Angular momentum and an invariant quasilocal energy in general relativity

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A key feature of the Brown-York definition of quasilocal energy is that, under local boosts of the fleet of observers measuring the energy, the quasilocal energy surface density transforms as one would expect based on the equivalence principle, namely, like E in the special relativity formula: $E^2 - \vec{p}^2 = m^2$. This paper provides physical motivation for the general relativistic analogue of this formula, and thereby arrives at a geometrically natural definition of an ''invariant quasilocal energy'' (IQE). In analogy with the invariant mass m, the IQE is invariant under local boosts of the fleet of observers on a given two-surface S in spacetime. A reference energy subtraction procedure is required, but in contrast with the Brown-York procedure, S is isometrically embedded in a *four*-dimensional reference spacetime of one's choosing. For example, it is well known that *any* sphere, round or not, can always be isometrically embedded into Minkowski space, even if its scalar curvature is not everywhere positive. So rather than embeddability being a concern, the problem now is that such embeddings are not unique, leading to an ambiguity in the reference IQE. However, in this codimension-two setting there are two curvatures associated with S: the curvature of its tangent bundle, and the curvature of its normal bundle. Taking advantage of this fact suggests a possible way to resolve the embedding ambiguity, which at the same time will be seen to incorporate angular momentum into the energy at the quasilocal level. The IQE is analyzed in the following cases: both the spatial and future null infinity limits of a large sphere in asymptotically flat spacetimes; a small sphere shrinking to a point along either spatial or null directions; and finally, a large sphere in asymptotically anti–de Sitter spacetimes. The last case reveals a striking similarity between the reference IQE and a certain counterterm energy recently proposed in the context of the conjectured AdS/CFT correspondence.

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I. INTRODUCTION

It is generally agreed that gravitational energy exists, but because of the equivalence principle it cannot be localized. The notion of quasilocal energy is currently one of the most promising descriptions of energy in the context of general relativity, and can be characterized simply as follows. The total energy, including both matter and gravitational contributions, contained in a finite spatial volume Σ can be defined only as the integral of some energy surface density over its two-surface boundary, $S = \partial\Sigma$. This implies that, strictly speaking, there is no such thing as a local energy volume density, except that which arises from the small S limit of quasilocal energy.[1] And even this local notion is not truly local because it cannot be integrated over a finite volume unless one is willing to ignore effects due to gravity. In short, energy is associated with closed spacelike two-surfaces in spacetime, not points.

There is also a growing consensus that the Arnowitt–Deser–Misner (ADM) and Bondi–Sachs masses are simply not enough. We need some definition of energy that is ''more local'' than these; i.e., a quasilocal definition that does not rely on the existence of an asymptotically flat region [2]. For example, recent proponents of this movement are Ashtekar *et al.* [3,4], who have introduced the quasilocal idea of an isolated horizon to describe a black hole. They articulate several reasons for this need, and it is useful to paraphrase here at least part of their argument: Let us accept that a black hole is a thermodynamic object, and so obeys the first law: $\delta E = T\delta S + \cdots$. Now suppose that the universe is asymptotically flat in spatial directions, and contains a single black hole. Then E in the first law is the ADM mass. But if there is anything else in the universe then E is not the ADM mass, and the question arises, What expression is to be used for E in the first law? In other words, we expect that we can put something else in the universe, say a galaxy somewhere, such that the black hole we started with, considered by itself, will still behave as more or less the same thermodynamic object, with the same mass, radiating at the same temperature as before, and with the same entropy equal to one quarter its area. This expectation requires the ability to compute the energy of a given system contained within a finite closed surface, rather than merely the total energy of all such systems comprising the whole universe.

Thus quasilocal energy lies between the notions of local energy density and total energy of an isolated system, in the sense that it is expected to give the energy contained in any volume, no matter how small or large. Although the equivalence principle precludes the existence of a local gravitational energy density, it does not prevent us from evaluating the (quasilocal) gravitational energy in an arbitrarily small but nonvanishing volume Σ. This is because no matter how

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[1]A notable exception is the Tolman density, which integrates to the Komar mass [1]. But it can be defined only when the spacetime possesses special properties, namely a timelike Killing vector field and an asymptotically flat spatial infinity, and so tells us little about the nature of energy in a general context.

small Σ is, $S=\partial\Sigma$ is not a point, but rather the boundary of some neighborhood of a point, and so we are always inherently making a ''tidal force measurement.'' In this sense quasilocal energy is distinct from attempts to define a local gravitational energy density based on certain symmetries of the action, and the concept of a Nöther charge.[2] At the other extreme is the Komar mass (or the closely related ADM mass), i.e., the total energy associated with the time translation symmetry of an isolated system. As emphasized in Ref. [6], this gravitational conserved charge is intimately connected with a lapse function, whereas quasilocal energy need not make any reference to a lapse function. The point is, the two are conceptually distinct [2,6], even though in some circumstances one might expect their numerical values to coincide.

Currently there are several contenders for a good definition of quasilocal energy (see Refs. [7–9] and the references therein). The two that interest us at the moment are the Brown-York ''canonical quasilocal energy'' (CQE) [7], and the various definitions based on the integral over S of the Witten-Nester two-form (the two-form used in Witten's proof of the positive energy theorem [10,11]). The latter approach uses spinorial methods, and the different definitions are distinguished by the choice of supplementary equation the S-spinors are supposed to satisfy, for example the Sen-Witten equation [11,12], the Dougan-Mason equation [13], or the Ludvigsen-Vickers equation [14]. The Brown-York definition of quasilocal energy has the form[3]

$$\mathrm{CQE}=-\frac{1}{8\pi}\int_S dS\,k-\mathrm{CQE}^{\mathrm{ref}}, \tag{1.1}$$

in geometrized units, with $G=c=1$. The CQE is supposed to be the energy of the gravitational and matter fields contained in a finite spatial volume Σ, whose boundary two-surface is $S=\partial\Sigma$. dS is the induced integration measure on S, and k is the trace of the extrinsic curvature of S as embedded in Σ. Thus, $-k/(8\pi)$ is the Brown–York quasilocal energy surface density. When Σ is asymptotically flat the integral in Eq. (1.1) (the unreferenced CQE) diverges as S is taken to infinity, and a reference term, denoted $\mathrm{CQE}^{\mathrm{ref}}$, is required to regulate the energy. The Brown-York prescription is to choose

$$\mathrm{CQE}^{\mathrm{ref}}=-\frac{1}{8\pi}\int_S dS\,k^{\mathrm{ref}}, \tag{1.2}$$

where k^{ref} is the trace of the extrinsic curvature of an isometric embedding of S into some reference space, usually taken to be flat $\mathbb{R}^3$. With this choice the resulting CQE reduces to the ADM mass when S is taken to infinity [7]. While the CQE has a host of desirable properties, neatly summarized in Ref. [15], the embedding prescription needed to evaluate $\mathrm{CQE}^{\mathrm{ref}}$ is not entirely satisfactory, because not all two-surfaces that arise in practice can be embedded into flat $\mathbb{R}^3$. A ready example is the horizon of the Kerr black hole, which fails to be embeddable in flat $\mathbb{R}^3$ when the angular momentum exceeds the irreducible mass (but the black hole is not yet extremal), and the two-sphere develops regions with negative scalar curvature [16]. While this is but one example, it is noteworthy that the breakdown of embeddability is in this case associated with angular momentum and negative scalar curvature. It is precisely such issues: embeddability, angular momentum, and negative scalar curvature, that will figure prominently in this paper, and will be seen to be subtly intertwined.

Although a relationship between the Brown-York quasilocal energy and the spinorial definitions based on the Witten-Nester integral is not immediately obvious, Lau [15] has shown that spinors may always be chosen so that the resulting spinorial definition is equal to the unreferenced Brown-York quasilocal energy in Eq. (1.1). Moreover, he shows that the role of the Sen-Witten equation is to provide a definite reference point for the energy, which is not in general the same as $\mathrm{CQE}^{\mathrm{ref}}$ in Eq. (1.2). The point being made here is twofold: (i) the unreferenced Brown-York quasilocal energy seems to be robust, and (ii) all of the problems lie in choosing a suitable reference energy. The various prescriptions are either not generally well defined, or they do not agree with each other.

I will now present a brief review of the Brown-York approach in a form that will be useful to us later. The classical stress-energy tensor of matter is a local concept, associated with a spacetime point. It is defined for any field theory residing on a nondynamical background spacetime (M,g) via the functional derivative of the (first order) matter action with respect to the metric, as follows:

$$2\,\delta_g I^{\mathrm{mat}}[\varphi,g]=\int_M d^4x\,\sqrt{-g}\,T^{ab}_{\mathrm{mat}}\delta g_{ab}. \tag{1.3}$$

Here φ denotes the matter field(s) in question, and the factor of two on the left is a convention. Usually, as we will assume here, there is no boundary term arising from this variation (for minimally coupled matter), but in case there is it does not change the essence of the following argument, it just adds an interesting dimension to it. T^{ab}_{mat} so defined is covariantly conserved, as follows from the matter Euler-Lagrange equations. This prescription for learning about matter stress-energy gives reasonable answers for all field theories, and so it is natural to try the same thing for gravity. In this case one finds, for the usual first order action [17],

$$2\,\delta_g I^{\mathrm{grav}}[g]=\int_M d^4x\,\sqrt{-g}\left(-\frac{1}{8\pi}G^{ab}\right)\delta g_{ab}+\int_{\mathcal{B}} d^3x\,\sqrt{-\gamma}\left(-\frac{1}{8\pi}\Pi^{ab}\right)\delta\gamma_{ab}. \tag{1.4}$$

Inspecting the bulk term one is thus tempted to define $T^{ab}_{\mathrm{grav}}:=-G^{ab}/(8\pi)$ as the local stress-energy tensor of the gravitational field, where G^{ab} is the Einstein tensor. And this is

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[2] However, a recent discussion of the connection between pseudotensor methods and the quasilocal idea can be found in Ref. [5].

[3] We use a sign convention for extrinsic curvatures opposite to that of Brown and York, hence the negative sign in front of this integral.

perfectly reasonable: it is covariantly conserved—in this case identically so, via the contracted Bianchi identity. Moreover, its on-shell value is zero, in full accord with the equivalence principle, i.e., there is no nontrivial local stress-energy tensor for the gravitational field.

In fact this absence of a nontrivial local stress-energy tensor is true not only for the gravitational field, but also for any system comprised of both matter *and* gravity. To see this we need only make the spacetime metric dynamical, in which case the matter action in Eq. (1.3) must be augmented by the gravity action. Adding Eqs. (1.3) and (1.4) one finds for the total action

$$2\,\delta_g I^{\mathrm{tot}}[\varphi,g]=\int_M d^4x\sqrt{-g}\left(T^{ab}_{\mathrm{mat}}-\frac{1}{8\pi}G^{ab}\right)\delta g_{ab}+\int_{\mathcal{B}} d^3x\sqrt{-\gamma}\left(-\frac{1}{8\pi}\Pi^{ab}\right)\delta\gamma_{ab}. \quad (1.5)$$

Thus one is led to identify $T^{ab}_{\mathrm{tot}}:=T^{ab}_{\mathrm{mat}}-G^{ab}/(8\pi)$ as the total local stress-energy tensor for matter plus gravity. It has the desirable property of being covariantly conserved, but turns out to be just zero by the Einstein equations. If this argument is taken seriously we learn that, as soon as we add gravity to any matter system, the notion of a nontrivial local stress-energy tensor disappears. Furthermore, one might interpret the Einstein equations, written in the form $T^{ab}_{\mathrm{mat}}+T^{ab}_{\mathrm{grav}}=0$, as a micro-balancing of local stress–energy at each spacetime point: wherever a component of matter stress–energy is positive, the corresponding component of gravitational stress–energy is negative, and vice versa, such that the total is always zero. The idea that $-G^{ab}/(8\pi)$ is the local stress-energy tensor of gravity is, of course, a very old idea, first put forward by Lorentz and Levi-Civita. It was rejected by Einstein, since it implies that the total energy of a closed system would always be zero, which is obviously problematic.[4] It is only with hindsight that we now realize why the problem was not resolved much sooner. People then did not think about boundary terms as much as they do today. Thanks to Brown and York we now know that what comes to the rescue is the boundary term in Eq. (1.5).

In this equation the spacetime is assumed to be the topological product of a three-space Σ and a real line interval. The boundary component $\mathcal{B}$ is a timelike tube, topologically the product of $S=\partial\Sigma$ and the real line interval. (The two spacelike end-cap boundary components of ∂M have been omitted, as they play no role in this discussion.) The quantity $-\sqrt{-\gamma}\Pi^{ab}/(16\pi)$, constructed in the usual way out of the extrinsic curvature of $\mathcal{B}$, is the gravitational momentum conjugate to the three-metric γ_{ab} induced on $\mathcal{B}$. Now, in the spirit of identifying the stress-energy tensor as the functional derivative of the action with respect to the metric, one reads off from Eq. (1.5) the stress-energy tensor $T^{ab}_{\mathcal{B}}:=-\Pi^{ab}/(8\pi)$, which is inherently associated with the boundary $\mathcal{B}$, rather than the bulk spacetime. Like any acceptable stress-energy tensor, $T^{ab}_{\mathcal{B}}$ is covariantly conserved (with respect to the derivative operator induced on $\mathcal{B}$); this follows from the analogue of the diffeomorphism constraint of general relativity for the three-surface $\mathcal{B}$ (rather than Σ), and assuming the appropriate components of T^{ab}_{mat} vanish on $\mathcal{B}$. Physically, $\mathcal{B}$ is to be thought of as the congruence of world lines of a two-parameter family of observers with four-velocity u^a, hypersurface orthogonal to a one-parameter foliation of $\mathcal{B}$ by spacelike two-surfaces with the topology of S. Within their three-dimensional spacetime $(\mathcal{B},\gamma)$, the observers measure a spatial energy density $T^{ab}_{\mathcal{B}}u_a u_b$, which is precisely $-k/(8\pi)$, and thus one is led to Eq. (1.1). Finally, observe that one can add to the action any covariant functional of the boundary three-metric γ_{ab} without affecting the previous argument. This is the source of the reference point ambiguity $\mathrm{CQE}^{\mathrm{ref}}$ in Eq. (1.1). This summarizes the central idea of the Brown-York approach [7].

Now if energy is really quasilocal, and calculated via a surface integral involving $T^{ab}_{\mathcal{B}}$, one comes to the conclusion that *a priori* neither T^{ab}_{mat} nor T^{ab}_{grav} has anything to do with energy. While this might be unsettling at first, it is reassuring to know that a satisfactory notion of local *matter* energy density can be recovered from the small sphere limit of quasilocal energy. For example, in Ref. [19] it is shown that, for a certain choice of reference term $\mathrm{CQE}^{\mathrm{ref}}$, the Brown-York quasilocal energy contained in an infinitesimal sphere of proper radius r is the volume of the sphere $(4\pi r^3/3)$ times the local matter energy density $T^{ab}_{\mathrm{mat}}u_a u_b$ (evaluated at the center of the sphere) that would be measured by an observer with four-velocity u^a. Moreover, this is a well-established property of most quasilocal energy definitions [19–24], so the result is quite robust. But at higher order in r, gravitational energy begins to appear, as will be discussed in detail later. The point is there is no contradiction between (i) the local stress-energy tensor $T^{ab}_{\mathrm{mat}}+T^{ab}_{\mathrm{grav}}$ being zero, and (ii) there being nonzero stress–energy in a finite spatial volume. This is because $T^{ab}_{\mathrm{mat}}+T^{ab}_{\mathrm{grav}}$ is *not* a local stress-energy tensor—indeed, if we accept the previous argument, there is no such thing. There is only $T^{ab}_{\mathcal{B}}$, associated with the fact that energy is fundamentally quasilocal.

The main purpose of this introduction is to emphasize, firstly, that energy is fundamentally quasilocal, i.e., associated with closed spacelike two-surfaces—not points—in spacetime; and secondly, there are strong reasons to believe that $-k/(8\pi)$ is the correct quasilocal energy surface density. The major unresolved problem is how to choose the right well-defined energy reference term, $\mathrm{CQE}^{\mathrm{ref}}$. Rather than address this problem *per se*, I will begin with $-k/(8\pi)$ as an energy surface density and construct a new definition of quasilocal energy based on analogy with the special relativity formula: $E^2-\vec{p}^{\,2}=m^2$. The new definition is both physically and geometrically natural, and lies somewhere between the Brown-York CQE and the Hawking [25] or Hayward [8] definitions. A reference subtraction procedure is still required, that involves a reference embedding, but this is a codimension-two embedding that is not subject to the prob-

[4] See the historical discussion given on pages 176 and 177 in Ref. [18]. Thanks to L. de Menezes for bringing this reference to my attention.

lem that afflicts the Brown-York embedding prescription. Moreover, there is a shift in the physics: the reference embedding is not associated with determining a reference energy so much as a reference angular momentum, so to speak. Why angular momentum? Because angular momentum contributes to energy, and the new definition can be seen as a precise formulation of this fact at the quasilocal level.

The paper is organized as follows. In Sec. II we introduce the geometrical quantities we will use later. Section III contains the physical and geometrical motivations behind the new definition of quasilocal energy (as well as the definition itself). The reference subtraction term is discussed in Sec. IV. In Sec. V both the spatial and future null infinity limits of the energy are examined; the small sphere limit is considered in Sec. VI. Finally, in Sec. VII we examine the new energy in the context of asymptotically anti–de Sitter spacetimes. A summary of results is found at the end of the paper, which also includes some additional discussion.

II. THE GEOMETRY OF TWO-DIMENSIONAL SPACELIKE SUBMANIFOLDS

Let (M,g) be a four-dimensional Lorentzian geometry with signature $+2$, and S be a closed two-dimensional spacelike submanifold. Let u^a and n^a be timelike and spacelike unit normals to S that are orthogonal to each other: $u^a u_a = -1$, $n^a n_a = 1$, and $u^a n_a = 0$. We will assume that S is orientable, and an open neighborhood of S in M is space and time orientable, so that u^a and n^a are globally well defined [26]. u^a and n^a are fixed up to an arbitrary local boost transformation:

$$\begin{aligned} u'^a &= u^a \cosh\lambda + n^a \sinh\lambda, \\ n'^a &= u^a \sinh\lambda + n^a \cosh\lambda. \end{aligned} \tag{2.1}$$

The physical picture to keep in mind is that of a finite spatial volume Σ, i.e., a three-dimensional spacelike submanifold, whose boundary is S. Although S need not be connected, nor simply connected, we will often think of Σ as having the topology of a three-ball, and S that of a two-sphere, and thus will sometimes refer to the direction of n^a (assumed outward directed) as the radial direction. Given such a three-surface Σ spanning S it is natural to choose u^a to be orthogonal to Σ (and future directed), in which case n^a is tangential to Σ. Physically, u^a is the instantaneous four-velocity of a two-parameter family of observers on S. With u^a thus tied to the spanning surface Σ, a deformation of Σ (preserving S) will in general effect a radial boost, Eqs. (2.1).

The remainder of this section is a summary of some standard facts about the geometry of the submanifold S, as can be found, e.g., in Ref. [27], except here we follow a notation similar to that used in Ref. [26]. The surface projection operator, $\mathcal{P}^a_b$, is a tensor defined on S by

$$\mathcal{P}^a_b := \delta^a_b + u^a u_b - n^a n_b. \tag{2.2}$$

(All raising and lowering of indices a,b,c, ... will be effected with the metric g_{ab}, or its inverse, g^{ab}.) A surface tensor is defined as a tensor on S that is left invariant under projection of all its indices with the surface projection operator. Obviously one such tensor is the spatial two-metric

$$\sigma_{ab} := \mathcal{P}^c_a \mathcal{P}^d_b g_{cd} = g_{ab} + u_a u_b - n_a n_b \tag{2.3}$$

induced on S. Another is the corresponding volume form on S, given by

$$\epsilon_{ab} := \epsilon_{abcd} u^c n^d, \tag{2.4}$$

where ϵ_{abcd} is the volume form on M. The symbol dS will be used in place of ϵ_{ab} as the integration measure for (S,σ).

If ∇_a denotes the Levi-Civita connection of (M, g), then $\mathcal{D}_a$, the Levi-Civita connection induced on (S,σ), is defined by

$$\mathcal{D}_a T^{b\cdots}_{c\cdots} = \mathcal{P}^d_a \mathcal{P}^b_e \mathcal{P}^f_c \cdots \nabla_d T^{e\cdots}_{f\cdots}, \tag{2.5}$$

where $T^{b\cdots}_{c\cdots}$ is any surface tensor. Then for any two surface vector fields X^a and Y^a, the Gauss formula reads

$$X^a \nabla_a Y^c = X^a \mathcal{D}_a Y^c + h^c{}_{ab} X^a Y^b, \tag{2.6}$$

where $h^c{}_{ab}$ is the second fundamental form. Its first index is normal to S, i.e., $\mathcal{P}^d_c h^c{}_{ab} = 0$, whereas the remaining two are surface tensor indices, that are symmetric under interchange [as can be easily seen by interchanging X and Y in Eq. (2.6) and subtracting the two equations]. Thus the second fundamental form can be decomposed into components along the two unit normals:

$$h^c{}_{ab} = u^c l_{ab} - n^c k_{ab}, \tag{2.7}$$

where the two extrinsic curvatures are (symmetric) surface tensors given by

$$\begin{aligned} l_{ab} &= -u_c h^c{}_{ab} = \mathcal{P}^c_a \mathcal{P}^d_b \boldsymbol{\nabla}_c u_d, \\ k_{ab} &= -n_c h^c{}_{ab} = \mathcal{P}^c_a \mathcal{P}^d_b \boldsymbol{\nabla}_c n_d. \end{aligned} \tag{2.8}$$

It is useful to decompose the extrinsic curvatures into trace and trace-free parts:

$$\begin{aligned} l_{ab} &= \tfrac{1}{2} l \sigma_{ab} + \tilde{l}_{ab} \\ k_{ab} &= \tfrac{1}{2} k \sigma_{ab} + \tilde{k}_{ab}, \end{aligned} \tag{2.9}$$

where $l = \sigma^{ab} l_{ab}$ and $k = \sigma^{ab} k_{ab}$, and a tilde appearing over any quantity in this paper will always mean trace-free part of. The mean curvature vector is then

$$H^c := \tfrac{1}{2} \sigma^{ab} h^c_{ab} = \tfrac{1}{2} (l u^c - k n^c), \tag{2.10}$$

and $H \cdot H = (k^2 - l^2)/4$ is the square of the mean curvature.

