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Maximum Entropy Spectral Analysis—Some Comments

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Summary. Study of a model data analysis situation with the help of computer experiments reveals that the Maximum Entropy Method of Spectral Analysis owes its popularity to a peak-sharpening property which is found to be a strong function of the level of background white noise present in the spectrum. In one limit, misleading results may be obtained by this technique. Some suggestions for a more fruitful use of the technique are made.

Key words: maximum entropy method — spectral analysis

1. Introduction

The Maximum Entropy Method (MEM) of estimating power spectra from measured auto-correlation functions (acf's) has been applied in several fields in recent years (Cohen and Lintz, 1974; Lacoss, 1971 and Ulrych, 1972). It has been claimed that the method has much superior resolution properties compared to conventional, linear methods. The reason given is that in maximizing the entropy it seeks to minimize one's assumptions (maximise one's ignorance) about unavailable data, thus making it the most rational method of spectral analysis (Ables, 1974; Ulrych and Bishop, 1975). Ables (1974) suggests application of the technique to aperture-synthesis in radio-astronomy.

Since the available resolutions claimed are substantially larger than those attainable by conventional methods, it challenges, in our opinion, a very intuitive principle in optics with which we are all familiar, namely that the resolving power of an optical system is proportional to the linear dimensions of the system or in the time domain, the simple principle that in order to see two frequencies separated by an interval $\Delta\nu$, one must measure the acf up to a lag τ, approximately equal to $1/\Delta\nu$. In order to find out if one has to pay some price for achieving this better resolution, we tried to evaluate the performance of the MEM algorithm with the help of simple computer experiments on several sets of truncated and sampled acf's corresponding to spectra known before hand. We wish to present here the results of one such set which illustrate several noteworthy features of MEM not discussed in the literature.

2. Description of Computations and Results

An acf corresponding to a true spectrum consisting of two narrow gaussian peaks of equal width separated by two half-widths sitting on top of a white noise background was taken. The two peaks in the spectrum are not resolved (Fig. 1a). This acf was sampled at regular intervals and 27 lag values were used to reconstruct the spectrum using the MEM algorithm on a computer. The exercise was repeated, each time with a different amount of power in the white-noise background. This is done trivially by varying the zero-lag value of the acf. These exercises correspond to acf's which have been measured to an arbitrary accuracy and questions of measurement error etc. are not dealt with here.

The observed behaviour of the MEM spectrum (Figs. 1c–1f) as a function of white noise can be classified into three regimes: (I) When the power in the white noise is small compared to that in each of the peaks, (II) when it is comparable and (III) when the white noise power is large compared to the power in each peak.

In regime (I), represented by Figures 1c and 1d, the MEM spectrum is qualitatively different from the true spectrum, showing a larger number of peaks. For extremely low levels of white noise, the spectrum is very sensitive to small perturbation of the acf and is in general unreliable.

In regime (II), represented by Figure (1e) the spectrum is qualitatively similar to the true spectrum in that it consists of two peaks at the correct frequencies, but each of the peaks is made narrower even than those in the true spectrum, so that the two peaks are seen separately. The spectrum in this regime is fairly stable. When the true spectrum was replaced by an almost similar-looking spectrum consisting of a single gaussian, twice as wide,

