Circuits for Distributing Quantum Measurement

P. K. Panigrahi*, M. Gupta†, A. Pathak† and R. Srikanth**

*Physical Research Laboratory, Navrangpura, Ahmedabad, 380 009, India
†Jaypee Institute of Information Technology, Noida, 201 307, India
**Raman Research Institute, Bangalore, 560012, India

Abstract. We provide explicit quantum circuits for the distributed measurement of Bell states in the Hilbert space C^{d^n}, where d is qudit dimension. We discuss a method for generalizing this to distributed measurements on any set of orthogonal states distributed among n parties. From the practical viewpoint, we show that such distributed measurements can help lower quantum communication complexity under certain conditions.

Keywords: Entangled states, Quantum circuits, Distributed measurement
PACS: 03.67.Hk, 03.67.Mn

INTRODUCTION

Entangled states play a key role in the transmission and processing of quantum information [1, 2]. Using an entanglement channel [3], an unknown state can be teleported [4] with local unitary operations, appropriate measurement and classical communication; one can achieve entanglement swapping through joint measurement on two entangled pairs [5]. Entanglement leads to increase in the capacity of the quantum information channel, known as quantum dense coding [6]. The bipartite, maximally entangled Bell states provide the most transparent illustration of these aspects, although three-particle entangled states like GHZ and W states are beginning to be employed for various purposes [7, 8]. Nonorthogonal states cannot be discriminated with certainty [9], while the discrimination of orthogonal states are in principle possible. A large number of results regarding distinguishing orthogonal states, shared between two or more parties, who may hold one or more copies of the states, and whose communication is restricted to local operations and classical communication (LOCC), have recently been established [10, 11, 12, 13, 14].

We consider the different problem of discriminating between a set of orthogonal states in which quantum communication between different parties is not excluded. The trivial solution is for the qudits to be brought together and measured. Alternatively, they may be separately brought in interaction with a common ancilla, which is then measured to obtain the relevant information. We refer to such an act of measuring an observable W of system A indirectly via an ancilla as 'outsourcing' of the measurement of W. In the former case, a number of theoretical and experimental results already exist in the area of unambiguous state discrimination [15, 16, 17, 18, 19]. A outsourced measurement on many qudits is said to be 'distributed' when it is broken up into a sequence of interactions between a common ancilla and qudits of the principal system, followed by a final measurement on the ancilla.

CP864, *Quantum Computing,* edited by D. Goswami

This work presents a general strategy for obtaining circuits that outsource and distribute measurements of (a generalization of) Bell states in C^{d^n}. The article is divided as follows. In Section , we present circuits for distributing Bell state discrimination for n qubits shared among n players, beginning with the case of conventional (2-qubit) Bell states. In Section , this result is generalized to construct circuits for Bell state discrimination among two or more qudits. We briefly point out the underlying mathematical structure that clarifies how our proposed circuits work. In principle, this can be used to further generalize our results of Section to distributed discrimination of any set of orthogonal states. In Section , we examine a specific situation where such methods of distributing measurements can be useful in computing and cryptography.

BELL STATE DISCRIMINATION IN C^{2^n} HILBERT SPACE

In principle, any set of orthogonal states can be discriminated in quantum mechanics, though LOCC may not be sufficient if the state is distributed among two or more players. Here we start with a C^{2^n} Hilbert space. To describe any state in this Hilbert space we need 2^n orthonormal basis vectors. The choice of the basis is not unique, but one choice of particular importance is the set of maximally entangled n-qubit generalization of Bell states given by:

$$|\psi_x^+\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |\bar{x}\rangle), \tag{1}$$

$$|\psi_x^-\rangle = \frac{1}{\sqrt{2}}(|x\rangle - |\bar{x}\rangle) \tag{2}$$

where x varies from 0 to $2^{n-1}-1$ and $\bar{x} \equiv 1^{\otimes n} \oplus x$ in modulo 2 arithmetic. The set of complete basis vectors (1,2) reduces to Bell basis for $n=2$ and to GHZ states for $n=3$. As an example, setting $n=2$ in Eq. (1,2) we get the usual Bell states

$$\begin{aligned}
|\psi_{00}\rangle = |\psi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\\
|\psi_{01}\rangle = |\phi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\\
|\psi_{10}\rangle = |\psi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),\\
|\psi_{11}\rangle = |\phi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{aligned} \tag{3}$$