Extrinsic curvature is a measure of how a unit normal vector rotates as it is parallelly propagated tangent to S in the ambient space (M, g). Two normal vectors means two extrinsic curvatures. However, from Eqs. (2.8) we see that l_{ab} and k_{ab} measure only the components of this rotation tangent to S. There is also a normal component, i.e., the component of the rotation of one normal vector along the other. Thus a complete characterization of the extrinsic geometry of S requires also the surface one-form

$$A_a := \mathcal{P}_a^b n^c \nabla_b u_c . \tag{2.11}$$

This is an SO (1,1) connection in the normal bundle of S, and its associated curvature two-form is

$$\mathcal{F}_{ab} := \mathcal{D}_a A_b - \mathcal{D}_b A_a . \tag{2.12}$$

We will see later that the curvature of the normal bundle of S plays a key role with regard to angular momentum.

While the second fundamental form (including H^c and $H\cdot H$) and the curvature of the normal bundle are invariant under local radial boosts, the extrinsic curvatures and the connection on the normal bundle are not. They transform as

$$\begin{aligned} l'_{ab} &= l_{ab}\cosh\lambda + k_{ab}\sinh\lambda, \\ k'_{ab} &= l_{ab}\sinh\lambda + k_{ab}\cosh\lambda, \\ A'_a &= A_a + \mathcal{D}_a\lambda. \end{aligned} \tag{2.13}$$

Observe that A_a is different from the other measures of extrinsic geometry in that it transforms as a gauge field.

Our sign conventions are such that the Riemann tensor of (M,g) is defined by $(\nabla_a\nabla_b - \nabla_b\nabla_a)X_c = R_{abc}{}^d X_d$, and similarly that of (S,σ) by $(\mathcal{D}_a\mathcal{D}_b - \mathcal{D}_b\mathcal{D}_a)X_c = \mathcal{R}_{abc}{}^d X_d$ (X_c is a surface one-form in the latter case). Appropriate projections of the Riemann tensor of (M,g) yield the Gauss equation

$$\begin{aligned} \mathcal{P}_a^e\mathcal{P}_b^f\mathcal{P}_c^g\mathcal{P}_d^h R_{efgh} = \mathcal{R}_{abcd} &+ (l_{ac}l_{bd} - l_{bc}l_{ad}) \\ &- (k_{ac}k_{bd} - k_{bc}k_{ad}), \end{aligned} \tag{2.14}$$

the Codazzi equations

$$\begin{aligned} \mathcal{P}_a^e\mathcal{P}_b^f\mathcal{P}_c^g u^h R_{efgh} &= (\mathcal{D}_a l_{bc} - \mathcal{D}_b l_{ac}) - (A_a k_{bc} - A_b k_{ac}), \\ \mathcal{P}_a^e\mathcal{P}_b^f\mathcal{P}_c^g n^h R_{efgh} &= (\mathcal{D}_a k_{bc} - \mathcal{D}_b k_{ac}) - (A_a l_{bc} - A_b l_{ac}), \end{aligned} \tag{2.15}$$

and the Ricci equation:

$$\mathcal{P}_a^e\mathcal{P}_b^f u^g n^h R_{efgh} = -\mathcal{F}_{ab} + (k_a{}^c l_{bc} - l_a{}^c k_{bc}). \tag{2.16}$$

These are the integrability conditions for the isometric embedding of (S,σ) into (M,g), and so by definition of S are necessarily satisfied.

III. THE INVARIANT QUASILOCAL ENERGY

A physical interpretation of the various geometrical quantities introduced in the preceding section can be given as follows. The expansion k measures the fractional expansion of the area of a small element of S when each point in the element is projected a unit distance radially outward. It will have a certain positive value if, for example, S is a round sphere enclosing a volume of flat $\mathbb{R}^3$. (For our present purposes, imagine flat $\mathbb{R}^3$ as a $t=$constant surface in Minkowski space.) Now if S is a round sphere of the same area enclosing some matter, then, according to the Einstein equations, the matter curves the space inside S in such a way that its volume is greater than one would infer by measuring just the area of the sphere and using Euclidean geometry. Thus the expansion measured at S must be smaller, i.e., the areas of spherical shells at larger radii will not increase as rapidly as expected. So we see that the unreferenced Brown-York quasilocal energy in Eq. (1.1) is greater (less negative) when S contains matter, than when it does not. This is an intuitive reason why k is a measure of the energy inside S. [It also explains the need to subtract off a reference energy of the form given in Eq. (1.2): $k^{\rm ref}$ is the nonzero value of k when S merely encloses a volume of flat $\mathbb{R}^3$, i.e., no energy.]

l is similar to k, except that it measures the expansion of S in time, i.e., in the direction of the observer's four-velocity u^a. Intuitively, if the observers tend to be moving radially outward then the area of the two-surface they are on will be expanding, i.e., $l>0$. Conversely, a radially inward motion corresponds to $l<0$. Thus l [more precisely, $l/(8\pi)$] can be interpreted as a radial momentum surface density [28]. In the case that (k^2-l^2) is positive, the observers can always make appropriate local radial boosts such that $l=0$ at each point of S, a situation corresponding to a quasilocal rest frame. I will comment on this notion more precisely at the end of this section.

The trace-free quantities $\tilde{k}_{ab}$ and $\tilde{l}_{ab}$ measure the shear of S, and are intimately connected with angular momentum (or at least $\tilde{l}_{ab}$ is). For example, consider a set of locally nonrotating observers who at coordinate time t are on a constant r, t sphere of the Kerr black hole in Boyer-Lindquist coordinates. Their four-velocity is given by

$$u^a = \frac{1}{N}\left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\phi}\right)^a, \tag{3.1}$$

where N is the lapse function, and $\omega(r,\theta) = -g_{t\phi}/g_{\phi\phi}$ is an observer's angular velocity as measured from infinity [29]. Starting at Eqs. (2.8) it is not difficult to show that in this case $l=0$, so here is an example of observers in a quasilocal rest frame as defined above. Furthermore, one can show that the nonvanishing components of the shear in the time direction are given by

$$\tilde{l}_{\theta\phi} = \tilde{l}_{\phi\theta} = \frac{g_{\phi\phi}}{2N}\frac{\partial\omega}{\partial\theta}. \tag{3.2}$$

Physically, a nonzero ω reflects the frame dragging caused by the rotating black hole. The fact that the degree of frame dragging depends on θ is what makes the observers at different latitudes of the sphere rotate at different rates relative to the distant stars, and more to the point, relative to each other. This causes a shear effect between observers at neighboring latitudes, which obviously disappears when the angular momentum is zero.

Furthermore, let the locally nonrotating observers label themselves with coordinates (θ',ϕ'), which at $t=0$ coincide with the Boyer-Lindquist (θ,ϕ) coordinates on S. Then although the observers always measure the same two-geometry of S as time t goes on, the components of the two-metric σ_{ab} in their (θ',ϕ') coordinates will differ from those in the (θ,ϕ) coordinates by a t- and θ-dependent diffeomorphism along the ϕ-direction. So although one usually associates shear with a geometrical deformation, for instance a round sphere evolving into an ellipsoid, one can also have a physically meaningful shear associated with a continuous parameter family of isometric surfaces. This fact plays an important role in understanding certain embedding equations we will encounter later, and will be discussed in detail elsewhere [30].

The last geometrical quantity to interpret is A_a, the connection in the normal bundle. In the Brown-York analysis the quantity $-A_a/(8\pi)$ is called the momentum surface density, and is denoted as j_a [7]. The momentum vector j^a is tangential to S, corresponding to a rotating two-surface, and thus should be associated with angular momentum. Indeed, this is correct: Let $\mathcal{B}$ denote the timelike three-surface that is the congruence of world lines belonging to the two-parameter family of observers on a two-sphere S. If $\mathcal{B}$ admits a Killing vector field ϕ^a, whose orbits lie in S, then one can define the angular momentum charge

$$J := \int_S dS\, \phi^a j_a, \tag{3.3}$$

which can be shown to coincide with the ADM angular momentum at infinity for asymptotically flat spacetimes [7]. Thus we expect both the shear and the connection in the normal bundle to play a role in angular momentum at the quasilocal level, and indeed we will see that this turns out to be the case.

Now the first goal of this paper is to provide a physical motivation for the general relativistic analogue of the special relativity formula: $E^2-\vec{p}^2=m^2$. First of all, this formula applies strictly to point particles (as opposed to extended objects). One imagines determining, say, the instantaneous three-velocity of such a particle by measuring its location in space at two closely separated points in time, in some inertial reference frame. In the spirit of the quasilocal idea, the analogue of this in general relativity would be to first replace measurements at a point with measurements on a closed spacelike two-surface S. But measurements of what? It would seem that measurements of the location of the point particle in some inertial frame is to be replaced with measurements of the two-geometry of S in a generic spacetime. These measurements are to be repeated at two closely separated points in time. In the point particle case this yields the three-velocity (or the three-momentum $\vec{p}$ if one also knows m); in the two-surface case it yields l_{ab}, the time component of the extrinsic curvature of S. Now I pointed out above that the trace of l_{ab}—more precisely $l/(8\pi)$—indeed has the interpretation of a momentum: it is the normal (or radial) momentum surface density [28]. So it seems reasonable to replace $\vec{p}$ with $l/(8\pi)$. What about the trace-free part of l_{ab}? It was argued above that $\tilde{l}_{ab}$ is associated with angular momentum. Insofar as angular momentum is qualitatively distinct from linear momentum, its role at least at this point of the argument is not clear, and we will simply drop it for now (however, its role will become clear later). Notice that dropping $\tilde{l}_{ab}$ is at least roughly consistent with being interested only in the two-geometry of S, i.e., the two-metric σ_{ab} modulo diffeomorphisms, since, as indicated above, $\tilde{l}_{ab}$ is in some cases associated with just diffeomorphisms of S.

Thus, in the expression $E^2-\vec{p}^2$ we will replace $\vec{p}$ with $l/(8\pi)$. What should replace E? Given the preceding discussion, the obvious answer is the Brown-York energy surface density, $-k/(8\pi)$. Clearly $l/(8\pi)$ and $-k/(8\pi)$ are on exactly the same geometrical footing, being proportional to the timelike and spacelike components of the mean curvature vector H^c [see Eq. (2.10)]. Thus we arrive at the generalization

$$E^2-\vec{p}^2 \rightarrow \frac{1}{(8\pi)^2}(k^2-l^2). \tag{3.4}$$

Now before we accept this generalization, let us observe that there is something unexpected about it. The four-momentum $(E,\vec{p})$ has become a two-momentum $(-k,l)/(8\pi)$. What happened to the other two components of spatial momentum? $l/(8\pi)$ is just the radial component; should not the Brown-York momentum surface density j^a [in our notation, $-A^a/(8\pi)$], which is tangent to S, be the analogue of the two missing components of $\vec{p}$? If so, then instead of Eq. (3.4) we should have

$$E^2-\vec{p}^2 \overset{?}{\rightarrow} \frac{1}{(8\pi)}(k^2-l^2-A^aA_a). \tag{3.5}$$

At first sight this expression is appealing because it manifestly includes a contribution from angular momentum, and it is known that in general relativity angular momentum contributes to mass. A simple example that illustrates this phenomenon is the Kerr black hole, where the ADM mass in excess of the irreducible mass is due to rotational energy. The precise relationship is [29]

$$M_{\mathrm{ir}}^2 = M^2 - \left(\frac{J}{2M_{\mathrm{ir}}}\right)^2, \tag{3.6}$$

where M is the ADM mass, M_{ir} the irreducible mass, and J the angular momentum of the black hole. Comparing the right-hand sides of the previous two equations suggests we conceptually identify $|A|/(8\pi)$ with the angular momentum term, $J/(2M_{\mathrm{ir}})$, which seems reasonable. This leaves $\sqrt{k^2-l^2}/(8\pi)$ to be interpreted as an object like M, viz., a total mass, total in the sense that it includes the contribution from angular momentum. But here then is the point: $\sqrt{k^2-l^2}/(8\pi)$ somehow implicitly already includes the angular momentum contribution to mass. Precisely how will become clear later, but to see immediately that this is at least plausible, consider the case $l=0$. Then $-\sqrt{k^2-l^2}/(8\pi)$ reduces to the Brown-York energy surface density, at least when k is non-negative, and it is known that the (referenced) Brown-York quasilocal energy yields the ADM mass at spatial infinity, which includes the correct angular momentum contribution to mass. So we do not need the A^aA_a term in Eq. (3.5). Besides, putting it in is counter to our goal of seeing if general relativity admits an analogue of the *invariant* mass, m: While the combination k^2-l^2 is invariant under radial boosts,[5] A^aA_a is not—see Eqs. (2.13). So from this point of view the right-hand side of Eq. (3.5) is defective, not to mention the generally unsavory fact that it *mixes* objects with different transformation properties.[6] The question of missing momentum components can also be thought about as follows. A point particle has three components of spatial momentum. Likewise, each point on a two-surface S also has three components of spatial momentum (more properly, momentum surface density): one normal, and two tangential to S. But being tangential, the latter two are associated with a rotating surface, and hence with angular momentum. In going from a point to a two-surface, two components of the linear momentum have become angular momenta. So they do not (directly at least) contribute to the expression for an invariant mass given on the right-hand side of Eq. (3.4) because, as claimed, this expression already inherently includes the contribution from angular momentum.

Thus we are led to propose the following definition of an invariant quasilocal energy (or IQE):

$$\mathrm{IQE}=-\frac{1}{8\pi}\int_S dS\,\sqrt{k^2-l^2}-\mathrm{IQE}^{\mathrm{ref}}, \tag{3.7}$$

where $\mathrm{IQE}^{\mathrm{ref}}$ is a reference subtraction term that will be defined later. The word invariant in IQE refers to the fact just mentioned, that k^2-l^2 is invariant under local radial boosts of the observers on S. And the word energy is used instead of mass—despite our analogy between $\sqrt{k^2-l^2}/(8\pi)$ and the mass m—because, as we will see in Sec. VII, the IQE behaves more like an energy than a mass. So the IQE can be thought of as the amount of rest energy contained in S, a quantity independent of the motion of the observers measuring it. Notice that the unreferenced IQE is negative. Nominally the reference energy $\mathrm{IQE}^{\mathrm{ref}}$ is *more* negative, so that the referenced IQE is positive.

It is useful to express the integrand of the unreferenced IQE in two other equivalent forms. Define the pair of null normals $\xi^a_{\pm}:=u^a\pm n^a$ on S, and the corresponding null expansions

$$\theta_{\pm}:=\sigma^{ab}\nabla_a\xi_{\pm b}=l\pm k, \tag{3.8}$$

cf. Eqs. (2.8) and (2.9). Then we have the following three equivalent expressions:

$$\frac{1}{8\pi}\sqrt{k^2-l^2}=\frac{1}{8\pi}\sqrt{-\theta_+\theta_-}=\frac{1}{4\pi}\sqrt{H\cdot H}. \tag{3.9}$$

For the last expression recall the definition of the mean curvature given after Eq. (2.10). Thus the unreferenced IQE in Eq. (3.7) has a very simple geometrical interpretation: up to a proportionality constant, it is just the mean curvature of S, averaged over S. Because of the square root it is defined only when $k^2-l^2\geqslant 0$, i.e., at each point of S the mean curvature vector H^c in Eq. (2.10) must be either spacelike or null, but never timelike. Roughly speaking, this means that the area of S changes more rapidly in a radial direction, than in time. For example, this condition is satisfied for the constant r,t two-spheres outside the horizon of a Schwarzschild black hole, but not for those inside; on the horizon the unreferenced IQE is zero.

In terms of the null expansions, recall that a future (past) trapped surface is one for which both ingoing and outgoing null expansions, θ_- and θ_+, are everywhere negative (positive) on S [1]. Thus, the unreferenced IQE is imaginary when S is a future or past trapped surface. It is real only when no point on S is trapped. Now a future trapped surface does not quite characterize a black hole, and more subtle characterizations have been proposed for a local definition of a black hole horizon [3,4,32]. For example, Hayward [32] has introduced the notion of a future outer trapping horizon, H, characterized by (i) $\theta_-|_H<0$ (in-going light rays converging), (ii) $\theta_+|_H=0$ (outgoing light rays instantaneously parallel on the horizon), (iii) $\theta_+|_{H^+}>0$ (outgoing light rays diverging just outside the horizon), and (iv) $\theta_+|_{H^-}<0$ (outgoing light rays converging just inside the horizon). According to this general definition of a black hole, the unreferenced IQE is nonzero just outside the horizon, zero on the horizon, and undefined (or imaginary) just inside the horizon. In this connection see also Ref. [33].

Furthermore, observe that the condition for the integrand of the unreferenced IQE to be real and nonzero, namely $k^2-l^2>0$, is precisely the same condition that ensures that the observers can always, by appropriate local boosts, go to a quasilocal rest frame in which $l=0$ at each point of S. Such a two-surface is analogous to a massive particle. The case $k^2-l^2=0$ everywhere on S, for instance when S is a future outer trapping horizon, is analogous to a massless particle, for which no quasilocal rest frame exists. And finally, the

[5]This was first noted in Ref. [28]. A further discussion of boosted observers in the Brown-York framework appears in Ref. [31].

[6]Hayward's [8] definition of quasilocal energy includes an angular momentum contribution of the form $-\omega^a\omega_a$, analogous to the $-A^aA_a$ term in Eq. (3.5). Hayward's ω_a is a suitably normalized anholonomicity, or twist, of the pair of null normals to S, and encodes essentially the same information as A_a. The important distinction is that, unlike the connection A_a, the object ω_a is boost invariant, and so representing angular momentum with a term proportional to $-\omega^a\omega_a$, as Hayward does, is perfectly acceptable. (The relationship between A_a and ω_a is discussed in Appendix B of Ref. [32].) However, there is no need, or even natural way for ω_a to enter our work here. For instance, the A^aA_a term in Eq. (3.5) cannot simply be replaced with $\omega^a\omega_a$, since the (tentative) inclusion of this term is suggested by the physical interpretation of A_a as a momentum surface density. This interpretation arises from Brown's and York's Hamilton-Jacobi analysis of the gravitational action [7], and it is not clear that a similar interpretation can be given to ω_a.

case $k^2-l^2<0$, say inside a future outer trapping horizon, corresponds to a superluminal particle.[7]

The situation is actually more subtle than indicated in the previous two paragraphs. The conditions for $\sqrt{-\theta_+\theta_-}$ to be real are reminiscent of the condition $\theta_-\leqslant 0$ required for the holomorphic case of the Dougan-Mason quasilocal energy to be non-negative. The conditions $\theta_+\geqslant 0$ and $\theta_-\leqslant 0$ essentially imply that the two-surface S is suitably convex [13]. To emphasize that ''suitably convex'' is not a serious restriction, in particular it does not mean that S cannot be concave, consider a two-parameter family of observers at rest in an inertial frame in flat spacetime. Suppose that at $t=0$ they lie on a two-sphere S that is round except for a small indentation. Then $l=0$ at each point of S, and k is positive everywhere except in a small region near the center of the indentation, where it is negative. Thus there will be a circle of points C at which $k=0$. So at each point of S we have $k^2-l^2\geqslant 0$, equality holding on C. One might worry that a radial boost at a point on C will make $l^2>0$, and hence $k^2-l^2<0$. But of course this will not happen: If we consider a second set of observers, boosted relative to the first, then $k=l=0$ on C implies $k'=l'=0$ on C. So we can consider the second set of observers to be boosted radially outward in the region of the indentation, such that the indentation, and its attendant set of fixed points C, smoothly disappear as the sphere evolves in time. k switches from negative to positive by passing through the origin of a k-l diagram. Thus we can imagine a wide class of two-surfaces, including ones with indentations, and dynamically changing in time, for which $k^2-l^2\geqslant 0$ everywhere on S. Moreover, bear in mind that the observers are allowed to accelerate, so there is a great deal of freedom for them to maintain a physically reasonable S. Nevertheless, what is needed here is a careful analysis based on Raychaudhuri equations for a two-parameter family of accelerated timelike curves. Such a detailed analysis is outside the scope set for this introductory paper.

IV. THE REFERENCE INVARIANT QUASILOCAL ENERGY

As in the Brown-York case, the unreferenced IQE diverges in an asymptotically flat spacetime as the two-surface S is taken to (spatial or null) infinity, and so must be regulated with a reference term, $\mathrm{IQE}^{\mathrm{ref}}$, as already anticipated in Eq. (3.7). To better understand the nature of our definition of the invariant quasilocal energy, and to help suggest a natural choice for $\mathrm{IQE}^{\mathrm{ref}}$, we now make use of the Gauss embedding equation given in Eq. (2.14). This equation has only one independent component. Transvecting both sides with $\sigma^{ac}\sigma^{bd}$ reduces it to the scalar equation

$$\sigma\sigma R=\mathcal{R}-\tfrac{1}{2}(k^2-l^2)+(\tilde{k}^2-\tilde{l}^2), \tag{4.1}$$

where $\sigma\sigma R$ is shorthand for $\sigma^{ac}\sigma^{bd}R_{abcd}$, $(\tilde{k}^2-\tilde{l}^2)$ is shorthand for $(\tilde{k}^{ab}\tilde{k}_{ab}-\tilde{l}^{ab}\tilde{l}_{ab})$, and $\mathcal{R}$ is the scalar curvature of (S,σ). Using this equation we can express the IQE given in Eq. (3.7) in the equivalent form

$$\mathrm{IQE}=-\frac{1}{8\pi}\int_S dS\sqrt{2[\mathcal{R}-\sigma\sigma R+(\tilde{k}^2-\tilde{l}^2)]}-\mathrm{IQE}^{\mathrm{ref}}. \tag{4.2}$$

We remark here that $\sigma\sigma R$ is a natural geometrical object called the sectional curvature of (S,σ) as embedded in (M,g) [27]. It will play an important role in what follows.

Now the definition of the unreferenced IQE is rooted in the extrinsic geometry of the submanifold (S,σ), thought of as a two-surface isometrically embedded in the spacetime (M,g). It is then natural to define the reference IQE to be of the same form as the unreferenced IQE in Eq. (4.2), i.e., to be the same geometrical object, except with (S,σ) now isometrically embedded in a different spacetime—some reference spacetime $(M^{\mathrm{ref}},g^{\mathrm{ref}})$. Thus $\mathrm{IQE}^{\mathrm{ref}}$ will be the integral in Eq. (4.2) [or Eq. (3.7)], except with all quantities referred to the reference spacetime, which we indicate with a superscript ref. Note that although the extrinsic geometry of S will be different in $(M^{\mathrm{ref}},g^{\mathrm{ref}})$, its intrinsic geometry, by assumption, will not. So in the $\mathrm{IQE}^{\mathrm{ref}}$ integral we are constructing we can set $dS^{\mathrm{ref}}=dS$, $\mathcal{R}^{\mathrm{ref}}=\mathcal{R}$, and $(\sigma\sigma R)^{\mathrm{ref}}=\sigma\sigma R^{\mathrm{ref}}$. Also note that, in general, $M^{\mathrm{ref}}\neq M$ (topologically). For example, S may be a two-sphere embedded in a black hole spacetime, with $M=\mathbb{R}^2\times S^2$, whereas the reference spacetime might be Minkowski space, with $M^{\mathrm{ref}}=\mathbb{R}^4$. With this understanding, we define

$$\begin{aligned}\mathrm{IQE}^{\mathrm{ref}}&=-\frac{1}{8\pi}\int_S dS\sqrt{(k^2-l^2)^{\mathrm{ref}}}\\&=-\frac{1}{8\pi}\int_S dS\sqrt{2[\mathcal{R}-\sigma\sigma R^{\mathrm{ref}}+(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}]}.\end{aligned} \tag{4.3}$$

The term $\sigma\sigma R^{\mathrm{ref}}$ is shorthand for $\sigma^{ac}\sigma^{bd}R^{\mathrm{ref}}_{abcd}$, where R^{ref}_{abcd} is the Riemann tensor of the reference spacetime.

