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Bosonic Stimulation as a Natural Realization of the Pólya Urn Problem

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The kinematics of Bose statistics for the scattering of incoming bosons into a set of non-overlapping, degenerate single-particle states, is shown to manifest a metrically non-transitive and symmetry-breaking behaviour similar to the Pólya urn problem, familiar in diverse situations ranging from population biology to economics. It is a purely statistical consequence of bosonic stimulation, which favours scattering into a state with an already higher occupation number. A possible experimental realization is suggested.

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Urn models capture the statistics of sampling accompanied by after-effects [1]. In such a model, an urn typically begins with b blue and r red balls. A ball is drawn at random and replaced, with, furthermore, c balls of the drawn color and d balls of the opposite color being added. In fact, these correspond to postive-feedback. Subject to total number of balls being non-negative, it is also allowed that $c, d < 0$, which corresponds to negative feed-back schemes. The most general replacement scheme is given by an $n \times n$ addition matrix of integers, where n is the number of colors or labels. A special case is the Pólya urn problem, where $d = 0, c > 0$ [2]. The Pólya urn problem is known in a number of contexts, including population biology, genetics, ecology, economics [2, 3] and the spread of contagious diseases [4]. Formally, the generalized Pólya urn problem corresponds to a class path-dependent stochastic processes with postive feed-back [5].

In this work, we point out that Bose statistics in the context of scattering of incoming bosons into two or more final states, and the statistics of the resulting populations, provide a natural, physical realization of the Pólya urn problem. To our knowledge, this is the first time that such a connection has been made. In the particular realization of the urn problem studied here, we consider the statistics of a beam of incoming Bose particles that are scattered into the degenerate (spatially) non-overlapping single-particle states. We find that in a given realization, or run, of the scattering process, the fractional population in any given final state tends to a limiting value, with fluctuations that vanish quickly as the number of the incident particles grows to infinity. The limiting value itself, however, varies randomly from one run to another. We suggest a possible experimental realization of this idea in a laser system.

This random non-degenerate limit for the fractional population of the non-overlapping states is indeed a case of metrical non-transitivity [7] and symmetry breaking. It is a purely kinematic consequence of Bose statistics (bosonic stimulation)– the $(1+N)$ Bose factor that favours populating the already populated state by preferentially scattering into it [8]. To include the Bose factor, we consider the following simple generalization of the urn problem: the probability to sample a ball of a given color need not be proportional to its actual proportion in the population, but to a value shifted by some fixed quantity. For example, the probability that the first two balls sampled are blue need not be $[b/(b+r)][(b+c)/(b+r+c)]$, but instead: $[(b+s_b)/(b+r+s_b+s_r)][(b+s_b+c)/(b+r+s_b+s_r+c)]$, where s_b and s_r are the blue and red 'shifts', respectively. Letting $b' \equiv b + s_b$ and $r' \equiv r + s_r$, and in terms of generalized binomial coefficients, the total probability that n_b blue balls and n_r red balls turn up out of $n = n_b + n_r$ draws is:

$$p_n(n_b) = \frac{\binom{n_b - 1 + (b'/c)}{n_b}\binom{n_r - 1 + (r'/c)}{n_r}}{\binom{n - 1 + (b'+r')/c}{n}} = \frac{\binom{-p/\gamma}{n_b}\binom{-q/\gamma}{n_r}}{\binom{-1/\gamma}{n}}, \tag{1}$$

where $p \equiv b'/(b'+r')$, $q \equiv r'/(b'+r')$ and $\gamma \equiv c/(b'+r')$ are positive real numbers such that $p+q = 1$. Eq. (1) is just the Polya distribution [6], with shifted coefficients b', r' instead of the normal ones b, r. We note that bosonic stimulation emulates the generalized Pólya urn problem with color shift $s_b = s_r \equiv s = 1$. It comes as a pleasant surprise that Bose statistics and the partitioning through scattering provide a natural setting for the physical realization of the generalized Pólya urn problem, and its irreproducible behaviour.

