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Luminosity limits for funnels in thick accretion discs

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Summary. A narrow, highly luminous funnel in a thick accretion disc is a common feature of many models for active galactic nuclei. We examine the constraints on the luminosity imposed by the effects of radiation forces on the funnel. Our treatment allows properly for the irradiation of any part of the funnel by the rest, an important effect for this problem. We find that the maximum luminosity of a funnel of small semi-angle ϕ is reduced below the Eddington limit L_{E} by a factor ϕ^2 if the funnel is to be in strict equilibrium. Even with allowance for flow induced viscous stresses, the luminosity cannot exceed $\sim L_{\mathrm{E}}\phi$. In contrast, current models have luminosities of the order of L_{E}/ϕ. We show that the resulting large unbalanced forces at the funnel surface cause a significant outflow of matter which should be incorporated in the model for consistency. These results do not depend on the detailed angular momentum distribution over the disc surface but only on the funnel geometry.

1 Introduction

A promising model for active galactic nuclei and quasars is based on a thick accretion disc around a massive black hole (Lynden-Bell 1977). The theory of thin accretion discs is well developed (Shakura & Sunyaev 1973; Pringle 1981) and shows that the inner regions begin to thicken when the total luminosity L approaches the Eddington (1926) luminosity L_{E} given by

$$L_{\mathrm{E}} = 4\pi GMc/\kappa. \tag{1}$$

Here M is the mass of the central object and κ ($\mathrm{cm}^2\,\mathrm{g}^{-1}$) the appropriate opacity (due to Thomson scattering for fully ionized matter). In the pioneering work of Paczynski & Wiita (1980, PW) a theory of thick discs was developed with the following features.

(i) The specific angular momentum distribution in the equatorial plane is not Keplerian (because of radial pressure gradients) and is provided as an input in model building, subject to the physical requirements of stability, accretion and matching on to thin discs at large radii.

(ii) The angular momentum is constant on cylinders and is thus known throughout the disc (von Zeipel's theorem for a barotropic fluid i.e., one in which the pressure is a function of density alone).

(iii) The funnel surface is an equipotential of gravity and centrifugal force combined; most of the models have a steep funnel.

(iv) The dominant force balancing gravity and centrifugal force is the gradient of radiation pressure; this makes it possible to calculate the radiation flux at any point and thus the luminosity of the disc which can be $\sim 10 L_E$ or more.

The subsequent work of Jaroszynski, Abramowicz & Paczynski (1980, JAP) and Abramowicz, Calvani, & Nobili (1980, ACN) incorporated general relativity and further physical consistency checks into this picture without changing the essential features; in particular, the super-Eddington luminosities remain. The large luminosity in these models is required to balance the large centrifugal force at the funnel surface implied by (i) and (ii) above. In the three papers cited, the details of how the luminosity escapes from the funnel are not considered and the emphasis is on equilibrium of the disc interior.

In this paper we consider the implications of the super-Eddington luminosity for the equilibrium of the funnel surface and, in particular, the effect of the radiation received by one part of the funnel from the rest. Sikora (1981) has recently treated this problem but his approach and aim are rather different from ours as discussed in Section 5. We first note that the current theory of thick discs (PW, JAP, ACN) bypasses the difficult problems of energy generation and radiative transfer within the disc. In spite of this, conditions (i)–(iv) above uniquely define the flux of radiation at each point in the disc. Taking the divergence of the flux, one easily shows that the radiation generated per unit volume is constant on cylinders. This is clearly unphysical and shows that von Zeipel's theorem will be strongly violated in models with realistic energy dissipation and transfer. Lacking such models at present, we take the pragmatic approach of considering funnel geometry as the basic input rather than angular momentum. It turns out that our results depend basically on the inclination ϕ of the funnel walls to the rotation axis and not on finer details.

Section 2 of this paper develops the connection between super-Eddington luminosities and re-entrant shapes such as funnels. It is then shown that strict equilibrium of the funnel surface restricts its maximum luminosity $L_{\max}$ to less than $\phi^2 L_E$. This estimate is far less than the luminosity in the PW model, given by L_E/ϕ. The funnel surface is thus clearly not in equilibrium in the PW and related models. Section 3 considers the effect of the excess tangential radiation forces on the funnel walls. One possibility is that these forces are balanced by viscous stresses induced by a surface flow pattern. This, however, leads to a maximum luminosity less than $\sim L_E \phi$. This limit, though model dependent, seems hard to exceed. We thus conclude that funnels with $L \gtrsim L_E$ must continuously blow off matter. Section 4 estimates the energy carried off by this outflow which is shown to be of the same order as the energy carried by the escaping radiation for $L > L_E$. We discuss the conclusions and implications of our work in Section 5.