Typically one is motivated to choose a reference spacetime of constant curvature, the geometrical reason being that then the Gauss, Codazzi, and Ricci embedding equations make no reference to where (S,σ) is embedded in $(M^{\mathrm{ref}},g^{\mathrm{ref}})$. In other words, the conditions placed on k^{ref}_{ab}, l^{ref}_{ab}, and A^{ref}_a by the reference version of Eqs. (2.14)–(2.16)—which are just integrability conditions for the reference embedding—do not depend on knowing the embedding itself [34]. This is a pleasing criterion because it keeps the reference spacetime abstract, rather than concrete. For a four-dimensional space of constant curvature we have

[7]As noted in the text, the spirit of the quasilocal idea is to replace measurements at a point (of certain aspects of a point particle, say) with measurements on a closed spacelike two-surface. If one takes seriously that Eq. (3.4) is the generalization of point particle rest mass, then one is quickly led to speculate that a closed spacelike two-surface is the generalization of a point particle. This is curiously reminiscent of string theory, except that the one-dimensional string is replaced by a two-dimensional surface.

$$R^{\rm ref}_{abcd}=\frac{C}{12}(g^{\rm ref}_{ac}g^{\rm ref}_{bd}-g^{\rm ref}_{bc}g^{\rm ref}_{ad}), \tag{4.4}$$

where C is the constant value of its scalar curvature. For this choice of reference spacetime one gets

$$\sigma\sigma R^{\rm ref}=\frac{C}{6}. \tag{4.5}$$

For example, for Minkowski space we have $C=0$, and for anti–de Sitter space we have $C=-12/\ell^2$, where ℓ is the radius of curvature of the anti–de Sitter space, and is related to the (negative) cosmological constant Λ by $\Lambda=-3/\ell^2$. We will return to these two examples later.

The idea of embedding (S,σ) into some reference space-(time) is in the same spirit as the Brown-York approach, but an important difference that arises out of using the invariant quantity $\sqrt{k^2-l^2}$, rather than k, deserves further comment. In the Brown-York approach k is the trace of the extrinsic curvature of (S,σ) as embedded in a three-geometry (Σ,h), where Σ is a spacelike three-surface spanning S, with induced metric h_{ab}. Thus it is natural to take $k^{\rm ref}$ as the trace of the extrinsic curvature of (S,σ) as embedded in some three-dimensional reference space, $(\Sigma^{\rm ref},h^{\rm ref})$. So the embeddings of S, for both the unreferenced and reference CQE, inherently have a *three*-dimensional target space. On the other hand, $\sqrt{k^2-l^2}$ is proportional to a geometrical invariant of S, namely its mean curvature, and makes no essential reference to a spanning three-surface Σ (making the IQE truly quasilocal in the sense that it depends on S alone[8]). As a consequence, the embeddings inherently have a *four*-dimensional target space(time).

The advantage of a three-dimensional target reference space, say flat $\mathbb{R}^3$, is that when the embedding exists, it is unique (up to translations and rotations), and so the Brown-York CQE$^{\rm ref}$ is unique. The disadvantage, as is well known, is that such embeddings do not exist for all (S,σ) of interest, and this problem is not limited to just a few isolated exceptional cases.

For a four-dimensional target reference spacetime, say Minkowski space, the situation is reversed: an embedding always exists, but it is not unique. Regarding the first half of this statement, Brinkmann [35] has shown, by a simple explicit construction, that any n-dimensional conformally flat Riemann space can be considered as a particular cut of a light cone in $(n+2)$-dimensional Minkowski space. And conversely, any cut of such a light cone gives an n-dimensional conformally flat Riemann space. Now any $n=2$ space is, of course, conformally flat, and thus any (S,σ) can always be so embedded, even if S has regions with negative scalar curvature. In the Introduction I mentioned the example of the horizon of the Kerr black hole, which cannot be globally embedded into flat $\mathbb{R}^3$ when the angular momentum exceeds the irreducible mass, which coincides with the two-sphere developing regions with negative scalar curvature [16]. However, it is a simple exercise to apply Brinkmann's construction and thus globally embed the horizon into a light cone of four-dimensional Minkowski space. I will omit the details of this calculation, and just note that the embedding is valid for all angular momentum J (up to and including the extremal black hole case), and changes smoothly with J, including at the critical point when J equals the irreducible mass.

On the other hand, in a codimension-two (versus codimension-one) embedding there is more elbow room, and consequently the embedding is not unique—there is a function worth of freedom (which will be discussed in detail in Ref. [30]). This results in an ambiguity in the reference energy, IQE$^{\rm ref}$, which enters via the reference shear term $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$—see Eq. (4.3). This is the only term in IQE$^{\rm ref}$ not yet determined, and the only one for which we require an explicit reference embedding. Observe that in the Brown-York approach the undetermined quantity is $k^{\rm ref}$, an expansion. Here it is $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$, a shear. I will now argue that it is precisely this term that plays the key role in properly incorporating angular momentum into the IQE. The basic idea is simple, but first we will introduce some notation.

In the preceding section we introduced the null normals $\xi^a_\pm=u^a\pm n^a$, and corresponding null expansions $\theta_\pm=l\pm k$ in Eq. (3.8). Similarly, the (trace-free) shears in the two null directions are defined by $s_{\pm ab}:=\tilde{l}_{ab}\pm\tilde{k}_{ab}$. The curvature of the normal bundle has only one independent component, and can be written as $\mathcal{F}_{ab}=(\mathcal{F}/2)\epsilon_{ab}$ for some scalar field $\mathcal{F}$, where ϵ_{ab} is the volume form on S defined earlier in Eq. (2.4). With this notation, and assuming that the reference spacetime is one of constant curvature, i.e., Eq. (4.5) holds, the Gauss, Codazzi, and Ricci embedding equations given at the end of Sec. II take the form

$$\frac{C}{6}=\mathcal{R}+\frac{1}{2}\theta^{\rm ref}_+\theta^{\rm ref}_- - s^{{\rm ref}a}_{+\ \ b}s^{{\rm ref}b}_{-\ \ a}, \tag{4.6}$$

$$0=\tfrac{1}{2}(\mathcal{D}_a\mp A^{\rm ref}_a)\theta^{\rm ref}_\pm-(\mathcal{D}_b\mp A^{\rm ref}_b)s^{{\rm ref}b}_{\pm\ \ a}, \tag{4.7}$$

$$0=\mathcal{F}^{\rm ref}+\tfrac{1}{2}\epsilon^a{}_b[s^{\rm ref}_+,s^{\rm ref}_-]^b{}_a. \tag{4.8}$$

In the Ricci equation, $[s^{\rm ref}_+,s^{\rm ref}_-]^b{}_a$ denotes the commutator of the shears: $s^{{\rm ref}b}_{+\ \ c}s^{{\rm ref}c}_{-\ \ a}-s^{{\rm ref}b}_{-\ \ c}s^{{\rm ref}c}_{+\ \ a}$. Notice that by using null directions, rather than u^a and n^a, the Codazzi equations have decoupled into a $+$ and a $-$ set.

Our task is thus: Given σ_{ab}, and hence $\mathcal{R}$, $\mathcal{D}_a$, and ϵ_{ab}, solve these embedding equations for the unknown quantities $\theta^{\rm ref}_\pm$, $s^{\rm ref}_{\pm ab}$, and $A^{\rm ref}_a$. (Of course $\mathcal{F}^{\rm ref}=2\epsilon^{ab}\mathcal{D}_aA^{\rm ref}_b$ is not an independent quantity.) In particular, we are interested in the solution for the boost invariant reference shear term

$$(\tilde{k}^2-\tilde{l}^2)^{\rm ref}\equiv -s^{{\rm ref}a}_{+\ \ b}s^{{\rm ref}b}_{-\ \ a} \tag{4.9}$$

[8]The CQE can also be made truly quasilocal, in a slightly different sense: by relaxing the restriction that the foliation of the spacetime (i.e., Σ) be orthogonal to the boundary $\mathcal{B}$ [7], it is shown in Ref. [31] that the resulting CQE no longer depends on Σ, but instead just depends on the foliation of $\mathcal{B}$.

appearing in the Gauss equation, which is to then be substituted into the second integral of Eq. (4.3). Or equivalently, solve for $(k^2-l^2)^{\mathrm{ref}} \equiv -\theta_+^{\mathrm{ref}}\theta_-^{\mathrm{ref}}$ and substitute the answer into the first integral of Eq. (4.3). This is how IQE$^{\mathrm{ref}}$ is determined.

However, as already noted, any solution we obtain is not unique. We can see this immediately by counting functional degrees of freedom. $\theta_\pm^{\mathrm{ref}}$ are two functions, $s_{\pm ab}^{\mathrm{ref}}$ are four (the two shears are symmetric and trace-free), and A_a^{ref} are two. These eight functions are subject to six equations: Gauss is one, Codazzi are four, and Ricci is one. This leaves two arbitrary functions in the solution. But owing to the invariance of the embedding equations under a local boost transformation [see Eqs. (2.13)], one of these functions is just the boost parameter, λ, leaving one nontrivial arbitrary function in the solution.

The question then arises, Is there a natural way to impose one additional functional condition on the unknowns so that the embedding, subject to this additional condition, is unique, and hence IQE$^{\mathrm{ref}}$ is unique? One of the central ideas in this paper is to impose the additional condition

$$\mathcal{F}^{\mathrm{ref}} = \mathcal{F}, \tag{4.10}$$

i.e., the curvature of the normal bundle of S as embedded in the reference spacetime $(M^{\mathrm{ref}},g^{\mathrm{ref}})$ should equal that of S as embedded in the original physical spacetime (M,g).

There are several reasons why it is geometrically natural to demand $\mathcal{F}^{\mathrm{ref}}=\mathcal{F}$. First, the two-surface S has two connections: one is an SO(2) connection on the tangent bundle of S, associated with the curvature $\mathcal{R}$, and the other is an SO(1,1) connection on the normal bundle of S, associated with the curvature $\mathcal{F}$. In fact both of these connections are metric connections, associated with the metrics in the tangent and normal bundles to S, respectively [27]. Furthermore, Szabados [26] has considered the two-dimensional version of the Sen connection for spinors and tensors on a submanifold such as S, and has found that the two-surface spinor curvature has, essentially, imaginary part equal to $\mathcal{R}$, and real part equal to $\mathcal{F}$. Finally, although A_a is a measure of extrinsic geometry, as pointed out earlier it is not really on the same footing as the extrinsic curvatures k_{ab} and l_{ab}, since its transformation law under local radial boosts is qualitatively different—see Eqs. (2.13). It transforms like the connection that it is, and gives rise to a curvature, and so arguably has more in common with $\mathcal{R}$ than with k_{ab} and l_{ab}. The point is, $\mathcal{F}$ is really on the same geometrical footing as $\mathcal{R}$. We have already demanded that $\mathcal{R}^{\mathrm{ref}}=\mathcal{R}$, as a necessary condition for the embedding of (S,σ) into $(M^{\mathrm{ref}},g^{\mathrm{ref}})$ to be isometric. So demanding also that $\mathcal{F}^{\mathrm{ref}}=\mathcal{F}$ is thus seen to be quite natural.

Unfortunately, implementing Eq. (4.10) seems like an intractable task. Embedding equations involving curvature of the normal bundle, i.e., codimension-two (and higher) embeddings, have, of course, been studied for a long time. With regard to solutions, although one expects to be able to express $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ in terms of $\mathcal{F}$, $\mathcal{R}$, and their derivatives, I am not aware of any such general results in the literature. In fact, much of the literature on such embeddings considers the case $\mathcal{F}=0$, which is not the case we are particularly interested in here (a notable exception is Ref. [27]). One possible way to proceed is as follows. Given $\mathcal{F}^{\mathrm{ref}}$ [$=\mathcal{F}$ by Eq. (4.10)], choose A_a^{ref} such that $\mathcal{F}^{\mathrm{ref}}=2\epsilon^{ab}\mathcal{D}_a A_b^{\mathrm{ref}}$. There may be a convenient gauge choice, such as $\mathcal{D}\cdot A^{\mathrm{ref}}=0$, or $l^{\mathrm{ref}}=0$. Then view the Codazzi equations (4.7) as a set of four linear partial differential equations for the four independent degrees of freedom in the two shears. These equations are of second order if one makes use of the fact that any trace-free symmetric tensor s_{ab} on a two-surface (S,σ) with two-sphere topology can be expressed as $s_{ab}=\mathcal{D}_a v_b+\mathcal{D}_b v_a-\sigma_{ab}\mathcal{D}\cdot v$ for some vector field v^a. Thus solve for $s_{\pm ab}^{\mathrm{ref}}$ in terms of the expansions, $\theta_\pm^{\mathrm{ref}}$, and their derivatives. The expressions one obtains at this stage are, in general, nonlocal. Then substitute these into the Gauss and Ricci equations (4.6) and (4.8), which are really just nonlinear algebraic constraints. But because the shears involve nonlocal operators acting on the expansions, one ends up with two nonlocal and nonlinear partial differential equations for the two expansions. Remarkably, it is almost possible to solve these equations, but in the end one encounters a certain combination of nonlocality and nonlinearity that makes the final step to a solution seem impossible. Nevertheless, it appears that the solution for $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$, if it can be found, almost certainly depends in a simple way on both $\mathcal{R}$ and $\mathcal{F}$, and derivatives (of a finite or possibly infinite order) of these two curvatures. I am suggesting that it is through this subtle presence of $\mathcal{F}$ in $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ that angular momentum is properly incorporated into the IQE.

So although a direct attack on the embedding equations has not yet yielded a solution, fortunately one can make some progress of a general nature by calculating the first and second order variations of $\mathcal{F}^{\mathrm{ref}}$ and $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ under isometric deformations of a given embedding. The idea is to see how both of these quantities change under such a deformation, and thereby infer how $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ depends on $\mathcal{F}^{\mathrm{ref}}$, and hence angular momentum. The results are somewhat involved, and will be given elsewhere [30]. For now let us start by making some simple observations regarding the enigmatic object $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$.

To begin with, one might object to our argument thus far because it implies that $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$, and hence IQE$^{\mathrm{ref}}$, depends on the extrinsic geometry of S as embedding in the physical spacetime. In particular, through Eq. (4.10) and the reference embedding equations, $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ depends on $\mathcal{F}$. On the other hand, it is often stated that a reference subtraction term should be a functional of only the intrinsic geometry of (S,σ). However, notice that there is no dependence on the extrinsic curvatures proper, i.e., l_{ab} and k_{ab}, only a dependence on $\mathcal{F}$, a quantity which I argued above is really on the same geometrical footing as the intrinsic quantity $\mathcal{R}$. Moreover, as discussed in the Introduction [refer to Eq. (1.5)], in the Brown-York approach one is free to add to the action any functional of the boundary *three*-metric, γ_{ab}, which contains information about the two-metric σ_{ab}, as well as information about how (S,σ) is embedded in the three-boundary $\mathcal{B}$. For instance, one could add to the action a boundary integral of the scalar curvature of $\mathcal{B}$, whose variation would add to Π^{ab} in Eq. (1.5) a term proportional to the

Einstein tensor of γ_{ab}, as is done in Ref. [36]. Such a term obviously depends on some extrinsic geometry of (S,σ). In the work of Brown and York this fact is of course recognized, but being in a Hamiltonian framework, they restrict the form of the arbitrary boundary functional such that the energy surface density $[-k/(8\pi)]$ and momentum surface density $[-A_a/(8\pi)]$ of S in a particular spacelike hypersurface Σ depend only on the canonical data on Σ. This effectively means that their reference subtraction term can depend only on σ_{ab} [7]. But as I emphasized earlier, our approach is based on the invariant object $\sqrt{k^2-l^2}$, and makes no essential reference to a three-surface Σ spanning S. The invariant quasilocal energy constructed here does not come out of a canonical analysis, so there is no reason that our subtraction term cannot depend on $\mathcal{F}$.

So the shear term $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ is allowed, but is it really necessary? Perhaps it is just an unsavory term resulting from a poor definition of the IQE. For instance, looking at the Gauss embedding equation (4.1) one might be tempted to write, instead of Eq. (3.4),

$$E^2-\vec{p}^2 \overset{?}{\rightarrow} \frac{1}{(8\pi)^2}[(k^2-l^2)-2({}^2\tilde{k}^2-\tilde{l}^2)], \qquad (4.11)$$

where the additional shear term on the right-hand side is perhaps the proper way to include angular momentum, somewhat like the A^aA_a term we attempted in Eq. (3.5). This would have the advantage of changing $\mathrm{IQE}^{\mathrm{ref}}$ in Eq. (4.3) to

$$\mathrm{IQE}^{\mathrm{ref}} \overset{?}{=} -\frac{1}{8\pi}\int_S dS\sqrt{2[\mathcal{R}-\sigma\sigma R^{\mathrm{ref}}]}, \qquad (4.12)$$

which is clearly unique, and moreover, requires no explicit reference embedding (no equations need to be solved). Equation (4.12) is a more general case of the zero point energy suggested by Lau [37] (except that his derivation of it requires an explicit reference embedding—we will return to this point later). But unfortunately it cannot be correct. For example, when the reference spacetime is Minkowski space, Eq. (4.5) tells us that $\sigma\sigma R^{\mathrm{ref}}=0$ and thus the radical reduces to $\sqrt{2\mathcal{R}}$, which is not defined for negative $\mathcal{R}$. Nor is this problem properly solved by taking $C\neq 0$ in Eq. (4.5), since this would put an *ad hoc* fixed lower bound on $\mathcal{R}$.

So the shear term $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ is not only allowed, it is necessary (or at least its absence leads to an unsatisfactory result). In fact its role seems to be *to keep non-negative what is under the square root* in Eq. (4.3). To see this more clearly, consider the special case that (S,σ) is embeddable in flat $\mathbb{R}^3$. If our reference spacetime is Minkowski space, we can then choose to embed (S,σ) in a $t=$constant slice and, within this slice, the embedding is essentially unique. In this case it is easy to see that we will have $l_{ab}^{\mathrm{ref}}=0$. And by assumption, $\sigma\sigma R^{\mathrm{ref}}=0$, so Eq. (4.3) reduces to

$$\mathrm{IQE}^{\mathrm{ref}}|_{l_{ab}^{\mathrm{ref}}=0} = -\frac{1}{8\pi}\int_S dS|k^{\mathrm{ref}}|$$
$$= -\frac{1}{8\pi}\int_S dS\sqrt{2[\mathcal{R}+(\tilde{k}^{\mathrm{ref}})^2]}. \qquad (4.13)$$

The uniqueness of the embedding means that k^{ref} and $\tilde{k}_{ab}^{\mathrm{ref}}$ are unique. Now clearly, no matter what the surface is, the spatial shear $\tilde{k}_{ab}^{\mathrm{ref}}$ must be such that what is under the square root in Eq. (4.13) is non-negative, because $|k^{\mathrm{ref}}|$ is real. [More properly, one should look at the ''reference version'' of Eq. (4.1), with $(l^{\mathrm{ref}})^2=(\tilde{l}^{\mathrm{ref}})^2=\sigma\sigma R^{\mathrm{ref}}=0$.] For example, consider a dumb-bell–shaped surface of revolution in flat $\mathbb{R}^3$. In a region near the throat of this surface $\mathcal{R}$ is negative, nevertheless at every point of the surface we have $\mathcal{R}+(\tilde{k}^{\mathrm{ref}})^2\geqslant 0$.

I emphasize that, even when (S,σ) can be embedded in flat $\mathbb{R}^3$, its embedding in Minkowski space need not be chosen to be in a $t=$const slice, as was done in the previous paragraph. One may also embed it in a light cone, or in a host of other ways—remember that there is a function-worth of freedom in our choice. I argued in the context of Eq. (4.10) that this freedom has to do with angular momentum, or more precisely, the curvature of the normal bundle. For the $t=$const embedding, $\tilde{l}_{ab}^{\mathrm{ref}}=0$ implies $s_{\pm ab}^{\mathrm{ref}}=\pm\tilde{k}_{ab}^{\mathrm{ref}}$, and so inspection of Eq. (4.8) reveals that in this case $\mathcal{F}^{\mathrm{ref}}=0$. However, it is not hard to see that starting with such a $t=$const embedding one can perform an infinitesimal isometric deformation of the embedding out of the $t=$const plane, i.e., in a direction $\varphi\partial/\partial t$, where φ is an arbitrary function. Furthermore, I show in Ref. [30] that after such an infinitesimal deformation $\mathcal{F}^{\mathrm{ref}}$ is no longer zero, and can in fact be made to be essentially any infinitesimal function we like by a suitable choice of φ. It is not hard to imagine (just hard to do) that by integrating such isometric deformations one may be able to achieve a two-geometry (S,σ) isometrically embedded with any desired curvature of the normal bundle. With this in mind, it would seem unnatural to reference-embed, e.g., a constant *r,t* two-sphere of the Kerr geometry (which is easily shown to have $\mathcal{F}\neq 0$) as a two-sphere in a Minkowski reference spacetime with $\mathcal{F}^{\mathrm{ref}}=0$ (say, in a $t=$const slice), when it seems possible to instead embed it with $\mathcal{F}^{\mathrm{ref}}=\mathcal{F}$. Note, however, that as the embedding is deformed out of the $t=$const surface, $\tilde{l}_{ab}^{\mathrm{ref}}$ will also cease to be zero, and so will introduce a negative contribution to the quantity under the square root in Eq. (4.3). This jeopardizes the non-negativity of this quantity. But at the same time we clearly cannot simply throw away the $-(\tilde{l}^{\mathrm{ref}})^2$ term, since this would violate a key property of the IQE, namely its invariance under local boosts. Short of solving the embedding equations for a generic $\mathcal{F}^{\mathrm{ref}}$ and explicitly checking, I do not know of any guarantee of non-negativity. Equivalently, in question here is the non-negativity of the quantity $(k^2-l^2)^{\mathrm{ref}}$, which is the same type of question as the non-negativity of (k^2-l^2) discussed at the end of Sec. III. I do not at present have a complete answer to either of these difficult questions.