To demonstrate urn-like behaviour, let us consider the case of a two-fold partitioning of the scattered beam, i.e., the beam can be scattered into any of two non-overlapping states, coupled equally to incoming state. Label the

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states as b and r and let the corresponding populations be initially set to $N_b(0)$ and $N_r(0)$. We now start populating these two outgoing states with the incident Bose particles as described above. To fix the idea, consider a simple illustration of an opticial analog where the bosons in question are the photons in a two-mode laser cavity of very high quality factor. (The latter is required to overcome the fact that photon number is not conserved, and may decay on time-scales relevant to this problem). Here the coherent de-excitation of the active atoms in the lasing medium injects photons into the two degenerate modes of the laser field. Now, given an initial population of each mode, how their subsequent growth proceeds is the question. Here the run-to-run statistical fluctuations and the limiting behaviour will be reflected in the pulse-to-pulse relative intensity in the two modes, as the laser is operated in the pulse-mode. We suggest that such a multi-mode laser system can serve as a potential realization of the Pólya urn problem as Bosonic stimulation.

Let the populations grow to $N_b(i)$ and $N_r(i)$ at the i^{th} incidence. (We, of course, assume identical coupling to the two distinct states). Bosonic stimulation then gives the probabilistic law for the evolution of the fragment populations:

$$(N_b(i), N_r(i)) \longrightarrow \begin{cases} (N_b(i)+1, N_r(i)) & \text{with probability } \frac{N_b(i)+1}{N_b(i)+N_r(i)+2} \\ \\ (N_b(i), N_r(i)+1) & \text{with probability } \frac{N_r(i)+1}{N_b(i)+N_r(i)+2} \end{cases} \tag{2}$$

The resulting algorithmic evolution, in probability, can now be obtained iteratively by generating a random number at each of the iterations. In Fig. 1, we have plotted the fractional population $N_b(i)$ for $i = 0$ to $i = 2000$, starting with $N_b(0) = N_r(0) = 1$, in three separate runs. For each of the runs, the relative fraction can be clearly seen to tend to a particular limiting value, with fluctuations about that value diminishing quickly to zero as i assumes large values. The limiting value itself, however, can be seen to fluctuate randomly from one run to another. We have tried several initial values for $N_b(0)$ and $N_r(0)$ and find the trend to remain the same. In fact, the distribution of the number of balls of drawn from a given color will be given by the Pólya distribution, Eq. (1), with parameters $n = 2000, b = N_b(0)$, $n_b = N_b(n)$, $r = N_r(0)$ and $n_r = N_r(n)$. If $N_b(0) = N_r(0)$, it follows from Eq. (1) that the distribution is symmetric with respect to both colors, and in particular, the outcome $N_b(n) = N_r(n)$ is the most likely. We note that we have ignored here the possibility of coherent superposition of the two distinct parts.

We will now derive explicitly the asymptotic probability distribution associated with the above Pólya urn problem. We note that a limiting distribution is guaranteed here by the strong law for such path dependent processes [5]. The limiting distribution as $n \mapsto \infty$, keeping fixed the fraction n_b/n, is given by the beta distribution with parameters $\beta \equiv b'/c$ and $\rho \equiv r'/c$. To see this, we rewrite the Polya distribution:

$$\begin{aligned} p_n(n_b) &= \frac{\Gamma(\beta+\rho)}{\Gamma(\beta)\Gamma(\rho)} \begin{pmatrix} n \\ n_r \end{pmatrix} \frac{(n_b+\beta-1)!(n_r+\rho-1)!}{(n+\beta+\rho-1)!} \\ &\approx \frac{\Gamma(\beta+\rho)}{\Gamma(\beta)\Gamma(\rho)} \frac{1}{n} (n_b/n)^{\beta-1} (n_r/n)^{\rho-1}, \end{aligned} \tag{3}$$

where the last expression uses the large n and Stirling approximation, and $\Gamma(\cdot)$ is the Gamma function. Letting $t \equiv n_b/n$, and going from the discrete to continuous case, we obtain the beta distribution:

$$f(t; \beta, \rho) = \frac{1}{B(\beta, \rho)} t^{\beta-1} (1-t)^{\rho-1}, \tag{4}$$

where $B(\beta, \rho) = \Gamma(\beta)\Gamma(\rho)/\Gamma(\beta+\rho)$ is the beta function. The plot in Fig. 1 corresponds to the case $b' = r' = 2$ and $c = 1$, so that $\beta = \rho = 2$, and the asymptotic distribution has the form $f(t) \sim t(1-t)$, centered about a maximum at $t = 1/2$.