2 Luminosity limits for equilibrium funnels

We first give a geometrical argument to show that super-Eddington luminosities need re-entrant shapes like funnels. Fig. 1 shows a surface element of a figure of revolution which radiates a maximum luminosity $dL_{\max}$. We compare this to the maximum luminosity $dL_{\max,s}$ which an element of a sphere subtending the same solid angle at the centre would radiate.

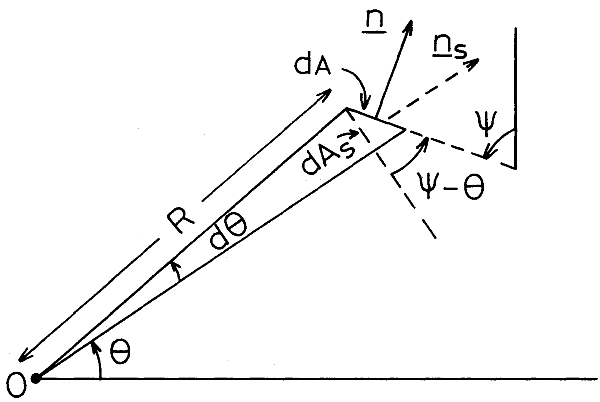

Figure 1. Surface element of a figure of revolution illustrating the radiation force acting on it. O is the central mass.

(i) The area dA of the element is related to the corresponding area dA_s of the sphere by

$$dA/dA_s = 1/\cos(\psi - \theta). \tag{2}$$

(ii) The vertical component (i.e. parallel to the rotation axis) of the unit normal n_z is related to that for the spherical element $n_{z,s}$ by

$$n_z/n_{z,s} = \sin\psi/\sin\theta. \tag{3}$$

(iii) We consider the equilibrium of a surface element in the vertical direction since this does not involve the centrifugal force. The vertical component of the radiation force per unit mass is proportional to n_z and the flux dL/dA, taking the net flux to be normal to the surface. The vertical component of gravity is the same for the sphere and a general surface element and this defines the maximum radiation force for both. Thus

$$n_z \frac{dL_{\max}}{dA} = n_{z,s} \frac{dL_{\max,s}}{dA_s}$$

which leads to

$$dL_{\max}/dL_{\max,s} = \frac{\sin\theta}{\cos(\psi-\theta)\sin\psi} = \frac{\cos(\psi-\theta)\sin\psi - \sin(\psi-\theta)\cos\psi}{\cos(\psi-\theta)\sin\psi}. \tag{4}$$

As is well known, the integral of $dL_{\max,s}$ over all solid angles gives the Eddington luminosity L_E of (1). From (4), we see that one obtains an enhancement over L_E for $\psi > 90°$ which corresponds to a re-entrant surface like a funnel.

The other case, $\psi - \theta < 0°$, does not lead to super-Eddington luminosities as can be checked by writing down the condition that centrifugal force acts outwards, which gives a more stringent limit than (4) for $\psi - \theta < 0°$.

Motivated by the above result, we now consider the idealized funnel of Fig. 2 rising from cylindrical coordinate r_i to r_f with a slope $\tan\theta$. We can apply the above arguments (vertical equilibrium) to evaluate the maximum luminosity. The result is

$$\frac{L_{\max}}{L_E} = \int_{r_i}^{r_f} \frac{\tan\theta\,(r-r_i)r\,dr}{\cos^2\theta\,[(r-r_i\sin^2\theta/\cos\theta)^2 + r_i^2\sin^2\theta]^{3/2}} \tag{5}$$

$$\simeq \tan\theta - 2 + \ln(2r_f/r_i\cos\theta)$$

for $\tan\theta$, (r_f/r_i) both $\gg 1$.