A circuit to distribute measurement of the generalized orthonormal entangled basis states (1,2) employing ancillas is shown in Fig. 1. To discriminate the members of the entangled, orthonormal basis set in C^{2^n}, we have to communicate and carry out measurements on n ancillary qubits in the computational basis. The first measurement is done on the state $|R_{nA_1}\rangle$, as shown in Eq. (4). This measurement determines the relative phase between $|x\rangle$ and $|\bar{x}\rangle$. It will give 0 for $\frac{1}{\sqrt{2}}(|x\rangle + |\bar{x}\rangle)$ and 1 for $\frac{1}{\sqrt{2}}(|x\rangle - |\bar{x}\rangle)$. The

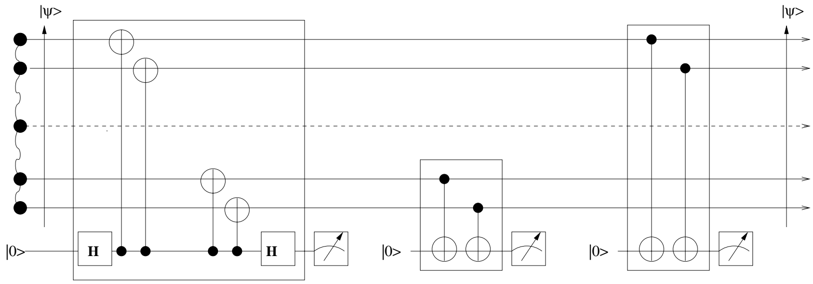

FIGURE 1. Diagram depicting the circuit for distribution of generalized orthonormal qubit Bell state discrimination. The first bounded box depicts an effective measurement of $X^{\otimes n}$, which yields the phase bit value. The second and third boxes depict an effective, measurement of $Z^{\dagger} \otimes Z$, which yields the relative parity between two consecutive qubits. To obtain the full relative parity information, $n-1$ relative parity measurements are required.

next measurement computes the parity between two consecutive bits and yields zero if the bits coincide and one, otherwise. This follows from Eq. (5), which shows the state for the complex of the system and the ith ancilla, where $2 \leq i \leq n$. Each ancilla A_i is sequentially interacted with the system and then measured. It can be shown (Section) that this action leaves the states $|\psi_x^{\pm}\rangle$ undisturbed. This means that the corresponding measurements, M_i, represent commuting observables. In general, M_1 gives the phase bit, and M_i gives the parity of the string comprising of the ith and i+1th qubits.

In a way clarified below, M_1 may be regarded as the distributed measurement of $X^{\otimes n}$ and M_i $(2 \leq i \leq n)$ that of measuring $Z(i-1) \otimes Z(i)$, so that the simultaneous measurability of any pair of M_i's follows from the fact that $[X^{\otimes n}, Z(j) \otimes Z(k)] = 0$ and $[Z(j) \otimes Z(k), Z(j') \otimes Z(k')] = 0$ where $Z(j)$ is the Pauli Z operator acting on the jth qubit.

A note on notation: the sign $Q(j \leftarrow k)$ signifies a C-NOT gate, with k being (ancilla) control index number, and j being (system) target index number. Conversely, $Q(j \rightarrow k)$ signifies a C-NOT gate with j being (system) control index number and k being (ancilla) target index number.

$$|R_{(n\times 2)A1}\rangle = \left[I_2^{\otimes n} \otimes H_2\right] \times \left[\bigotimes_{j=1}^{n} Q(j \leftarrow 1)\right] \times \left[I_2^{\otimes n} \otimes H_2\right] (|\Psi\rangle_{1\cdots n} \otimes |0\rangle_{A1}), \quad (4)$$

$$|R_{(n\times 2)Ai}\rangle = \left[Q([i-1] \rightarrow i) \otimes Q(i \rightarrow i)\right] (|\Psi\rangle_{1\cdots n} \otimes |0\rangle_{Ai}), \quad (5)$$

where $2 \leq i \leq n-1$. Therefore, all together we need n measurements on n ancillary qubits to discriminate 2^n orthonormal, entangled basis states of the form (1,2). Furthermore, we require $3n-2$ applications of CNOT gates. The question of quantity of quantum communication required, which depends on the topology of the quantum communication network, is discussed in Section in detail.