Finally, one might guess that it is possible to avoid the embedding problem entirely by simply setting $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}=(\tilde{k}^2-\tilde{l}^2)$, which is in the same spirit as Eq. (4.10) in that, like $\mathcal{F}$, the shear term has something to do with angular momentum. But this does not work. For example, it is not hard to show that, although $(\tilde{k}^2-\tilde{l}^2)$ is in general nonvanishing on constant r,t spheres of the Kerr geometry, it happens to vanish on the horizon. So if we set $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}=(\tilde{k}^2-\tilde{l}^2)$, then in calculating IQE$^{\rm ref}$ for the Kerr horizon example we would run into the same problem with negative $\mathcal{R}$ as we did in Eq. (4.12). So such a prescription must not be valid, and we cannot avoid the embedding problem this way.

Let us conclude this section by addressing the following question: What is the relationship between the IQE defined here and the Brown-York CQE? We begin by supposing that the following four conditions are satisfied: (i) (S,σ) is a two-surface in the physical spacetime such that $k^2-l^2>0$; (ii) $k>0$; (iii) (S,α) is such that it can be embedded in flat $\mathbb{R}^3$; and (iv) for the embedding in (iii), $k^{\rm ref}\geqslant 0$. As discussed earlier, condition (i) ensures that the unreferenced IQE is well defined—roughly speaking, S is not inside a black hole. Then it is always possible to go to a quasilocal rest frame where $l=0$ on S, and the integrand in Eq. (3.7) is just $|k|$. Given condition (ii), the unreferenced IQE thus reduces to the unreferenced CQE in Eq. (1.1), provided the observers in the Brown-York case are in a quasilocal rest frame. Condition (iii) ensures that the Brown-York prescription is well defined, and allows us to choose a $t=$const embedding in Minkowski space, as above, and get Eq. (4.13). The first integral in this equation, together with condition (iv), shows that our IQE$^{\rm ref}$ reduces to the Brown-York CQE$^{\rm ref}$ in Eq. (1.2). So if these conditions hold, and we choose to use a $t=$const embedding to calculate IQE$^{\rm ref}$, then our invariant quasilocal energy is the same as the Brown-York rest energy. In most applications considered in the literature these conditions are satisfied, and the IQE will then share all of the desirable properties of the CQE. For example, it will be the thermodynamic energy that appears in the first law of black hole thermodynamics for Schwarzschild black holes, as considered in Ref. [7].

On the other hand, I emphasize that conditions (ii) and (iv) are easily violated. One need only think of a round sphere with a small indentation, an example discussed at the end of Sec. III (except here the embedding spacetime is generic). So in general, the IQE defined here is not simply the Brown-York rest energy. Furthermore, we need not choose a $t=$const embedding to calculate IQE$^{\rm ref}$. Indeed, as I have argued, such a choice is unnatural when $\mathcal{F}\neq 0$. In short, the invariant quasilocal energy defined here is not quite the same object as the Brown–York quasilocal energy. Note that the aforementioned thermodynamic nature of the CQE is derived in Ref. [7] assuming that S is a round sphere. It would be interesting to extend this analysis to indented spheres, for instance, and determine which, if either of the CQE or IQE, is the correct thermodynamic energy.

V. THE LARGE SPHERE LIMIT OF THE IQE

Let us now assume that the spacetime (M,g) is asymptotically flat, and evaluate the limit of the invariant quasilocal energy as (S,σ) tends to a large sphere at infinity. At spatial and null infinity we might expect these limits to be the ADM and Bondi-Sachs masses, respectively. Let us see if this is so.

Let τ be a time function on M such that $\tau=\tau_*$ defines a spacelike (respectively, null) hypersurface $\mathcal{H}_{\tau_*}$ of topology $\mathbb{R}\times S^2$ extending to spatial (respectively, null) infinity. Letting the parameter τ_* vary over some range gives a foliation of a part of M. Let r be a function on M such that $r=r_*$ defines a hypersurface that intersects each leaf $\mathcal{H}_{*\tau}$ (over the allowed range of τ_*) in a spacelike two-sphere, S_{τ_*,r_*}. The parameter r_* ranges to infinity, and over its range the surfaces S_{τ_*,r_*} provide a foliation of $\mathcal{H}_{\tau_*}$. We are interested in the limit $r_*\to\infty$, with τ_* arbitrary but fixed. In a rather benign abuse of notation we will refer to S_{τ_*,r_*} as simply S, and take the limit as $r\to\infty$ with τ fixed. The metric induced on S will, as usual, be denoted as σ_{ab} (in abstract index notation).

Now assume that the functions τ and r have been chosen such that (S,σ) tends to a round sphere at infinity. Thus the components of its metric in spherical coordinates $x^i=(\theta,\phi)$ have an asymptotic expansion of the form

$$\sigma_{ij}=r^2\begin{pmatrix}1 & 0\\ 0 & \sin^2\theta\end{pmatrix}+2r\begin{pmatrix}X & Y\sin\theta\\ Y\sin\theta & Z\sin^2\theta\end{pmatrix}+O_<(r). \tag{5.1}$$

In this expansion, X, Y, and Z are each arbitrary functions of τ, θ, and ϕ. The symbol $O_<(r^{-n})$ denotes a term that falls off *faster* (or grows more slowly, depending on the sign of n) than r^{-n}, but not necessarily according to a power of r. For example, rather than $O(1)$, the remainder term $O_<(r)$ in Eq. (5.1) might grow as $\ln r$. The motivation for this increased generality will be explained below when we consider the large sphere limit at null infinity. Furthermore, we can choose (the function) r to be an areal radius, in which case we may take $\sqrt{\sigma}=r^2\sin\theta$, where $\sigma=\det\sigma_{ij}$. It is easy to see that this requires $Z=-X$ in Eq. (5.1).

The scalar curvature of a round sphere of areal radius r is $2/r^2$. Since the metric in Eq. (5.1) differs from that of a round sphere by a term one power lower in r, we immediately have that its scalar curvature $\mathcal{R}$ is given by

$$\mathcal{R}=\frac{2}{r^2}+\frac{\Delta_{\mathcal{R}}}{r^3}, \tag{5.2}$$

where the remainder term $\Delta_{\mathcal{R}}$ is of order one.[9] In our asymptotically flat spacetime the components of the Riemann tensor fall off as $1/r^3$, and so the same will be true of $\sigma\sigma R$, the sectional curvature of (S,σ). In the present context the ap-

[9] In Ref. [38] it is shown that $\Delta_{\mathcal{R}}=\mathcal{D}\cdot v+O_<(1)$ for some vector field v in S. In other words, to leading order $\Delta_{\mathcal{R}}$ is a divergence. This plays a crucial role in some of the results in Ref. [38]. However, we will not need to use this fact, except to provide some insight into our discussion of the 'solution' of the Ricci embedding equation in Sec. V B.

propriate reference spacetime is Minkowski space, and so $\sigma\sigma R^{\rm ref}=0$. The only other terms to consider in Eqs. (4.2) and (4.3) are the shear terms, $(\tilde{k}^2-\tilde{l}^2)$ and $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$. We will see below that these, too, fall off at least as fast as $1/r^3$ in both the spatial and null infinity limits. In the large sphere limit the unreferenced IQE thus behaves as

$$\begin{aligned}\mathrm{IQE}^{\rm unref}&=-\frac{1}{8\pi}\int_S dS\sqrt{2\left[\frac{2}{r^2}+\frac{\Delta_{\mathcal{R}}}{r^3}-\sigma\sigma R+(\tilde{k}^2-\tilde{l}^2)\right]}\\&=-\frac{1}{8\pi}\int_S dS\,\frac{2}{r}\left\{1+\frac{r^2}{4}\left[\frac{\Delta_{\mathcal{R}}}{r^3}-\sigma\sigma R\right.\right.\\&\qquad\left.\left.+(\tilde{k}^2-\tilde{l}^2)\right]+O(r^{-2})\right\}. \qquad (5.3)\end{aligned}$$

The reference IQE behaves similarly, except we have $\sigma\sigma R^{\rm ref}=0$. Thus

$$\begin{aligned}\mathrm{IQE}^{\rm ref}=&-\frac{1}{8\pi}\int_S dS\,\frac{2}{r}\left\{1+\frac{r^2}{4}\left[\frac{\Delta_{\mathcal{R}}}{r^3}-0+(\tilde{k}^2-\tilde{l}^2)^{\rm ref}\right]\right.\\&\left.+O(r^{-2})\right\}. \qquad (5.4)\end{aligned}$$

In forming the difference of the previous two expressions it is important to observe that not only do the divergent terms coming from the $2/r^2$ piece of $\mathcal{R}$ cancel, but also the (finite) remainder terms $\Delta_{\mathcal{R}}$ are the same in both, and thus also cancel, independent of what $\Delta_{\mathcal{R}}$ is. Thus we find that the large sphere behavior of the (referenced) IQE is given by

$$\begin{aligned}\mathrm{IQE}=&\frac{1}{16\pi}\int_S dS\, r[\sigma\sigma R-(\tilde{k}^2-\tilde{l}^2)+(\tilde{k}^2-\tilde{l}^2)^{\rm ref}\\&+O(r^{-4})]. \qquad (5.5)\end{aligned}$$

Of course if the shear terms fall off as $1/r^4$ or faster they get absorbed into the $O(r^{-4})$ remainder term. It is worth emphasizing that in the large sphere limit the square root in the IQE is eliminated by the fact that $\mathcal{R}$ dominates over the other terms. The areal radius factor r outside the brackets in Eq. (5.5) is really $\sqrt{2/\mathcal{R}}$.[10] In Sec. VI we will see a similar mechanism at work in the small sphere limit. But in the intermediate regime the IQE is, in general, an integral of the difference of two radicals.

As a quick check of Eq. (5.5) let us evaluate the right-hand side for the Schwarzschild geometry. In the usual Schwarzschild coordinates r is an areal radius, and it is a simple exercise to compute the sectional curvature of a t, r $=$const two-sphere. The result is $\sigma\sigma R=4M/r^3$. The shear terms obviously vanish, and with $dS=r^2\,d\Omega$ ($d\Omega$ the measure on the unit round sphere) one immediately gets IQE $=M$, the ADM mass of the black hole. Now the main task is to investigate in detail the shear terms in Eq. (5.5), which we will do separately for the spatial and null infinity limits, respectively.

[10]This mechanism works even when the sphere is not asymptotically round. In this case the shear terms contribute at a higher order, viz. $1/r^2$, in an effort to keep what is under the square root sign positive, as discussed earlier. In other words, the factor $\sqrt{2/\mathcal{R}}$ is modified in such a way that negative $\mathcal{R}$ is likely not a problem. We will not consider this more complicated case here.

A. The spatial infinity limit

Rather than proceed with complete generality, it is more instructive to consider an asymptotically flat metric that exhibits angular momentum explicitly, and then see how this angular momentum works its way into the shear terms. (We will be completely general in the more interesting null infinity limit case.) The spacetime far from any isolated stationary (nonradiating) rotating source is described asymptotically by the Kerr metric (see Secs. 19.3 and 33.3 of Ref. [29]), so let us take (M,g) to be the Kerr spacetime. We choose the following basis of orthonormal one-forms:

$$\begin{aligned}&e^0=N\,dt,\quad e^1=\frac{\rho}{\sqrt{\Delta}}dr,\quad e^2=\rho\, d\theta,\\&e^3=\sqrt{g_{\phi\phi}}(d\phi-\omega\, dt), \qquad (5.6)\end{aligned}$$

which are associated with locally nonrotating observers. The corresponding basis of orthonormal vector fields is

$$\begin{aligned}&e_0=\frac{1}{N}(\partial_t+\omega\partial_\phi),\quad e_1=\frac{\sqrt{\Delta}}{\rho}\partial_r,\quad e_2=\frac{1}{\rho}\partial_\theta,\\&e_3=\frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi. \qquad (5.7)\end{aligned}$$

The notation used is standard: $x^a=(t,r,\theta,\phi)$ are Boyer-Lindquist coordinates, $\omega(r,\theta)=-g_{t\phi}/g_{\phi\phi}$ is an observer's angular velocity as measured from infinity, $N=\sqrt{\omega^2 g_{\phi\phi}-g_{tt}}$ is the lapse function, etc. (see, e.g., Sec. 33.4 of Ref. [29]). Let $A,B,\ldots$ be indices labeling the basis vectors and one-forms, ranging from 0 to 3, and $I,J,\ldots$ denote the subset of these taking values 2 and 3. These indices are raised and lowered with the flat Lorentz metric $\eta_{AB}=\eta^{AB}=\mathrm{diagonal}(-1,1,1,1)$.

The vector fields $e_I{}^a$ are tangent to any r, $t=$constant two-sphere S, and so $u^a:=e_0{}^a$ and $n^a:=e_1{}^a$ are, respectively, timelike and spacelike unit vectors orthogonal to S. From Eqs. (2.8) we see that the orthonormal basis components of the extrinsic curvatures l_{ab} and k_{ab} are given by

$$l_{IJ}=e_I{}^a e_J{}^b\nabla_a u_b=-\omega_{0JI}, \qquad (5.8)$$

$$k_{IJ}=e_I{}^a e_J{}^b\nabla_a n_b=-\omega_{1JI}. \qquad (5.9)$$

Here $\omega_{CBA}=-e_A{}^a e_B{}^b\nabla_a e_{Cb}$ are Ricci rotation coefficients.

Working out these coefficients[11] I find that the trace-free parts of l_{IJ} and k_{IJ} are given by

$$\tilde{l}_{IJ}=\alpha\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \text{where} \quad \alpha=\frac{\sqrt{g_{\phi\phi}}}{2N\rho}\,\partial_\theta\omega, \tag{5.10}$$

$$\tilde{k}_{IJ}=\beta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \text{where} \quad \beta=\frac{\sqrt{\Delta}}{2\rho}\,\partial_r \ln\frac{\rho}{\sqrt{g_{\phi\phi}}}. \tag{5.11}$$

Geometrically, the coefficient α is just $e_1\cdot\Omega^{(\text{precess})}$, i.e., the radial component of the angular velocity vector $\Omega^{(\text{precess})}$ that measures the precession of a gyroscope carried by a locally nonrotating observer, relative to the observer's orthonormal frame [see Eq. (33.24) of Ref. [29]; compare also with Eq. (3.2) above, and the discussion following it]. Thus, the unreferenced shear term in Eq. (5.5) is given by

$$(\tilde{k}^2-\tilde{l}^2)=2(\beta^2-\alpha^2)=\frac{a^4}{2r^6}\sin^4\theta+O\left(\frac{1}{r^7}\right), \tag{5.12}$$

where the last expression on the right-hand side is the large r asymptotic expansion. So clearly, being of order $1/r^6$, $(\tilde{k}^2-\tilde{l}^2)$ does not contribute to the large sphere limit of the IQE at spatial infinity.

What about the reference term $(\tilde{k}^2-\tilde{l}^2)^{\text{ref}}$? It is plausible that the reference term is of the same order in $1/r$ as the unreferenced term, viz. $1/r^6$, or less, and so also does not contribute. However, to be certain one needs to solve the embedding equations (4.6)–(4.8), subject to the condition $\mathcal{F}^{\text{ref}}=\mathcal{F}$, as argued in Sec. IV. We will not attempt to do so here, but it is instructive to at least work out what $\mathcal{F}$ is for the Kerr geometry. From Eq. (2.11) we see that the orthonormal basis components of the connection in the normal bundle are given by

$$A_I=e_I{}^b n^c\nabla_b u_c=-\omega_{01I}. \tag{5.13}$$

Evaluating these Ricci rotation coefficients reveals that the one-form $A=A_Ie^I$ (pulled back to the two-sphere S) is given by

$$A=\gamma d\phi,$$

where

$$\gamma=\frac{g_{\phi\phi}\sqrt{\Delta}}{2N\rho}\,\partial_r\omega=-\frac{3aM}{r^2}\sin^2\theta+O\left(\frac{1}{r^4}\right). \tag{5.14}$$

Recall that $\omega=\omega(r,\theta)$ is a measure of the frame dragging produced by the rotating geometry. While the shear in the time direction measures the θ dependence of ω [see α in Eq. (5.10)], Eq. (5.14) shows that the connection in the normal bundle measures its r dependence. Both are measures of angular momentum.

Calculating the exterior derivative of A leads to the curvature in the normal bundle:

$$\mathcal{F}=\frac{1}{\sqrt{g_{\phi\phi}}\rho}\,\partial_\theta\left(\frac{g_{\phi\phi}\sqrt{\Delta}}{2N\rho}\,\partial_r\omega\right)=-\frac{6aM}{r^4}\cos\theta+O\left(\frac{1}{r^6}\right), \tag{5.15}$$

which is of order $1/r^4$. Inspection of the Ricci equation (4.8), or Eq. (2.16) with the left-hand side set to zero, reveals that a solution to the reference embedding equations, subject to Eq. (4.10), requires that $\tilde{l}_{IJ}^{\text{ref}}$ and $\tilde{k}_{IJ}^{\text{ref}}$ be two matrices whose commutator is of order $1/r^4$. On the other hand, one might guess that the trace of the difference of their squares, $(\tilde{k}^2-\tilde{l}^2)^{\text{ref}}$, might be of order $1/r^6$, as suggested above. It is not difficult to convince oneself that these two conditions are not incompatible, so the reference embedding equations at least do not obviously forbid the reference shear term $(\tilde{k}^2-\tilde{l}^2)^{\text{ref}}$ from being of the same order of magnitude as $(\tilde{k}^2-\tilde{l}^2)$, such that neither contributes to the IQE.

In any case, assuming just that $(\tilde{k}^2-\tilde{l}^2)^{\text{ref}}$ is at most $O_<(r^{-3})$, which is almost certainly true, we find that the large sphere limit of the IQE at spatial infinity is given by

$$\lim_{r\to\infty}\text{IQE}=\lim_{r\to\infty}\frac{1}{16\pi}\int_S dS\, r\,\sigma\sigma R. \tag{5.16}$$

This limit of the IQE thus has a simple geometrical interpretation: apart from a factor proportional to the areal radius r, it is just the average over S of the sectional curvature of (S,σ) as embedded in the physical spacetime (M,g).

Now let us assume that (M,g) is vacuum $(R_{ab}=0)$ near spatial infinity, so that there the Riemann tensor reduces to the Weyl tensor, C_{abcd}. From the definition of the two-surface metric given in Eq. (2.3), and the fact that the Weyl tensor is traceless, one immediately gets

$$\sigma\sigma R=2E_{ab}n^a n^b. \tag{5.17}$$

Thus the sectional curvature of (S,σ) is just (twice) the radial-radial (Coulomb) component of the electric part of the Weyl tensor, $E_{ab}:=-C_{acbd}u^c u^d$. Inserting this result into Eq. (5.16) we see that in this limit the IQE is precisely the coordinate-independent expression of the ADM mass given by Ashtekar and Hansen [39].[12]

One more remark is in order here: Hayward's work on quasilocal energy [8] resembles what is done here in the sense that the Gauss embedding equation plays a central role, and that the analysis is boost invariant in spirit, i.e., no reference is made to a spacelike three-surface spanning S, with

[11]The easiest way to do this is to recognize that, with $Z=0$ or 1, $\omega_{ZJI}=\alpha_{(IJ)Z}$, where $\alpha^C{}_{AB}=i_{e_B}i_{e_A}de^C$. Thus we need only compute the exterior derivative of e^I.

[12]In Ref. [39] a different definition of E_{ab} is used, namely $E_{ab}=C_{acbd}n^c n^d$, but accounting for this difference in notation the two results agree.

its attendant preferred time direction on S, and so on. However, Hayward's quasilocal energy is distinct from the IQE here: it does not involve a square root. Basically, Hayward starts with an integral over S of the $2+2$ Hamiltonian density, which yields a dimensionless quantity, and then multiplies this quantity by the areal radius of S to correct this defect, i.e., give it the dimensions of energy. This is in the same spirit as the areal radius factor appearing in the Hawking mass [25]. For the large sphere limit at spatial infinity Hayward arrives at the same result given in Eq. (5.16), except with r *outside* the integral, so to speak. His quasilocal energy has the very appealing feature of not requiring a reference subtraction term, at least when the sectional curvature $\sigma\sigma R$ falls off as $1/r^3$ [however it diverges if, e.g., the spacetime is asymptotically anti–de Sitter space—recall Eq. (4.5)]. In our case, the square root in Eq. (4.2) ensures that the IQE has the dimensions of energy, but the price paid is that a reference subtraction term is needed. [Without the square root the large sphere limit of the unreferenced IQE would just be (negative) the Euler number of S, which is finite, but carries no information about energy.] The areal radius factor r in Eq. (5.16) appears inside the integral: as mentioned before, it arises from the dominant scalar curvature term $\mathcal{R}$, and is really $\sqrt{2/\mathcal{R}}$. I emphasize that this is a geometrically natural mechanism—r is not put in by hand. Finally, while one might feel that there is something unattractively ad hoc about a reference subtraction term, the flexibility it affords makes it possible to deal with the wide range of boundary conditions possible in general relativity. In Sec. VII we will consider an interesting example of this.

B. The null infinity limit

We now suppose that the physical spacetime (M,g) is asymptotically flat at future null infinity. As in the preceding subsection we begin with the generic large sphere form of the IQE given in Eq. (5.5), except now we take the large S limit in the future null direction. More precisely, on M we introduce the Bondi coordinates [40] $x^a=(w,r,\theta,\phi)$, and as before, denote the subset $(\theta,\ \phi)$ of spherical coordinates by x^i. The retarded time w labels a one-parameter family of outgoing null hypersurfaces, and r is an areal radius (luminosity parameter) along the outgoing null geodesic generators of these hypersurfaces. The $w,r=$constant surfaces are topologically two-spheres, any one of which we denote as S. This setup is the same as discussed at the beginning of Sec. V, where w here is what we there called the time coordinate, τ. We are interested in the one-parameter family of two-spheres S in the limit as $r\rightarrow\infty$, with w arbitrary but fixed.