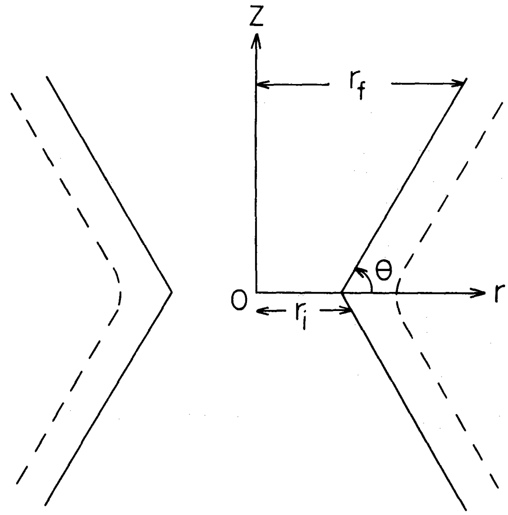

Figure 2. Geometry of a model funnel whose maximum luminosity is analysed in the text. The dotted lines show a surface a few optical depths below the free surface where the external radiation is negligible.

Equation (5), although based on a simplified picture, can be applied to the inner portion of the model shown in fig. 4 of PW which can be approximated by a funnel with $r_i = 8$, $r_f = 40$, $\tan\theta = 5$. Equation (5) gives a maximum luminosity $6.9\,L_E$, which can be compared with $6.1\,L_E$ (for $r_f = 40$) from the detailed calculation of PW. This confirms that the simplified funnel geometries used in this paper do capture the essential features of the problem.

So far, the argument has been based on the equilibrium between the pressure of outgoing radiation, centrifugal force and gravity. It applies not to the surface of the funnel but to a parallel surface several optical depths below (shown by dotted lines in Fig. 2). When we consider the surface proper, it is necessary to include the incoming radiation from other parts of the funnel. Fig. 3(a) shows, for definiteness, a conical funnel. We can no longer

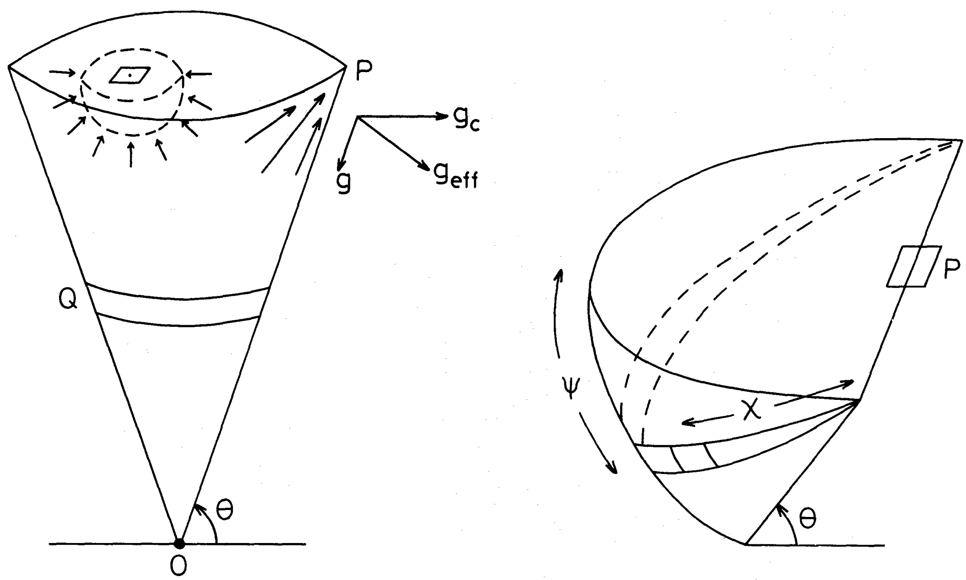

Figure 3. (a) A simplified funnel with conical geometry. The element at P on the rim receives external radiation from points such as Q. The surface element on the cap illustrates the calculation of the total centrifugal force g_c. (b) Geometry of incoming rays at P on the rim of the cone in (a). The flux through the horizontal surface element P gives the vertical force.

discuss the equilibrium at a given surface point P in terms of local conditions but need to use information about radiation coming in from other points such as Q.

We can divide the radiation field near P into two parts. The first, made up of all rays travelling into a hemisphere centred on the outward normal at P, is called the self flux and originates locally. The second consists of all rays directed into the opposite hemisphere and is called the external flux. A unit mass at P experiences an upward force due to the external radiation received from points such as Q below it. We make the plausible assumption that the brightness of the funnel decreases with height. Replacing it by a constant brightness distribution with the same total luminosity thus involves moving brightness upwards. It can be verified that for a fixed total luminosity the vertical force at P decreases in this process and is thus underestimated. Further, the self force due to the radiation emitted at P has a positive vertical component. In what follows, we calculate the upward force at P due only to the external part of the radiation and that too from a uniformly bright cone. This estimate clearly gives a lower bound to the force and balancing this force against gravity therefore gives an upper bound to the luminosity.