A proof that the circuit described in Eq. (4,5), and depicted in Fig. 1 achieves the required Bell state discrimination is deferred to Section . Here we simply illustrate it using the specific example of the usual Bell states (3). Since (1,2) reduces to (3) for

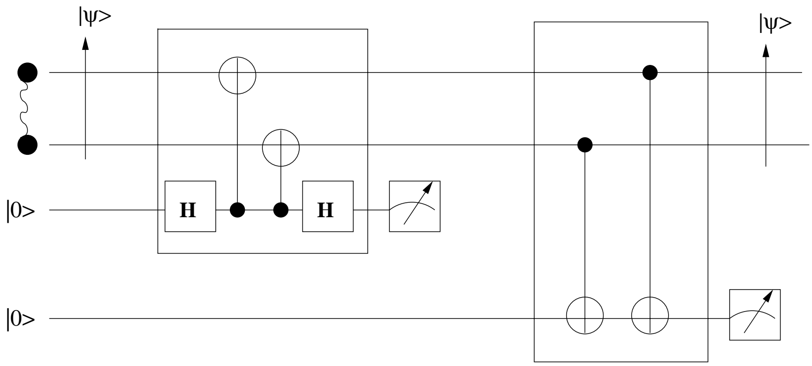

FIGURE 2. A special case of Fig. 1: the diagram depicting the circuit for Bell state discriminator.

$n = 2$, our generalized circuit reduces to that shown in Fig. 2, where one needs only two ancillary qubits, four CNOT gates, two measurements and two qubits of quantum communication.

In Table 1, we have shown the results of the measurements on both the ancillas when different Bell states are present in the given circuit (Fig. 2). Just before measurement, the states can be explicitly written as,

$$\begin{aligned} |R_{(2\times2)A1}\rangle &= [I_2 \otimes I_2 \otimes H_2] \times [Q(1 \leftarrow 1) \otimes Q(2 \leftarrow 1)] \\ &\times [I_2 \otimes I_2 \otimes H_2]\,(|\Psi\rangle_{12} \otimes |0\rangle_{A1}) \qquad (6) \\ |R_{(2\times2)A2}\rangle &= [Q(1 \rightarrow 2) \otimes Q(2 \rightarrow 2)]\,(|\Psi\rangle_{12} \otimes |0\rangle_{A2})\,. \qquad (7) \end{aligned}$$

TABLE 1. Results of outsourced measurements on two ancilla for the Bell states 3

Bell State	**Measurement A_1**	**Measurement A_2**
$\|\psi^+\rangle$	0	0
$\|\psi^-\rangle$	1	0
$\|\phi^+\rangle$	0	1
$\|\phi^-\rangle$	1	1

Thus we have provided a circuit for distributing qubit Bell state discrimination shared between two or more parties, in which joint measurement is replaced by indirect measurement mediated by an ancilla. These results can be straightforwardly generalized, as shown in the following Section.

GENERALIZED BELL STATE DISCRIMINATION IN C^{d^n}

The results of the preceding Section can be generalized to entangled states of n qudits. To this end, we replace the regular Pauli matrices with their d-dimensional analogs [20].

We generalize X and Z gates to X_d and Z_d, respectively, which have the action:

$$\begin{aligned} Z_d|j\rangle &\mapsto e^{2\pi\imath j/d}|j\rangle & (8)\\ X_d|j\rangle &\mapsto |j-1\rangle, & (9)\end{aligned}$$

where the arithmeticin the ket is in mod d. The operators X_d and Z_d are related by a Fourier transform $X_d = H_d Z_d H_d^\dagger$, where H_d is the generalized Hadamard transformation given by:

$$(H_d)_{jk} = \frac{1}{\sqrt{d}} e^{2\pi\imath j\cdot k/d}. \qquad (10)$$

Unlike the qubit case, the unitary operators Z_d, X_d and H_d are not Hermitian.

