), a variable amount cbits between $n-1$ and $2(n-1)$ will need to be communicated depending upon the structure of the EPR spanning tree. A quantitative study, based on the number of non-pendant vertices of the spanning tree would be an interesting problem, useful for an actual spanning tree network of EPR pairs (see [26]).

Open question 4 *Given an arbitrary EPR spanning tree on n vertices, determine the minimum amount of classical communication required to generate a pure n-CAT state shared between the n vertices in terms of combinatorial properties of the spanning tree. Address the same question for arbitrary EPR graphs and EC hypergraphs too.*

We end this section by sketching a generalization of the circuit in Figure 3 for generating an $(m+n-1)$-CAT state from an n-CAT state and an m-CAT state. Here a party A, shares its qubit $a1$ ($a2$) in the n-CAT (m-CAT) with $n-1$ ($m-1$) other parties, and also uses an ancilla qubit $a3$. The generalized protocol and its analysis remains similar; the ancilla $a3$ and the qubit $a2$ are measured creating 2 units of entropy and 2 cbits are communicated for determination of the pure $(m+n-1)$-CAT state generated.

We note here that the Zeilinger et al. [32] result for the same problem as presented in [6] uses no ancilla qubit and measures only one qubit. Therefore, in the creation of tripartite entanglement from two EPR pairs, the final entropy gain is only 1 unit, from the only qubit measured. This method of Zeilinger is therefore more efficient than the standard teleportation method or the one above using dynamic involvement of multiple parties.

INCOMPARABLE ENTANGLEMENT CONFIGURATIONS

In this section we consider LOCC incomparability between entanglement configurations. Given two (incomparable) EC hypergraphs $H1$ and $H2$, we wish to show that neither $H2 <_{LOCC} H1$ nor $H1 <_{LOCC} H2$ holds. We follow the paradigm of *bicolored merging* as developed in [25, 27]. To illustrate this technique we start with a very simple example, Theorem 1 from [27].

Theorem 1 *[27]. Starting from a GHZ state shared amongst three parties in a communication network, two EPR pairs cannot be created between any two sets of two parties using only LOCC.*

We sketch the proof of this theorem. One EPR is shared by A with each of B and C in EC hypertree $H1 = (S, F1)$ where $F1 = \{\{A,B\},\{A,C\}\}$. Let $H2 = (S, F2)$ be another EC hypertree where $F2 = \{\{A,B,C\}\}$, representing a GHZ state. Let EC hypertree $H3 = (S, F3)$ where $F3 = \{\{A,B\},\{B,C\}\}$. For the sake of contradiction, suppose it is possible to create two EPR pairs from a single GHZ. First create $H2$ from $H1$ by LOCC. Then (without loss of generality), convert $H2$ to $H3$ using LOCC, creating two EPR pairs so that B shares one pair each with A and C. Now consider the bicoloring where A and C are given the same color and B is given a different color. We collapse A and C into one single party and B into another single party. Now there are two edges $\{A,B\}$ and $\{C,B\}$ between the two merged parties in $H3$. With the same bipartition, $H1$ has a collapsed edge $\{A,C\}$ and a surviving edge $\{A,B\}$. We observe that by LOCC we have transformed $H1$ to $H3$, and in the process, have increased the marginal entropy of B by one unit, an impossibility. This approach of Singh et al. [27] for demonstrating LOCC incomparability, as initially developed in [25], is motivated by the marginal entropy criterion of [5, 4] that average bipartite entanglement or partial entropy of bipartite states cannot increase under LOCC.

Using this result of impossibility of the conversion of a single GHZ into two EPRs, Singh et al. [27] show that *selective teleportation* of two unknown quantum states ψ_1 and ψ_2, from A to B and C, respectively, cannot be performed using only a single GHZ between A, B and C. [In selective teleportation, multiple unknown quantum states are teleported to different destinations from a central party, say A.]

We can also consider a generalization of the problem of the creation of two EPR pairs from one GHZ as follows: Given $(n-2)$ copies of the n-CAT state shared between n parties, construct an EPR spanning tree of the n parties by LOCC. Singh et al. [27] and Singh [25] present a novel proof of this result (see Theorem 7 in [27]), again by using the technique of bicolored merging elegantly. The following impossibility result about a general version of selective teleporation follows from Theorem 7 in [27].