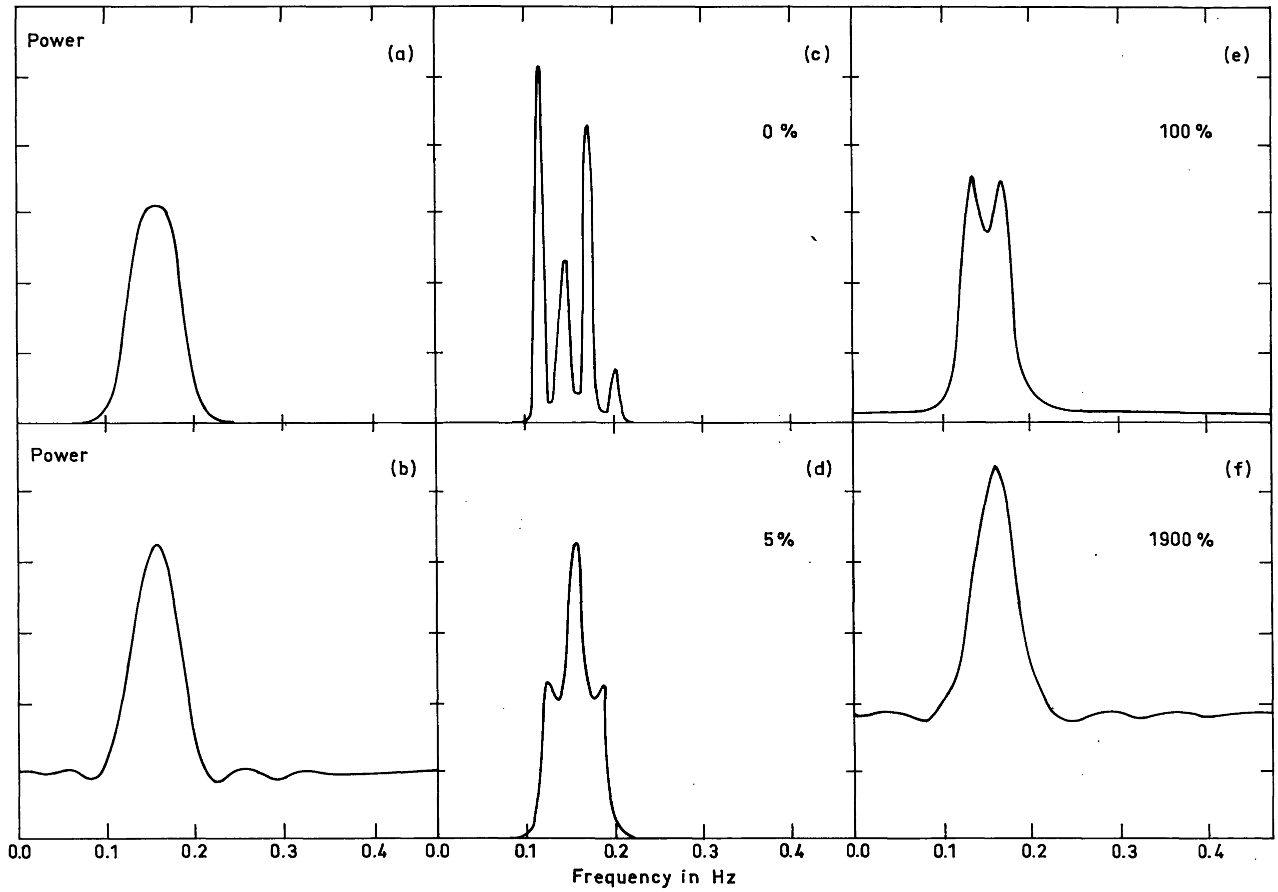

Fig. 1. a Shows the true spectrum in arbitrary units of power as a function of frequency. **b** shows the Blackman-Tukey spectrum with a rectangular window for 27 lags and a sampling interval of 0.5 s. All spectra extend up to 1 Hz, but only interesting portions are shown. **c**—**f** show MEM spectra for 27 lags and 0.5 s, sampling interval for varying amounts of white noise indicated against each curve. The numbers indicate the percentage ratio of the total white-noise power and total power in a single peak. The scale on the power axis is the same in all curves except (c), where it is reduced by a factor of 2.

centred at the middle, sitting on the same amount of white noise and the whole exercise was repeated, the resulting MEM spectrum did not show two peaks. This is therefore not just a case of superior resolution, but one of 'super-resolution', for even an infinite length of a measured acf would not yield two separate peaks in a conventional analysis.