Fig. 3(b) shows the polar angles χ and ψ defining a ray incident at P. The external vertical flux at P from the cone of constant brightness b is given by

$$F_{\mathrm{v}} = b \int_0^\theta \int_0^\pi \sin\psi \sin^2\chi \, d\chi \, d\psi = \frac{\pi b}{2}(1-\cos\theta). \tag{6}$$

The net escaping flux at any point on the cap of the cone (Fig. 4a) is given by integrating over a hemisphere, i.e., putting $\theta = \pi$ in (6). The luminosity of two cones is thus

$$L = 2\pi r^2 \times \pi b. \tag{7}$$

The vertical force balance at P gives

$$GM \cos^2\theta \sin\theta / r^2 = \pi b \kappa (1-\cos\theta)/2c. \tag{8}$$

The upper bound to the maximum luminosity then follows from (7) and (8)

$$L_{\mathrm{max}}/L_{\mathrm{E}} \leqslant \frac{\sin\theta \cos^2\theta}{(1-\cos\theta)}.$$

For small opening angles

$$\phi = \frac{\pi}{2} - \theta$$

$$L_{\mathrm{max}} \leqslant L_{\mathrm{E}}\phi^2. \tag{9}$$

Note that this limit is of the same order as $L_{\mathrm{Edd}}(\phi)$, the Eddington limit appropriate to the $2\pi\phi^2$ solid angle of the two cones viz.,

$$L_{\mathrm{Edd}}(\phi) = L_{\mathrm{E}}(2\pi\phi^2/4\pi) = L_{\mathrm{E}}\phi^2/2. \tag{10}$$

The argument just given assumes a convex funnel, i.e., one in which the radiation from any part can reach the rim. It is, however, difficult to achieve a larger luminosity by hiding the rim from the hottest parts of the funnel. One could then choose a point immediately above the region of the funnel which dominates the luminosity and apply similar arguments. Thus, the requirement of equilibrium of the funnel surface always leads to a bound with the same structure as (10).

3 Luminosity limit with surface stresses

Clearly, (9) shows that funnels with $L \sim L_E$ or greater have a large unbalanced tangential force on the surface layers. This point is particularly clear in the work of Sikora (1981) who has carried out a detailed calculation of the radiation field within the funnel including general relativistic effects. After imposing the condition of equilibrium normal to the funnel surface, he noted the presence of unbalanced tangential forces. For example, in fig. 3 of his paper, the net vertical force at $z = 300$ due to radiation is 0.0058 (in units with $G = M = c = 1$) while the downward force due to gravity is only $300/(300^2 + 30^2)^{3/2} = 1.1 \times 10^{-5}$, more than 500 times lower.

We now consider the effect of this large tangential radiation force on the surface of the funnel. One possibility is that this force could set up a surface flow which in turn could generate balancing shear stresses. To check this, we compare the tangential force $F_t\kappa/c$ to the effective gravity g_{eff} at P in Fig. 3(a). g_{eff} is the resultant of the gravity $g = GM/r^2$ and the centrifugal force g_c as shown. If we assume that the net flux of radiation in the interior is normal to the funnel surface as in PW, $g_{eff} \simeq g/\phi$. We take ϕ to be small and it is therefore not necessary to distinguish between the vertical and tangential directions at P. Although a detailed theory of the surface flow would be difficult, the following limit on the tangential flux F_t for a steady flow seems physically reasonable.

$$F_t\kappa/c \leqslant \alpha g_{eff} = \alpha g/\phi. \tag{11}$$

The parameter α in (11) is like a coefficient of dynamic friction. The effect of the new criterion (11) instead of $F_t\kappa/c \leqslant g$ (as in Section 2) is to modify the luminosity limit (9) to

$$L_{max} \leqslant \alpha\phi L_E. \tag{12}$$

One could also take the more conservative approach of comparing $F_t\kappa/c$ with the largest force acting locally which is $F_s\kappa/c$, the self-radiation force.

$$F_t\kappa/c \leqslant \alpha\frac{F_s\kappa}{c} = \alpha\left(\frac{F_n\kappa}{c} + g_{eff}\right). \tag{13}$$