In the Bondi coordinates our asymptotically flat metric takes the standard form [41,42]

$$g_{ab}\,dx^a\,dx^b=-UV\,dw^2-2U\,dw\,dr+\sigma_{ij}(dx^i+W^i\,dw)\times(dx^j+W^j\,dw). \tag{5.18}$$

We assume the following expansions for the various terms in this metric:[13]

$$V=1-2mr^{-1}+O_<(r^{-1}), \tag{5.19}$$

$$U=1-\tfrac{1}{2}\,(X^2+Y^2)r^{-2}+O_<(r^{-2}), \tag{5.20}$$

$$W^\theta=(2X\cot\theta+\partial_\theta X+\csc\theta\partial_\phi Y)r^{-2}+O_<(r^{-2}),$$

$$W^\phi=\csc\theta(2Y\cot\theta+\partial_\theta Y-\csc\theta\partial_\phi X)r^{-2}+O_<(r^{-2}), \tag{5.21}$$

$$\sigma_{ij}=r^2\begin{pmatrix}1 & 0\\ 0 & \sin^2\theta\end{pmatrix}+2r\begin{pmatrix}X & Y\sin\theta\\ Y\sin\theta & -X\sin^2\theta\end{pmatrix}+O_<(r). \tag{5.22}$$

The function V contains the mass aspect, $m(w,\theta,\phi)$. Observe that the metric on S is of the same form given in Eq. (5.1) (with $Z=-X$ because r is an areal radius here), except now $X(w,\theta,\phi)$ and $Y(w,\theta,\phi)$ have a significant physical interpretation: they are the real and imaginary parts of Sachs' complex asymptotic shear $c=X+iY$ [41]. Thus the scalar curvature of (S,σ) will be given by Eq. (5.2), and we can begin our discussion of the IQE at Eq. (5.5). Our first task is to compute the unreferenced shear term $(\tilde{k}^2-\tilde{l}^2)$.

Inspecting the metric in Eq. (5.18), we choose the following basis of one-forms:

$$e^-=\tfrac{1}{2}\,U\,dw,\quad e^+=dr+\tfrac{1}{2}\,V\,dw,\quad e^I=\gamma^I{}_i(dx^i+W^i\,dw), \tag{5.23}$$

where indices $I,J,\ldots$ take the values 2 and 3, and $\gamma^I{}_i$ is defined by demanding that $\sigma_{ij}=\delta_{IJ}\gamma^I{}_i\gamma^J{}_j$. A suitable choice for $\gamma^I{}_i$ is given by

$$\gamma^I{}_i=\begin{pmatrix}r+X & 0\\ 2Y & (r-X)\sin\theta\end{pmatrix}+O_<(1). \tag{5.24}$$

In this matrix expression, $I(i)$ is a row (column) index. Let the indices A, B, ... take values in the set $\{-,+,I\}$. Then the metric is given by $g_{ab}=\eta_{AB}e^A{}_a e^B{}_b$, where

[13]We are following closely the notation used in Ref. [38], as well as the spirit of the discussion in their footnote 2. The meaning of the notation $O_<(r^{-n})$ was described following Eq. (5.1). The motivation for this level of generality is that Chruściel *et al.* [42] have recently shown that one can allow polyhomogeneous terms of the form $r^{-n}\ln^m r$ in these expansions and still have a consistent framework for solving the Bondi-Sachs–type characteristic initial value problem. Allowing only expansions in powers of inverse r is tantamount to Sachs' outgoing radiation condition [41], which they argue is overly restrictive. However, besides making the calculations tighter as regards remainder terms, and slightly more general, we would get the same results had we assumed Sachs' outgoing radiation condition.

$$\eta_{AB}=\begin{pmatrix} 0 & -2 & 0 & 0\\ -2 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{5.25}$$

This matrix, and its inverse, η^{AB}, are used to raise and lower the the basis indices. The vector fields dual to the one-forms in Eq. (5.23) are given by $e_A{}^a=\eta_{AB}g^{ab}e^B{}_b$, or explicitly

$$e_-=\frac{2}{U}\left(\partial_w-\frac{1}{2}V\partial_r-W^i\partial_i\right),\quad e_+=\partial_r,\quad e_I=\gamma_I{}^i\partial_i, \tag{5.26}$$

where $\gamma_I{}^i$ is defined by $\gamma_I{}^i=\delta_{IJ}\sigma^{ij}\gamma^J{}_j$, σ^{ij} being the inverse of the matrix σ_{ij}.

Inspection of e_I in Eq. (5.26) shows that these vectors are tangent to S, and so $e_\pm$ are two null normals to S. Since their normalization is such that $e_+\cdot e_-=-2$, we can set $\xi^a_\pm=e^a_\pm$, where $\xi^a_\pm$ was previously defined by $\xi^a_\pm:=u^a\pm n^a$ [see Eq. (3.8)]. Thus, from Eqs. (2.8) we have the following result, in basis components:

$$l_{IJ}\pm k_{IJ}=e_I{}^a e_J{}^b\nabla_a e_{\pm b}=-\omega_{\pm JI}. \tag{5.27}$$

Working out the required Ricci rotation coefficients[14] I find

$$l_{IJ}+k_{IJ}=B_{IJ},\quad l_{IJ}-k_{IJ}=\frac{2}{U}\left(A_{IJ}-\frac{V}{2}B_{IJ}-\mathcal{D}_{(I}W_{J)}\right), \tag{5.28}$$

where

$$A_{IJ}=\gamma_{(I}{}^i\dot{\gamma}_{J)i}=\frac{1}{r}\begin{pmatrix}\dot{X} & \dot{Y}\\ \dot{Y} & -\dot{X}\end{pmatrix}+O_<(r^{-1}), \tag{5.29}$$

$$B_{IJ}=\gamma_{(I}{}^i\gamma'_{J)i}=\frac{1}{r}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\frac{1}{r^2}\begin{pmatrix}X & Y\\ Y & -X\end{pmatrix}+O_<(r^{-2}). \tag{5.30}$$

Here we use an overdot (prime) to denote differentiation with respect to $w(r)$. Observe that $\mathcal{D}_{(I}W_{J)}$ in Eq. (5.28) is of order $1/r^2$. Taking the trace-free part of Eqs. (5.28) gives us the basis components of the shears in the two null directions: $s_{\pm IJ}=\tilde{l}_{IJ}\pm\tilde{k}_{IJ}$. Explicitly,

$$s_{+IJ}=-\frac{1}{r^2}\begin{pmatrix}X & Y\\ Y & -X\end{pmatrix}+O_<(r^{-2}),$$

$$s_{-IJ}=\frac{2}{r}\begin{pmatrix}\dot{X} & \dot{Y}\\ \dot{Y} & -\dot{X}\end{pmatrix}+O_<(r^{-1}). \tag{5.31}$$

Thus the unreferenced shear term in Eq. (5.5) is given by

$$(\tilde{k}^2-\tilde{l}^2)\equiv -s_{+IJ}s_-{}^{IJ}=\frac{4}{r^3}(X\dot{X}+Y\dot{Y})+O_<(r^{-3}) \tag{5.32}$$

[cf. Eq. (4.9)].

We thus learn that, in contrast to the spatial infinity limit [see Eq. (5.12)], in the null infinity limit the unreferenced shear term is of order $1/r^3$, and so *does* contribute to the IQE. We will argue below that the reference shear term $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$ is also of order $1/r^3$, but that it is a total derivative and therefore does not contribute. So as not to interrupt the flow or our discussion, for the moment let us assume this is true, in which case Eq. (5.5) becomes

$$\lim_{r\to\infty}\mathrm{IQE}=\lim_{r\to\infty}\frac{1}{16\pi}\int_S dS\, r\left[\sigma\sigma R-\frac{4}{r^3}(X\dot{X}+Y\dot{Y})\right]. \tag{5.33}$$

Because of the $c\dot{\bar{c}}+\dot{c}\bar{c}=2(X\dot{X}+Y\dot{Y})$ term, this result looks like it could very well be the Bondi-Sachs mass [41]. To see that in fact it is, a straightforward calculation of the Riemann tensor of g_{ab} projected into S gives the following sectional curvature of (S,σ):

$$\sigma\sigma R=\frac{4}{r^3}(m+X\dot{X}+Y\dot{Y})+O_<(r^{-3}). \tag{5.34}$$

Inserting this result into Eq. (5.33) we see that the shear terms cancel, leaving only the mass aspect, m:

$$\lim_{r\to\infty}\mathrm{IQE}=\frac{1}{4\pi}\int_S d\Omega\, m(w,\theta,\phi). \tag{5.35}$$

In obtaining this expression, recall that because r is an areal radius we can (and did) take $\sqrt{\sigma}=r^2\sin\theta$, and so $dS=r^2\,d\Omega$, where $d\Omega=\sin\theta\, d\theta\, d\phi$ is the measure on the unit round sphere.[15] Thus the future null infinity limit of the IQE is the Bondi-Sachs mass [41].

Now there is an important lesson to be learned from this result. The unreferenced shear term $(\tilde{k}^2-\tilde{l}^2)$ is solely responsible for producing the all-important $c\dot{\bar{c}}+\dot{c}\bar{c}$ term that accounts for the mass loss due to gravitational radiation. Hence this term is *necessary* under the square root in Eq. (4.2), and so there is no natural way to avoid $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$ in $\mathrm{IQE}^{\rm ref}$, and its attendant embedding problem. Moreover, we learn that these shear terms are not only associated with angular momentum, as I have been stressing, but also encode information about gravitational radiation. We will see precisely the same phenomenon emerge in the small sphere

[14] See footnote 11, with $Z=\pm$.

[15] On this note, it might be helpful to point out an important detail in the calculation of the sectional curvature given in Eq. (5.34). One of the terms that arises in the calculation is the trace of A_{IJ} in Eq. (5.29), which appears to be of order $O_<(r^{-1})$. If this were so it would be problematic. But in fact it is zero, because $A_I{}^I=\gamma_I{}^i\dot{\gamma}^I{}_i=(1/2)\dot{\sigma}/\sigma=0$, since $\sigma=r^4\sin^2\theta$ does not depend on w.

limit in Sec. VI. Furthermore, it is emphasized in Ref. [43] that it is easy to construct, *ab initio*, an integral expression involving the Riemann tensor (e.g., an integral of $\sigma\sigma R$ over S) that is conserved under certain circumstances. One is thus tempted to interpret such a conserved quantity as an energy. However, such attempts fail to produce, in the null infinity limit, the crucial null-surface-dependent shear terms seen in Eq. (5.33), and it is difficult to see how to modify them in a *covariant* way to produce these terms [43]. The shear term $(\tilde{k}^2-\tilde{l}^2)$ is precisely such a covariant modification. Moreover, it arises naturally from simply replacing the Brown-York k with the boost invariant quantity $\sqrt{k^2-l^2}$. [Of course a similar observation can be made concerning, say, the Hawking mass [25], which has the same large sphere limit as in Eq. (5.5), except without the *reference* shear term.]

These clean results rely on our assumption that the reference shear term $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$ does not contribute to the null infinity limit of the IQE. I claimed above that this is so because it is a total derivative. To prove this would require solving the embedding equations (4.6)–(4.8), which we know is a very difficult task. However, I will now present a heuristic solution of the Ricci equation that leads to a substantiation of this claim. Moreover, we will see how demanding $\mathcal{F}^{\rm ref}=\mathcal{F}$ plays a crucial role in achieving this result, which provides our first bit of indirect but concrete evidence that this condition is required to properly account for angular momentum (and as we now see, also gravitational radiation).

To begin we need to calculate the connection in the normal bundle, A_a, and then its corresponding curvature, $\mathcal{F}$. Proceeding as we did in the spatial infinity case [see Eq. (5.13)] we find that the basis components of A_a are given by

$$A_I=\frac{1}{2}\,\omega_{+-I}=\frac{1}{rU}\,W_I+\frac{1}{2}\,e_I(\ln U), \tag{5.36}$$

where $e_I(\ln U)$ denotes the derivative of $\ln U$ along the vector field e_I. This term is pure gauge. As for the other term, since we only know W_I to leading order we can put $U=1$ here—see Eqs. (5.20) and (5.21). Thus, up to a gauge transformation, the normal bundle connection A_I is just ($1/r$ times) W_I. And the curvature is thus proportional to the curl of W:

$$\mathcal{F}=\frac{2}{r}\,\epsilon^{IJ}\mathcal{D}_I W_J=\frac{2}{r}\,(\mathcal{D}_2W_3-\mathcal{D}_3W_2). \tag{5.37}$$

Keep in mind that the numerical indices here refer to basis components, not coordinate components. It is easy to see that $\mathcal{F}$ is of order $1/r^3$. It is interesting to compare $\mathcal{F}$ with the scalar curvature $\mathcal{R}$, whose form was given in Eq. (5.2). Using the metric in Eq. (5.22) it is not difficult to evaluate the remainder term, $\Delta_{\mathcal{R}}$. The net result is [38]

$$\mathcal{R}=\frac{2}{r^2}+\frac{2}{r}\,\mathcal{D}\cdot W+O_<(r^{-3}), \tag{5.38}$$

Note from Eq. (5.21) that W^i is of order $1/r^2$, so the term above involving W is, indeed, of order $1/r^3$, as it should be. Thus we see that $\mathcal{R}$ is associated with the divergence of W, and $\mathcal{F}$ with its curl. This is an explicit example of a point made earlier, namely that both curvatures are on the same geometrical footing: To capture the two pieces of information in W—its divergence and its curl—requires precisely both $\mathcal{R}$ and $\mathcal{F}$.

Let us now turn to the null shears of S as they appear in the physical spacetime, Eq. (5.31). The form of these shears suggests we make the following ansatz for the null shears of S in the reference (Minkowski) spacetime:

$$s^{\rm ref}_{+IJ}=\frac{1}{r^2}\begin{pmatrix}\alpha & \beta\\ \beta & -\alpha\end{pmatrix}+O_<(r^{-2}),$$

$$s^{\rm ref}_{-IJ}=\frac{1}{r}\begin{pmatrix}\gamma & \delta\\ \delta & -\gamma\end{pmatrix}+O_<(r^{-1}). \tag{5.39}$$

Observe that we might expect the pair (α,β) to play a role distinct from the pair (γ,δ). Comparing Eqs. (5.39) and (5.31) suggests that α and β will be like X and Y in that they have something to do with the *intrinsic* geometry of (S,σ). The more important terms will be γ and δ, because they occur at the dominant power of inverse r. Also, we expect them to be related to the *extrinsic* geometry of S, since their counterparts, $\dot{X}$ and $\dot{Y}$, measure how σ_{ij} changes as a function of the retarded time w—they are the two ''news'' functions [41].

With these observations in mind, we will now heuristically solve the Ricci embedding equation (4.8). We first compute (in basis components)

$$-\frac{1}{2}\,\epsilon^I{}_J[s^{\rm ref}_+,s^{\rm ref}_-]^J{}_I=\frac{2}{r^3}\,(\alpha\delta-\beta\gamma)=\frac{2}{r^3}\det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}. \tag{5.40}$$

Next we impose the condition $\mathcal{F}^{\rm ref}=\mathcal{F}$, and observe that $\mathcal{F}$ in Eq. (5.37) can also be expressed as a determinant, i.e.,

$$\mathcal{F}^{\rm ref}=\mathcal{F}=\frac{2}{r}\det\begin{pmatrix}\mathcal{D}_2 & \mathcal{D}_3\\ W_2 & W_3\end{pmatrix}. \tag{5.41}$$

The Ricci embedding equation instructs us to equate the two previous determinant expressions. One solution is to make the identifications $\alpha\leftrightarrow r\mathcal{D}_2$ and $\beta\leftrightarrow r\mathcal{D}_3$ (which are consistent with our expectation that α and β be associated with intrinsic geometry), together with $\gamma\leftrightarrow rW_2$ and $\delta\leftrightarrow rW_3$ (which are consistent with γ and δ being associated with extrinsic geometry, since W is proportional to the connection in the normal bundle—a measure of extrinsic geometry). Notice that this means $s^{\rm ref}_{+IJ}$ is a derivative operator. To make this more palatable one may go to a Fourier transform space, where the derivative operators $\mathcal{D}_I$ become momenta, $\mathcal{K}_I$. Accepting these heuristic identifications, and recalling Eq. (4.9), the key point now is to observe that

$$
\begin{aligned}
(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}} &\equiv -s_{+}^{\mathrm{ref}I}{}_{J} s_{-}^{\mathrm{ref}J}{}_{I} \\
&= -\frac{2}{r^3}(\alpha\gamma+\beta\delta) \\
&= -\frac{2}{r}(\mathcal{D}_2 W_2+\mathcal{D}_3 W_3) = -\frac{2}{r}\mathcal{D}\cdot W,
\end{aligned} \tag{5.42}
$$

the result we desired: being a total derivative, $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ does not contribute to the IQE.

Nevertheless, the result in Eq. (5.42) might seem peculiar. It has been stressed that by solving the reference embedding equations subject to the condition $\mathcal{F}^{\mathrm{ref}}=\mathcal{F}$, $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ will carry information about $\mathcal{F}$ into IQE$^{\mathrm{ref}}$ [see Eq. (4.3)], and it is through this mechanism that important information about angular momentum is envisioned to enter the IQE. But according to Eq. (5.42), $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ not only does not contribute to IQE$^{\mathrm{ref}}$, it is in fact functionally independent of $\mathcal{F}$ (the former depends on the divergence of W, and the latter the curl of W). The explanation is that in this simple case there is no need for $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ to carry any information about angular momentum (or gravitational radiation) because all of the relevant information is already carried in the *unreferenced* shear term $(\tilde{k}^2-\tilde{l}^2)$. This is not to say that the condition $\mathcal{F}^{\mathrm{ref}}=\mathcal{F}$ has not played an important role here. It is precisely this condition that leads to $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ being a total derivative, without which $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ might have spoiled the delicate Bondi-Sachs mass result.

Thus the null infinity limit is a simple case that only minimally exercises the consequences of the condition $\mathcal{F}^{\mathrm{ref}}=\mathcal{F}$. In the generic strong field case $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$ will almost certainly depend on $\mathcal{F}$, and play a nontrivial role.

VI. THE SMALL SPHERE LIMIT OF THE IQE

Having considered the large sphere case, we now turn our attention to evaluating the IQE when (S,σ) is a small sphere. The large and small sphere limits are similar in that in both cases S approaches an asymptotically flat region of (M,g). In the latter case, the asymptotically flat region is the infinitesimal neighborhood of a generic spacetime point $p\in M$, which is the center of our shrinking sphere. For simplicity we will suppose that (S,σ) is asymptotically round. Another feature in common with the large sphere limit is that in this codimension-two setting, the limit can be approached from different directions, either spatial or null. More precisely, fix a set of Riemann normal coordinates (t,x^i) about the point p, set $r^2:=\delta_{ij}x^i x^j$, and define S_* by the condition $(r,t)=(r_*,\alpha r_*)$, where α is a direction parameter. Then consider the limit of S_* as $r_*\to 0$. As before, we will henceforth omit the subscript $*$. The case $\alpha=0$ is a spatial limit, since then S always lies entirely in the $t=0$ spacelike three-surface containing p. $\alpha=\pm 1$ is the null limit, in which S lies in the future/past light cone of the point p. The latter case was considered by Horowitz and Schmidt [20] in their classic work on the small sphere limit of the Hawking mass. Brown, Lau, and York [19] also consider this same limit of the Brown-York quasilocal energy. We will be borrowing some results from these two references.

Explicitly, for a given value of the parameter r, S is defined as a submanifold of (M,g) by embedding a topological two-sphere with coordinates (θ,ϕ) into the Riemann normal coordinate system, as follows: $t=\alpha r$, $x^1=r\sin\theta\cos\phi$, $x^2=r\sin\theta\sin\phi$, $x^3=r\cos\theta$. Since (with $t=\alpha r$) the deviation from the flat metric in Riemann normal coordinates is $O(r^2)$, the induced metric σ_{ab} on S will differ from that of the round sphere to this same order, and so the scalar curvature of (S,σ) will have an expansion in r of the form

$$
\mathcal{R}=\frac{2}{r^2}+\mathcal{R}^{(0)}+r\mathcal{R}^{(1)}+r^2\mathcal{R}^{(2)}+O(r^3), \tag{6.1}
$$

where each of the coefficients $\mathcal{R}^{(n)}$ is a function of θ, ϕ, and the parameter α. To evaluate the IQE we will also need similar expansions for the other quantities appearing in Eqs. (4.2) and (4.3). These are written as follows:

$$
\sigma\sigma R=\sigma\sigma R^{(0)}+r\sigma\sigma R^{(1)}+r^2\sigma\sigma R^{(2)}+O(r^3), \tag{6.2}
$$

$$
(\tilde{k}^2-\tilde{l}^2)=r^2(\tilde{k}^2-\tilde{l}^2)^{(2)}+O(r^3), \tag{6.3}
$$

$$
(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}=r^2(\tilde{k}^2-\tilde{l}^2)^{(2)\mathrm{ref}}+O(r^3), \tag{6.4}
$$

where each of the coefficients on the right-hand side is similarly a function of θ, ϕ, and α. Since the appropriate reference spacetime in this case is Minkowski space, we have $\sigma\sigma R^{\mathrm{ref}}=0$. Substituting these expansions into Eq. (4.2) we find that in the small sphere limit the unreferenced IQE behaves as

$$
\begin{aligned}
\mathrm{IQE}^{\mathrm{unref}} = -\frac{1}{8\pi}\int_S dS\,\frac{2}{r}\Bigg\{1+\frac{r^2}{4}\Bigg[&(\mathcal{R}^{(0)}-\sigma\sigma R^{(0)}) \\
&+r(\mathcal{R}^{(1)}-\sigma\sigma R^{(1)}) \\
&+r^2\Bigg(\mathcal{R}^{(2)}-\sigma\sigma R^{(2)}+(\tilde{k}^2-\tilde{l}^2)^{(2)} \\
&-\frac{1}{8}(\mathcal{R}^{(0)}-\sigma\sigma R^{(0)})^2\Bigg)\Bigg]+O(r^5)\Bigg\}.
\end{aligned} \tag{6.5}
$$

Similarly, the reference IQE behaves as

$$
\begin{aligned}
\mathrm{IQE}^{\mathrm{ref}} = -\frac{1}{8\pi}\int_S dS\,\frac{2}{r}\Bigg\{1+\frac{r^2}{4}\Bigg[&\mathcal{R}^{(0)}+r\mathcal{R}^{(1)} \\
&+r^2\Bigg(\mathcal{R}^{(2)}+(\tilde{k}^2-\tilde{l}^2)^{(2)\mathrm{ref}}-\frac{1}{8}(\mathcal{R}^{(0)})^2\Bigg)\Bigg]+O(r^5)\Bigg\}.
\end{aligned} \tag{6.6}
$$

Notice that, unlike in the large sphere limit, neither the unreferenced nor reference energies diverge as $r\to 0$. Nevertheless, the reference subtraction procedure is still necessary to eliminate the leading $O(r)$ term in IQE$^{\mathrm{unref}}$, which has nothing to do with energy. Thus, forming the difference of the previous two expressions we find that the small sphere behavior of the (referenced) IQE is given by

$$\mathrm{IQE}=\frac{1}{16\pi}\int_S dS\, r[\sigma\sigma R-(\tilde{k}^2-\tilde{l}^2)+(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}} + \tfrac{1}{8}\, r^2\sigma\sigma R^{(0)}(\sigma\sigma R^{(0)}-2\mathcal{R}^{(0)})+O(r^3)]. \quad (6.7)$$

The sectional curvature and shear terms have been resummed according to Eqs. (6.2)–(6.4), and the resulting expression is valid to the order indicated. Notice that, as in the large sphere limit, the scalar curvature $\mathcal{R}$ dominates the other terms under the square root, allowing us to expand the radical about $\sqrt{4/r^2}=2/r$.[16] And after the reference subtraction is performed what remains again is the sectional curvature of (S,σ) as the dominant term contributing to the energy. Comparing Eqs. (6.7) and (5.5) we observe that both the small and large sphere limits of the IQE are very nearly formally identical.