Theorem 2 *Suppose A shares $(n-2)$ n-CAT states with $(n-1)$ other parties. It is not possible for A to selectively teleport $(n-1)$ unknown qubit states to the $(n-1)$ parties using only LOCC.*

Proof: Assuming that multi-pronged selective teleportation is possible, we note that A would be able to create $(n-1)$ EPR pairs, one with each of the other $(n-1)$ parties by first creating the EPR pairs locally and then teleporting them. This results in creation of a (star) EPR spanning tree, an impossibility by Theorem 7 in [27]. [In this special case a simple bicoloring too works. Consider the bipartition resulting due to bicoloring, where A gets a color different from the rest. Collapsing the two parties based on this bicoloring, we note that staring with $(n-2)$ edges between the two partitions, selective teleportation results in $(n-1)$ edges, a contradiction.]

□

It is likely that similar selective teleportation impossibility results can be proved using LOCC incomparable results on EC hypergraphs.

We now develop a generalization to EPR graphs of Theorem 7 of [27]. Let $G=(V,E)$ be an EPR graph. For a subset S of V, the cut $E(S,\bar{S})$ is defined to be the set of edges from S to $\bar{S}(=V\setminus S)$. More formally, $E(S,\bar{S})=\{\{u,v\}\in E:$ either $u\in S, v\in\bar{S}$ or $v\in S, u\in\bar{S}\}$. Let us denote by $q(G)$ the size of the maximum cut in the graph G i.e. $q(G)=max_{S\subset V}|E(S,\bar{S})|$. The following is a then a natural generalization of Theorem 7 in [27] which we prove by using bicolored merging.

Theorem 3 *$q(G)$ is a lowerbound on the number of copies of $n-CAT$ required to prepare a single copy of the EPR graph $G=(V,E)$ under LOCC where $n=|V|$.*

Proof: First of all we observe that any bicolored merging of a $n-CAT$ reduces to a single edge. Therefore, there will be m edges after (any valid) bicolored merging if we use m copies of $n-CAT$ states. Choose a $S\subset V$ such that $|E(S,\bar{S})|=q(G)$. Now assign color A to vertices in S and color B to vertices in $\bar{S}$ and perform bicolored merging. Clearly all the edges across the cut $(S,\bar{S})$ (i.e. in $E(S,\bar{S})$) will be retained during this bicolored merging. Therefore the number of edges in G after this bicolored merging will be $q(G)$ hence it immediately follows that $m\geq q(G)$.

□

Note that for the particular case when G is a spanning EPR tree, $q(G)=n-1$ and therefore we obtain the same bound as in Theorem 7 of [27] However, in the case of general EPR graph we do not know whether we can actually acheive this bound.

Another line of argument is to generalize the simple 3-party example to connected 2-uniform EC hypertrees on n vertices (EPR spanning trees). In the above 3-party example $H1$ and $H3$ are two mutually incomparable 2-uniform EC hypertrees. Generalizing to any two n-vertex EPR spanning trees, the technique of bicolored merging can be used to show their LOCC incomparability as in Theorem 8 in [27]. This immediately leads to the observation that there are an exponential number of (actually n^{n-2}) EPR spanning trees of n labelled vertices [10]. This count of the number of EPR spanning trees is due to what is popularly called Cayley's theorem [10, 18]. It gives the unique coding of vertex labelled spanning trees called *Prüfer coding*. Naturally, we may ask similar questions about 2-uniform EC hypergraphs (also call EPR graphs) of different kinds.

Unlike trees, graphs may have cycles. So, counting the number of LOCC incomparable EPR graphs is a challenging combinatorial problem; the study of this problem may be suitably aided by the lattice partial orders defined in this paper. Also, we may not restrict our investigations to EPR graphs and extend the study to EC hypergraphs in general.

Open question 5 *Determine the number of EPR graphs on n vertices that are mutually LOCC incomparable. Determine the number of EC hypergraphs on n vertices that are mutually LOCC incomparable.*

As already mentioned above, initial work in this direction by Singh [25] and Singh et al. [27] shows that *r*-uniform hypertrees are mutually LOCC incomparable. This is true even if the pendant sets of vertices in the two distinct *r*-uniform hypertrees are identical. In the more general case they have the following interesting combinatorial result.

Theorem 4 *[27] Let $H_1 = (S, F_1)$ and $H_2 = (S, F_2)$ be two entangled hypertrees. Let P_1 and P_2 be the set of pendant vertices of H_1 and H_2 respectively. If the sets $P_1 \setminus P_2$ and $P_2 \setminus P_1$ are both nonempty then the multi-partite states represented by H_1 and H_2 are necessarily LOCC-incomparable.*

As may be anticipated, two distinct hypertrees H_1 and H_2 may or may not be LOCC incomparable if their pendant sets of vertices viz., P_1 and P_2 are identical.

We concentrate now on the following combinatorial result; the non-trivial proof of the incomparability of *r*-uniform hypertrees in [25, 27] uses this result.