The d generalized Bell states are

$$|\Psi_{pq}\rangle = \frac{1}{\sqrt{d}} \sum_j e^{2\pi\imath jp/d}|j\rangle|j+q\rangle, \qquad (0 \le p, q \le d-1) \qquad (11)$$

which form an orthogonal, complete basis of maximally entangled vectors for the d^2 dimensional "qudit" space [21]. The parameter p denotes phase and q the generalized parity. The states $|\Psi_{pq}\rangle$ are d-dimensional analogs of Bell states (3) in that they are eigenstates of the operator $X_d \otimes X_d$, which is equivalent to the phase observable, whose eigenvalues are p or some function $f(p)$, and $Z_d^\dagger \otimes Z_d$, which is equivalent to the parity observable, whose eigenvalues are q or some function $f(q)$. Therefore, measurements equivalent to (i.e., compatible with) these operators guarantee a complete characterization of the generalized Bell states. Furthermore, the set of generalized Bell states remains closed under the action $H_d^\dagger \otimes H$ or $H_d \otimes H_d^\dagger$ or (cf. Appendix).

The generalization of the CNOT that we require is the one, whose action we define by

$$\mathscr{C}_X : |j\rangle|k\rangle \longmapsto |j\rangle|j-k\rangle, \qquad (12)$$

which reduces to the conventional CNOT for qubits. We use the following notation: $\mathscr{C}_X(j \leftarrow k)$ signifies a generalized CNOT gate with k being (ancilla) control index number, and j being (system) target index number; $\mathscr{C}_X(j \rightarrow k)$ signifies a generalized CNOT gate with the control-target order reversed. A similar terminology extends to the two-qudit gate $\mathscr{C}_X^\dagger$, whose action is given by either $|j\rangle|k\rangle \longmapsto |j\rangle|k-j\rangle$ or $|j\rangle|k\rangle \longmapsto |j-k\rangle|j\rangle$, depending on whether the system or ancilla is the control register.

A direct generalization of Eq. (4,5) to d-dimension of Eq. (6,7) is

$$\begin{aligned} |R_{(2\times d)A1}\rangle &= [I_d \otimes I_d \otimes H_d] \times [\mathscr{C}_X(1 \leftarrow 1)\mathscr{C}_X(2 \leftarrow 1)] \\ &\times [I_d \otimes I_d \otimes H_d^\dagger](|\Psi\rangle_{12} \otimes |0\rangle_{A1}). & (13)\\ |R_{(2\times d)A2}\rangle &= \left[\mathscr{C}_X(1 \rightarrow 2)\mathscr{C}_X^\dagger(2 \rightarrow 2)\right](|\Psi\rangle_{12} \otimes |0\rangle_{A2}). & (14)\end{aligned}$$

We will denote the observables corresponding to circuits (13) and (14) as M_1 and M_2, respectively. M_1 will yield the 'phase value' p, and M_2 the generalized parity, q. Obserables M_1 and M_2 correspond, respectively, to the unitary operations $X_d \otimes X_d$

