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Undamped oscillations of homogeneous collisionless stellar systems

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Summary. We use the Lewis invariant of the time-dependent harmonic oscillator to construct exact time-dependent, uniform density solutions of the collisionless Boltzmann equation. The spatially bound solutions are time-periodic.

1 Introduction

The dynamics of stars in galaxies are governed by the collisionless Boltzmann equation (CBE).

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}-\frac{\partial \phi}{\partial \mathbf{r}}\cdot\frac{\partial f}{\partial \mathbf{v}}=0, \tag{1}$$

where $f(\mathbf{r}, \mathbf{v}, t)\, d^3r\, d^3v$ is the mass in $d^3r\, d^3v$ at time $= t$, and $\phi(\mathbf{r}, t)$ the gravitational potential determined self-consistently by the Poisson equation

$$\nabla^2\phi=4\pi G\rho(\mathbf{r}, t)=4\pi G\int f(\mathbf{r}, \mathbf{v}, t)\, d^3v. \tag{2}$$

Many static solutions (and solutions static in a rotating frame) to the CBE are known (e.g. Binney & Tremaine 1987, hereafter BT; Fridman & Polyachenko 1984, hereafter FP).

While the time-dependent behaviour of solutions of the CBE is poorly understood, there is a long-standing belief (see Lynden-Bell 1967) – made more precise by Tremaine, Henon & Lynden-Bell (1986) – that most solutions tend to relax to a time-independent state. Although some numerical simulations (e.g. White 1979) seem to support the conjecture of relaxation, others show long-lived oscillations (see references in Louis & Gerhard 1988). Louis & Gerhard (1988) have constructed numerically a self-consistent model of a spherical galaxy undergoing small but non-linear radial oscillations.

We present an analytic method of constructing time-dependent self-consistent models. The models all share the property that at any time the density is uniform over some (in general ellipsoidal) region of space and zero outside. The gravitational potential within the region is quadratic in the spatial coordinates. The shape and size of the region change with time (while the region remains ellipsoidal), leading to a time-dependence in the strength of the potential.

Quadratic time-dependent potentials admit exact integrals of motion. The invariant for the time-dependent harmonic oscillator (TDHO) was discovered by Lewis (1968).

The left-hand side of equation (1) is the convective derivative of f. Any function of the isolating integrals of motion (Jeans' Theorem – see BT) solves equation (1). In particular, functions of the Lewis invariant/s are solutions of equations (1) and (2) so long as they reproduce the time-dependent potential for which the Lewis invariant is an integral of motion. In Section 2 we use this invariant to construct a time-dependent model in one spatial dimension (1D). The same approach may be carried over to three dimensions. For simplicity, spherical models are considered in Section 3.

2 The 1D model

The equation of motion for the TDHO is

$$\ddot{x}+\omega^2(t)\,x=0. \tag{3}$$

The Lewis invariant,

$$I(x,v,t)=\frac{x^2}{2\xi^2}+\tfrac{1}{2}(\xi v-\dot{\xi}x)^2, \tag{4}$$

is an integral of motion, where $\xi(t)$ is any solution of

$$\ddot{\xi}+\omega^2(t)\,\xi-\frac{1}{\xi^3}=0; \qquad \xi>0. \tag{5}$$

The reader is referred to Goldstein (1980) and references therein for a comprehensive discussion.

In 1D the CBE describes the evolution of a self-gravitating system of plane-parallel sheets. All steady-state models have distribution functions that are functions of energy alone. The distribution function for a model with uniform density within some interval and zero outside is

$$f(E)=K\theta(E_{\rm m}-E)(E_{\rm m}-E)^{-1/2}, \tag{6}$$

where K, $E_{\rm m}$ are constants and

$$E=\frac{v^2}{2}+\phi. \tag{7}$$

We omit the θ-function in later work, understanding that $f=0$ when the argument of the square-root is negative.

The time-dependent model is constructed by replacing E by the Lewis invariant, K and $E_{\rm m}$ by some other constants K' and $I_{\rm m}$:

$$f(x,v,t)=K'(I_{\rm m}-I)^{-1/2}. \tag{8}$$

The density

$$\begin{aligned}\rho(x,t)=\int f\,dv&=\pi\sqrt{2}\,K'/\xi; &\qquad |x|&<\xi\sqrt{2I_{\rm m}}\\ &=0 & |x|&>\xi\sqrt{2I_{\rm m}}.\end{aligned} \tag{9}$$

The potential in the interior is

$$\phi(x,t)=\omega^2(t)\,\frac{x^2}{2}$$

where

$$\omega^2(t)=4\pi G\rho \quad =4\pi^2 G\surd 2K'/\xi \quad =A/\xi \tag{10}$$

We recall that ξ is any solution of equation (5) with $\omega^2(t)$ given by equation (10). Therefore,

$$\ddot{\xi}+A-\frac{1}{\xi^3}=0. \tag{11}$$

This equation describes a one-dimensional anharmonic oscillator in a potential $[A\xi+1/(2\xi^2)$.

Thus all solutions are periodic functions of time. We have a one-parameter family of oscillating models (with given total mass and energy), where the parameter may be taken as the first integral of equation (11). In the limit $A\rightarrow\infty$ we recover cold homologous collapse of a system of sheets.

