Comment on Holst's Lagrangian formulation

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We make two observations about Holst's derivation of Barbero's Hamiltonian formulation from a covariant Lagrangian. While Holst's derivation does appear to be correct, there are two points in the derivation which may be worth clarifying. These concern the choice of time gauge and the manner in which the Hamiltonian variables are defined in terms of the covariant ones. We emphasize that our observations in no way affect the validity of Holst's result.

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Recent work in loop quantum gravity [1] is based on a Hamiltonian formulation due to Barbero [2]. The variables in Barbero's Hamiltonian formulation [2,3] are a *real* $SO(3)$ connection A_a^i and a *real* densitized triad $\tilde{E}_i^a$, which are canonically conjugate. These variables are subject to the following constraints:

$$\mathcal{D}_a \tilde{E}_i^a \approx 0 \tag{1}$$

$$\tilde{E}^{ai} F_{abi} \approx 0 \tag{2}$$

$$\epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} - 4 \tilde{E}_{[i}^a \tilde{E}_{j]}^a (A_a^i - \Gamma_a^i)(A_b^j - \Gamma_b^j) \approx 0, \tag{3}$$

where $\mathcal{D}$ is the covariant derivative associated with the connection A_a^i, F its curvature and Γ_a^i the Ricci rotation coefficients of the spatial triad variables. The main advantage of Barbero's formulation is that all variables are real, even for Lorentzian general relativity. A good deal of current work is based on Barbero's formulation, which was arrived at by performing a canonical transformation on the extended [4] phase space of general relativity.

Given that Barbero's Hamiltonian formulation has a foundational place in current work on loop quantum gravity, it would be useful to understand it from diverse points of view. In particular, it is natural to seek a Lagrangian formulation. Such a formulation has been given by Holst [5]. Our purpose in this paper is to make two observations about Holst's formulation.

The first point concerns the choice of "time gauge" in [5] in the argument following Eq. (9). After foliating the space-time manifold $\mathcal{M}$ in the standard way into $\Sigma \times \mathbb{R}$ where $t \in \mathbb{R}$ is time and Σ_t is "space at an instant of time," Holst chooses the "time gauge." The tetrad e_μ^I is rotated by an $SO(3,1)$ transformation at each point of Σ_t so that e_μ^0 is normal to Σ_t. If n_μ is the unit normal to Σ_t, n_I is rotated into the form $n_I = (1,0,0,0,)$ so that $n_\mu = n_I e_\mu^I = e_\mu^0$. Plugging this gauge choice into the Lagrangian, Holst arrives at his equation (10), which is the "gauge fixed" Lagrangian. He then proceeds to use the gauge fixed Lagrangian to derive the constraints of the theory.

We observe that this procedure is incorrect: *it is not permissible to fix the gauge before performing the Legendre transformation*. That this procedure leads in general to incorrect conclusions can be seen in electromagnetism by choosing the temporal gauge before performing the constraint analysis: one loses the Gauss law constraint. The correct procedure is to fix the gauge only after the constraint analysis is performed. This procedure has been followed in recent papers by Alexandrov [6] and Barros e Sa [7]. From their work it is clear that Holst's conclusion is correct: Barbero's Hamiltonian formulation does derive from Holst's covariant formulation.

The second point concerns the definition of the Hamiltonian variables in terms of the Lagrangian ones. One of the variables in Holst's Lagrangian formulation is an $SO(3,1)$ connection A^{IJ}. One of the Hamiltonian variables of Barbero's formulation is a real $SU(2)$ connection ${}^{-}\mathcal{A}$ defined in terms of the Lagrangian one in Eq. (12) of [5]. Notice that in this definition, the Barbero connection ${}^{-}\mathcal{A}^k$ is *not* the pullback to Σ_t of the space-time connection A^{IJ} form. Rather, specific components of the space-time connection are defined to be components of Barbero's $SU(2)$ connection ${}^{-}\mathcal{A}^k$. As a result, the holonomy of Barbero's connection depends on the slicing [8].

This feature is in contrast to the old Ashtekar variables. In a Lagrangian derivation [9] of these variables, the Ashtekar connection was defined as the *pullback* to a spatial slice of the space-time connection of the Lagrangian formulation.The holonomy of the Ashtekar connection along a loop γ depends only on the loop and not on the slicing.

Neither of these remarks in any way invalidates the main claim of [5]. The first remark shows that in spite of a gap in his logic, Holst's conclusion stands. The second remark clarifies the nature of Barbero's connection: unlike the Ashtekar connection, it depends on the slicing.

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