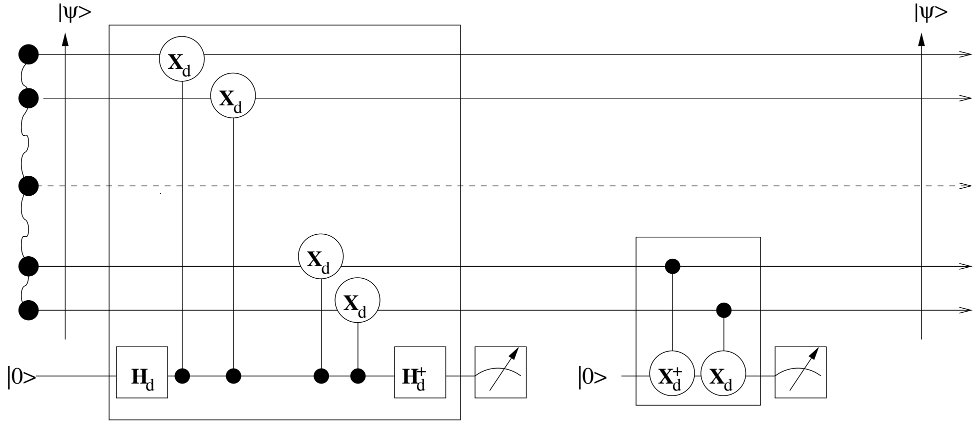

FIGURE 3. Diagram depicting the circuit for distribution of generalized orthonormal qudit Bell state discrimination. The first bounded box depicts the outsourced measurement of an observable that is compatible with $X_d^{\otimes n}$, which for the generalized Bell states yields the global phase value p. The second box depicts the outsourced measurement of an observable compatible with $Z_d^\dagger \otimes Z_d$, which yields the relative parity between two consecutive qudits. To obtain the full relative parity information, $n-1$ such relative parity measurements are needed.

and $Z^\dagger \otimes Z$, so that the simultaneous measurability of M_1 and M_2 can be shown as a consequence of the fact that $[X_d \otimes X_d, Z_d^\dagger \otimes Z_d] = 0$. More directly, we will show that both measurements have $|\Psi_{pq}\rangle$ as eigenstate.

A complete, maximally entangled Bell basis for the Hilbert space C^{d^n} can be given by:

$$|\Psi_{pq_1q_2\cdots q_{n-1}}\rangle = \sum_{j=0}^{d-1} e^{2\pi\iota j\cdot p/d}|j, q_1+j, q_2+j, \cdots, q_{n-1}+j\rangle. \tag{15}$$

We call them Bell states in the sense that any state $|\Psi_{pq_1q_2\cdots q_{n-1}}\rangle$ is an eigenstate of $X_d^{\otimes n}$ and $Z_d(j) \otimes Z_d^\dagger(j+1)$ $(1 \leq j \leq (n-1))$, which correspond to observables with eigenvalues p and $q_{j+1} - q_j$ respectively, the latter being called the *generalized relative parity*.

A generalization of Eq. (13,14) to n qudits is Eq. (16,17), which describes a circuit to measure phase information p and generalized parity information $q_1, q_2, \cdots, q_{n-1}$ of such states. The circuit is depicted in Fig. 3. The required ancilla are n qudits.

$$|R_{(n\times d)A1}\rangle = \left[I_d^{\otimes n} \otimes H_d^\dagger\right] \times \left[\Pi_{j=1}^n \mathscr{C}_X(j \leftarrow 1)\right] \times \left[I_d^{\otimes n} \otimes H_d\right] (|\Psi\rangle_{1\cdots n} \otimes |0\rangle_{A1}), \tag{16}$$

$$|R_{(n\times d)Ai}\rangle = \left[\mathscr{C}_{X_d}([i-1] \rightarrow i)\mathscr{C}_X^\dagger(i \rightarrow i)\right] (|\Psi\rangle_{1\cdots n} \otimes |0\rangle_{Ai}). \tag{17}$$

We will denote the measurements realized by these circuits, via ancilla A_i, by M_i $(1 \leq i \leq n)$. To see that the M_i's are compatible, it turns out to be sufficient to note that $[X_d^{\otimes n}, Z_d(j) \otimes Z_d^\dagger(k)] = 0$ $(j \neq k)$ and $[Z_d(j) \otimes Z_d^\dagger(k), Z_d(j') \otimes Z_d^\dagger(k')] = 0$, which indeed follows from the fact the states $|\Psi_{pq_1q_2\cdots q_{n-1}}\rangle$ are eigenstates of $X_d^{\otimes n}$ and $Z_d^\dagger(j)\otimes Z_d(k)$, which is shown below explicitly.