3 Spherical models

A homogeneous static sphere corresponds to a polytrope of index zero which is not realizable in a stellar system with isotropic velocity dispersion (Vandervoort 1980). So a distribution function that is a function of energy alone will not describe a uniform sphere. The simplest alternative is to seek a function of both E and L^2 ($\mathbf{L}$ is angular momentum $=\mathbf{r}\times\mathbf{v}$). Given ρ, the inversion of the integral equation for f is not unique. Two functions that describe a uniform sphere are (see FP and references therein)

$$f\sim\left(\frac{L^2}{2R^2}+\phi(R)-E\right)^{-1/2}$$

and

$$f\sim\delta(v_{\mathrm{r}})\,\delta[v_{\perp}-v_{\mathrm{c}}(r)]$$

[$v_{\mathrm{c}}(r)$ is the circular velocity at radius r]. Any f is a member of a two- (or more) parameter family of functions characterized by total mass and radius (R). If the interior potential

$$\phi(r)=\omega_0^2\,\frac{r^2}{2}, \tag{12}$$

we can take the parameters to be ω_0 and R. We write the distribution function in the form

$$f=f(E/\omega_0, L^2;\ \omega_0, R). \tag{13}$$

Then

$$\rho=\int f\,d^3v \quad =\frac{\nabla^2\phi}{4\pi G} \quad =\frac{3\omega_0^2}{4\pi G}; \qquad r<R$$

$$=0; \qquad r>R.$$

Therefore, defining $\mathbf{v}' = \mathbf{v}/\sqrt{\omega_0}$, $\mathbf{r}' = \mathbf{r}\sqrt{\omega_0}$

$$\int f\left(\frac{v'^2}{2}+\frac{r'^2}{2},\ |\mathbf{r}'\times\mathbf{v}'|^2;\ \omega_0,\ R\right) d^3v' = \frac{3\sqrt{\omega_0}}{4\pi G};\qquad r' < R\sqrt{\omega_0}$$
$$= 0;\qquad r' > R\sqrt{\omega_0} \tag{14}$$

Each harmonic oscillator has a Lewis invariant (I_x, I_y, I_z). Choosing the same function $\xi(t)$ for all of them, and adding, we define

$J = I_x + I_y + I_z$.

Therefore

$$J = \frac{r^2}{2\xi^2} + \tfrac{1}{2}|\xi\mathbf{v} - \dot{\xi}\mathbf{r}|^2. \tag{15}$$

The time-dependent model is constructed by replacing E/ω_0 in equation (13) by J:

$$f = f(J, L^2;\ \omega_0, R). \tag{16}$$

The density

$$\rho(r, t) = \int f\left(\tfrac{1}{2}|\xi\mathbf{v} - \dot{\xi}\mathbf{r}|^2 + \frac{r^2}{2\xi^2},\ |\mathbf{r}\times\mathbf{v}|^2;\ \omega_0, R\right) d^3v$$
$$= \frac{1}{\xi^3}\int f\left(\frac{u^2}{2}+\frac{a^2}{2},\ |\mathbf{a}\times\mathbf{u}|^2;\ \omega_0, R\right) d^3u, \tag{17}$$

where

$$\mathbf{u} = \xi\mathbf{v} - \dot{\xi}\mathbf{r},$$
$$\mathbf{a} = \mathbf{r}/\xi. \tag{18}$$

From equation (14), (17) and (18)

$$\rho(r, t) = 3\sqrt{\omega_0}/4\pi G\xi^3;\qquad r < R\sqrt{\omega_0}\,\xi$$
$$= 0;\qquad r > R\sqrt{\omega_0}\,\xi. \tag{19}$$

The interior potential

$$\phi(r, t) = \omega^2(t)\frac{r^2}{2}. \tag{20}$$

Since $\nabla^2\phi = 4\pi G\rho$, using equation (19) and (20)

$$\omega^2(t) = \sqrt{\omega_0}/\xi^3. \tag{21}$$

ξ is any solution of equation (5) with $\omega^2(t)$ given by equation (21). Therefore ξ satisfies

$$\ddot{\xi} + \frac{\sqrt{\omega_0}}{\xi^2} - \frac{1}{\xi^3} = 0. \tag{22}$$

The solutions are either time periodic or those that eventually approach infinity as $t \to \infty$. These correspond to spheres with total energy negative and positive, respectively.

4 Discussion

It is important to note that anharmonicity is absent in these models. It is precisely the lack of mixing in the angle variables that allowed us to construct these models. Nevertheless, the models show that time-dependence alone does not necessarily cause relaxation to a steady state. Numerical simulations of the 1D model have been carried out. They show long-lived oscillations and no violent instability is apparent. If these models are really stable, then solutions in their vicinity do not relax. The method developed in this paper also applies to other homogeneous models. Spheroidal systems and the connection between the Virial Theorem and the equation of motion for ξ will be discussed in a later paper.

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References

Binney, J. & Tremaine, S., 1987. *Galactic Dynamics,* Princeton University Press, Princeton, NJ.
Fridman, A. M. & Polyachenko, V. L.,1984. *Physics of Gravitating Systems I,* Springer-Verlag, New York.
Goldstein, H., 1980. *Classical Mechanics,* 2nd edn, Addison-Wesley, Singapore.
Lewis, H. R., 1968. *J. math. Phys.,* **9,** 1976.
Louis, P. D. & Gerhard, O. E., 1988. *Mon. Not. R. astr. Soc.,* **233,** 337.
Lynden-Bell, D., 1967. *Mon. Not. R. astr. Soc.,* **136,** 101.
Tremaine, S., Henon, M. & Lynden-Bell, D., 1986. *Mon. Not. R. astr. Soc.,* **219,** 285.
Vandervoort, P. O., 1980. *Astrophys. J.,* **240,** 478.
White, S. D. M., 1979. *Mon. Not. R. astr. Soc.,* **189,** 831.