We will split the IQE into three pieces, each to be discussed separately: $\mathrm{IQE}=\mathrm{IQE}_1+\mathrm{IQE}_2+\mathrm{IQE}_3+O(r^6)$, where

$$\mathrm{IQE}_1=\frac{1}{16\pi}\int_S dS\, r[\sigma\sigma R-(\tilde{k}^2-\tilde{l}^2)], \quad (6.8)$$

$$\mathrm{IQE}_2=\frac{1}{128\pi}\int_S dS\, r^3[\sigma\sigma R^{(0)}(\sigma\sigma R^{(0)} -2\mathcal{R}^{(0)})], \quad (6.9)$$

$$\mathrm{IQE}_3=\frac{1}{16\pi}\int_S dS\, r[(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}]. \quad (6.10)$$

We begin with IQE_1, and show that this piece is essentially the Hawking mass [25]. To see this, we combine Eqs. (4.1) and (2.10) to get

$$\sigma\sigma R-(\tilde{k}^2-\tilde{l}^2)=\mathcal{R}-\tfrac{1}{2}\,(k^2-l^2)=\mathcal{R}-2H\cdot H. \quad (6.11)$$

Replacing the integrand of IQE_1 with the last expression, and using the Gauss-Bonnet theorem to integrate the $\mathcal{R}$ term, we find

$$\mathrm{IQE}_1=\frac{1}{4\pi}\sqrt{\frac{A}{4\pi}}\left[2\pi-\frac{1}{2}\int_S dS\, H\cdot H\right], \quad (6.12)$$

where we pulled r outside the integral and replaced it with $\sqrt{A/(4\pi)}$, where A is the area of (S,σ). This form of IQE_1 is precisely the expression of the Hawking mass given in Ref. [20]. [Our H^c in Eq. (2.10) is their $N^c/2$, and the sign of their metric signature is opposite to ours.] Comparing Eq. (6.12) with Eqs. (3.7) and (3.9) we observe that, while the unreferenced IQE involves the mean curvature itself, $\sqrt{H\cdot H}$, the Hawking mass is constructed from the square of the mean curvature. As mentioned above, the square root in $\sqrt{H\cdot H}$ effectively disappears in the small (and large) sphere limits due to the presence of the dominant scalar curvature term, and consequently the leading order contribution to the IQE reduces to essentially the Hawking mass.

There are two subtleties worth mentioning: (i) Replacing r with $\sqrt{A/(4\pi)}$ is in general not valid because it requires that r be an areal radius which, in general, it is not. However, it certainly *is* to lowest order in r, which will be sufficient for our purposes here. But to higher order, IQE_1 and the Hawking mass will in general give different results. (ii) It is well known that the Hawking mass runs into difficulties when (S,σ) is not a round sphere [8,20], a problem that was addressed by Hayward in Ref. [8]. It might be that this problem is a result of having to insert by hand the factor $\sqrt{A/(4\pi)}$ outside the integral, versus having r inside the integral generated automatically by $\sqrt{2/\mathcal{R}}$. A related issue was discussed at the end of Sec. V A in connection with Hayward's definition of quasilocal energy.

The connection between IQE_1 and the Hawking mass allows us to borrow some results from Ref. [20], which are calculated for the null limit case $(\alpha=1)$. When matter is present Horowitz and Schmidt find that, to lowest order in r, the Hawking mass is

$$\mathrm{IQE}_1=\left(\frac{4}{3}\pi r^3\right)T^{\mathrm{mat}}_{ab}u^a u^b\big|_p+O(r^4). \quad (6.13)$$

Here T^{mat}_{ab} is the stress-energy tensor of matter, and the expression is to be evaluated at the point p, where the unit timelike vector u^a is just $(\partial/\partial t)^a$ in our Riemann normal coordinates. This is a standard result in the literature on quasilocal energy [19–24], and a very significant one. As emphasized in the Introduction, the quasilocal idea asserts that the time-time component of the stress-energy tensor of matter *a priori* has nothing to do with energy. It is only from the small sphere limit of the quasilocal energy that we learn this quantity is an energy volume density, i.e., multiplying it by the volume factor $4\pi r^3/3$ gives the energy in an infinitesimal sphere of proper radius r. However, integrating this energy volume density over a finite volume to determine the total energy inside is not, in general, valid unless one wishes to ignore gravitational effects which, as we will see in moment, come at higher order in r.[17] It is in this sense that the quasilocal idea implies that even energy due to matter is not localizable in the context of general relativity.

[16] As remarked in footnote 10, if the sphere is not asymptotically round the shear terms will contribute at order $1/r^2$, and the $2/r$ term outside the braces in Eqs. (6.5) and (6.6) will be modified accordingly. We will not consider this more complicated case here.

[17] A well-established example of this phenomenon is the Tolman density, which integrates to the Komar mass, and is defined in the special case that the spacetime is stationary and asymptotically flat [1]. It is noteworthy that is not T_{ab} that appears in the Tolman density, but rather the combination $T_{ab}-(1/2)Tg_{ab}$. The extra term involving the trace of T_{ab} is associated with gravitational effects—see Ref. [44], and problems 4 and 5 in Chap. 11 of Ref. [1].

Now let us assume the spacetime is vacuum in the neighborhood of p. Then the leading order contribution to the Hawking mass is [20]

$$\mathrm{IQE}_1 = \frac{1}{90} r^5 T_{abcd} u^a u^b u^c u^d \big|_p + O(r^6), \tag{6.14}$$

where T_{abcd} is the Bel-Robinson tensor [45]. Thus gravitational energy begins to appear at $O(r^5)$. This same result is obtained for the Brown-York quasilocal energy for a suitable choice of reference embedding [19]. However, this is not a universal result in the literature [21–24]. For example, Hayward's quasilocal mass gives a similar result as above, but with the numerical factor 1/90 replaced with $-2/45$ [22]. Given that gravitational energy is such a difficult problem it is not surprising that a consensus has not yet been reached.

We now turn our attention to the second contribution to the IQE, namely IQE_2 given in Eq. (6.9). This quantity represents a deviation from the Hawking mass due to the fact that the IQE is roughly the square root of the former. Actually, it is clearer to compare with the Brown-York quasilocal energy, since in constructing the IQE we simply replaced the Brown-York k with $\sqrt{k^2 - l^2}$. In the context of our generalization given in Eq. (3.4), we therefore have the heuristic comparison:

$$m = \sqrt{E^2 - \vec{p}^2} = E - \frac{\vec{p}^2}{2E} - \cdots \rightarrow \frac{1}{8\pi} \sqrt{k^2 - l^2}$$

$$= \frac{1}{8\pi} \left(k - \frac{l^2}{2k} - \cdots \right). \tag{6.15}$$

Therefore IQE_2 might be thought of as the analogue of the term $-l^2/(2k)$, and as such would be expected to reduce the magnitude of the IQE from the result given in Eq. (6.14).

In order to calculate IQE_2 we need to evaluate the quantities $\mathcal{R}^{(0)}$ and $\sigma\sigma R^{(0)}$ in Eq. (6.9). To do so we appeal to the Gauss equation (4.1). Up to zeroth order in r, this equation reads

$$\sigma\sigma R^{(0)} = \frac{2}{r^2} + \mathcal{R}^{(0)} - \frac{1}{2}(k^2 - l^2) + O(r), \tag{6.16}$$

where we made use of Eq. (6.1). We will show later that

$$(k^2 - l^2) = \frac{4}{r^2} + \frac{4}{3}(1 + 2\alpha^2) E_{ab} n^a n^b + O(r), \tag{6.17}$$

where α is the direction parameter introduced at the beginning of this section. $E_{ab} n^a n^b$ is the radial-radial component of the electric part of the Weyl tensor, which we saw earlier in Eq. (5.17). This quantity $(E_{ab} n^a n^b)$ is to be evaluated at the point p, where in our Riemann normal coordinates the radial unit vector n^a has only spatial components, given by $n^i = x^i/r$. We also need dS to lowest order, which is just $r^2\, d\Omega$, $d\Omega$ being the measure on the unit sphere. Putting these results together we have

$$\mathrm{IQE}_2 = -\frac{1}{96\pi}(5 + 4\alpha^2) r^5 E_{ij} E_{kl} \int d\Omega\, n^i n^j n^k n^l. \tag{6.18}$$

By symmetry, the integral over the product of radial vectors must be proportional to $\delta^{(ij}\delta^{kl)}$. Transvecting both this term and the integral in question with $\delta_{ij}\delta_{kl}$, we easily obtain that the proportionality constant is $4\pi/5$. Using the fact that E_{ab} is symmetric, trace-free, and orthogonal to u^a, we find that

$$\mathrm{IQE}_2 = -\tfrac{1}{180}(5 + 4\alpha^2) r^5 E_{ab} E^{ab}. \tag{6.19}$$

So IQE_2 is negative, as expected, and this negative contribution is to be added to IQE_1 in Eq. (6.14). Of course we can only consider the $\alpha = 1$ case, since this is the case assumed in Eq. (6.14). Recall that the time component of the Bel-Robinson tensor can be expressed in terms of the electric and magnetic parts of the Weyl tensor [45]: $T_{abcd} u^a u^b u^c u^d = E_{ab} E^{ab} + B_{ab} B^{ab}$, so IQE_1 is non-negative. Inspection of Eq. (6.19) shows that adding to IQE_1 the $\alpha = 1$ value of IQE_2 makes the energy have indefinite sign. It is positive (negative) if the magnetic (electric) part dominates. This seems like a strange result, but it is only an intermediate result. We have not yet considered the last contribution, IQE_3, involving the reference shear term. But unfortunately at present I do not know how to solve the embedding equations to determine this term.

Now one can construct a heuristic argument much like the one given at the end of Sec. V B, which suggests that $(\tilde{k}^2 - \tilde{l}^2)^{\mathrm{ref}}$ is a total derivative, and so does not contribute. However, the argument is much less believable in this case. In contrast to Eq. (5.39) it turns out that, because $\mathcal{F}$ is $O(1)$ in r (as we shall see later), we must expand the reference shears $s_{\pm}^{\mathrm{ref}}$ over *three* orders of magnitude in r, from $O(1)$ to $O(r^2)$. One might trust a heuristic argument working to leading order, but believing the higher order corrections is less palatable. In short, I do not know what energy prediction the IQE gives at $O(r^5)$, and until a solution to the embedding equations is found there is no sense in speculating.

However, before leaving this section I will provide an intriguing interpretation of how a definition of quasilocal energy such as the Hawking mass (or the IQE) provides a measure of the gravitational energy contained inside a small sphere.

In the $\alpha = 1$ null limit case, the lowest order contribution to the gravitational energy, namely $(1/90) r^5 T_{abcd} u^a u^b u^c u^d |_p$ in Eq. (6.14), originates in the $O(r^2)$ terms inside the square brackets of the integrand of IQE_1 in Eq. (6.8). In the terminology of Eqs. (6.2) and (6.3), this means we are interested in the coefficients $\sigma\sigma R^{(2)}$ and $(\tilde{k}^2 - \tilde{l}^2)^{(2)}$. Inspecting the Appendix of Ref. [19] reveals that these two coefficients differ only by a numerical factor. They are both proportional to $\Psi_0 \overline{\Psi}_0 |_p$ (in Newman-Penrose notation), and the two numerical factors conspire to produce the 1/90 factor in the final result. Thus, to understand how the integrand of IQE_1 encodes information about gravitational energy it suffices to study the shear term $(\tilde{k}^2 - \tilde{l}^2)^{(2)}$. We will now compute this term for arbitrary $\alpha\in[-1,1]$ to see how it behaves in both the spatial and null limit cases of the small sphere.

Denoting our Riemann normal coordinates (t,x^i) collectively as x^a, the metric in these coordinates take the form

$$g_{ab}=\eta_{ab}-\tfrac{1}{3}J_{abcd}x^c x^d+O(x^3), \tag{6.20}$$

where $J_{abcd}=(R_{acbd}+R_{adbc})/2$ is the Jacobi curvature tensor [29]. We first construct a pair of mutually orthogonal unit normal vector fields u^a and n^a on S, with u^a normal to the $t=$const surface passing through S. These are given by

$$u^0=\frac{1}{N},\quad u^i=-\frac{1}{3}r^2\beta^{i0}+O(r^3),\quad u_0=-N,\quad u_i=0, \tag{6.21}$$

$$n^0=0,\quad n^i=\rho\left[\frac{x^i}{r}+\frac{1}{3}r^2\beta^{ij}\frac{x_j}{r}+O(r^3)\right],$$

$$n_0=-\frac{1}{3}r^2\beta_{0j}\frac{x^j}{r}+O(r^3),\quad n_i=\rho\frac{x_i}{r}, \tag{6.22}$$

where

$$N=1+\tfrac{1}{6}r^2\beta_{00}+O(r^3), \tag{6.23}$$

$$\rho=1-\frac{1}{6}r^2\beta_{ij}\frac{x^ix^j}{r^2}+O(r^3), \tag{6.24}$$

$$\beta_{ab}=\alpha^2J_{ab00}+2\alpha J_{ab0j}\frac{x^j}{r}+J_{abij}\frac{x^ix^j}{r^2}. \tag{6.25}$$

Since the Jacobi tensor in Eq. (6.20) is evaluated at the coordinate origin p, its indices, and thus those of β_{ab}, are raised and lowered with the flat spacetime metric $\eta_{ab}=\eta^{ab}=\text{diagonal}(-1,1,1,1)$. Similarly, $x_i:=\delta_{ij}x^j$.

Now define on S a pair of mutually orthogonal unit tangent vector fields $e_I{}^a$, where indices $I,J,\ldots$ take the values 2 and 3. The set $\{e_0{}^a:=u^a,\ e_1{}^a:=n^a, e_I{}^a\}$ thus comprises an orthonormal basis adapted to S. Let basis indices $A,B,\ldots$ run from 0 to 3, and $\alpha,\beta,\ldots$ from 1 to 3. Beginning with this setup it is straightforward to compute the basis components of the extrinsic curvatures defined in Eqs. (2.8). I find

$$l_{IJ}=e_I{}^ae_J{}^b\nabla_a u_b=-\tfrac{2}{3}r[\alpha J_{00IJ}+J_{01IJ}]+O(r^2), \tag{6.26}$$

$$k_{IJ}=e_I{}^ae_J{}^b\nabla_a n_b=\frac{1}{r}\delta_{IJ}-\frac{1}{3}r\left[J_{11IJ}-\alpha^2\left(J_{00IJ}-\frac{1}{2}J_{0011}\delta_{IJ}\right)\right]+O(r^2). \tag{6.27}$$

In these equations a quantity such as J_{01IJ} means $[e_0{}^ae_1{}^be_I{}^ce_J{}^dJ_{abcd}]|_p$, which is a function of only the angles θ and ϕ on S.

Since we are interested in purely gravitational energy we shall restrict ourselves to the vacuum case. In our basis components the electric and magnetic parts of the Weyl tensor are defined by [46]

$$E_{\alpha\beta}=-C_{0\alpha0\beta}\quad\text{and}\quad B_{\alpha\beta}=-{*C}_{0\alpha0\beta}, \tag{6.28}$$

where ${*C}_{ABCD}=(1/2)\epsilon_{AB}{}^{EF}C_{EFCD}$. These are symmetric trace-free three-dimensional tensors associated with $t=$const spacelike hypersurfaces. As such, each has five independent components, which together comprise the 10 independent components of the Weyl tensor. In terms of these fields, the components of the Jacobi curvature tensor relevant to Eqs. (6.26)–(6.27) read:

$$J_{0011}=-E_{11},\quad J_{00IJ}=-E_{IJ},\quad J_{01IJ}=-{*\tilde{B}}_{IJ},$$
$$J_{11IJ}=-E_{IJ}-E_{11}\delta_{IJ}. \tag{6.29}$$

Here $\tilde{B}_{IJ}$ is the trace-free part of B_{IJ}, and ${*\tilde{B}}_{IJ}=\epsilon_I{}^K\tilde{B}_{KJ}$ is its dual in (S,σ), which is also trace-free. The trace of E_{IJ} is $\delta^{IJ}E_{IJ}=\delta^{\alpha\beta}E_{\alpha\beta}-E_{11}=-E_{11}$, since $E_{\alpha\beta}$ is trace-free. Thus, the trace of the extrinsic curvatures in Eqs. (6.26)–(6.27) is found to be

$$l=-\tfrac{2}{3}r\alpha E_{11}+O(r^2), \tag{6.30}$$

$$k=\frac{2}{r}+\frac{1}{3}r(1+2\alpha^2)E_{11}+O(r^2). \tag{6.31}$$

Squaring these and forming their difference leads to Eq. (6.17) written earlier. The trace-free parts are

$$\tilde{l}^{IJ}=\tfrac{2}{3}r(\alpha\tilde{E}_{IJ}+{*\tilde{B}}_{IJ})+O(r^2), \tag{6.32}$$

$$\tilde{k}_{IJ}=\tfrac{1}{3}r(1-\alpha^2)\tilde{E}_{IJ}+O(r^2). \tag{6.33}$$

Now $\tilde{E}_{IJ}$ and $\tilde{B}_{IJ}$ each have two independent components, and it is useful to introduce the notation

$$\tilde{E}_{IJ}=\begin{pmatrix}\tfrac{1}{2}(E_{22}-E_{33}) & E_{23}\\ E_{23} & -\tfrac{1}{2}(E_{22}-E_{33})\end{pmatrix}=:\begin{pmatrix}e^2 & e^3\\ e^3 & -e^2\end{pmatrix}, \tag{6.34}$$

$$\tilde{B}_{IJ}=\begin{pmatrix}\tfrac{1}{2}(B_{22}-B_{33}) & B_{23}\\ B_{23} & -\tfrac{1}{2}(B_{22}-B_{33})\end{pmatrix}=:-\begin{pmatrix}b^2 & b^3\\ b^3 & -b^2\end{pmatrix}, \tag{6.35}$$

and thus define a pair of two-vectors $\vec{e}:=e^Ie_I{}^a$ and $\vec{b}:=b^Ie_I{}^a$ tangent to S. Since $\vec{e}$ and $\vec{b}$ represent the pullbacks to S of $E_{\alpha\beta}$ and $B_{\alpha\beta}$, respectively, they are to be thought of as gravitoelectric and magnetic fields induced on S by the Weyl curvature that S is embedded in. It is easy to see that under a rotation of the basis vectors $e_I{}^a$ through an angle γ, the components of $\vec{e}$ and $\vec{b}$ rotate through an angle 2γ, so $\vec{e}$ and $\vec{b}$ are not true (spin one) vectors, but rather spin two objects, as one would expect.

Thus we arrive at the results we are interested in. To lowest order in r we have

$$\tilde{l}^2 = \tfrac{4}{9}\, r^2(\alpha\tilde{E} + *\tilde{B})^2 = \tfrac{8}{9}\, r^2(\alpha^2\vec{e}\cdot\vec{e} + \vec{b}\cdot\vec{b} - 2\alpha\vec{e}\times\vec{b}), \tag{6.36}$$

$$\tilde{k}^2 = \tfrac{1}{9}\, r^2(1-\alpha^2)^2\tilde{E}^2 = \tfrac{2}{9}\, r^2(1-\alpha^2)^2\vec{e}\cdot\vec{e}. \tag{6.37}$$

The difference of these two gives the $O(r^2)$ piece of the unreferenced shear term appearing in the integrand of IQE_1 in Eq. (6.8). Notice that it is the appearance of $*\tilde{B}_{IJ}$ (rather than $\tilde{B}_{IJ}$) that gives rise to the cross product term $\vec{e}\times\vec{b} = e^2b^3 - e^3b^2$ in Eq. (6.36).

We first consider the case $\alpha=1$, in which S lies in the future light cone of the point p. Then $\tilde{k}^2=0$ and so the shear term $(\tilde{k}^2-\tilde{l}^2)$ is proportional to $r^2(\tilde{E}+*\tilde{B})^2$. As a quick check, it is easy to verify that $(\tilde{E}+*\tilde{B})^2$, in turn, is proportional to $\Psi_0\overline{\Psi}_0|_p$, in agreement with Ref. [19]. I mentioned above that in this case the $\sigma\sigma R$ term in Eq. (6.8) also contributes a term proportional to $r^2\Psi_0\overline{\Psi}_0|_p$ [19]. Putting in the numerical factors I find that

$$\mathrm{IQE}_1 = \int_S dS\, \frac{r^3}{9}\left[\frac{1}{8\pi}(\vec{e}\cdot\vec{e}+\vec{b}\cdot\vec{b}) - \frac{1}{4\pi}\vec{e}\times\vec{b}\right] + O(r^6). \tag{6.38}$$

Now $(\vec{e}\cdot\vec{e}+\vec{b}\cdot\vec{b})/(8\pi)$ looks like the energy *surface* density of the gravitoelectromagnetic field, but we must be careful about its dimension. $(\vec{e}\cdot\vec{e}+\vec{b}\cdot\vec{b})/(8\pi)$ has dimension L^{-4}, where L means length, which is not correct. However, the additional factor of $r^3/9$ in the integrand suggests that it is really $\mathcal{E} := r^3(\vec{e}\cdot\vec{e}+\vec{b}\cdot\vec{b})/(72\pi)$ that is the proper energy surface density. $\mathcal{E}$ has dimension L^{-1}, consistent with it being interpreted as the gravitoelectromagnetic energy per unit area of S. Besides giving the right dimension, the additional r^3 factor forces $\int_S dS\,\mathcal{E}$ to go to zero as r^5, consistent with the fact that there can be no gravitational energy at order r^3. We interpret $\int_S dS\,\mathcal{E}$ as the total gravitoelectromagnetic energy that was on S at $t=0$, or equivalently, the total gravitational energy that was in the small volume spanning S at $t=0$.