To this end, we note that the action of the first two (boxed) operations in Eq. 16 on a state $|\Psi_{pq_1q_2\cdots q_{n-1}}\rangle|k\rangle$ is

$$\begin{aligned}
|\Psi_{pq_1,q_2,\cdots,q_{n-1}}\rangle|k\rangle &= \left[\sum_{j=0}^{d-1} e^{2\pi\iota j\cdot p/d}|j,q_1+j,q_2+j,\cdots,q_n+j\rangle\right]|k\rangle \\
&\longrightarrow \left[\sum_{j=0}^{d-1} e^{2\pi\iota j\cdot p/d}|j,q_1+j-k,q_2+j-k,\cdots,q_n+j-k\rangle\right]|k\rangle \\
&= \left[\sum_{j'=0}^{d-1} e^{2\pi\iota j'\cdot p/d}|j',q_1+j',q_2+j',\cdots,q_n+j'\rangle\right]|k\rangle \\
&= e^{2\pi\iota k\cdot p/d}|\Psi_{pq_1,q_2,\cdots,q_{n-1}}\rangle|k\rangle, \qquad (18)
\end{aligned}$$

from which it follows that the full effect of the operation described in Eq. (16) produces the state:

$$\begin{aligned}
|\Psi_{pq_1,q_2,\cdots,q_{n-1}}\rangle H_d|k\rangle &= |\Psi_{pq_1,q_2,\cdots,q_{n-1}}\rangle\left(\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle\right) \\
&\longrightarrow |\Psi_{pq_1,q_2,\cdots,q_{n-1}}\rangle\left(\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{2\pi\iota p\cdot j/d}|j\rangle\right) \\
&\longrightarrow |\Psi_{pq_1,q_2,\cdots,q_{n-1}}\rangle|p\rangle. \qquad (19)
\end{aligned}$$

This yields the phase bit upon the ancilla being measured.

It is easily seen that the action (17) non-destructively extracts the relative parity information. For,

$$\begin{aligned}
&\left[\mathscr{C}_{X_d}([i-1]\to i)\mathscr{C}^\dagger_{X_d}(i\to i)\right]|\Psi_{pq_1,q_2,\cdots,q_{n-1}}\rangle|0\rangle_i \\
&= \mathscr{C}_{X_d}([i-1]\to i)\sum_{j=0}^{d-1}e^{2\pi\iota j\cdot p/d}|j,q_1+j,q_2+j,\cdots,q_{n-1}+j\rangle|q_{i+1}+j\rangle_i \\
&= \sum_{j=0}^{d-1}e^{2\pi\iota j\cdot p}|j,q_1+j,q_2+j,\cdots,q_{n-1}+j\rangle|q_{i+1}-q_i\rangle_i \\
&= |\Psi_{pq_1,q_2,\cdots,q_{n-1}}\rangle|q_{i+1}-q_i\rangle_i. \qquad (20)
\end{aligned}$$

The operation $\left[\mathscr{C}_{X_d}([i-1]\to i)\mathscr{C}^\dagger_{X_d}(i\to i)\right]$ serves to entangle and then disentangle the input Bell state and the ancilla, such that the relative parity of the two concerned qudits can be read off the latter in the computational basis. This also proves that the circuits given in Eqs. (4,5), (6,7) and (13,14) perform distributed Bell state discrimination in dimensions 2^n, 2×2 and $d\times d$, respectively, for they are all special cases of the circuit described in Eq. (16,17).

Note that although the circuit for qubits in Fig. 1 and for qudits in Fig.3 use relative parity measurements on consecutive pairs of qudits, they need not do so. Given any set of $n-1$ relative parity values $q_j - q_k$ that suffice to fully determine the q_j's in a state $|\Psi_{pq_1q_2\cdots q_{n-1}}\rangle$, our distributed measurements are such that the generalized Bell states are eigenstates of such operators, and hence form a complete set of compatible observables. Such relative parity measurements correspond to an observable compatible with $Z_d^\dagger(j)\otimes Z_d(k)$ (in the $d=2$ case, the observable is identical with $Z(j)\otimes Z(k)$). Depending on the topology of the quantum communication network available, the choice of relative parity measurements can vary. For example, if the communication network has a star topology, as in Fig. 4(a), then the set of observables can correspond to $Z_d^\dagger(1)\otimes Z_d(j)$, where 1 is the hub index (marked A in the figure), and j runs through the remaining vertices. Since the operators $X_d^{\otimes n}$ and $Z_d^\dagger(j)\otimes Z_d(k)$ commute, the distributed versions of measurements compatible with them can be simultaneously determined.