Theorem 5 *[27] Given two distinct r-uniform hypertrees $H_1 = (S, F_1)$ and $H_2 = (S, F_2)$ with $r \geq 3$, there exist vertices $u, v \in S$ such that u and v belong to same hyperedge in H_2 but necessarily to different hyperedges in H_1.*

Note that the parties (or vertices) in S are labelled. The *r*-uniform hypertrees H_1 and H_2 are distinct because the hyperedge sets F_1 and F_2 are distinct. If u and v are parties in S that share a hyperedge in exactly one of H_1 and H_2, then we call such a pair $\{u, v\}$ a *witness*. The above theorem states that two distinct *r*-uniform hypertrees must have a witness. Naturally, the contrapositive would imply the equality of two (verex labelled) *r*-uniform hypertrees if there is no witness. Now consider the binary relation $R(H)$ of *all possible pairs* of vertices (u, v) in an (unknown) hypergraph H, where u and v share a hyperedge in H. We call this relation $R(H)$, the *vertex pairing* relation of the hypergraph H. Theorem 5 implies that $R(HT)$ uniquely encodes an *r*-uniform hypertree HT. Note that the relation $R(H)$ is reflexive and symmetric but not transitive for an arbitrary hypergraph H. Also note that $R(H)$ does not uniquely encode hypergraphs that are not *r*-uniform hypertrees. In the following example we consider the hypergraph of 5 vertices with vertex pairing relation including the following pairs, viz., $\{1,2\}$, $\{1,5\}$, $\{1,3\}$, $\{1,4\}$, $\{3,4\}$, $\{2,3\}$ and $\{2,4\}$. We show that this vertex pairing relation can represent either a hypertree or a cyclic hypergraph. The hypertree in this case has hyperedges $\{1,2,3,4\}$ and $\{1,5\}$. This is not a uniform hypertree as one hyperedge has 4 vertices and the other has only 2. The cyclic hypergraph has hyperedges corresponding to all the 7 pairings in the vertex pairing relation and is certainly 2-uniform, although it has a cycle in the hyperedges $\{1,2\}$, $\{2,3\}$ and $\{3,1\}$. So, we note that the vertex pairing relation has two reconstructions, one giving a non-uniform hypertree, and the other giving a uniform but cyclic hypergraph. We summarize the following corollary to Theorem 5.

Corollary 1 *Let HT_1 and HT_2 be two r-uniform hypertrees defined on the same set of vertices. If the vertex pairing relations $R(HT_1)$ and $R(HT_2)$ are identical then $HT_1 = HT_2$.*

This corollary immediately suggests that given the vertex pairing relation of a r-uniform hypertree, we must be able to uniquely (and possibly) efficiently reconstruct the hypertree. A r-uniform hypertree has $\frac{n-1}{r-1}$ hyperedges and therefore can be encoded in $\frac{n}{1-1/r}$ integers. The vertex pairing relation has $\frac{r}{2}(r-1)\frac{n-1}{r-1} \geq \frac{n}{1-1/r}$ integers. So, the straightforward listing of the vertex pairing relation requires a larger number of integers than the size of the hypergraph representation itself. Determining a more efficient encoding of r-uniform hypertrees is an interesting problem.

Open question 6 *Determine an encoding of r-uniform hypertrees with less than $\frac{n}{1-1/r}$ integers.*

EC hypertrees are minimal hypergraphs connecting all the n nodes in the network. Furthermore r-uniform hypertrees have hyperedges where each hyperedge represents an r-CAT state. Such structures are important because they are simpler than general hypergraphs and can be used to generate n-CAT states between all the n parties. In this context, coding and counting of such structures are important problems. Coding and counting of vertex labelled spanning trees is well settled. The well known Prüfer coding technique for labelled spanning trees (2-uniform hypertrees) leads to a proof of Cayley's theorem for counting labelled spanning trees [18]. Renyi and Renyi [24] developed Prüfer-like codes for graphs called *partial k-trees* and some counting techniques.

CONCLUDING REMARKS

The problems and results considered in this paper relate to pure multipartite entanglement states. The partial ordering of LOCC transformations between entanglement configurations are also accordingly restricted to EC hypergraphs with hyperedges permitting the representation of only multipartite GHZ or CAT states. The present study with respect to EC hypergraphs may be generalized in a number of ways: considering other inequivalent n-particle states, mixed states and non-maximal entanglement. The combinatorial approach of [25, 27] may be studied in general stochastic LOCC (SLOCC) settings, where problems mentioned in this paper may be generalized suitably.

Acknowledgments: The authors thank Sudhir Kumar Singh for his stimulating ideas and insightful comments and suggestions.

REFERENCES

1. H. Buhrman, R. Cleve and W. van Dam, Quantum entanglement and communication complexity. *SIAM Journal of Computing*, 30(6):1829–1841, 2001.