In passing, we may consider the following *negative* feed-back strategy, by switching the probabilities in Eq. (2):

$$(N_b(i), N_r(i)) \longrightarrow \begin{cases} (N_b(i)+1, N_r(i)) & \text{with probability } \frac{N_r(i)+1}{N_b(i)+N_r(i)+2} \\ \\ (N_b(i), N_r(i)+1) & \text{with probability } \frac{N_b(i)+1}{N_b(i)+N_r(i)+2} \end{cases} \tag{5}$$

As expected, simulation shows that the asymptotic populations tend to the same value $N_b(n) = N_r(n) = (b+r+n)/2$.

It is a simple exercise now to generalize Eq. (2) to the case of m final states, with the corresponding probabilistic recursion relation given by

$$N_j(i) \longrightarrow \begin{cases} N_j(i)+1 & \text{with probability } p_{ij} = \frac{N_j(i)+1}{m+\sum_j N_j(i)} \\ N_j(i) & \text{with probability } 1-p_{ij} \end{cases} \tag{6}$$

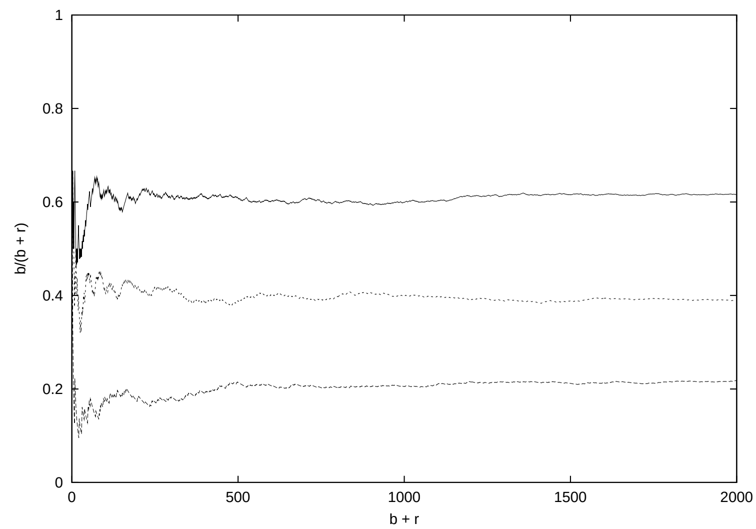

FIG. 1: Asymptotic limit of the population of the bosonic states: Three different runs of the algorithm of Eq. (2), with the same initial condition $N_b(0) = N_r(0) = 1$, for 2000 iterations.

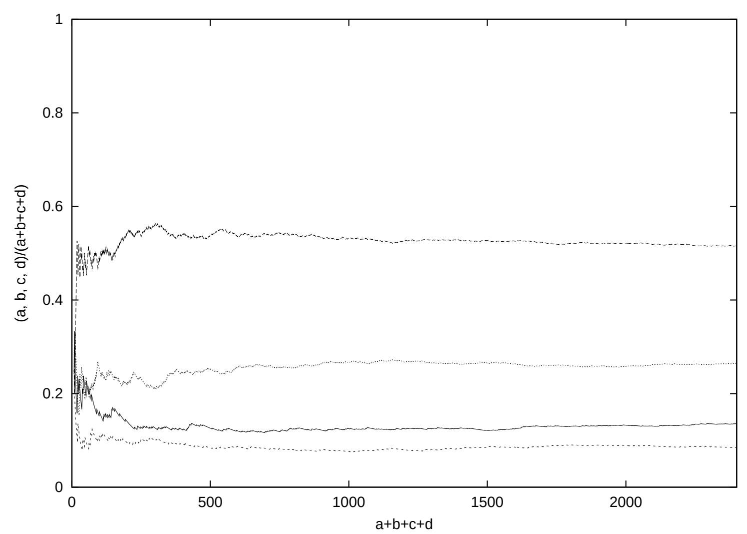

FIG. 2: Asymptotic limit of the population of the bosonic states: A run of the algorithm of Eq. (6) generalized to four final states, with the same initial condition $N_a(0) = N_b(0) = N_c(0) = N_d(0) = 1$, for 2400 iterations.

for $1 \leq j \leq m$. The asymptotic levelling out of the fractional population for each final state was verified in several runs for this case. A run using $m = 4$ is depicted in Figure 2. This irreproducibility of the non-degenerate limiting value of the relative population of the outgoing states– the strangeness of proportion– is characteristic of Polya urn game.