To further justify this interpretation we now turn to the radiation term in Eq. (6.38). Clearly $\vec{e}\times\vec{b}/(4\pi)$ might be thought of as the gravitational analogue of the electromagnetic Poynting flux, directed radially outward from S. But again, its dimension is wrong. Of course the factor of $r^3/9$ will fix this problem, as before, but the situation is more interesting this time. Multiplying by $r^2/9$ we get the proper Poynting flux, $\mathcal{P} := r^2\vec{e}\times\vec{b}/(36\pi)$. $\mathcal{P}$ has dimension L^{-2}, consistent with interpreting it as the gravitoelectromagnetic energy per unit time per unit area. So $\int_S dS\,\mathcal{P}$ gives the gravitoelectromagnetic energy per unit time radiating from (or through) the surface S. The factor of r^2 indicates that the efficiency of a small volume to radiate gravitationally grows in proportion to its surface area, in analogy with an electromagnetic antenna. But there is one more factor of r, which one might imagine is the r outside the brackets in Eq. (6.8), i.e., $\sqrt{2/\mathcal{R}}$. This distinction between rs is suggested by the close analogy between $\mathcal{P}$ and the shear term responsible for radiation in the null infinity limit—see Eqs. (5.32)–(5.33). This additional factor of r has the interpretation of a time lapse, i.e., $r\int_S dS\,\mathcal{P}$ is the amount of electromagnetic energy radiated from S between time $t=0$ and $t=r$. Thus, the following picture has emerged regarding Eq. (6.38). The gravitoelectromagnetic energy on the surface S (or equivalently, the gravitational energy in the volume spanning S) at time $t=r$ is the energy at $t=0$ minus the amount of energy radiated during this time interval. (Keep in mind that $\vec{e}$ and $\vec{b}$ are evaluated at p, and hence at $t=0$.)

The case $\alpha=-1$ is similar, except now S lies in the past light cone of the point p. Inspection of Eq. (6.36) reveals that the radiation term in Eq. (6.38) now appears with the opposite sign. The fact that this sign change comes out correctly is reassurance that our picture is correct: The energy at time $t=-r$ is the energy at $t=0$ *plus* the amount of energy that is radiated from the sphere during the interval from $t=-r$ to $t=0$.[18]

Finally, we consider the spatial limit case, $\alpha=0$. According to our discussion above we would expect IQE_1 to be the same as in Eq. (6.38), except with the radiation term absent. Inspection of Eqs. (6.36)–(6.37) reveals that this is not the case. However, it is only when $\alpha=1$ (and presumably also when $\alpha=-1$) that we know that $\sigma\sigma R^{(2)}$ in Eq. (6.8) is proportional to $(\tilde{k}^2-\tilde{l}^2)^{(2)}$, in which case it suffices to consider only the shear term. Unfortunately, it is not possible to compute $\sigma\sigma R$ to $O(r^2)$ within the framework of our $O(r^2)$ Riemann normal coordinates, so we cannot learn if this simple proportionality between the two persists when $|\alpha|<1$. One might guess that it almost certainly does not, but I will leave this question for future work. Nevertheless, since we expect $\tilde{l}^2$ to play the key role with regards to radiation, Eq. (6.36) is still of some qualitative value when $|\alpha|<1$. From this equation we see that the radiation term is zero when α is zero, and turns on in proportion to α, precisely as it should since the time lapse is now αr, instead of r.

As satisfying as it might seem, the picture just given is not truly quasilocal, in the sense that $\vec{e}$ and $\vec{b}$ are evaluated at the point p. To be truly quasilocal we need gravitoelectromagnetic fields, call them $\vec{E}$ and $\vec{B}$, evaluated on S. This is where observers reside, and measurements are made, according to the quasilocal idea. Such a quasilocal picture is achieved

[18]The reader may have noticed that $\vec{b}$ in Eq. (6.35) was defined with an awkward minus sign. This sign was chosen to give the picture just described. Reversing the sign is equivalent to replacing α with $-\alpha$. Insofar as $\vec{e}$ and $\vec{b}$ (like $\vec{E}$ and $\vec{B}$ in electromagnetism) are defined by their physical interpretation, choosing the sign of $\vec{b}$ to give a result with the correct interpretation is legitimate. But this assumes we know what the correct interpretation is, and it is not certain we do. For example, I mentioned above that at $O(r^5)$ Hayward's quasilocal energy gives a negative gravitational energy [22]. If this is correct, then we should replace the definition of $\vec{b}$ with $-\vec{b}$.

very naturally as follows. The basic idea is that $\vec{e}$ and $\vec{b}$ are certain components of the Weyl tensor evaluated at p ($r=0$). But this information is contained in the $O(r)$ piece of certain connection coefficients evaluated on S ($r>0$). Thus we expect the desired $\vec{E}$ and $\vec{B}$ fields to be associated with connection coefficients.

For simplicity we will restrict ourselves to the case $\alpha=1$, which also allows us to borrow some results from Ref. [19]. We first observe that

$$\Psi_0|_p = -2[(e^2-b^3)+i(e^3+b^2)]. \tag{6.39}$$

On the left-hand side is a component of the Weyl tensor in Newman–Penrose (NP) notation. Using Eqs. (6.28) and (6.34)–(6.35), Ψ_0 is easily converted to the expression given on the right-hand side. From Eq. (B5b) of Ref. [19] we have

$$\sigma = \frac{r}{3}\Psi_0|_p + O(r^2), \tag{6.40}$$

where σ is one of the NP spin coefficients. Thus we find that

$$\frac{1}{4}\sigma\bar{\sigma} = \frac{r^2}{9}[\vec{e}\cdot\vec{e}+\vec{b}\cdot\vec{b}-2\vec{e}\times\vec{b}]+O(r^3). \tag{6.41}$$

Comparing this with the integrand in Eq. (6.38) we are led to define

$$\vec{E} := \frac{r}{3}\vec{e}+O(r^2), \quad \vec{B} := \frac{r}{3}\vec{b}+O(r^2), \tag{6.42}$$

or in other words,

$$\sigma = -2[(E^2-B^3)+i(E^3+B^2)]+O(r^2). \tag{6.43}$$

Thus, to $O(r)$, $\vec{E}$ and $\vec{B}$ are related to connection coefficients, as we expected.

Substituting Eq. (6.42) into Eq. (6.38) we have

$$\mathrm{IQE}_1 = \int_S dS\, r\left[\frac{1}{8\pi}(\vec{E}\cdot\vec{E}+\vec{B}\cdot\vec{B}) - \frac{1}{4\pi}\vec{E}\times\vec{B}\right]+O(r^6). \tag{6.44}$$

Observe that the mysterious $r^2/9$ factor has disappeared, and the analogy with electromagnetism is improved: $\vec{E}$ and $\vec{B}$ now have their usual dimension (L^{-1}), as do the energy density and Poynting flux terms. I emphasize again that, in contrast to $\vec{e}$ and $\vec{b}$ in Eq. (6.38), $\vec{E}$ and $\vec{B}$ are fields measured by observers residing on S, in the true quasilocal spirit. $\vec{E}$ is clearly associated with tidal forces *tangential* to S, and $\vec{B}$ is a measure of frame dragging effects. Notice that $\vec{E}$ and $\vec{B}$ vanish as $r\to 0$, in accord with the equivalence principle.

To conclude this section we consider the connection in the normal bundle and its associated curvature. I find that

$$A_J = \frac{2}{3}r\left[\alpha E_{1J}+\frac{1}{2}{*B_{1J}}\right]+O(r^2), \tag{6.45}$$

where ${*B_{1J}} = \epsilon_J{}^K B_{1K}$, and

$$\mathcal{F} = 2B_{11}+O(r). \tag{6.46}$$

So to leading order the curvature of the normal bundle is (twice) the radial-radial component of the magnetic part of the Weyl tensor, and is thus associated with gravitational magnetic charge. There are both local and global dimensions to this result. Locally, $\mathcal{F}$ is associated with frame dragging, a ready example being $\mathcal{F}$ for the Kerr black hole given in Eq. (5.15), which is proportional to the angular momentum. Globally, it is known that in exact analogy with the scalar curvature $\mathcal{R}$, the integral of $\mathcal{F}$ over S is proportional to the Euler number of the normal bundle [27]. For a Euclidean-signature spacetime the normal bundle is an SO(2) [rather than SO(1, 1)] bundle, and there can be a nontrivial winding number, corresponding to a gravitational magnetic monopole. In the case of the Kerr spacetime there is no monopole present since, as is obvious from inspection of Eq. (5.15), the integral of $\mathcal{F}$ is zero. It might be interesting to explore topologically nontrivial cases in the context of the IQE.

The result in Eq. (6.46) can actually be obtained immediately by inspection of Eq. (2.16), assuming that the shear terms are higher order in r than $\mathcal{F}$ is. To lowest order in r we then see that $\mathcal{F} = -2R_{0123}$. But $R_{0123} = C_{0123}$ is identically true, and thus we are led to Eq. (6.46). So this equation is true whether or not matter is present, and is also independent of α. It is instructive to compare this result with the sectional curvature of S in vacuo:

$$\sigma\sigma R = 2E_{11}+O(r). \tag{6.47}$$

This is essentially the same as the spatial infinity limit result given in Eq. (5.17), and is derived similarly. Comparing the previous two equations we see a striking electric/magnetic duality between the sectional curvature of S (electric), and the curvature of its normal bundle (magnetic). When matter is present, the right-hand side of the equation above acquires an additional term, and it is precisely this term that is responsible for the $O(r^3)$ matter contribution seen in Eq. (6.13). So the sectional curvature is the dominant term in the energy that encodes information about the matter content of the spacetime. It seems reasonable to expect inertial effects (frame dragging) produced by this matter to also play a role in the energy. But consideration of such effects is subtle, because the magnetic part of the Weyl tensor has no Newtonian gravity analogue. I have argued that the procedure suggested in Sec. IV is a geometrically natural way to incorporate such inertial effects into the energy: one demands $\mathcal{F}^{\mathrm{ref}} = \mathcal{F}$, and then solves the embedding equations for the reference shear term, present in the reference energy. In this way the inertia information contained in $\mathcal{F}$ makes its presence felt in the energy. Moreover, by inspection of the purely spatial ($\alpha=0$) case of Eq. (6.45), one observes that the set of magnetic quantities: $\mathcal{F}$, A_J, and $\vec{b}$, precisely encode the five independent components of the magnetic part of the Weyl tensor. It seems likely that the phenomenon of gravitational energy is subtle enough to be sensitive to this full set. Out of this set, in this section we have seen only the role of $\vec{b}$. To see whether or not the other components play a role [via $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$] will have to wait until a solution to the embedding equations is found.

VII. ASYMPTOTICALLY ANTI–DE SITTER SPACETIMES

In this last section we will explore the significance of the $\sigma\sigma R^{\rm ref}$ term in Eq. (4.3). Suppose our physical spacetime (M,g) is asymptotically anti–de Sitter space. The $\sigma\sigma R^{\rm ref}$ term in $\mathrm{IQE}^{\rm ref}$ gives us the freedom to specify the Riemann tensor of a reference spacetime, which in this case is naturally the Riemann tensor of anti–de Sitter space. Thus, according to Eq. (4.5) we have $\sigma\sigma R^{\rm ref}=-2/\ell^2$, and so

$$\mathrm{IQE}^{\rm ref}=-\frac{1}{8\pi}\int_S dS\,\sqrt{2\left[\frac{2}{\ell^2}+\mathcal{R}+(\tilde{k}^2-\tilde{l}^2)^{\rm ref}\right]}. \quad (7.1)$$

In the large sphere limit it is clear that the cosmological constant term will dominate, rather than $\mathcal{R}$, and the behavior of the IQE is qualitatively different from that for asymptotically flat spacetimes.

Let us now specialize to the case that (M,g) is the AdS_4-Schwarzschild spacetime, so that our main argument is not obscured by consideration of the shear terms, which will obviously be just zero. The line element in this case is given by

$$ds^2=-N^2\,dt^2+\frac{1}{f^2}dr^2+r^2\,d\Omega^2,$$

where

$$N(r)=f(r)=\left(\frac{r^2}{\ell^2}+1-\frac{2M}{r}\right)^{1/2}, \quad (7.2)$$

and $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$ is the line element on the unit round sphere. Let S be a t, $r=$constant two-sphere. Its scalar curvature is $\mathcal{R}=2/r^2$, and a simple calculation shows that its sectional curvature is given by

$$\sigma\sigma R=-\frac{2}{\ell^2}+\frac{4M}{r^3}, \quad (7.3)$$

the dominant term coming from the anti-de Sitter ''background.'' Substituting these results into Eqs. (4.2) and (7.1) we find

$$\mathrm{IQE}=-\frac{1}{8\pi}\int_S dS\,\sqrt{2\left[\frac{2}{\ell^2}+\frac{2}{r^2}-\frac{4M}{r^3}\right]}+\frac{1}{8\pi}\int_S dS\,\sqrt{2\left[\frac{2}{\ell^2}+\frac{2}{r^2}\right]}=\frac{M\ell}{r}+O\left(\frac{1}{r^3}\right). \quad (7.4)$$

The divergent terms due to the cosmological constant cancel, so the limit of the IQE as $r\to\infty$ exists, and this limit is zero. This would be the expected result if the IQE had the interpretation of an *energy*, which should be redshifted to zero by the cosmological horizon. In contrast, we do not expect an invariant mass to be redshifted. This is why the terminology invariant quasilocal energy was chosen rather than invariant quasilocal mass, even though the IQE is the analogue of the mass m in the formula: $m=\sqrt{E^2-\vec{p}^2}$.

However, one can easily modify the definition of the IQE—to give it the interpretation of mass—by multiplying the right-hand side of Eq. (3.4) by a lapse function. Thus one replaces Eq. (4.2) with

$$\mathrm{IQE}[N_{\mathcal{B}}]=-\frac{1}{8\pi}\int_S dS\,N_{\mathcal{B}}\sqrt{2[\mathcal{R}-\sigma\sigma R]}+\frac{1}{8\pi}\int_S dS\,N_{\mathcal{B}}^{\rm ref}\sqrt{2[\mathcal{R}-\sigma\sigma R^{\rm ref}]} \quad (7.5)$$

(ignoring the shear terms). Here the smearing function, $N_{\mathcal{B}}$, is the lapse function in the timelike three-boundary, $\mathcal{B}$, swept out by the two-parameter family of observers [cf. Eqs. (11) and (13) in Ref. [36]]. In the AdS_4–Schwarzschild example $\mathcal{B}$ is an $r=$const surface, and $N_{\mathcal{B}}=N$. The question arises, What are we to put for $N_{\mathcal{B}}^{\rm ref}$? The answer that works is $N_{\mathcal{B}}^{\rm ref}=N_{\mathcal{B}}$, which is intuitively justified as follows: we are already isometrically embedding (S,σ) into $(M^{\rm ref},g^{\rm ref})$, and the condition $N_{\mathcal{B}}^{\rm ref}=N_{\mathcal{B}}$ represents the next would-be step towards an isometric embedding of $(\mathcal{B},\gamma)$ into $(M^{\rm ref},g^{\rm ref})$, where γ_{ab} is the three-metric in $\mathcal{B}$. [''Would-be'' step in the sense that, while the lapse carries some information about γ_{ab}, we still only need to embed (S,σ) into $(M^{\rm ref},g^{\rm ref})$, not $(\mathcal{B},\gamma)$ into $(M^{\rm ref},g^{\rm ref})$.] By comparing Eq. (7.5) with (7.4), and using the fact that the lapse function goes as r/ℓ for large r, it is easy to see that with $N_{\mathcal{B}}^{\rm ref}=N_{\mathcal{B}}$ we get $\lim_{r\to\infty}\mathrm{IQE}[N_{\mathcal{B}}]=M$ for the AdS_4-Schwarzschild case. Thus $\mathrm{IQE}[N_{\mathcal{B}}]$ has the interpretation of a mass, as claimed. Unlike the original IQE, it is not redshifted to zero, and is thus a different physical quantity. Regarding the comment at the end of the previous paragraph, since special relativity does not know about lapse functions, the generalization given in Eq. (3.4) is open to this ambiguity: one can define both an invariant quasilocal energy and an invariant quasilocal mass.

It is instructive to evaluate $\mathrm{IQE}[N_{\mathcal{B}}]$ also at the horizon, $r=r_+$ [where $N_{\mathcal{B}}(r_+)=0$], and compare with what one gets using the unsmeared IQE. The following results for the AdS_4–Schwarzschild example are easily established:

$$\mathrm{IQE}=\begin{cases}\sqrt{2Mr_+} & \text{at } r=r_+,\\ 0 & \text{at } r=\infty,\end{cases} \quad (7.6)$$

$$\mathrm{IQE}[N_{\mathcal{B}}]=\begin{cases}0 & \text{at } r=r_+,\\ M & \text{at } r=\infty.\end{cases} \quad (7.7)$$

We thus learn that the IQE *decreases* with increasing r, which can be interpreted as the result of negative binding energy—another reason to think of the IQE as an energy. On the other hand, $\mathrm{IQE}[N_{\mathcal{B}}]$ *increases* with r, which might be interpreted as saying that, for larger r, more mass is enclosed; it starts from zero at the horizon (no mass in the interior of the black hole) and accumulates to M at infinity. This is reminiscent of the old notion that the substance of mass is nothing but the curvature of spacetime itself. A similar behavior is observed for the usual Schwarzschild case:

$$\mathrm{IQE}=\begin{cases} 2M & \text{at } r=2M, \\ M & \text{at } r=\infty, \end{cases} \tag{7.8}$$

$$\mathrm{IQE}[N_{\mathcal{B}}]=\begin{cases} 0 & \text{at } r=2M, \\ M & \text{at } r=\infty. \end{cases} \tag{7.9}$$

The only qualitative difference occurs at $r=\infty$, where $\mathrm{IQE}[N_{\mathcal{B}}]=\mathrm{IQE}$ in the Schwarzschild case because, of course, the lapse function goes to one in this limit. There is no cosmological horizon.

Despite these appealing features of $\mathrm{IQE}[N_{\mathcal{B}}]$, it is unsatisfactory from the point of view taken here because the presence of the lapse function means it depends on a choice of three-surface passing through S. The situation might be improved by replacing k with $N_{\mathcal{B}}k$ and l with $N_{\Sigma}l$ in Eq. (3.4), and then proceeding as before. Here $N_{\mathcal{B}}$ and N_{Σ} are time and radial lapse functions, respectively (equal to N and $1/f$ in the AdS_4-Schwarzschild example). Admittedly such a procedure is ad hoc, and unless it can be improved upon we are not particularly interested in $\mathrm{IQE}[N_{\mathcal{B}}]$. It was introduced simply to illustrate the distinction between mass and energy, but for the remainder of this section we will return to the original definition of the IQE.

The main point of this section is to draw attention to a remarkable similarity between the reference subtraction term given in Eq. (7.1), and a certain counterterm action recently suggested in the context of the conjectured AdS/CFT correspondence. We begin by observing that when the reference shear term vanishes, Eq. (7.1) reduces to precisely the same reference subtraction term suggested by Lau [37], in the context of the Brown-York quasilocal energy (except that Lau's expression has a lapse function present in the manner discussed above). However, our derivations of this expression are different. Lau employs a light cone reference embedding of (S,σ), together with a rest frame assumption, $l^{\mathrm{ref}}=0$, to derive an expression for k^{ref}, which is then used to construct his reference subtraction term. In our case we get the same end result, but we get it without any recourse to an explicit reference embedding. This is because the cosmological constant term in Eq. (7.1) comes from the direct dependence of $\mathrm{IQE}^{\mathrm{ref}}$ on the Riemann tensor of the reference spacetime, i.e., the term $\sigma\sigma R^{\mathrm{ref}}$ in Eq. (4.3). An explicit reference embedding of (S,σ) into $(M^{\mathrm{ref}},g^{\mathrm{ref}})$ is required only to evaluate the reference shear term, $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$. This higher order correction—which I have argued accounts for angular momentum—is not present in Lau's reference subtraction term. Also, his additional rest frame assumption is not required here because the IQE is already naturally a rest frame energy.

Let us now return to Eq. (1.5). We know that when space is noncompact the boundary (or quasilocal) stress-energy tensor $T^{ab}_{\mathcal{B}}=-\Pi^{ab}/(8\pi)$ diverges in general as $\mathcal{B}$ is taken to infinity.[19] To render it finite, Brown and York suggest the use of a reference subtraction term that involves an isometric embedding of $(\mathcal{B},\gamma)$ into a suitable reference spacetime. However, like their prescription to embed (S,σ) into a suitable three-dimensional reference space, this prescription suffers from the drawback that such a codimension-one embedding does not always exist. Recently, Balasubramanian and Kraus [36] have proposed an alternative procedure: Since it is always possible to add to the action a local functional of the intrinsic geometry of the boundary without affecting the equations of motion or the symmetries (but of course this alters $T^{ab}_{\mathcal{B}}$), their idea is to choose this functional such that its divergences cancel those of the original $T^{ab}_{\mathcal{B}}$, rendering the improved boundary stress-energy tensor finite as $\mathcal{B}$ is taken to infinity. No recourse is made to a reference embedding. This procedure was first applied to spacetimes that are asymptotically AdS space, in which case the required counterterms amounted to a simple finite polynomial in the curvature invariants of $\mathcal{B}$ [36]. This idea is exactly analogous to the standard prescription for removing ultraviolet divergences in quantum field theory by adding to the Lagrangian a finite polynomial in the fields. Moreover, the conjectured AdS/conformal field theory (CFT) correspondence [47] implies that the two procedures are not merely analogous, they are one and the same [36].

Now since flat spacetime is recovered from AdS space by taking ℓ to infinity, one might expect that in this same limit the counterterms found by Balasubramanian and Kraus would produce counterterms suitable for asymptotically flat spacetimes. It is not obvious that this is so [36]. However, Mann [48] has suggested the following generalization of their counterterm action:

$$I_{ct}=\frac{1}{8\pi}\int_{\mathcal{B}_\infty} d^3x\sqrt{-\gamma}\sqrt{2\left[\frac{2}{\ell^2}+R(\gamma)\right]}, \tag{7.10}$$

where $R(\gamma)$ is the scalar curvature of $(\mathcal{B},\gamma)$, and $\mathcal{B}_\infty$ indicates that we are to take the limit as $\mathcal{B}$ goes to infinity. For small ℓ Mann's formula reduces to the one given by Balasubramanian and Kraus, but in addition it has a smooth flat spacetime limit as $\ell\to\infty$. Moreover, Mann showed that in many explicit examples it leads to a cancellation of all divergences, and the remaining finite part agrees with that obtained using the reference spacetime procedure [48,49]. While a counterterm action and a reference energy are not the same thing, the resemblance between the expressions in Eqs. (7.10) and (7.1) is nevertheless striking.[20]

To see that the connection between $\mathrm{IQE}^{\mathrm{ref}}$ in Eq. (7.1) and the AdS/CFT-inspired counterterm action is probably much

[19]As elsewhere in this paper, we use the symbol $\mathcal{B}$ loosely to refer to either a timelike three-surface in the interior of M, bounding a finite spatial region, or the boundary at infinity. The meaning should be clear from the context in which it is used.