In concluding this Section, we will briefly mention the basic mathematical structure underlying our circuits, reported in detail elsewhere [22]. In so doing, we will be able to indicate how to adapt the ideas of the preceding Sections to the case of distributing any orthonormal state discrimination. As pointed out earlier, the generalized Bell states are eigenstates of the unitary operators $X_d^{\otimes n}$ and $Z_d^\dagger\otimes Z_d$, where $d,n\geq 2$. A key observation is that the measurement of M_1, effected through the ancilla A_1, is equivalent to measuring an observable compatible with the unitary operators $X_d^{\otimes n}$, while the measurement M_i $(2\leq i\leq n)$, effected through the ancilla A_i, is equivalent to measuring an observable compatible with the unitary operators $Z_d^\dagger\otimes Z_d$. In the case of $d=2$, of course, the observable and the unitary operator, given by the $X\otimes X$ and $Z\otimes Z$, are identical though in general this need not be the case.

More generally, it can be shown that, given unitary operator U and an observable W compatible with it, measurement of W can be outsourced to an ancilla using the controlled operation given by $\mathscr{C}_U\equiv\sum_j |j\rangle\langle j|\otimes U^j$, where $\{|j\rangle\}$ is the possibly degenerate, simultaneous eigenbasis of U and W [22].

Next we consider the problem of distributing measurement. Observation of the examples presented in the preceding section shows that if the operator U (eg., $X\otimes X$) is a product of operations on subsystems, then each term in U represents an interaction of the ancilla with a system qudit via a control operation. More generally, it can be shown that, given observable W which is compatible with unitary operator $U=\bigotimes_m U_m$, where m $(=1,2,\cdots,n)$ labels subsystems, its measurement can be distributed by means of separate controlled operations on the individual subsystems j from the same ancilla. The control-operations may be performed in any order or together [22]. The direction of control-operations may be reversed using circuit identities of the type depicted in Figure (10) of Ref. [23], by introducing Hadamards, and generalizable to higher dimensions [22].

SOME APPLICATIONS

Such distributed state discrimination can be useful in distributed quantum computing, especially when there are restrictions coming from the topology of the quantum communication network. Unlike their classical counterparts, quantum channels are expected to be expensive and not amenable to change to suit a problem at hand. Rather, it is worthwhile to use protocols that minimize quantum communication complexity, that is, the quantity of quantum information that must be communicated between different parties to perform a computation or process some information, in a given network.

A simple way to perform Bell state discrimination is for all other members to communicate their qudits to single station, whose member (called, say Alice) performs a joint measurement on all n qubits or qudits to determine the state. She then re-creates the measured state and transmits them for further use. Actually, in the present situation, instead of a joint measurement on all qubits, Alice can apply a string of $n-1$ $\mathscr{C}_X^\dagger$ operations on each consecutive pair of qudits in the Bell state $|\Psi_{pq_1q_2\cdots q_{n-1}}\rangle$ and $H_d^\dagger$ finally on the first qudit. It is easily seen that each application of $\mathscr{C}_X^\dagger$ will disentangle the controlled qudit from the rest. For the Bell states, this procedure effects the transformation:

$$|\Psi_{pq_1q_2\cdots q_{n-1}}\rangle \longmapsto |p\rangle|q_2-q_1\rangle\cdots|q_{n-1}-q_{n-2}\rangle. \qquad (21)$$

Subsequent measurement of each qudit in the computational basis completely characterizes the Bell state. The Bell state thus being discriminated, the above procedure can be reversed to re-create the state $|\Psi_{pq_1q_2\cdots q_{n-1}}\rangle$ and transmit it back to the remaining players.

Irrespective of network topology, such a disentangle-and-reentangle strategy requires in all $2(n-1)$ two-qudit gates to be implemented. In our method, the number of two-qudit gates is the sum of n two-qudit gates for determining phase parameter p and $2(n-1)$ for determining the (relative) parities, giving $3n-2$ two-qudit gates. From this viewpoint of consumption of nonlinear resources, our method does not offer any advantage. However, this turns out not to be the case from the viewpoint of quantum communication complexity.