Suppose we have m states, with initial population in each state given by $k_1, k_2, \cdots, k_m$ and respective shifts by $s_1, s_2, \cdots, s_m$. Analogous to Eq. (4), the limiting distribution as $n \mapsto \infty$, is given by the Dirichlet distribution, the multivariate generalization of the beta distribution, with parameters $\kappa_1 \equiv (k_1 + s_1)/c$,$\kappa_2 \equiv (k_2 + s_2)/c$ and so on. In place of Eq. (3), we obtain the multivariate Pólya distribution:

$$
\begin{aligned}
p_n(n_1, n_2, \cdots, n_m) &= \frac{\Gamma\left(\sum_j \kappa_j\right)}{\Pi_j\left[\Gamma(\kappa_j)\right]} \frac{n!}{n_1! n_2! \cdots n_m!} \frac{\Pi_j\left[(n_j+\kappa_j-1)!\right]}{(n+\sum_j(\kappa_j)-1)!} \\
&\approx \frac{\Gamma\left(\sum_j \kappa_j\right)}{\Pi_j\left[\Gamma(\kappa_j)\right]} \frac{1}{n^{m-1}} \Pi_j\left[(n_j/n)^{\kappa_j-1}\right], \qquad (7)
\end{aligned}
$$

obtained analogously to Eq. (4). Letting $t_j \equiv n_j/n$, and going from the discrete to continuous case, we obtain the Dirichlet distribution:

$$
f(t_1, \cdots, t_m; \kappa_1, \cdots, \kappa_m) = \frac{1}{B(\kappa_1, \cdots, \kappa_m)} \Pi_j\left(t_j^{\kappa_j-1}\right), \qquad (8)
$$

where $\sum_j t_j = 1$ and $B(\kappa_1, \cdots, \kappa_m) = \Pi_j\left(\Gamma(\kappa_j)\right)/\Gamma(\sum_j \kappa_j)$ is the normalization factor.

It is worth noting that the distributions (4) and (8) represent the global probability distributions for ensembles over many different runs. A different and rather subtle problem is that of determining the distribution in a *given* run as a function of N by following prescriptions Eqs. (2) or (6). In this work, we content ourselves with giving a qualitative insight into the behaviour brought out by the simulations. For simplicity, consider Eq. (2). A given run, though initially fluctuating, tends to stabilize in a particular state ratio $N_b(i)/N_r(i)$ for large i. Intuitively, this can be understood as follows. for $N \mapsto \infty$, the two transition probabilities in Eq. (2) are asymptotically $p \equiv N_b(i)/N$ and $1-p = N_r(i)/N$, respectively. In this limit, the immediate modification of the urn's state ratio due to adding new balls can be ignored, turning the problem into one of random tosses of a biased coin with probabilities p and $1-p$. Thus variance goes as $p(1-p)/N$, so that standard deviation of fluctuations die out as $\sim 1/\sqrt{N}$.

We would like to conclude here by pointing out that the phenomenon considered here provides a physically interesting example of symmetry breaking which is purely of statistical origin and distinct from the thermodynamic spontaneous symmetry breaking, well known in the context of phase transitions. The former derives ultimately from the kinematics of Bosonic stimulation (the $N+1$ factor), while the latter from the dynamics of cooperative behaviour. Clearly, an interplay of the two is possible. The statistical partitioning of the population into multiple states may be relevant in the context of a fragmented Bose-Einstein condensate (BEC). We would like to emphasize that we are envisaging a mode of fragmentation where incoming bosons outcoupled from a master condensate is scattered into two or more spatially non-overlapping traps. This could be approximated by the populating of traps in a microchip with ultracold atoms outcoupled from a reservoir. This is, of course, quite different from the usual fragmented condensates realized in thermodynamic equilibrium in a trap. Presence of weak coupling between the fragments can, however, cause coarsening of the fragments into a single BEC whole, which will require a thermodynamic treatment. The competition between the bosonic stimulation favouring random fragmentation and the weak inter-particle interaction favouring a single BEC whole poses an interesting problem beyond the scope of this work.

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