Figure 1f shows the MEM spectrum in regime (III), where the total white-noise power is large compared to power in each of the peaks. This is almost identical to the corresponding Blackman-Tukey spectrum (Fig. 1b). In fact it can be easily shown that it should be so. In this limit, the zero-lag value of the acf ρ_0 is large compared to all other values $\rho_1, \rho_2, \ldots, \rho_{N-1}$. Let us take therefore, $\rho_1 = \varepsilon_1 \rho_0$, $\rho_2 = \varepsilon_2 \rho_0, \ldots, \rho_{N-1} = \varepsilon_{N-1} \rho_0$, where $\varepsilon_1, \varepsilon_2, \ldots, \ll 1$. The MEM spectrum $S(\nu)$ is given by (Ables, 1974)

$$S(\nu) = \alpha_0 \Delta \left| \sum_{n=0}^{N-1} \alpha_n \exp(2\pi i n \nu \Delta) \right|^{-2}. \tag{1}$$

where Δ is the sampling interval and where the column vector $\{\alpha_n\}$ is obtained by solving the equation

$$R\{\alpha_n\} = \text{col.}\,\{1, 0, 0, \ldots\}; \tag{2}$$

R is the autocorrelation matrix given by

$$R_{ij} = \rho_{|i-j|}\,. \tag{3}$$

In the limit under consideration,

$$R = \rho_0 (I + \varepsilon) \tag{4}$$

where I is the unit matrix and ε is a matrix, all elements of which are small. Hence,

$$\{\alpha_n\} = R^{-1}\,\text{col.}\,\{1, 0, 0, \ldots\} \simeq \rho_0^{-1}(I - \varepsilon)\,\text{col.}\,\{1, 0, 0, \ldots\}$$
$$= \rho_0^{-1}\,\text{col.}\,\{1, -\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_{N-1}\}. \tag{5}$$

Substituting in (1), we get to $0(\varepsilon)$,

$$S(\nu) = \Delta \left[\rho_0 + 2 \sum_{m=1}^{N-1} \rho_m \cos(2\pi m \nu \Delta) \right]. \tag{6}$$

MEM, therefore, offers no advantage over simple linear methods in problems involving a small non-white signal buried in a large amount of white-noise.

3. Conclusions

The above observations, in our opinion, suggest the following empirical hypothesis: "Out of all possible spectra consistent with known a.c. values MEM selects the one with the sharpest and the largest number of peaks". The peaks can sometimes be sharper or even larger in number than in the true spectrum. This peak-sharpening property can no doubt be used to advantage in data analysis as in Figure 1e or as in several published applications of MEM, particularly where one is not interested in the information contained in the width of the peaks. One must be careful, however, to make sure that MEM has not over-done its job by splitting the spectrum into a larger number of peaks. A comparison of Figures 3 and 4 in (Cohen and Lintz, 1974) provide support for these assertions in a real data analysis situation. Every feature in the spectrum of Figure 4, obtained by a periodogram analysis, has been sharpened in the MEM spectrum of Figure 3 and a spurious feature at 0.02 cycles yr^{-1} has appeared. The small difference in the data length ($\sim 20\%$) in the two cases does not explain these differences.

Finally, we would like to suggest, that in any application of MEM, the behaviour of the spectrum as a function of the zero-lag value of the acf, considered as a variable, must be studied. Although it violates the basic premise (Ables, 1974) of MEM in that it amounts to tampering with known data, it turns out nevertheless to be a useful exercise. In case of a spectrum in regime (III), for example, reducing the zero-lag value of the acf can sharpen the peaks in the spectrum compared to the Blackman-Tukey spectrum and in case of a spectrum in regime (I), increasing the zero-lag value can overcome the instabilities leading to splitting of the spectra. In any case, judgement on the part of the analyst as to which peaks are spurious and which ones are genuine is a necessary requirement in any application of the technique.

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