[20]I am indebted to R. B. Mann for pointing out to me the significance of the $\sigma\sigma R^{\mathrm{ref}}$ term in $\mathrm{IQE}^{\mathrm{ref}}$, and emphasizing that it provides at least some measure of geometrical motivation for his expression in Eq. (7.10).

deeper than mere resemblance, we now turn to recent work done by Kraus *et al.* [50]. Besides providing an independent derivation of Mann's formula, and its generalization to higher dimensions, of most interest to us here is their geometrical argument suggesting what the counterterm for Π^{ab} in Eq. (1.5) should be in order to cancel divergences. Their result is an expansion in powers of ℓ. Denoting their counterterm ($\widetilde{\Pi}_{ab}$) as $\Pi^{\rm ct}_{ab}$, and specializing their result to a three-dimensional boundary $\mathcal{B}$, they find

$$\Pi^{\rm ct}_{ab}=-\frac{2}{\ell}\gamma_{ab}+\ell(R_{ab}-\tfrac{1}{2}\gamma_{ab}R)+\ell^3\{\tfrac{1}{2}\gamma_{ab}(R_{cd}R^{cd}-\tfrac{3}{8}R^2)+\tfrac{3}{4}RR_{ab}-2R^{cd}R_{acbd}+\tfrac{1}{4}D_aD_bR-\Box R_{ab}+\tfrac{1}{4}\gamma_{ab}\Box R\}+O(\ell^5). \quad (7.11)$$

The curvatures and covariant derivatives in this expression all refer to the induced timelike three-metric γ_{ab} on $\mathcal{B}$.

Now consider our usual two-surface (S,σ) in the physical spacetime (M, g), the latter an asymptotically AdS space. As we did at the end of Sec. IV, suppose that S is such that $(k^2-l^2)>0$ and $k>0$. Then we can always find a timelike unit vector u^a normal to S such that $l=0$, and so $\sqrt{k^2-l^2}=k=\Pi_{ab}u^au^b$ is the Brown-York energy surface density [modulo the factor of $-1/(8\pi)$]. In other words, our unreferenced IQE reduces to the unreferenced Brown-York CQE, which would be called the unrenormalized energy in Ref. [50]. The counterterm required to renormalize the energy surface density is thus $\Pi^{\rm ct}_{ab}u^au^b$, which we will denote at $E_{\rm ct}$. Hence, our task is to compare $E_{\rm ct}:=\Pi^{\rm ct}_{ab}u^au^b$ with the integrand of IQE$^{\rm ref}$ in Eq. (7.1); we expect to see at least some measure of agreement between the two.

This comparison will not be straightforward, however, because on the one hand we expect the integrand of IQE$^{\rm ref}$ to depend on $\mathcal{R}$, $\mathcal{F}$, and their derivatives in S, as discussed previously, whereas on the other hand $E_{\rm ct}$ depends on the three-metric γ_{ab}. Nevertheless, let us see how far we can get. Let $\mathcal{B}$ be a three-surface in (M,g) passing through S in a direction tangent to u^a on S. Different choices of $\mathcal{B}$ satisfying these conditions will lead to different induced metrics γ_{ab}, but this ambiguity will not affect our considerations. At least γ_{ab} on S is uniquely determined, and some information about γ_{ab} in the neighborhood of S is determined by the condition $l=0$. Our choice of $\mathcal{B}$ means that l_{ab} defined in Eqs. (2.8) is the extrinsic curvature of (S,σ) as embedded in $(\mathcal{B},\gamma)$, and so the corresponding codimension-one Gauss embedding equation reads

$$\mathcal{P}^e_a\mathcal{P}^f_b\mathcal{P}^g_c\mathcal{P}^h_dR_{efgh}=\mathcal{R}_{abcd}+(l_{ac}l_{bd}-l_{bc}l_{ad}). \quad (7.12)$$

This is just a truncated version of Eq. (2.14), except here R_{efgh} is the Riemann tensor of $(\mathcal{B},\gamma)$, not (M,g).

Now let $E^{(n)}_{\rm ct}$ denote the term in $E_{\rm ct}$ of order ℓ^n. Inspection of Eq. (7.11) shows that $E^{(-1)}_{\rm ct}=2\ell$. The term $E^{(1)}_{\rm ct}$ can be written in terms of G_{ab}, the Einstein tensor of $(\mathcal{B},\gamma)$, and we have

$$E^{(1)}_{\rm ct}=\ell G_{ab}u^au^b=\frac{\ell}{2}\sigma^{ac}\sigma^{bd}R_{abcd}=\frac{\ell}{2}(\mathcal{R}-\tilde{l}^2). \quad (7.13)$$

The second equality is an easily derived identity (valid in any codimension-one setting) relating the uu component of the Einstein tensor to the sectional curvature of the hypersurface orthogonal to u^a (in this case the hypersurface S in $\mathcal{B}$). The third equality follows from contracting Eq. (7.12) with $\sigma^{ac}\sigma^{bd}$, and using the fact that $l=0$ by our choice of u^a (and as usual, $\tilde{l}^2$ is shorthand for $\tilde{l}_{ab}\tilde{l}^{ab}$). Thus, to order l we have

$$E_{\rm ct}=\frac{2}{\ell}+\frac{\ell}{2}(\mathcal{R}-\tilde{l}^2)+O(\ell^3)=\sqrt{2\left[\frac{2}{\ell^2}+\mathcal{R}-\tilde{l}^2+O(\ell^2)\right]}. \quad (7.14)$$

Comparing the last expression with Eq. (7.1) suggests the correspondence:

$$(\tilde{k}^2-\tilde{l}^2)^{\rm ref}\leftrightarrow-\tilde{l}^2+O(\ell^2). \quad (7.15)$$

Immediately we see something odd: we are identifying a boost invariant quantity with one that is not, i.e., it seems that a $\tilde{k}^2$ is missing from the right-hand side. I will comment on this shortly. Let us assume for the moment that the right-hand side reads $(\tilde{k}^2-\tilde{l}^2)+O(\tilde{\ell}^2)$, in which case Eq. (7.15) seems reasonable: it suggests that, if we solve the embedding equations (4.6)–(4.8) for $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$ we will find that, to lowest order in ℓ, the reference shear term is the same as the unreferenced shear term, the difference to be seen at a higher order in ℓ. On the other hand, this seems like a problem: Would it not mean, e.g., that the shear terms in Eq. (5.5) basically cancel, thus ruining the Bondi-Sachs mass result in Eq. (5.35), which depends so crucially on $(\tilde{k}^2-\tilde{l}^2)$? The answer is No, because Eq. (5.5) is valid in the asymptotically flat case, not the asymptotically AdS case. To make a statement that is valid in the asymptotically flat case ($\ell\rightarrow\infty$) we need to know $E_{\rm ct}$ to all orders in ℓ, then sum the infinite series, and finally take the limit $\ell\rightarrow\infty$. So being at the other end of the series, Eq. (7.15) has nothing to say about the asymptotically flat case. But we also expected $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$ to depend on $\mathcal{R}$, $\mathcal{F}$, and their derivatives. Why do we not see these quantities on the right-hand side of Eq. (7.15)? The answer is, we will—we just have to calculate $E_{\rm ct}$ to the next order in ℓ.

But before doing so I will comment on the missing $\tilde{k}^2$ in Eq. (7.15). Kraus *et al.* [50] have devised an algorithm to compute the extrinsic geometrical quantity $\Pi^{\rm ct}_{ab}$ from the intrinsic geometry of $\mathcal{B}$. Insofar as l_{ab} (and thus $\tilde{l}^2$) depends only on the metric γ_{ab}, there is no doubt that the $-\tilde{l}^2$ term in Eq. (7.15) is correct. On the other hand, k_{ab} (and thus $\tilde{k}^2$) depends on the extrinsic geometry of $\mathcal{B}$, being just a certain projection of Π_{ab} into S. The algorithm of Kraus *et al.* relies on the fact [51] that the *divergent* part of the derivative of Π_{ab} in the direction normal to $\mathcal{B}$ can be expressed in terms of just the intrinsic geometry of $\mathcal{B}$. In essence, their algorithm is designed precisely to compute the *divergent* part of Π_{ab}. The correctness of the accompanying finite part is a subtle issue. For example, in a slightly different context they discuss two different counterterm actions that both properly cancel divergences, but that lead to different finite terms in the action. Furthermore, they point out that their algorithm, when carried to all orders in ℓ, might imply singularities in the bulk spacetime, but that this is of no concern because they truncate their counterterm expressions to a finite number of terms, enough at least to cancel the divergences. In our case we have the quasilocal idea in mind, and so are interested in *all* of the finite terms—it matters what happens in the bulk. But going further with this discussion will take us beyond the scope set for this simple comparison. I will just conclude by saying that, insofar as the shear terms almost certainly represent a finite contribution to the energy, we do not *necessarily* expect the algorithm of Kraus *et al.* to produce a $\tilde{k}^2$ term on the right-hand side of Eq. (7.15). Our goals are slightly different, and it is too much to expect exact agreement between $E_{\rm ct}$ and the integrand of IQE$^{\rm ref}$.

Nevertheless, it is still instructive to proceed with the comparison to the next order in ℓ. In light of my previous remarks, we will make the simplifying assumption that the metric on $\mathcal{B}$ has a product structure: $\gamma_{ab}\,dx^a\,dx^b = -N^2\,dt^2 + \sigma_{ij}(x)dx^i\,dx^j$, where $x^a=(t,x^i)$ are local coordinates on $\mathcal{B}$, N is a constant lapse, and $\sigma_{ij}(x)$ is the metric on any $t=$const two-surface S. The idea is that $E_{\rm ct}$ and the integrand of IQE$^{\rm ref}$ should agree *at least* in their dependence on the intrinsic geometry of (S,σ). Assuming such a product structure for γ_{ab} is a convenient was to isolate this dependence, and ignore everything else.

In this case the only nonvanishing components of the Riemann tensor of γ_{ab} are $R_{ijkl}=\mathcal{R}_{ijkl}$, the Riemann tensor of σ_{ij}. And clearly $l_{ab}=0$. Thus $E^{(1)}_{\rm ct}$ in Eq. (7.13) reduces to $\ell\mathcal{R}/2$, and it is a simple exercise to work out $E^{(3)}_{\rm ct}$. The net result is

$$E_{\rm ct}=\frac{2}{\ell}+\frac{\ell}{2}\mathcal{R}-\frac{\ell^3}{16}(\mathcal{R}^2-4\Delta\mathcal{R})+O(\ell^5)$$
$$=\sqrt{2\left[\frac{2}{\ell^2}+\mathcal{R}-\frac{\ell^2}{2}\Delta\mathcal{R}+O(\ell^4)\right]}, \qquad (7.16)$$

where Δ is the Laplacian in (S,σ). Comparing the last expression with Eq. (7.1) we now have the higher order correspondence:

$$(\tilde{k}^2-\tilde{l}^2)^{\rm ref}\leftrightarrow -\frac{\ell^2}{2}\Delta\mathcal{R}+O(\ell^4), \qquad (7.17)$$

Thus we begin to see how a solution to our embedding equations might yield an expression for $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$ in terms of $\mathcal{R}$, $\mathcal{F}$, and their derivatives, as we have expected all along.

To conclude this section we make two general observations. First, it is especially clear from the higher order expression in Eq. (7.16) that the AdS/CFT-inspired counterterm energy is, in fact, the square root of some quantity. This is not surprising, since the algorithm of Kraus *et al.* [50] is a means of solving a Gauss embedding equation for $\Pi^{\rm ct}_{ab}$, and this equation is quadratic in $\Pi^{\rm ct}_{ab}$. But it is significant. Beginning simply with the definition of the quasilocal stress-energy tensor as the functional derivative of the action with respect to the boundary metric [7], in which there is no square root in sight, the counterterm energy required to cancel divergences unmistakably involves a square root. Moreover, it concurs with the square root introduced here, in the context of the IQE, as the general relativistic analogue of the special relativistic formula: $m=\sqrt{E^2-\vec{p}^2}$. I believe it is unlikely this is a mere coincidence. Given that it is nonanalytic, a square root is too unusual an object to occur without good reason.

Second, under the square root (in our case) is $2/\ell^2+\mathcal{R}+(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$. In the case of the AdS/CFT-inspired counterterm energy [50], it is $2/\ell^2+\mathcal{R}+X$, where X is an infinite series in increasing powers of ℓ. That X is clearly not zero lends strong support for our additional term $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$, which is thus seen to be a necessary generalization of Lau's suggestion [37]. I have argued that its necessity is closely linked to the proper inclusion of angular momentum in the energy. Given that angular momentum is a subtle notion in general relativity, especially so at the quasilocal level we envision here, it is not surprising that our biggest difficulty lies in evaluating $(\tilde{k}^2-\tilde{l}^2)^{\rm ref}$. In light of the algorithm given by Kraus *et al.* [50], work is currently in progress to try to apply similar techniques to solve the embedding equations (4.6)–(4.8). Since these embedding equations are manifestly boost invariant, I expect at least to recover the missing $\tilde{k}^2$ term in Eq. (7.15), and hopefully the entire series.

VIII. SUMMARY AND DISCUSSION

In this paper I have introduced a new definition of quasilocal energy that is a simple modification of the Brown-York quasilocal energy. I just replace their energy surface density k with $\sqrt{k^2-l^2}$, where l is the radial momentum surface density. [For ease of exposition here I will omit the $-1/(8\pi)$ factors.] The principle motivation for doing this stems from an analogy with the formula: $m=\sqrt{E^2-\vec{p}^2}$ in special relativity. Identifying E with k (which are both energies), and $\vec{p}$ with l (both momenta), identifies m with $\sqrt{k^2-l^2}$. Like m, $\sqrt{k^2-l^2}$ is a boost invariant quantity, and hence the integral of $\sqrt{k^2-l^2}$ over a spacelike two-surface S gives rise to an invariant quasilocal energy, or IQE. In what follows I will refer to the Brown-York quasilocal energy as the CQE—canonical quasilocal energy.

There are several important consequences of replacing k with $\sqrt{k^2-l^2}$:

(1) While k is always well defined for any spacelike two-surface S, $\sqrt{k^2-l^2}$ is not. Roughly speaking, it is real when S lies in the exterior region of a black hole, zero when it is on the horizon, and imaginary in the black hole interior. Thus (again roughly speaking) the IQE asserts that energy is real only outside of a black hole.

(2) Both the CQE and the IQE require a reference energy subtraction procedure. Since k is associated with a spacelike

three-surface spanning S, the reference space into which S is to be isometrically embedded is inherently three-dimensional. Such a codimension-one embedding does not always exist, but when it does, it is essentially unique. This means the CQE, when it is defined, is unique. In contrast, $\sqrt{k^2-l^2}$ makes no reference to a three-surface spanning S, and so the reference space(time) is inherently four dimensional. Such codimension-two embeddings (at least of a generic nonround sphere into Minkowski space) always exist [35], but are not unique. However, in this situation there are two curvatures associated with S: its scalar curvature $\mathcal{R}$, and the curvature of its normal bundle, $\mathcal{F}$. A necessary condition for an isometric embedding is that $\mathcal{R}^{\mathrm{ref}}=\mathcal{R}$. I argued that demanding also $\mathcal{F}^{\mathrm{ref}}=\mathcal{F}$ is both a means to make the embedding essentially unique, and at the same time, a geometrically natural way to properly incorporate angular momentum into energy at the quasilocal level. Indeed, since angular momentum is associated with rotational kinetic energy, it *should* contribute to the energy in some way.

(3) While $\mathrm{CQE}^{\mathrm{ref}}$ is associated with a reference *energy* density k^{ref}, $\mathrm{IQE}^{\mathrm{ref}}$ is concerned with a reference *shear* term $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$. ($\tilde{k}_{ab}$ and $\tilde{l}_{ab}$ are the trace-free parts of the two extrinsic curvatures of S.) In a certain sense, the IQE already inherently contains the correct reference energy, without recourse to a reference embedding. The reference embedding is required only to determine the reference shear term, which is a higher order correction to the energy associated with angular momentum.

(4) The CQE is sensitive to the sign of k, whereas since it involves $\sqrt{k^2-l^2}$, the IQE is not. Thus one can easily construct simple examples for which the two energies give different results, even when $l=0$. Thus the IQE is not simply the rest energy version of the CQE. Note: the IQE naturally assigns zero energy to *any* two-surface in flat spacetime. This is because the natural reference spacetime in this case is the very same spacetime, namely flat spacetime. So obviously one can always reference-embed the two-surface identically (up to Poincaré transformations) to the way it is embedded in the physical spacetime, and get IQE=0. The only subtlety that may arise is if the two conditions: isometric embedding and $\mathcal{F}^{\mathrm{ref}}=\mathcal{F}$ do not uniquely determine $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$. Then the flat spacetime result (IQE=0) may be reduced to a choice, rather than a necessary fact. To properly address this subtlety requires an in depth understanding of the embedding equations. But in any case, the fact that IQE=0 in flat spacetime is independent of the motion of the observers. In contrast, moving observers in flat spacetime could measure nonzero energy in the Brown-York approach [31]. This is because under a radial boost the Brown-York energy surface density dilates by a Lorentz factor, as in special relativity, whereas the reference energy surface density does not. According to Ref. [7] the latter depends only on the intrinsic geometry of S, and therefore does not know about the time derivative of this geometry.[21]

We examined both the large and small sphere limits of the IQE, taking S to be asymptotically round for simplicity. In an asymptotically flat spacetime, the large sphere limit of the IQE in a spatial direction yields the ADM mass. In the future null direction it reduces to the Bondi-Sachs mass, provided the reference shear term is a total divergence. Short of solving the embedding equations, I gave a heuristic argument which shows that is. It is significant that this argument relies on the condition $\mathcal{F}^{\mathrm{ref}}=\mathcal{F}$, since this provides evidence that the curvature of the normal bundle is involved in quasilocal energy, albeit its involvement in this simple example is minimal.

The quantity $\sqrt{k^2-l^2}$ is proportional to the mean curvature of S as a two-surface embedded in the physical spacetime, and so the IQE is a natural geometrical invariant of S. Since the Hawking mass [25] is constructed using (k^2-l^2), the IQE can be thought of roughly as the square root of the Hawking mass. In the small sphere limit the square root disappears, and to leading order the IQE reduces to the Hawking mass (but differs from it at higher order). Thus, when matter is present, the lowest order contribution to the IQE gives the standard result: $(4\pi r^3/3)T^{\mathrm{mat}}_{ab}u^a u^b$, i.e., the expected matter energy contained in a small sphere of proper radius r. Note that u^a here is not necessarily the four-velocity of any observer on S, since the IQE is boost invariant, and so independent of the observers' velocities on S. Rather, u^a is the four-velocity that observers would have *if* they were in the rest frame determined by S. More precisely, in the small sphere limit we considered, namely a $t,r=$const two-sphere in Riemann normal coordinates (with $t\propto r$), $u^a=(\partial/\partial t)^a$ evaluated at the center of the sphere. In the limit $r\to 0$, the four-velocity $(\partial/\partial t)^a$ corresponds to observers who at each point on S have zero radial momentum, i.e., $l=0$. In general, since the IQE is an energy rather than a mass, the question arises, In whose rest frame is the energy measured?[22] The answer is, The quasilocal rest frame determined by the condition $l=0$ at each point on S. Whenever $(k^2-l^2)>0$, observers on S can always achieve this state of motion by appropriate local radial boosts. This [or more precisely, $(k^2-l^2)\geqslant 0$] is the same condition required for the unreferenced IQE to be well defined in the first place—refer to the paragraph numbered (1) above.

Returning to the small sphere case, in vacuo the leading order contribution due to gravitational energy occurs at order r^5. At this order the IQE results are inconclusive because it is expected that the reference shear term will play a significant role, and without a solution to the embedding equations (which is an extremely difficult problem) this term cannot be determined. Nevertheless, it was possible to show that in the small sphere limit, the Hawking mass, which in this case is closely related to the IQE, can be understood as a measure of the gravitational energy contained in S by considering certain tangential gravitoelectromagnetic fields $\vec{E}$ and $\vec{B}$ induced on S by the Weyl curvature S is embedded in. In terms of $\vec{E}$ and $\vec{B}$, gravitational energy and radiation are essentially identical

[21] Thanks to R. B. Mann and I. S. Booth for this remark on the Brown-York case.

[22] I thank A. Ashtekar for posing this question.

in nature to their counterparts in electromagnetism, except for one crucial difference: the density $r(\vec{E}\cdot\vec{E}+\vec{B}\cdot\vec{B})/(8\pi)$ is integrated over the *surface* S to determine the gravitational energy contained in the spatial volume that S encloses (which necessitates the additional factor of areal radius r). Notice that this measurement of gravitational energy in a volume is truly quasilocal, taking place on the surface of the volume, S.

The IQE was analyzed in the context of asymptotically anti–de Sitter spacetimes. The fact that $\mathrm{IQE}^{\mathrm{ref}}$ depends explicitly on the Riemann tensor of the reference spacetime (naturally taken to be anti–de Sitter space) was seen to play a significant role. A connection was established between $\mathrm{IQE}^{\mathrm{ref}}$ and a certain counterterm energy that has recently been proposed [50] in the context of the conjectured AdS/CFT correspondence. Two similarities are striking: (i) Both energies involve a square root, and (ii) the two leading terms under the square root match. The remaining term under the square root in our case is the reference shear term, $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$; in the case of Ref. [50] it is an infinite series in increasing powers of ℓ, where ℓ is the radius of curvature of the AdS space. It was shown that the first two nontrivial terms of this series (i.e., to the highest order given in Ref. [50]) can plausibly be identified with $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$. This agreement is impressive because $\mathrm{IQE}^{\mathrm{ref}}$ and the AdS/CFT-inspired counterterm energy are independently motivated, and derived quite differently. It might be possible to use techniques developed in Ref. [50] to solve our embedding equations for $(\tilde{k}^2-\tilde{l}^2)^{\mathrm{ref}}$. The present lack of a solution to these equations is the main outstanding obstacle to further understanding the nature of the IQE.

A final remark is in order. Most definitions of quasilocal energy, including the IQE, assume that energy is associated with a closed spacelike two-surface, S. Given such a two-surface one can always find a timelike unit normal vector field u^a, which at each point on S is supposed to correspond to an observer's instantaneous four-velocity. But this may not be a general enough setting. While a two-parameter family of observers will always sweep out a timelike three-surface $\mathcal{B}$, the two-surface elements orthogonal to their world lines in $\mathcal{B}$ are not, in general, integrable. Thus a shift in emphasis from S to $\mathcal{B}$, i.e., from Eulerian to Lorentzian observers [28], might lead to a deeper understanding of quasilocal energy, in particular of gravitational radiation at the quasilocal level. This shift would also bring the quasilocal energy idea closer in line with the conjectured AdS/CFT correspondence. Whether or not this is the right direction, the results in Sec. VII strongly suggest that this is the direction the IQE is pointing in.

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