In (13) F_n is the normal component of the external flux and we have used the normal equilibrium condition. We note that the inequality in (13) is quite close in spirit to the Shakura–Sunyaev (1973) estimate of the shear stress in thin discs as a multiple α of the normal stress (both being dominated by radiation). It is expected that $\alpha < 1$. Now, the normal external radiation F_n can be estimated by arguments similar to those leading to equation (6). For small ϕ and a uniformly bright cone, $F_n \simeq F_t$. The uniform brightness approximation underestimates F_t as argued in Section 2 and also clearly overestimates F_n. We thus have

$$F_n = xF_t \qquad (x < 1). \tag{14}$$

For example, $x = ½$ at a point above 90 per cent of the luminosity in fig. 5 of Sikora (1981). Using (14) in (13), we obtain

$$F_t\kappa/c \leqslant \alpha g_{eff}/(1-\alpha x) = \frac{\alpha g}{\phi(1-\alpha x)}. \tag{15}$$

Comparing (15) and (11) one again obtains a luminosity limit similar to (12), but with an extra factor $(1-\alpha x)^{-1}$ which is, however, not large because of the restrictions on α and x.

We thus conclude that the luminosity of a funnel cannot exceed $\sim \phi L_E$ if steady flow is to be maintained at its surface. While this limit can be larger than the estimate of equation (9) (obtained assuming strict equilibrium) and is model dependent because of the parameter

α, these results show quite clearly that funnels with super-Eddington luminosities must accelerate the matter in their walls, as discussed below.

4 Ejection of matter from a funnel with super-Eddington luminosity

In this section, we concentrate, as before, on the region of the funnel just above that which dominates the luminosity and make a semi-quantitative estimate of the energy which must be carried out as a stream of matter when $L \gtrsim L_E$. The tangential radiation flux F_t is obtained from (6) and (7) and we write it in the form (for small ϕ)

$$F_t = L/4\pi r^2 \tag{16}$$

where r is the radius of the funnel at the level of interest, at a distance R from the central mass. A force $F_t\kappa/c$ acting over a distance R will accelerate matter to a velocity v given by

$$v^2 = 2F_t\kappa/c \times R. \tag{17}$$

Introducing the dimensionless radiation luminosity $l_r = L/L_E$ we have

$$v^2/c^2 = l_r \frac{Rr_s}{r^2} \tag{18}$$

where r_s is the Schwarzschild radius associated with the mass M. Of course, (18) is consistent only if the calculated v^2/c^2 is much less than 1. It can also be checked that v^2 is much greater than GM/R when $l_r > 1$ so that the matter escapes.

The unbalanced tangential force acts on a layer of thickness $\sim 1/\kappa\rho$ where ρ is the surface density. The matter thus flows out over an area $2\pi r/\kappa\rho$ with velocity v, carrying away energy L_p given by

$$l_p = L_p/L_E = \frac{r}{2r_s}\left(\frac{l_r R r_s}{r^2}\right)^{3/2}. \tag{19}$$

Note the 3/2 power of l_r which means that the particle luminosity l_p overtakes the radiation luminosity of the funnel when

$$l_r > \frac{4r^4}{r_s R^3}. \tag{20}$$

For typical values like $r \sim 100$, $R = 1000$, $r_s = 2$, the right-hand side of (20) is about 0.2. The numerical factor in (20) of course depends on the precise distance over which the acceleration takes place (which was set equal to R in 17) as well as the detailed variation of the tangential force over the funnel. It seems clear, however, that a significant fraction of luminosity is in the form of outflowing matter for $L \gtrsim L_E$.

We note that Sikora (1981) has mentioned the possibility of a 'disc wind' and Sikora & Wilson (1981, SW) have studied the motion of test particles in the radiation field of a funnel. They concluded that the particle luminosity is limited since the optical depth of the blown off matter in the longitudinal direction, i.e., along the funnel axis, must be of order unity or less. However, this is only a condition for the consistency of the radiation field computed by SW. The arguments given earlier show that a layer with *transverse* optical depth of order unity is out of equilibrium and thus accelerated by the escaping radiation. If the matter in this layer fills the funnel, the longitudinal optical depth would certainly be greater than unity. This would imply even more efficient coupling between the radiation and the matter and the conclusion about the importance of the particle luminosity is reinforced.