Suppose a quantum communication network with a star topology and n members is given, as for example in Fig. 4(a). For all members to transmit their qudits to Alice (at A), and for her to transmit them back would require $2(n-1)$ qudits to be communicated, where the factor 2 comes from the two-way requirement. In our protocol, one way quantum communication suffices. For measuring the 'phase observable' M_1, the number of qudits communicated is seen to be $2(n-1)$, since the ancilla must pass through the hub to reach each member on a single-edge vertex; and if measured edgewise, the communication complexity for relative parity measurement is n qudits. In all, this requires $3n-2$ qudits to be communicated, which is larger than that required for a plain disentangle-reentangle method.

However consider a linear configuration of the communication network, as in Fig. 4(b), where members are linked up in a single series. In the disentangle-reentangle method, if Alice is located at one end, the communication complexity is seen to be $n(n-1)$ qudits; it is $(n^2-1)/2$ if she is in the middle. In either case, it is of order $O(n^2)$. In contrast, our distributed method can be implemented using $n-1$ qudits communicated both for phase and relative parity measurement, requiring in all only $2(n-1)$ qudits to be communicated, so that the required communication is only of order $O(n)$. Thus our method gives a quadratic saving in quantum communication complexity.

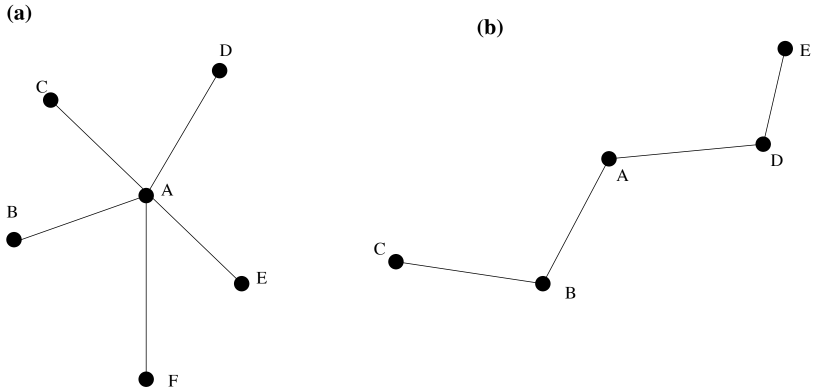

FIGURE 4. Two possible configurations of the quantum communication network: (a) In a star topology, a set of 'relative parity' measurements could be along each edge; (b) in the linear configuration, the strategy of observing consecutive qubits or qudits, as given in Eqs. (5) or (14), can be used.

A further advantage, that may be of some importance in certain situations, is that our method divides the required resources in terms of applying nonlinear gates and of measurements equally among the various members. In a real life situation, this may facilitate the distribution of quantum information processing resources among the various members.

ACKNOWLEDGMENTS

We are thankful to Prof. J. Pasupathy, V. Aravindan and H. Harshavardhan, Dr. Ashok Vudayagiri, Dr. Ashoka Biswas, Dr. Shubhrangshu Dasgupta for useful discussions.

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CLOSURE OF GENERALIZED BELL STATES UNDER HADAMARDS

The action of $H \otimes H^\dagger$ on $|\Psi_{pq}\rangle$ on the states in Eq. (13,14) produces the effect of effectively interchanging the indices pq of $|\Psi_{pq}\rangle$:

$$\begin{aligned}
(H \otimes H^\dagger)|\Psi_{pq}\rangle &= \frac{1}{\sqrt{d}}\sum_{j,k,l} e^{(2\pi\imath/d)(j[p+k-l]-ql)}|k\rangle|l\rangle \\
&= \frac{1}{\sqrt{d}}\sum_{j,l} e^{(2\pi\imath/d)(-ql)}|l-p\rangle|l\rangle \\
&= \frac{1}{\sqrt{d}}\sum_{j} e^{(2\pi\imath/d)([d-q]l)}|j\rangle|j+p\rangle, \\
&= |\Psi_{q'p}\rangle,
\end{aligned} \tag{22}$$

where $q' = (d-q) \bmod d$ and the second step follows from noting that the only non-zero contributions come for the case $p+k-l=0$, and an overall phase factor has been dropped in the third step. Similarly, one finds $(H \otimes H^\dagger)|\Psi_{pq}\rangle = |\Psi_{qp'}\rangle$, where $p' = d-p$ mod d.