2. H. Buhrman, W. van Dam, P. Hoyer and A. Tapp, Multiparty quantum communication complexity. Phys. Rev. A 60(4):2737, 1999.
3. C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. 70, 1895 (1993).
4. C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A 54, 3824 (1996); eprint quant-ph/9604024 (1996).
5. C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Exact and asymptotic measures of multipartite pure-state entanglement, Phys. Rev. A 63, 012307 (2000).
6. S. Bose, V. Vedral, and P. L. Knight, Multiparticle generalization of entanglement swapping, Phys. Rev. A 57, 822 (1998).
7. D. Bouwmeester, A. Ekert and A. Zeilinger (Eds.), *The Physics of Quantum Information*, Springer (2000).
8. D. Bruss, Characterizing Entanglement, J. Math. Phys. 43, 4237 (2002).
9. R. Cleve and H. Buhrman, Substituting quantum entanglement for communication. *Physical Review A*, 56(2):1201–1204, 1997.
10. N. Deo, *Graph Theory: With Applications to Engineering and Computer Science*, Prentice Hall (1974).
11. L. Henderson and V. Vedral, Information, relative entropy of entanglement and irreversibility, Phys. Rev. Lett. 84, 2263 (2000).
12. M. Horodecki, Entanglement measures, Quantum Information and Computation, 1, 3 (2001).
13. M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: Necessary and sufficient conditions, Phys. Lett. A 223, 1 (1996).
14. J. Kempe, Multiparticle entanglement and its applications to cryptography, Phys. Rev. A 60, 910 (1999).
15. N. Linden, S. Popescu, On multi-partite entanglement, Fortsch. Phys. 46, 567-578 (1998).
16. N. Linden, S. Popescu and A. Sudbery, Nonlocal parameters for multiparticle density matrices, Phys. Rev. Lett. 83, 242 (1999).
17. D. Liu, Guoying Lu, J. P. Draayer, A simple entanglement measure for multipartite pure states, Int. J. Theor. Phys. 43, 1241 (2004); eprint quant-ph/0405133.
18. L. Lovász, J. Pelikán and K. Vesztergombi, *Discrete Mathematics: Elementary and Beyond*, Springer, 2003.
19. M. A. Nielsen, Conditions for a Class of Entanglement Transformations, Phys. Rev. Lett. 83, 436 (1999).
20. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2002).
21. S. P. Pal, S. Das and S. Kumar, Constant communication complexity protocols for multiparty accumulative boolean functions, eprint quant-ph/0510050, 2005.
22. M. H. Partovi, Universal Measure of Entanglement, Phys. Rev. Lett. 92, 077904 (2004).
23. E. M. Rains, Rigorous treatment of disillable entanglement, Phys. Rev. A 60, 173 (1999); Erratum: Phys. Rev. A 63, 173 (1999).
24. C. Rényi and A. Rényi, *The Prüfer code for k-trees*, Combinatorial Theory and its Applications III, pp. 945-971, (Eds.) P. Erdös, A. Rényi and Vera T. Sós, North-Holland Publsihing Comapny, 1970.
25. S. K. Singh, *Combinatorial Approaches in Quantum Information Theory*, M.Sc. Thesis, Dept. of Mathematics, IIT Kharagpur, India, eprint quant-ph/0405089 (2004).
26. S. K. Singh, S. Kumar and S. P. Pal, Characterizing the combinatorics of distributed EPR pairs for multi-partite entanglement, eprint quant-ph/0306049 (2003).
27. S. K. Singh, S. P. Pal, S. Kumar and R. Srikanth, A combinatorial approach for studying local operations and classical communication transformations of multipartite states, J. Math. Phys. 46, 122105 (2005); eprint quant-ph/0406135 v3.
28. G. Vidal, W. Dür, J. I. Cirac, Entanglement cost of bipartite mixed states, Phys. Rev. Lett. 89, 027901 (2002); eprint quant-ph/0112131.
29. G. Vidal and R.F. Werner, A computable measure of entanglement, Phys. Rev. A 65. 032314, (2002), eprint; quant-ph/0102117.
30. W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, Phys. Rev. Lett. 80, 2245 (1998).

31. A. C. C. Yao, Quantum communication complexity, Proceedings of the 34th IEEE Symposium on Foundations of Computer Science, pp. 352-361, 1993.
32. A. Zeilinger, M. A. Horne, H. Weinfurter, and M. Zukowski, Three-particle entanglements from two entangled pairs, Phys. Rev. Lett. 78, 3031 (1997).
33. M. Žukowski, A. Zeilinger, and H. Weinfurter, Entangling independent pulsed photon sources, Ann. N. Y. Acad. Sci. 755, 91 (1995).