The calculations given above are for the non-relativistic case. The most important effects of relativistic motion are the aberration of the transverse fluxes F_s and F_n and the Doppler redshift of the accelerating flux F_t. Both of these act to brake the motion of the walls. As an illustration, if we have $F_n = F_s = F_t$, then the total force falls to zero for $v/c = 0.27$ i.e., for a Lorentz factor $\gamma = 1.04$. In fact, SW find from detailed calculations that normal matter can be accelerated to $\gamma - 1 = 0.1$ or so (which is greater than 0.04 probaby because in the outer regions of the funnel $F_t > F_n$, F_s). We thus find the following limit for the particle luminosity

$$L_{p,\max} = \frac{2\pi r}{\kappa\rho} \times \frac{v}{c} \times \rho c \times (\gamma - 1)c^2 \simeq 0.25\, rc^3/\kappa \quad \left(\text{for } \frac{v}{c} \simeq 0.4, \gamma - 1 \simeq 0.1\right), \tag{21}$$

i.e.

$$l_{p,\max} = \frac{L_{p,\max}}{L_E} = \frac{1}{25}\left(\frac{r}{r_s}\right).$$

For the parameters used earlier ($r \sim 100$) we have $l_{p,\max} = 4$ which is still substantial.

5 Discussion and conclusion

The models of thick accretion discs studied by PW, JAP and ACN have two attractive features: (i) the super-Eddington luminosity and (ii) the funnels which collimate it. However, in their present form, these models do not incorporate the processes of energy generation and energy transfer within the disc. Instead one uses a postulated distribution of angular momentum on the surface to make definite predictions about the shape and luminosity of the funnel. Further, the equilibrium condition imposed is valid only in the interior of the disc, while external radiation is a new factor which has to be taken into account at the surface of the funnel. We believe that the above mentioned uncertainties in these models justify the more phenomenological approach taken in this paper.

We characterize the funnel geometry by the semi-angle ϕ and calculate the radiation force acting on the matter in the walls in terms of the luminosity L. Strict equilibrium at the surface requires $L \leqslant \phi^2 L_E$ while a steady flow with viscous stresses balancing excess radiation forces seems possible only for $L \lesssim \phi L_E$. In contrast, the PW family of models have $L = L_E/\phi$. The effect of the external radiation on the funnels of JAP has been studied in a fully general-relativistic framework by Sikora (1981). The main result of his work is a brightness distribution and associated radiation field consistent with equilibrium normal to the surface. The radiation field is greatly enhanced by the 'reflection effect' i.e., the exchange of radiation between different parts of the funnel. This effect is of course already included in our calculations. As expected from our arguments, there is a large unbalanced force in the tangential direction in Sikora's calculations, but this receives only passing mention. From our point of view, it is this tangential force which determines the behaviour of the funnel. For luminosities of the order of L_E or greater, we find that the matter in the walls is accelerated to much greater than escape velocities. The total luminosity carried away by the particle flow grows faster than the radiation luminosity in the non-relativistic limit. The two become of the same order as L crosses L_E, though our estimate is probably not accurate enough to fix the critical cross-over value of L precisely. The particle luminosity, however, saturates at a few times L_E due to aberration and Doppler shift effects. There is some overlap between this part of our work and the calculations of SW (1981) who calculated the motion of test particles in the intense radiation field of the funnel. They concluded that the longitudinal optical depth (parallel to the axis) would limit the luminosity emerging as a particle beam to insignificant values. In our calculations it is only the transverse optical depth of the surface layer which has to be less than unity and the resulting luminosity is therefore significant – a possibility that is mentioned by SW without elaboration.

An intense, collimated stream of particles is not unwelcome from the point of view of observations. However, if the flow carries a significant part of energy output, it is not at all clear that the original starting point (PW, ACN, JAP) which neglects the flow is a good one. We have shown that a considerable flow must exist in any funnel with $L \sim L_E$ or greater on rather simple physical and geometric grounds, independently of detailed models for the disc interior. This result suggests (at least!) two possible alternatives for thick disc models with a realistic distribution of frictionally generated energy. One is quasi-spherical accretion with quasi-radial energy flow near the black hole and a luminosity limited to $\sim L_E$ by matter outflow. This picture was in fact suggested by Shakura & Sunyaev (1973). The second possibility is that super-Eddington luminosities survive even when the energy generation and transport as well as flow dynamics are properly included. It would appear that the question can be answered only with detailed calculations on realistic models. This formidable problem now seems unavoidable because of the gross failure of equilibrium and significant outflow of matter which we have demonstrated in this paper.

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